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Local decay estimates for Schrödinger operators with long range potentials

by

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ABSTRACT. – For a class of long range potentials, sharp propagation estimates of the corresponding Schrödinger evolution groups are obtained without low-energy cut-off technique. Instead of low-energy cut-off, an explicit condition is given on the vanishing order in the L^2 sense at zero energy of initial states.

RÉSUMÉ. – Une estimation de propagation optimale est obtenue pour le groupe d'évolution d'une équation de Schrödinger avec un potentiel à longue portée sans le recours à la technique de coupure à basse énergie. Nous proposons à la place, des conditions explicites sur l'ordre d'annulation des états initiaux au sens de la topologie L^2 à énergie nulle.

1. INTRODUCTION

In this paper we consider the local time-decay of scattering solutions for Schrödinger operators with long-range potentials. Concerning this problem, explicit use of the low-energy cut-off function is often made in the statement of the previous results (*see* [1], [3], [8], [10], [12], [14] and Remarks 2, 3 below). A major reason consists in the fact that the low-energy part of quantum states, even in the short-range case, propagates slower than the high-energy part and prevents the full dynamics from behaving as a classical motion with velocity supported away from zero to provide a sharp propagation estimate (*see* [9], [11], [15]). When one observes the propagation of a particular state, however, the introduction of low-energy cut-off function imposes the condition that the Fourier transform of the state must vanish identically in a neighborhood of the origin. In this paper we present an explicit condition on the vanishing order at the origin of the Fourier transform of states which ensures sharp propagation estimates. Instead of the states with energy vanishing near zero, we turn our attention to the states in the range of a modified wave operator of Dollard type and consider the condition on vanishing order at the orgin of the Fourier transform of the inverse image of the states by the modified wave operator. In the usual case the range of the modified wave operator coincides with the absolute continuous spectral subspace (*see* [12], [17] and references therein), and moreover, the evolution of the states in the range of the modified wave operator turns out to be explicit as time tends to infinity provided that the right comparison dynamics is constructed. In this respect, the method in this paper is inspired by Cycon [2]. As compared to [2], we adopt a different choice of comparison dynamics which seems much easier to handle. Concerning the construction of the comparison dynamics, we follow [4], [16] though the choice of the phase function in the comparison dynamics requires a furher modification in order to exhibit a sharp approximation for the full dynamics.

Let $H = -\frac{1}{2}\Delta + V$ be a Schrödinger operator on $L^2(\mathbb{R}^n)$ with real potential V. Throughout the paper we suppose that the potential V satisfies the following condition.

Assumption (A). – V is a multiplication operator by a real-valued function on $\mathbb{R}^n$ for which the corresponding Schrödinger operator $H = -\frac{1}{2}\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n \backslash \{0\})$. Moreover, V is decomposed as $V = V^{(0)} + V^{(1)} + V^{(2)} + V^{(3)}$ with the following property:

$$V^{(0)} \in C^3(\mathbb{R}^n), \qquad V^{(1)} \in C^2(\mathbb{R}^n). \tag{A1}$$

There exist measurable functions ω_j on $\mathbb{R}_+$, $j = 0, 1, 2$, such that

$$|\partial^\alpha V^{(1)}(x)| \leq \omega_{|\alpha|}(|x|) \text{ for all multi-indices } \alpha \text{ with } |\alpha| \leq 2,$$
$$\int_0^\infty r^j \omega_j(r)\, dr < \infty \qquad \text{for} \quad j = 0, 1, 2. \tag{A2}$$

There exists a real constant μ with $1/2 < \mu \leq 1$ such that

$$\begin{gathered}\langle x\rangle^{\mu+|\alpha|}\,\partial^\alpha\, V^{(0)} \in L^\infty(\mathbb{R}^n) \qquad \text{for all} \quad \alpha \text{ with } 1 \leq |\alpha| \leq 3,\\ |x|^{1+\mu}\, V^{(2)},\ |x|^2\, V^{(3)} \in L^\infty(\mathbb{R}^n),\end{gathered} \tag{A3}$$

where $\langle x\rangle = (1 + |x|^2)^{1/2}$.

Example. – Let $V(x) = \lambda_1 |x|^{-\mu} + \lambda_2 |x|^{-\nu}$ with $\lambda_1,\ \lambda_2 \in \mathbb{R}$ and $1/2 < \mu \leq 1 < \nu < \min(n/2,\ 2)$. We distinguish three cases.

