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Bohm Aharonov effects for bounded states in the case of systems

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ABSTRACT. – We study the comparison problem for the eigenvalues of the covariant Laplacian with electric potential acting on the sections of vector bundle with structure group $\mathcal{U}(m)$.

RÉSUMÉ. – On s'intéresse dans cet article à un problème de comparaison de valeurs propres pour le Laplacien covariant, avec potentiel électrique, agissant sur les sections d'un fibré vectoriel de groupe structural $\mathcal{U}(m)$ $(m \in \mathrm{N}^*)$.

INTRODUCTION

Let (M, g) be an n-dimensional connected orientable Riemannian manifold with (possibly empty) boundary ∂M, $(E, (\ ,\ ))$ be a Hermitian (C^∞) bundle over M with rank m. We denote by $A^0(M, E) = C^\infty(M, E)$ the set of C^∞ sections of E. More generally we denote by $A^p(M)$ the set of the $C^\infty - p$ forms on M and by $A^p(M, E)$ the set of $E-$valued $C^\infty - p$ forms on M.

As usual, we put

$$A_0^p(M, E) = \{ \Theta \in A^p(M, E) : \operatorname{supp} \Theta \subset \operatorname{int}(M) = M \setminus \partial M \}$$

and we introduce on $A_0^0(M, E)$, $A_0^1(M, E)$ the inner products $[\ ,\ ]_0$, $[\ ,\ ]_1$ defined by:

$$[\xi, \xi']_0 = \int_M (\xi, \xi')(x)\, dv, \qquad \text{for} \quad \xi, \xi' \in A_0^0(M, E),$$

$$[\Theta, \Theta']_1 = \int_M \langle \Theta, \Theta' \rangle (x)\, dv, \qquad \text{for} \quad \Theta, \Theta' \in A_0^1(M, E).$$

where $\langle\ ,\ \rangle$, dv denote the natural metric in $T^*M \otimes E$ induced by g and the Riemannian volume element, respectively. Let $\nabla : C^\infty(M, E) \to A^1(M, E)$ be a connection on E, compatible with the Hermitian structure (*cf.* [13]). The dual operator

$$\nabla^* : A_0^1(M, E) \to C_0^\infty(M, E)$$

of $\nabla_{|C_0^\infty}(M, E)$ is defined by:

$$\forall \Theta \in A_0^1(M, E), \quad [\nabla^* \Theta, \xi]_0 = [\Theta, \nabla \xi]_1 \ \forall \xi \in C_0^\infty(M, E). \qquad (0.1)$$

We consider a positive C^∞ function V on M and we introduce the two following positive formally self-adjoint elliptic operators $H_{\nabla, V}$, H_V defined by:

$$\operatorname{Dom}(H_{\nabla, V}) = C_0^\infty(M, E), \qquad H_{\nabla, V} = \nabla^* . \nabla + V,$$
$$\operatorname{Dom}(H_V) = C_0^\infty(M), \qquad H_V = d^* . d + V.$$

In the case $\partial M \neq \varnothing$, the Bochner-Laplace (resp. Laplace) operator $H_{\nabla, V}^{M, E}$ (resp. H_V^M) is the Dirichlet realization for M in the completion $L^2(M, E)$ [resp. $L^2(M)$] of the pre-Hilbert space

$$(A_0^0(M, E), [\ ,\ ]_0) \text{ (resp. } C_0^\infty(M)$$

with the usual scalar product). If $\partial M = \varnothing$, we denoted by $H_{\nabla, V}^{M, E}$, H_V^M the unique self-adjoint extension (the closure) [7] of operators $H_{\nabla, V}$, H_V in the space $L^2(M, E)$ and $L^2(M)$, respectively. The problem we want to address in this work is, assuming to simplify $H_{\nabla, V}^{M, E}$ and H_V^M with compact resolvent, is the following:

Under which conditions on E and ∇ do the operators $H_{\nabla, V}^{M, E}$, H_V^M admit the same first eigenvalue or more generally the same spectrum.

We shall consider two cases:

Case I. – $E = M \times \mathbb{C}^m$ and M satisfies one of the following properties:

(P1) M is compact

(P2) M is the closure of an open set (possibly unbounded) Q of $\mathbb{R}^n$ with regular bounded boundary ∂Q,

(P3) $M = \mathbb{R}^n$.

We assume, in the case when M is not compact, that the electric potential V verifies:

$$V(x) \to +\infty \quad \text{as } |x| \to +\infty. \tag{0.2}$$

Case II. – E is not necessarily trivial but M is compact.

It is well known ([10], [11], ...) that if M is compact, the spectra of $H^{M, E}_{\nabla, V}$ and H^{M}_{V} are increasing sequences of positive eigenvalues tending to $+\infty$. When M is not compact, this follows from the condition (0.2) (*See* [11] for H^{M}_{V}; and Theorem 2.3 of [6], Theorem 1.2 of [1] for the operator $H^{M, E}_{\nabla, V}$ with $E = M \times \mathbb{C}^m$). As we shall see, the comparison problem for the spectra of two such operators is naturally related to the gauge transformations. In section 2 of this work, we discuss briefly this idea and we give a characterization for the trivial connections. We present in section 2 comparison theorems for the **case I** generalizing results obtained by Helffer [5], Shigekawa [12] in the scalar case and Manabe-Shigekawa [10] in the case of systems. We study the **case II** in section 4 and we give a theorem extending results of Kuwabara [8].

I would like to thank my adviser Bernard Helffer who suggested me this study.

1. GAUGE TRANSFORMATIONS AND TRIVIAL CONNECTIONS

Let $e_B = (e^1_B, \ldots, e^m_B)$ be a local orthonormal frame over an open set B of M, *i.e.*, $e^i \in C^\infty(B, E_{|B})$ for $1 \leqq i \leqq m$ such that $(e^i_B(x))_{1 \leqq i \leqq m}$ is an orthonormal basis of a fibre E_x for each $x \in B$. Then,

$$\nabla e^i_B = \Sigma_s \, \omega_{si} \otimes e^s, \qquad \text{where} \quad \omega_{si} \in A^1(B) \quad \text{for} \quad 1 \leqq i, s \leqq m. \tag{1.1}$$

We call the matrix 1-form $\omega = [\omega_{is}]_{i, s}$ the connection form of ∇ with respect to the frame e_B. Because ∇ is compatible with the metric $(\ ,\ )_E$, ω takes values in the Lie algebra $\mathscr{M}_{m, a}$ of the unitary group $\mathscr{U}(m)$. Let $\xi \in A^0(M, E)$ and $\xi_{B_i} = {}^t(\xi_1, \ldots, \xi_m)$ be the (local) trivialization of ξ with respect to e_B (defined by $\xi_{|B} = \Sigma_i \, \xi_i \, e^i_B$).

If $f_B = (f^1_B, \ldots, f^m_B)$ is another orthonormal frame over B and if $T = [t_{il}]$ is the $\mathscr{U}(m)$-valued function on B such that: $f^i_B = \Sigma_s \, t_{si} \, e^s_B$, or in matrix-notations $f_B = e_B . T$, then, the connection form ω' of ∇ and the trivialization ξ'_B of ξ with respect to f_B are given by:

$$\omega' = T^* \omega T + T^* dT, \tag{1.2}$$

$$\xi'_B = T^* \xi_B. \tag{1.3}$$

Transformations of the form (1.2) and (1.3) are called (local) gauge transformations. If E is trivializable and if e_M, f_M are (global) frames of E over M, then, for $\xi \in A^0_0(M, E)$, we have (with the notations of (2.2) and

(2.3)):

$$H_{\omega, V}(\xi_M) = (T \,.\, H_{\omega', V} \,.\, T^*)(\xi'_M), \tag{1.4}$$

where $T \in C^\infty(M, \mathcal{U}(m))$ and $H_{\omega, V} = (d+\omega)^* \,.\, (d+\omega) + V \otimes 1$ is the representation of $H_{\nabla, V}$ with respect to the frame e_M. Consequently, in the **case I**, $H^{M, E}_{\nabla, V}$ is nothing but a Schrödinger operator $H^M_{\omega, V}$ with magnetic potential $\omega \in A^1(M, \mathcal{M}_{m, a})$.

Properties (2.4) and (2.2) say that the operators $H^M_{\omega, V}$ and $H^M_V \otimes 1$ are unitary equivalent if there exists $S \in C^\infty(M, \mathcal{U}(m))$ such that $dS = \omega \,.\, S$ on M. A such form ω is called trivial.

Our problem is now to find caracterizations of such forms. Let $\omega \in A^1(M, \mathcal{M}_{m, a})$. We call ω flat if its curvature $K(\omega) = d\omega + \omega \Lambda \omega$ vanishes. It is easy to see that a trivial 1-form ω is flat. Let $\gamma : [0, 1] \to M$ be a closed curve in M, $\gamma^*(\omega) = A_{\gamma, \omega}(t)\, dt$ be the pull-back of ω by γ, and consider the associated system of differential equations:

$$\psi' = \psi \,.\, A_{\gamma, \omega}, \qquad \Psi(0) = I_m. \tag{1.5}$$

It is well known (*See* for example [2]) that a system (1.5) has a unique solution g in $C^1([0, 1], \mathcal{U}(m))$. Let us define the holonomy class of ω with respect to γ by:

$$U_\gamma(\omega) = \{ U \in \mathcal{U}(m) \text{ such that: } U \text{ and } g(1) \text{ are unitary equivalent} \}.$$