(*a*) When $\nu < 1 + \mu$, we take $V^{(0)}(x) = \lambda_1 \langle x\rangle^{-\mu}$, $V^{(1)}(x) = \lambda_2 \langle x\rangle^{-\nu}$, $V^{(2)} = V - V^{(0)} - V^{(1)}$, and $V^{(3)} \equiv 0$. Indeed,

$$\begin{aligned}|V^{(2)}(x)| &\leq C\big||x|^{-\mu} - \langle x\rangle^{-\mu}\big| + C\big||x|^{-\nu} - \langle x\rangle^{-\nu}\big|\\ &\leq C \int_0^1 (r + |x|^2)^{-\mu/2-1}\, dr + C \int_0^1 (r + |x|^2)^{-\nu/2-1}\, dr\\ &\leq C|x|^{-1-\mu} \int_0^1 r^{-1/2}\, dr + C|x|^{-1-\mu}\\ &\quad\times \int_0^1 r^{-(1+\nu-\mu)/2}\, dr = C|x|^{-1-\mu}.\end{aligned}$$

(*b*) When $\nu = 1 + \mu$, we take $V^{(0)}(x) = \lambda_1 \langle x\rangle^{-\mu}$, $V^{(1)} \equiv 0$, $V^{(2)} = V - V^{(0)}$, $V^{(3)} \equiv 0$.

(*c*) When $\nu > 1 + \mu$, we take $V^{(0)}(x) = \lambda_1 \langle x\rangle^{-\mu}$, $V^{(1)} \equiv 0$, $V^{(2)}(x) = \lambda_1 |x|^{-\mu} - V^{(0)}(x) + \lambda_2 |x|^{-\nu}(1 - \chi(x))$, $V^{(3)}(x) = \lambda_2 |x|^{-\nu}\chi(x)$, where χ is the characteristic function of the unit ball.

Under the assumption (A), the modified wave operator of Dollard type for positive time

$$\Omega = s\text{-}\lim_{t\to+\infty} e^{itH}\, U(t) \exp\left(-i \int_0^t V^{(0)}(-i\tau\,\nabla)\, d\tau\right) \tag{1.1}$$

exists as a strong limit on $L^2(\mathbb{R}^n)$, (*see* the proof below, or [17], [18] and references therein), where $U(t) = \exp(i\,(t/2)\,\Delta)$ is the free Schrödinger evolution group. To measure the vanishing order at zero energy of states, we introduce the following scale of function spaces X_m with $m > 0$.

$$X_m = \{\phi \in L_2^2(\mathbb{R}^n);\ |\xi|^{-m}\hat{\phi} \in L^2(\mathbb{R}^n)\}$$

where $L_s^2(\mathbb{R}^n)$, $s \in \mathbb{R}$, is the weighted L^2 space given by

$$L_s^2(\mathbb{R}^n) = \{\phi \in L_{\text{loc}}^2(\mathbb{R}^n)\,;\ \langle x\rangle^s \phi \in L^2(\mathbb{R}^n)\}$$

with norm $\|\phi\|_{L^2_s} = \|\langle x\rangle^s \phi\|_{L^2}$, and $\wedge$ denotes the Fourier transform given formally by

$$\hat{\phi}(\xi) = F\phi(\xi) = (2\pi)^{-n/2} \int \exp(-ix\cdot\xi)\, \phi(x)\, dx.$$

We equip the norm $\|\cdot\|_{X_m}$ by

$$\|\phi\|_{X_m}^2 = \|\phi\|_{L^2_2}^2 + \||\xi|^{-m}\, \hat{\phi}\|_{L^2}^2.$$

In terms of the homogeneous Sobolev spaces, $X_m = L^2_2 \cap \dot{H}^{-m}$. We summarize basic properties of X_m.

PROPOSITION 1. – (1) *X_m is a Hilbert space.*

(2) $X_m \hookrightarrow X_{m'}$ *for* $m > m' > 0$.

(3) $X_m = L^2_2$ *with equivalent norms if* $0 < m < \min(2, n/2)$.

(4) $X_m \hookrightarrow L^2_m$ *if* $2 \leq m < n/2$, $n \geq 5$.

(5) $F^{-1}(C_0^\infty(\mathbb{R}^n\backslash\{0\}))$ *is dense in* X_m *for any* $m \geq 2$.

We now state our main results. For $t \in \mathbb{R}$ we put $\langle t\rangle = (1+t^2)^{1/2}$.