For example, we have for a closed 1-form ω in $A^1(M, \mathcal{M}_{1, a})$:

$$U_\gamma(\omega) = \left\{ \exp\left(\int_\gamma \omega \right) \right\}.$$

One can verify (*See* [4]) that, if ω is flat, then $U_\gamma(\omega)$ depends only on the homotopy class of γ and that for $T \in C^1(M, \mathcal{U}(m))$, $\omega_T = T^* \,.\, \omega \,.\, T + T^* \,.\, dT$; we have:

$$K(\omega_T) = T^* \,.\, K(\omega) \,.\, T = 0, \tag{1.6}$$

$$U_\gamma(\omega_T) = U_\gamma(\omega). \tag{1.7}$$

The following theorem is probably classical (*See* [4])

THEOREM 1.1. – *For* $\omega \in A^1(M, \mathcal{M}_{m, a})$. *The following conditions* (i) *and* (ii) *are equivalent:*

(*a*) ω *is trivial,*

(*ii*) (*a*): ω *is flat,* (*b*): $U_\gamma(\omega) = \{ I_m \}$, *for each closed curve* γ *in* M.

COROLLARY 1.2. – *If* M *is simply connected. Then,* ω *is trivial if and only if it is flat.*

Let us look at the more general case of connections and consider a system $(B_\alpha, e_\alpha)_{\alpha \in I}$ of local trivializations of E, *i.e.*, $(B_\alpha)_\alpha$ is an open connected cover of M and e_α is an orthonormal frame over B_α for each $\alpha \in I$. For $B_{\alpha\beta} = B_\alpha \cap B_\beta \neq \varnothing$, the $\mathcal{U}(m)$-valued functions $g_{\alpha\beta}$ on $B_{\alpha\beta}$ such that

$e_\beta = e_\alpha g_{\alpha\beta}$ are called transition functions. If ω_α is the connection form of ∇ with respect to e_α, $K(\omega_\alpha)$ is called the curvature form of ∇ with respect to e_α. By (2.2), we have:

$$\omega_\beta = g^*_{\alpha\beta} \cdot \omega_\alpha \cdot g_{\alpha\beta} + g^*_{\alpha\beta} \cdot dg_{\alpha\beta}, \tag{1.8}$$

$$K(\omega_\beta) = g^*_{\alpha\beta} \cdot K(\omega_\alpha) \cdot g_{\alpha\beta} \quad \text{on } B_{\alpha\beta}. \tag{1.9}$$

The property (1.9) says that the condition, $K(\omega_\alpha) = 0$ for each $\alpha \in I$, depends only on the connection ∇. Connections which satisfy this condition are called flats. We say that ∇ is trivial if there exist a system of local trivializations $(B_\alpha, f_\alpha)_{\alpha \in I}$ of E such that the corresponding transition functions (resp. connection forms) $g'_{\alpha\beta}$ (resp. ω'_α) are all identity functions (resp. zero forms). As a necessary condition, E is trivializable and ∇ is flat. We start from these conditions and we consider the connection form ω of ∇ with respect to a given global frame e_M of E. It is clear, by (1.6) and (1.9), that for a closed curve γ in M, the class $U_\gamma(\omega)$ is independent of a choice of e_M. We define the holonomy class of ∇ with respect to γ by: $U_\gamma(\nabla) = U_\gamma(\omega)$. We can then state Theorem 1.1 as follows:

THEOREM 1.1. – *Suppose that* E *is trivializable. Then, the following conditions are equivalent:*

(i) ∇ *is trivial,*

(ii) *(a):* ∇ *is flat, (b):* $U_\gamma(\nabla) = \{I_m\}$, *for each closed curve* γ *in* M.

REMARK 1.3. – Let ∇ be a flat connection on E (unnecessarily trivializable). Using the fact that a flat connection is locally trivial, we construct in [4] a holonomy class $U_\gamma(\nabla)$, which coincides in the case of a trivializable vector bundle E with the class defined above, and such that, if $U_\gamma(\nabla) = \{I_m\}$, then E is trivializable and ∇ is trivial.

2. COMPARISON THEOREMS, CASE I

Through this section, we assume that $E = M \times \mathbb{C}^m$ and that M satisfies one of the properties (P1), (P2), (P3) mentioned in Section 1. If $A^0(M, E)$ is identified (in a natural way) with $C^\infty(M, \mathbb{C}^m)$, then $H_{\nabla, V}$ can be regarded [by (1.4)] as a Schrödinger operator $H_{\omega, V} = \nabla^*_\omega . \nabla_\omega + V$, where $\nabla_\omega = d + \omega$, with a (fixed) magnetic potential ω in $A^1(M, \mathscr{M}_{m,a})$ and electric potential V. Recall that if $\partial M \neq \varnothing$, $H^M_{\omega, V}$ is the Friedrichs' extension [11] associated to the positive sesquilinear form $q_{\omega, V}$ defined on $C^\infty_0(M, \mathbb{C}^m)$ by:

$$q_{\omega, V}(\varphi, \psi) = \int_M (\langle \nabla_\omega \varphi, \nabla_\omega \psi \rangle + (V\varphi, \psi))(x)\, dv,$$

$$\text{for } \varphi \text{ and } \psi \quad \text{in } C^\infty_0(M, \mathbb{C}^m).$$

Let λ_ω^M (resp. λ_0^M) be the first eigenvalue of $H_{\omega, V}^M$ (resp. H_V^M). As we know by the Kato's inequality (given in [6] for the case of systems), we have:

$$\lambda_0^M \leqq \lambda_\omega^M. \tag{2.1}$$

Let u_0 be the first eigenfunction of H_V^M attached to λ_0^M. We know that u_0 can be chosen such that $u_0 > 0$ on int(M) and $\| u_0 \|_0 = 1$. Using elementary computations and the fact that ω is skew Hermitian, we get the following lemma (due essentially to Lavine-O'Caroll [9]):

LEMMA 2.1. – *For* $\varphi \in C_0^\infty(M, \mathbb{C}^m)$,

$$\| \nabla_\omega \varphi - du_0 . \varphi/u_0 \|_1^2 = q_{\omega, V}(\varphi, \varphi) - \lambda_0^M \| \varphi \|_0^2.$$

The first consequence is of course that we get, as in [5], another proof of (2.1). Suppose now that $\lambda_\omega^M = \lambda_0^M$ and consider a normalized eigenfunction u_ω of $H_{\omega, V}^M$ attached to λ_ω^M. We deduce from Lemma 2.1 and using a minimizing sequence tending to u_ω in $L^2(M, \mathbb{C}^m)$ that:

$$[\nabla_\omega u_\omega - du_0 . u_\omega/u_0, \alpha]_1 = 0, \text{ for each } \alpha \in A_0^1(M, \mathbb{C}^m). \tag{2.2}$$

Consequently,

$$u_0 . \nabla_\omega(u_\omega/u_0) = \nabla_\omega u_\omega - du_0 . u_\omega/u_0 = 0,$$

on int(M) (since u_ω and u_0 are C^∞ on M).

That is to say,

$$\nabla_\omega(u_\omega/u_0) = 0, \text{ on int(M)}. \tag{2.3}$$

Now, let $\lambda_{\omega, 1}^M, \lambda_{\omega, 2}^M, \dots, \lambda_{\omega, k}^M$ $(k \leqq m)$ be the k-first eigenvalues of $H_{\omega, V}^M$. Then, we have

PROPOSITION 2.2. – *If* $\lambda_{\omega, 1}^M = \lambda_{\omega, 2}^M = \dots = \lambda_{\omega, k}^M = \lambda_0^M$. *Then, there exists* $\varphi_1, \varphi_2, \dots, \varphi_k$ *in* $C^\infty(M, \mathbb{C}^m)$ *such that, for each* $x \in M$, $(\varphi_q(x))_q$ *form an orthonormal system of* $\mathbb{C}^m$ *with:*

$$\nabla_\omega \varphi_q = 0, \text{ for each } 1 \leqq q \leqq k. \tag{2.4}$$

Proof. – Let $(u_{\omega, q})_{1 \leqq q \leqq k}$ be a system of k normalized eigenfunctions of $H_{\omega, V}^M$ attached to λ_ω^M, and define $\varphi_q = u_{\omega, q}/u_0$ on int(M). It is clear that $(\varphi_q)_{1 \leqq q \leqq k}$ satisfies (2.4) on int(M). On the other hand, using maximum principle (Lemma 3.4 in [3] applied to $\Delta - V$ and $-u_0$), we get that:

$$\partial u_0/\partial N = \nabla u_0 . N < 0 \text{ on } \partial M,$$

where $N : \partial M \to \mathbb{R}^n$ is the outward normal vector field to ∂M (note that ∂M is a regular bounded set). Then, let us define $\varphi_q(x_0)$, for $x_0 \in \partial M$ and $1 \leqq q \leqq k$, by:

$$\varphi_q(x_0) = \lim_{t \to 0, t > 0} \{ u_{\omega, q}(x_0 - t N(x_0))/u_0(x_0 - t N(x_0)) \}$$

$$= (\partial u_{\omega, q}/\partial N)(x_0)/(\partial u_0/\partial N)(x_0).$$

In order to show that φ_q verifies (2.4) on ∂M, it is sufficient to consider the case $M=\bar{Q}$. Let $\mathscr{V}$ be a neighbourhood of ∂Q and Φ in $C^\infty(\mathscr{V})$ such that:

$$\partial Q=\{x\in\mathscr{V}:\Phi(x)=0\} \quad \text{and} \quad (\nabla\Phi)(x)\neq 0 \quad \text{for} \quad x\in\mathscr{V}.$$