THEOREM 2. – (1) *Let* $1/2 < \mu < 1$ *and* $0 < \sigma \leq \mu$. *If* $0 < \sigma \leq 2\mu - 1$, *then there is a constant C such that for all* $t \in \mathbb{R}_+$ *and* $\phi \in X_4$

$$\|e^{-itH}\, \Omega\phi\|_{L^2_{-\sigma}} \leq C\, \langle t\rangle^{-\sigma}\, \|\phi\|_{X_4}. \tag{1.2}$$

If $2\mu - 1 < \sigma \leq \mu$, *then for any* $m > 8$ *there is a constant C such that for all* $t \in \mathbb{R}_+$ *and* $\phi \in X_m$

$$\|e^{-itH}\, \Omega\phi\|_{L^2_{-\sigma}} \leq C\, \langle t\rangle^{-\sigma}\, \|\phi\|_{X_m}. \tag{1.3}$$

(2) *Let* $0 < \sigma \leq \mu = 1$. *If* $0 < \sigma < 1$, *then there is a constant C such that for all* $t \in \mathbb{R}_+$ *and* $\phi \in X_4$

$$\|e^{-itH}\, \Omega\phi\|_{L^2_{-\sigma}} \leq C\, \langle t\rangle^{-\sigma}\, \|\phi\|_{X_4}. \tag{1.4}$$

If $\sigma = 1$, *then for any* $m > 8$ *there is a constant C such that for all* $t \in \mathbb{R}_+$ *and* $\phi \in X_m$

$$\|e^{-itH}\, \Omega\phi\|_{L^2_{-1}} \leq C\, \langle t\rangle^{-1}\, \langle \log\langle t\rangle\rangle \|\phi\|_{X_m}. \tag{1.5}$$

Remark 1. – In the Coulomb case $V(x) = \lambda|x|^{-1}$ with $\lambda \in \mathbb{R}\backslash\{0\}$ and $n \geq 3$, Cycon [2] proved

$$\|e^{-itH}\, \Omega\phi\|_{L^2_{-2n}} \leq C\, \langle t\rangle^{-1}\, \langle\log\langle t\rangle\rangle^{6n} \sum_{|\alpha|\leq 6n} \left\| \langle\xi\rangle^{4n} \exp\left(\frac{1}{|\xi|}\right) \partial^\alpha \hat{\phi} \right\|_{L^\infty}.$$

In the repulsive case $\lambda > 0$, it has been shown in [5] that for $t \in \mathbb{R}$ with $|t| \geq 1$

$$\||x|^{-1} e^{-itH} \phi\|_{L^2} \leq C|t|^{-1} \|\phi\|_{L^2_2}.$$

This indicates that the positivity or repulsivity may provide sharper estimates. But this problem is outside the purpose of this paper.

Remark 2. – In [1], [14] there are related estimates of $e^{-itH} \Omega f(H_0)$ in the short-range case, where Ω is the ordinary wave operator, $H_0 = -\frac{1}{2} \Delta$, and f is a low-energy cut-off function, *i.e.* $f \in C^\infty(\mathbb{R}, \mathbb{R})$ with $f \equiv 1$ on $[\lambda_0, \infty)$ and $f \equiv 0$ on $(-\infty, \lambda_0/2]$ for some $\lambda_0 > 0$.

Remark 3. – Estimates of the form

$$\|e^{-itH} f(H)\|_{B(L^2_\sigma, L^2_{-\sigma})} \leq C \langle t \rangle^{-\sigma'}$$

have been obtained in [3], [8], [10], [12] for long-range potentials, where f is a low-energy cut-off function, $B(L^2_\sigma, L^2_{-\sigma})$ is the space of bounded operators from L^2_σ to $L^2_{-\sigma}$, and $0 < \sigma' \leq \sigma$.

Remark 4. – Theorem 2 implies estimates of the form

$$\|e^{-itH} \Omega f(H_0)\|_{B(L^2_\sigma, L^2_{-\sigma})} \leq C \langle t \rangle^{-\sigma}$$

for any low-energy cut-off function f in the case where $1/2 < \mu < 1$ with $0 < \sigma \leq \mu$, or $0 < \sigma < \mu = 1$. Results of this type follow from the estimate in Remark 3 and the boundedness of $\Omega f_1(H_0)$ in L^2_σ for some low-energy cut-off function f_1 satisfying $f = f f_1$. For the possibility of the latter boundedness, *see* [13].

Some of the results in Theorem 2 are optimal. In fact:

THEOREM 3. – *Let* $1/2 < \mu \leq 1$ *and* $0 < \sigma < \mu$. *Let* $\phi \in X_4$. *When* $2\mu - 1 \leq \sigma < \mu$, *assume further that* $\phi \in X_m$ *for some* $m > 8$. *Then*

$$\lim_{t \to \infty} t^\sigma \|e^{-itH} \Omega \phi\|_{L^2_{-\sigma}} = \||\xi|^{-\sigma} \hat{\phi}\|_{L^2}. \tag{1.6}$$

Remark 5. – It seems that the formula (1.6) is new. A weak from of the lower bound estimates for the Schrödinger evolution group is obtained in [5], [6].

Under some restrictive assumption on ϕ and V, one could obtain the exponential decay results with the $L^2_{-\sigma}$ norm replaced by the norm with exponential weights such as $\exp(-|x|)$ or by the norm $\|\cdot\|_{L^2(K)}$ for some compact $K \subset \mathbb{R}^n$ (*see* [19], [20]). But this is outside the purpose of this paper.

The method of proof of Theorems 2 and 3 is roughly illustrated as follows. By the definition of the modified wave operator (1.1), the full dynamics $e^{-itH}\,\Omega\phi$ looks like the Dollard dynamics

$$U_D(t)\,\phi = U(t)\,\exp\left(-i\int_0^t V^{(0)}(-i\tau\,\nabla)\,d\tau\right)\phi \qquad (1.7)$$

in $L^2(\mathbf{R}^n)$ as $t \to \infty$. In order to observe the large time behavior of the full dynamics $e^{-itH}\,\Omega\phi$ in a more explicit way, we replace the Dollard dynamics $U_D(t)\,\phi$ by another comparison dynamics $W(t)\,\phi$, given by

$$(W(t)\,\phi)(x) = (it)^{-n/2}\,\exp\left(i|x|^2/2t - iS(t,\,x/t)\right)\hat{\phi}(x/t) \qquad (1.8)$$

with appropriate real function $S(t,\,\xi)$. With this form, we have $|(W(t))\phi(x)| = t^{-n/2}|\hat{\phi}(x/t)|$ and therefore the right behavior of $e^{-itH}\,\Omega\phi$ in $L^2_{-\sigma}$ is obtained through

$$\lim_{t\to\infty} t^{\sigma}\|W(t)\,\phi\|_{L^2_{-\sigma}} = \||\xi|^{-\sigma}\,\hat{\phi}\|_{L^2} \qquad (1.9)$$

provided that $W(t)\,\phi$ approximates $e^{-itH}\,\Omega\phi$ better than the order $O(t^{-\sigma})$ as $t \to \infty$. For this purpose we prove

$$\|e^{-itH}\,\Omega\phi - W(t)\,\phi\|_{L^2} \leq \int_t^{\infty} \|(i\partial_\tau - H)\,W(\tau)\,\phi\|_{L^2}\,d\tau \qquad (1.10)$$

and estimate the rate of decay of the integral in (1.10) as $t \to \infty$, which depends on the choice of phase function S and on the vanishing order at the origin of $\hat{\phi}$. By a direct calculation, we have

$$\begin{aligned}\|(i\partial_t - H)\,W(t)\,\phi\|_{L^2} \leq & \|(\partial_t S - V(t\xi) - (1/2)\,t^{-2}|\nabla S|^2)\,\hat{\phi}\|_{L^2} \\ & + t^{-2}\left(\|(\Delta S)\,\hat{\phi}\|_{L^2} + \|\nabla S\cdot\nabla\hat{\phi}\|_{L^2}\right. \\ & \left. + \|\Delta\hat{\phi}\|_{L^2}\right). \end{aligned} \qquad (1.11)$$

As in the original choice of Dollard, one might take

$$S(t,\,\xi) = \int_0^t V^{(0)}(\tau\xi)\,d\tau.$$

This choice, however, leads to a rather unsatisfactory result since this imposes restrictive conditions on the short-range part and implies a weak decay rate of (1.11). In other to obtain a better approximation, we take

$$S(t, \xi) = \int_0^t V^{(0)}(\tau\xi)\, d\tau - \int_t^\infty V^{(1)}(\tau\xi)\, d\tau \tag{1.12}$$