Then, the field $\vec{N}$ defined on $\mathscr{V}$ by: $N(x)=(\nabla\varphi)/|(\nabla\varphi)|$, is C^∞ on $\mathscr{V}$ and extend N on $\bar{Q}$. Let $\vec{A}=(A_1, A_2, \ldots, A_n)\in C^\infty(\bar{Q}, \mathscr{M}_{m,a})$ such that: $\omega=\Sigma_j A_j\,dx_j$ on $\bar{Q}$, $1\leqq q\leqq k$, and $x_0\in\partial Q$. By a simple computation, we see that, on a suitable neighbourhood of x_0, we have:

$$\varphi_q=(\nabla u_{\omega,q}.\nabla\Phi)/(\nabla u_0.\nabla\Phi)+((\vec{A}.\nabla\Phi)/(\nabla u_0.\nabla\Phi))\,u_{\omega,q}.$$

In particular,

$$\varphi_q\in C^\infty(\bar{Q}, \mathbb{C}^m)$$

$$\text{and} \quad (\nabla\varphi_q+\vec{A}\varphi_q)(x_0)=0, \quad \text{for} \quad 1\leqq q\leqq k \quad \text{and} \quad x_0\in\partial Q.$$

Now, we show the second part of this proposition. Let us remark that as a consequence of the Cauchy uniqueness theorem for linear systems of differential equations, we have:

LEMMA 2.3. – *If* $x_0\in M$, $\alpha\in A^1(M, \mathscr{M}_{m,a})$ *and* $\psi\in C^1(M, \mathbb{C}^m)$ *such that:* $\nabla_\alpha\psi=0$, $\psi(x_0)=0$. *Then,* $\psi=0$ *on* M.

By this lemma, we obtain easily that, for $x\in M$, the $(\varphi_q(x))_q$ are linearly independent in $\mathbb{C}^m$. Let us verify that, for $x\in M$ and $1\leqq p$, $q\leqq k$, $(\varphi_p(x), \varphi_q(x))=\delta^p_q$ (where δ^p_q is the Kronecker delta).

By differentiation of the application $S^p_q=(\varphi_p, \varphi_q)$ [which is in $C^\infty(M, \mathbb{C}^m)$] and using the fact that ω is skew Hermitian, we obtain:

$$\begin{aligned} dS^p_q &= \langle d\varphi_p, \varphi_q\rangle_0+\langle\varphi_p, d\varphi_q\rangle_0 \\ &= \langle -\omega.\varphi_p, \varphi_q\rangle_0+\langle\varphi_p, -\omega.\varphi_q\rangle_0=0. \end{aligned}$$

Here it is understood that the inner products on the right are defined by the requirement that: $\langle\Theta, \varphi\rangle_0=\Sigma_s\bar{\varphi}_s\theta_s\in A^1(M)$, for

$$\Theta=(\theta_s)_s\in A^1(M, \mathbb{C}^m) \text{ and } \varphi=(\varphi_s)_s\in A^0(M, \mathbb{C}^m).$$

Then, S^p_q is equal to a constant c^p_q on M (note here that M is connected) and finally

$$\delta^p_q=\int_M (u_{\omega,p}, u_{\omega,q})(x)\,dv=\int_M |u_0|^2(x)(\varphi_p, \varphi_q)(x)\,dv=c^p_q.$$

Let us translate this result on the curvature of ω. By differentiation of (2.4), we obtain:

$$K(\omega)\varphi_q=0, \quad \text{for} \quad 1\leqq q\leqq k. \tag{2.5}$$

Let us define the kernel of $K(\omega)$ as the subset of the trivial bundle $M \times \mathbb{C}^m$:

$$\ker K(\omega) = \{ (x, v) \in M \times \mathbb{C}^m : K(\omega)(x)[\partial_j(x), \partial_l(x)], \\ v = 0, \quad \text{for} \quad 1 \leqq j, l \leqq n \},$$

where $\{ \partial_j(x) \}_j$ is the natural basis of $T_x M$. Note here that $\ker K(\omega)$ defined in this way is independent of a choice of a basis in $T_x M$. Moreover, it is invariant under global gauge transformations. Suppose that $\lambda^M_{\omega, 1} = \lambda^M_{\omega, 2} = \ldots = \lambda^M_{\omega, k} = \lambda^M_0$, and consider k-functions $(\varphi_q)_q$ satisfing the above proposition. Let $\mathscr{K}$ be the trivial subbundle of $M \times \mathbb{C}^m$ generated by $(\varphi_q)_q$, and $\mathscr{K}^\perp$ the orthogonal fiber subbundle to $\mathscr{K}$. Condition (2.5) says that ker $K(\omega)$ contains $\mathscr{K}$. More precisely, we have:

LEMMA 2.4. – *Assume that* $\lambda^M_{\omega, 1} = \lambda^M_{\omega, 2} = \ldots = \lambda^M_{\omega, k} = \lambda^M_0$. *Then, the following equivalent conditions are satisfied:*

(i): ∇_ω restricted to $A^0(M, \mathscr{K})$ takes values in $A^1(M, \mathscr{K})$,

(ii): ∇_ω restricted to $A^0(M, \mathscr{K}^\perp)$ takes values in $A^1(M, \mathscr{K}^\perp)$.

In other words, the restriction of ∇_ω to $A^0(M, \mathscr{K})$ define a connection $\nabla_{\omega, \mathscr{K}}$ on $\mathscr{K}$.

Proof. – The equivalence between (i) and (ii) results from the following relation:

$$\langle \nabla_\omega f, \psi \rangle_0 = - \langle f, \nabla_\omega \psi \rangle_0, \quad \text{for } f \in A^0(M, \mathscr{K}) \text{ and } \psi \in A^0(M, \mathscr{K}^\perp).$$

Let us prove (i). Consider $f = \Sigma_q (f, \varphi_q) . \varphi_q \in A^0(M, \mathscr{K})$ and using (2.4), we obtain:

$$\nabla_\omega = \Sigma_q [d(f, \varphi_q) . \varphi_q + (f, \varphi_q) . d\varphi_q + (f, \varphi_q) . \omega . \varphi_q] \\ = \Sigma_q d(f, \varphi_q) . \varphi_q \in A^1(M, \mathscr{K}).$$

Let us give the main theorem of this section.

THEOREM 2.5. – The following three conditions are equivalent:

(i) $\lambda^M_{\omega, 1} = \lambda^M_{\omega, 2} = \ldots = \lambda^M_{\omega, k} = \lambda^M_0$

(ii) $\ker K(\omega)$ contains a trivial subbundle $\mathscr{K}$ of $M \times \mathbb{C}^m$ of rang k, such that:

(*a*): $\nabla_{\omega \mid} A^0(M, \mathscr{K}) : A^0(M, \mathscr{K}) \to A^1(M, \mathscr{K})$,

(*b*): $\nabla_{\omega, \mathscr{K}}$ is flat,

(*c*): $U_\gamma(\nabla_{\omega, \mathscr{K}}) = \{ I_k \}$, for each closed curve γ in M.

(iii) $k . \mathrm{Sp}(H^M_V) \subset \mathrm{Sp}(H^M_{\omega, V})$, where

$$k . \mathrm{Sp}(H^M_V) = \mathrm{Sp}(H^M_V) \cup \mathrm{Sp}(H^M_V) \cup \ldots \cup \mathrm{Sp}(H^M_V), (k \text{ times}).$$

Proof. – The assertion (i) $\Rightarrow$ (ii) is an easy consequence of Proposition 2.3, Lemma 2.4 and Theorem 1.1′. Let us prove (ii) $\Rightarrow$ (iii), which is the non trivial part of the statements. Consider a frame $\mathscr{E} = (e_q)_q$,

$e_q \in C^\infty(M, \mathbb{C}^m)$ for $1 \leqq q \leqq k$, of $\mathcal{K}$ over M. Using (*a*), we can write:

$$\nabla_\omega e_q = \Sigma_s \langle (d+\omega) e_q, e_s \rangle_0 . e_s.$$

This means that the 1-form $\omega_{\mathcal{K}} = {}^t([\langle (d+\omega) e_i, e_l \rangle_0]_{1 \leqq i, l \leqq k})$ is the connection form of $\nabla_{\omega, \mathcal{K}}$ with respect to $\mathcal{E}$. Now, conditions (*b*) and (*c*) say that $\nabla_{\omega, \mathcal{K}}$ is trivial:

$$\exists W \in C^\infty(M, \mathcal{U}(k)) : dW = W . \omega_{\mathcal{K}}. \qquad (2.6)$$

Using elementary computations, we see that if $(\eta_s)_s$ [resp. $(\delta_l)_l$] is the canonical basis of $\mathbb{C}^k$ (resp. $\mathbb{C}^m$) and $E = {}^t([e_i, \delta_l)]_{1 \leqq i \leqq k, 1 \leqq l \leqq m}$, then

$$(dE + \omega . E - E . W^* . dW) \eta_s \in A^1(M, \mathcal{K}) \cap A^1(M, \mathcal{K}^\perp), \qquad \text{for} \quad 1 \leqq s \leqq k.$$

Consequently,

$$dE + \omega . E - E . W^* . dW = 0 \quad \text{in } A^1(M, \mathcal{M}_{m \times k}), \qquad (2.7)$$

where $\mathcal{M}_{m \times k_M}$ is the set of $m \times k$-matrix.