for $0 < \sigma \leq 2\mu - 1$ with $1/2 < \mu < 1$ and $0 < \sigma < \mu = 1$. This choice requires the conditon $\phi \in X_4$ since the short-range part of $|\nabla S|^2$ creates singularity of the form $|\xi|^{-4}$. If $2\mu - 1 < \sigma \leq \mu$ with $1/2 < \mu < 1$ or $\sigma = \mu = 1$, the preceding choice (1.12) turns out insufficient since the long-range part of $|\nabla S|^2$ grows like $O(t^{2-2\mu})$ for $1/2 < \mu < 1$ and like $O((\log t)^2)$ for $\mu = 1$, both of which are larger than the expected rate $O(t^{1-\mu})$. To cancel the growth property of the quadratic contribution, we choose

$$S(t, \xi) = \int_0^t V^{(0)}(\tau\xi)\, d\tau - \int_t^\infty s^{-2} \left| \int_0^s \tau (\nabla V^{(0)})(\tau\xi)\, d\tau \right|^2 ds - \int_t^\infty V^{(1)}(\tau\xi)\, d\tau. \tag{1.13}$$

As for the decay rate in time of (1.11), the choice (1.13) gives an optimal result for $2\mu - 1 < \sigma \leq \mu$ but not for $\sigma = \mu = 1$, still in the latter case this improves the result with the previous choice (1.12) by one logarithmic power. The restriction $m > 8$ of X_m comes from the second term of the right hand side of (1.13) at the cost of better rate of decay in time of (1.11).

This paper is organized as follows. In Section 2 we collect several preliminary estimates, including the proof of Proposition 1.1. Section 3 is devoted to the estimates associated with the comparison dynamics $W(t)$. In Section 4 we prove Theorems 2 and 3.

We conclude this introduction by giving notations freely used in this paper. $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\nabla = (\partial_1, \dots, \partial_n)$, $\partial_j = \partial/\partial x_j$, $\partial_t = \partial/\partial t$. The variables in the Fourier transform are usually denoted by $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and the corresponding derivatives $\partial/\partial \xi_j$ might be denoted simply by ∂_j when no confusion could arise. The time variable t is usually taken to be positive and might be omitted in course of calculations. The norm without subscripts $\|\cdot\|$ denotes the $L^2(\mathbb{R}^n)$ norm for scalar $\mathbb{C}$ valued functions and for vector $\mathbb{C}^n$ valued functions as well. Different positive constants might be denoted by the same letter C, and if necessary, by $C(*, \dots, *)$ in order to indicate the dependence on the quantities in the parentheses.

2. PRELIMINARIES

We first give preliminary estimates associated with the norm in X_m.

LEMMA 1. – *Let* $m > 0$ *for* $n \geq 2$ *and let* $m \in (0, 1) \cup [2, \infty)$ *for* $n = 1$. *Then* $|\xi|^{-m/2} \nabla \hat{\phi} \in L^2$ *for all* $\phi \in X_m$. *Moreover,*

$$\| |\xi|^{-m/2} \nabla \phi \| \leq C(m, n) \|\phi\|_{L^2_2} \quad \text{for} \quad 0 < m < \min(2, n), \tag{2.1}$$

$$\| |\xi|^{-m/2} \nabla \phi \| \leq (m(m+2)/2) \|\phi\|_{X_m} \quad \text{for} \quad m \geq 2. \tag{2.2}$$

Proof. – By Hardy type inequality [7], for $0 < m < n$ we have

$$\| |\xi|^{-m/2} \nabla \hat{\phi} \| \leq C(m, n) \|(-\Delta)^{m/4} \nabla \hat{\phi}\| = C(m, n) \| |x|^{m/2+1} \phi \|.$$

This proves (2.1). Let $m \geq 2$ and $\phi \in X_m$. Since $|\nabla \hat{\phi}|^2 = (1/2)\, \Delta\, |\hat{\phi}|^2 - \mathrm{Re}(\bar{\hat{\phi}} \Delta \hat{\phi})$, by integration by parts and Hölder's inequality we have for $\varepsilon > 0$

$$\begin{aligned}
&\int (|\xi|^2 + \varepsilon)^{-m/2} |\nabla \hat{\phi}|^2 \, d\xi \\
&\leq (m(m+2-n)/2) \int (|\xi|^2 + \varepsilon)^{-m/2-2} \, |\xi|^2 \, |\hat{\phi}|^2 \, d\xi \\
&\quad + \int (|\xi|^2 + \varepsilon)^{-m/2} \, |\hat{\phi} \Delta \hat{\phi}| \, d\xi \\
&\leq (m/2) \max(m+2-n, 0) \, \| |\xi|^{-m/2-1} \hat{\phi} \|^2 + \| |\xi|^{-m} \hat{\phi} \| \, \|\Delta \hat{\phi}\| \\
&\leq (m/2) \max(m+2-n, 0) \, \| |\xi|^{-m} \hat{\phi} \|^{1+2/m} \|\phi\|^{1-2/m} + \|\phi\|_{X_m}^2 \\
&\leq (m(m+2)/2) \|\phi\|_{X_m}^2.
\end{aligned}$$

By the monotone convergence theorem, this implies (2.2).

Q.E.D.

We now prove Proposition 1.

Proof of Proposition 1. – For part (1), the only nontrivial issue is the completeness but the proof is standard and straightforward. Details are omitted. Part (2) follows by Hölder's inequality. Parts (3) and (4) follow from Hardy type inequality [7] of the form

$$\| |\xi|^{-m} \hat{\phi} \| \leq C(m, n) \|(-\Delta)^{m/2} \hat{\phi}\|$$

for $0 < m < n/2$. For part (5), let $\phi \in X_m$ with $m \geq 2$. Let two functions $f, g \in C^\infty(\mathbb{R})$ satisfy $0 \leq f, g \leq 1$, $f \equiv 0$ on $(-\infty, 1/2]$, $f \equiv 1$

on $[1, \infty)$, $g \equiv 1$ on $(-\infty, 1]$, $g \equiv 0$ on $[2, \infty)$. We set $\phi_j = F^{-1}\psi_j$, $j \geq 1$, where $\psi_j(\xi) = f(j|\xi|^2)\, g(j^{-1}|\xi|^2)\, \hat{\phi}(\xi)$. Then $\psi_j \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, $\psi_j \to \hat{\phi}$ and $|\xi|^{-m}\psi_j \to |\xi|^{-m}\hat{\phi}$ in L^2 as $j \to \infty$. It remains to prove $\Delta\psi_j \to \Delta\hat{\phi}$ in L^2 as $j \to \infty$. This follows from the fact that $|\xi|^{-2}\hat{\phi} \in L^2$ and $|\xi|^{-1}\nabla\hat{\phi} \in L^2$ [*see* (2.2)].

Q.E.D.

We next collect estimates for phase functions $S^{(0)}$ and $S^{(1)}$, defined by

$$S^{(0)}(t, \xi) = \int_0^t V^{(0)}(\tau\xi)\, d\tau, \tag{2.3}$$

$$S^{(1)}(t, \xi) = -\int_t^\infty V^{(1)}(\tau\xi)\, d\tau. \tag{2.4}$$

LEMMA 2. – (1) *Let* $1/2 < \mu < 1$ *and* $0 \leq \theta \leq \mu$. *Let* $1 \leq j \leq 3$. *Then there is a constant* C *independent of* t *and* ξ *such that*

$$\sum_{|\alpha|=j} |\partial^\alpha S^{(0)}(t, \xi)| \leq C t^{1-\theta} |\xi|^{-j-\theta}. \tag{2.5}$$

(2) *Let* $0 \leq \theta \leq \mu = 1$. *Let* $1 \leq j \leq 3$. *Then there is a constant* C *independent of* t *and* ξ *such that*

$$\sum_{|\alpha|=j} |\partial^\alpha S^{(0)}(t, \xi)| \leq C t^{1-\theta} (\log\langle t\rangle)^\theta (1 + |\xi|^{-j-\theta}). \tag{2.6}$$

(3) *Let* $0 \leq j \leq 2$. *Then there is a constant* C *independent of* t *and* ξ *such that*

$$\sum_{|\alpha|=j} |\partial^\alpha S^{(1)}(t, \xi)| \leq C |\xi|^{-1-j} \int_{t|\xi|}^\infty r^j \omega_j(r)\, dr. \tag{2.7}$$

Proof. – By definition and assumption, for $1/2 < \mu < 1$ we have

$$\begin{aligned}
&\sum_{|\alpha|=j} |\partial^\alpha S^{(0)}(t, \xi)| \\
&\quad \leq \sum_{|\alpha|=j} \int_0^t |\tau^j \partial^\alpha V^{(0)}(\tau\xi)|\, d\tau \\
&\quad \leq C \int_0^t \tau^j \langle\tau\xi\rangle^{-j-\mu}\, d\tau = C|\xi|^{1-j} \int_0^t \tau (\tau|\xi|)^{j-1} \langle\tau\xi\rangle^{-j-\mu}\, d\tau \\
&\quad \leq C|\xi|^{1-j} \int_0^t \tau \langle\tau\xi\rangle^{-1-\mu}\, d\tau = C|\xi|^{1-j}\, t^2 \int_0^1 \langle r^{1/2} t\xi\rangle^{-1-\mu}\, dr \\
&\quad \leq C|\xi|^{1-j}\, t^2 \int_0^1 (r^{1/2}\, t|\xi|)^{-1-\theta}\, dr = C t^{1-\theta} |\xi|^{-j-\theta} \int_0^1 r^{-(1+\theta)/2}\, dr,
\end{aligned}$$