Let $\lambda \in Sp(H_V^M)$, u an associated eigenfunction of H_V^M, and set:

$$u_q = u E . W^* . \eta_q \in C^\infty(M, \mathbb{C}^m), \qquad \text{for} \quad 1 \leqq q \leqq k.$$

Then, u_q's are independent in $L^2(M, \mathbb{C}^m)$ and we have for $1 \leqq q \leqq k$:

$$H_{\omega, V}^M(u_q) = EW^* . (H_V^M \otimes 1) u . \eta_q = \lambda u_q,$$

using (2.7). This means that λ is also an eigenvalue of $H_{\omega, V}^M$ with multiplicity greater or equal to k.

As a consequence, we have:

THEOREM 2.6. – *The following three conditions are equivalent:*

(i) $\lambda_{\omega, 1}^M = \lambda_{\omega, 2}^M = \ldots = \lambda_{\omega, m}^M = \lambda_0^M$,

(ii) $H_{\omega, V}^M$ and $H_V^M \otimes 1$ are unitary equivalent,

(iii) (*a*): $K(\omega) = 0$, (*b*): $U_\gamma(\omega) = \{ I_m)$, for each closed curve γ in M.

3. COMPARISON THEOREMS, CASE II

We look here at the **case II** and we fix a finite system of local trivializations $(B_\alpha, e_\alpha)_{\alpha \in I}$ of E, with B_α connected for each $\alpha \in I$. Let ω_α be the connection form of ∇ with respect to (B_α, e_α) and u_0 (resp. λ_0^M) the first eigenfunction (resp. eigenvalue of H_V^M as in Lemma 2.1.

Let us first remark that, using a partition of unity subordinate to the covering $\{ B_\alpha \}_\alpha$, we can formulate (*see* [4] for the detail of the proof) this lemma in this case as follow:

LEMMA 3.1. – $\| \nabla \xi - du_0 \otimes \xi / u_0 \|_1^2 = [H_{\nabla, V}^{M, E}(\xi), \xi]_0 - \lambda_0^M \| \xi \|_0^2,$

for $\xi \in A_0^0(M, E)$.

As a consequence of this lemma and the min-max principle [11], we have:

$$\lambda_0^{\mathrm{M}} \leqq \lambda_\nabla^{\mathrm{M, E}}, \qquad \text{where} \quad \lambda_\nabla^{\mathrm{M, E}} \text{ is the first eigenvalue of } \mathrm{H}_{\nabla, \mathrm{V}}^{\mathrm{M, E}}.$$

In order to formulate Proposition 2.1 in this case, we can get using local trivializations the following lemma:

LEMMA 3.2. — *If* $\xi \in \mathrm{A}^0(\mathrm{M}, \mathrm{E})$, $x_0 \in \mathrm{M}$ *such that* $\nabla \xi = 0$ *and* $\xi(x_0) = 0$. *Then*, $\xi = 0$.

Now, let us denote by $\lambda_{\nabla, 1}^{\mathrm{M, E}}, \lambda_{\nabla, 2}^{\mathrm{M, E}}, \ldots, \lambda_{\nabla, k}^{\mathrm{M, E}}$ the k-first eigenvalues of $H_{\nabla, \mathrm{V}}^{\mathrm{M, E}}$, and recall that ∇ is supposed compatible with the Hermitian structure of E. Namely,

$$d(\xi, \zeta) = \langle \nabla \xi, \zeta \rangle_0 + \langle \xi, \nabla \zeta \rangle_0, \qquad \text{for} \quad \xi, \zeta \in \mathrm{A}^0(\mathrm{M}, \mathrm{E}). \tag{3.1}$$

Then, using (3.1) and Lemma 3.2, we can obtain in the same way as in Proposition 3.2 the:

PROPOSITION 3.3. — *If* $\lambda_{\nabla, 1}^{\mathrm{M, E}} = \lambda_{\nabla, 2}^{\mathrm{M, E}} = \ldots = \lambda_{\nabla, k}^{\mathrm{M, E}} = \lambda_0^{\mathrm{M}}$, then, there exists k-sections (ξ_s) of E over M such that $\{\xi_s(x)\}_s$ is an orthonormal system of E_x for each $x \in \mathrm{M}$, and that:

$$\nabla \xi_s = 0 \quad \text{in } \mathrm{A}^1(\mathrm{M}, \mathrm{E}), \qquad \text{for} \quad 1 \leqq s \leqq k. \tag{3.2}$$

COROLLARY 3.4. — Under conditions: $\lambda_{\nabla, 1}^{\mathrm{M, E}} = \lambda_{\nabla, 2}^{\mathrm{M, E}} = \ldots = \lambda_{\nabla, k}^{\mathrm{M, E}} = \lambda_0^{\mathrm{M}}$, we have:

(i) $\mathrm{E} = \mathscr{K} \oplus \mathscr{K}^\perp$ (Whitney sum), where $\mathscr{K}$ is a trivializable subbundle of E with rank k,

(ii) $\nabla = \nabla_{\mathscr{K}} \oplus \nabla_{\mathscr{K}^\perp}$, where $\nabla_{\mathscr{K}}$ is a flat connection on $\mathscr{K}$ such that: $\mathrm{U}_\gamma(\nabla_{\mathscr{K}}) = \{\mathrm{I}_k\}$, for each closed curve γ in M.

Let us give the main theorem of this section.

THEOREM 3.5. — The three following conditions are equivalent:

(i) $\lambda_{\nabla, 1}^{\mathrm{M, E}} = \lambda_{\nabla, 2}^{\mathrm{M, E}} = \ldots = \lambda_{\nabla, m}^{\mathrm{M, E}} = \lambda_0^{\mathrm{M}}$,

(ii) $\mathrm{Sp}(\mathrm{H}_{\nabla, \mathrm{V}}^{\mathrm{M, E}}) = m . \mathrm{Sp}(\mathrm{H}_{\mathrm{V}}^{\mathrm{M}})$,

(iii) (*a*): E is trivializable, (*b*): the curvature of ∇ vanishes, (*c*): $\mathrm{U}_\gamma(\nabla) = \{\mathrm{I}_m\}$, for each closed curve γ in M.

Proof. — The implication (ii) $\Rightarrow$ (i) is trivial.

The assertion (i) $\Rightarrow$ (iii) follows directly from Corollary 3.4.

Let us prove (iii) $\Rightarrow$ (ii). We start from (iii) (*a*) and we consider a family $\{r_\alpha\}_\alpha$ of applications (*i.e.*, a trivialization of E) such that:

$$r_\alpha \in \mathrm{C}^\infty(\mathrm{B}_\alpha, \mathscr{U}(m)), r_\alpha = g_{\alpha\beta} . r_\beta \quad on\ B_{\alpha\beta}, \qquad \text{for} \quad \alpha, \beta \in \mathrm{I}. \tag{3.3}$$

Let $(\xi_\alpha)_\alpha$ be the local trivializations of a section ξ in the system $(\mathrm{B}_\alpha, e_\alpha)$. By (3.3), we have

$$r_\alpha^* \xi_\alpha = r_\beta^* \xi_\beta \quad \text{on } \mathrm{B}_{\alpha\beta}, \qquad \text{for} \quad \alpha, \beta \in \mathrm{I}. \tag{3.4}$$

Then for each $\xi \in A^1(M, E)$, define $F_\xi \in C^\infty(M, \mathbb{C}^m)$ by: $F_{\xi \mid B_\alpha} = r_\alpha^* . \xi_\alpha$ for $\alpha \in I$. It is easy to see that the application T defined by: $T(\xi) = F_\xi$ is one to one.

Moreover,

$$\operatorname{supp} \xi = \operatorname{supp} F_\xi, \tag{3.5}$$

$$[\xi, \xi']_0 = \int_M (F_\xi, F_{\xi'})\, dv \equiv [F_\xi, F_{\xi'}], \quad \text{for} \quad \xi, \xi' \in A^0(M, E). \tag{3.6}$$

On the other hand, if $\omega \in A^1(M, \mathcal{M}_{m,a})$ is the connection form of ∇ [which is trivial by the conditions (*b*), (*c*)] with respect to the frame defined by $(r_\alpha)_\alpha$, *i.e.*,

$$\omega_{\mid B_\alpha} = r_\alpha^* . \omega_\alpha . r_\alpha + r_\alpha^*\, dr_\alpha;$$

and if $\mathcal{H}^M_{\omega, V}$ is the Schrödinger operator with magnetic potential ω. Then, by a direct computation (and using the min-max principle for the hereunder (C.3) property) we obtain the following properties:

$$(C.1): d\xi_\alpha + \omega_\alpha \xi_\alpha = r_\alpha (dF_\xi + \omega . F_\xi)_{\mid B_\alpha} \quad \text{for} \quad \alpha \in I,$$

$$(C.2): [H_{\nabla, V}(\xi), \xi']_0 = [\mathcal{H}_{\omega, V}(F_\xi), F_{\xi'}], \quad \text{for} \quad \xi, \xi' \in A^0_0(M, E),$$

$$(C.3): \operatorname{Sp}(H^{M, E}_{\nabla, V}) = \operatorname{Sp}(\mathcal{H}^M_{\omega, V}).$$

Now, the condition (ii) results from (C.3) and Theorem 2.6, respectively.

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