where we have made the change of variable $\tau \mapsto r = t^{-2}\tau^2$. This proves (2.5). Similarly, for $\mu = 1$

$$\sum_{|\alpha|=j} |\partial^\alpha S^{(0)}(t, \xi)| \leq C|\xi|^{1-j} \int_0^t \tau \langle \tau\xi \rangle^{-2} \, d\tau.$$

The last integral is estimated by

$$\begin{aligned}\int_0^t \tau \langle \tau\xi \rangle^{-2} \, d\tau = |\xi|^{-2} \int_0^{t|\xi|} r(1+r^2)^{-1} \, dr \\ \leq C \min(t|\xi|^{-1}, (\log \langle t \rangle)(1+|\xi|^{-2})) \\ \leq C t^{1-\theta} (\log \langle t \rangle)^\theta (1+|\xi|^{-1-\theta}).\end{aligned}$$

This proves (2.6). Similar estimates show (2.7).

Q.E.D.

3. ESTIMATES FOR COMPARISON DYNAMICS

In this section we prove several estimates for the comparison dynamics $W(t)$. Throughout this section we assume $t \geq 10$ and $\hat{\phi} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ for simplicity. We first recall the factorization of the free Schrödinger evolution $U(t) = \exp(i(t/2)\Delta)$ [4], [16]

$$U(t) = M(t) D(t) FM(t), \tag{3.1}$$

where

$$M(t) = \exp(i|x|^2/2t),$$
$$(D(t)\psi)(x) = (it)^{-n/2} \psi(x/t).$$

Equivalent definition of $W(t)$ in (1.8) is then given by

$$\begin{aligned}W(t) &= U(t) M(-t) \exp(-iS(t, -i\nabla)) \\ &= M(t) D(t) F \exp(-iS(t, -i\nabla)) \\ &= M(t) D(t) \exp(-iS(t, \cdot)) F,\end{aligned} \tag{3.2}$$

where phase function S is defined by

$$S(t, \xi) = S^{(0)}(t, \xi) + S^{(1)}(t, \xi), \tag{3.3}$$

if $0 < \sigma \leq 2\mu - 1$ or $0 < \sigma < \mu = 1$,

$$S(t, \xi) = S^{(0)}(t, \xi) + S^{(1)}(t, \xi) - \int_t^\infty \tau^{-2} |\nabla S^{(0)}(t, \xi)|^2 \, d\tau \quad (3.4)$$

if $2\mu - 1 \leq \sigma \leq \mu$. In the limiting case $2\mu - 1 = \sigma < 1$, we need both (3.3) and (3.4) for technical reasons. We now consider

$$(i\partial_t - H) W(t) \phi = \left(i\partial_t + \frac{1}{2} \Delta \right) W(t) \phi - VW(t) \phi. \quad (3.5)$$

For the first term on the right hand side of (3.5), following [4], we compute

$$\begin{aligned} &\left(i\partial_t + \frac{1}{2} \Delta \right) W(t) \phi \\ &= \left(i\partial_t + \frac{1}{2} \Delta \right) U(t) M(t) \exp(-iS(t, -i\nabla)) \phi \\ &= U(t) i\partial_t M(-t) \exp(-iS(t, -i\nabla)) \phi \\ &= U(t) M(-t) \left(i\partial_t - \frac{1}{2} t^{-2} x^2 \right) \exp(-iS(t, -i\nabla)) \phi \\ &= U(t) M(-t) F^{-1} \left(i\partial_t - \frac{1}{2} t^{-2} \Delta \right) \exp(-iS(t, \cdot)) \hat{\phi} \\ &= U(t) M(-t) F^{-1} \left((\partial_t S)(t, \cdot) + \frac{1}{2} t^{-2} \Delta \right) \\ &\quad \times \exp(-iS(t, \cdot)) \hat{\phi}. \end{aligned} \quad (3.6)$$

For the second term, we compute

$$\begin{aligned} VW(t) \phi &= VM(t) D(t) \exp(-iS(t, \cdot)) \hat{\phi} \\ &= M(t) D(t) V(t\xi) \exp(-iS(t, \xi)) \hat{\phi} \\ &= U(t) M(-t) F^{-1} V(t\xi) \exp(-iS(t, \xi)) \hat{\phi}. \end{aligned} \quad (3.7)$$

By (3.6) and (3.7),

$$\begin{aligned}(i\partial_t - H)\, W(t)\, \phi &\\ = U(t)\, M(-t)\, F^{-1} \exp(-iS(t, \cdot))&\\ \cdot \left(\left(\partial_t S - V(t\xi) - \frac{1}{2} t^{-2} (|\nabla S|^2 + i\Delta S) \right) \hat{\phi} \right.&\\ \left. - it^{-2} \nabla S \cdot \nabla \hat{\phi} + \frac{1}{2} t^{-2} \Delta \hat{\phi} \right).& \quad (3.8)\end{aligned}$$

This gives

$$\begin{aligned}\|(i\partial_t - H)\, W(t)\, \phi\| &\\ \leq \left\| \left(\partial_t S - V^{(0)}(t\xi) - V^{(1)}(t\xi) - \frac{1}{2} t^{-2} |\nabla S|^2 \right) \hat{\phi} \right\|&\\ + \|(V^{(2)}(t\xi) + V^{(3)}(t\xi))\, \hat{\phi}\|&\\ + t^{-2} \|(\Delta S)\, \hat{\phi}\| + t^{-2} \|\nabla S \cdot \nabla \hat{\phi}\| + t^{-2} \|\Delta \hat{\phi}\|.& \quad (3.9)\end{aligned}$$

We estimate the terms on the right hand side of (3.9). We distinguish four cases.

(i) $1/2 < \mu < 1$, $0 < \sigma \leq 2\mu - 1$.

(ii) $1/2 < \mu < 1$, $2\mu - 1 \leq \sigma \leq \mu$.

(iii) $0 < \sigma < \mu = 1$.

(iv) $\sigma = \mu = 1$.

(i) When $1/2 < \mu < 1$ and $0 < \sigma \leq 2\mu - 1$, we have by (3.3) that $\partial_t S(t, \xi) = V^{(0)}(t\xi) + V^{(1)}(t\xi)$. By assumption,

$$\|(V^{(2)}(t\xi) + V^{(3)}(t\xi))\, \hat{\phi}\| \leq C^{t-1-\mu} (\||\xi|^{-1-\mu}\, \hat{\phi}\| + \||\xi|^{-2} \hat{\phi}\|). \quad (3.10)$$

By Lemma 2,

$$t^{-2} \||\nabla S|^2\, \hat{\phi}\| \leq C t^{-2\mu} (\||\xi|^{-2-2\mu}\, \hat{\phi}\| + \||\xi|^{-4}\, \hat{\phi}\|), \quad (3.11)$$

$$t^{-2} \|(\Delta S)^2\, \hat{\phi}\| \leq C t^{-1-\mu} (\||\xi|^{-2-2\mu}\, \hat{\phi}\| + \||\xi|^{-3}\, \hat{\phi}\|), \quad (3.12)$$

$$t^{-2} \|\nabla S \cdot \nabla\, \hat{\phi}\| \leq C t^{-1-\mu} (\||\xi|^{-1-\mu}\, \nabla \hat{\phi}\| + \||\xi|^{-2}\, \nabla \hat{\phi}\|). \quad (3.13)$$

Collecting (3.9)-(3.13) and using Lemma 1, we obtain

$$\|(i\partial_t - H)\, W(t)\, \phi\| \leq C t^{-2\mu} \|\phi\|_{X_4}. \quad (3.14)$$

(ii) When $1/2 < \mu < 1$ and $2\mu - 1 \leq \sigma \leq \mu$, we have by (3.4) that

$$\partial_t S = V^{(0)}(t\xi) + V^{(1)}(t\xi) + \frac{1}{2} t^{-2} |\nabla S^{(0)}|^2, \tag{3.15}$$

$$\begin{aligned} |\nabla S|^2 = {} & |\nabla S^{(0)}|^2 + |\nabla S^{(1)}|^2 + \left| \int_t^\infty \tau^{-2} \nabla^2 S^{(0)} \cdot \nabla S^{(0)} d\tau \right|^2 \\ & - 2(\nabla S^{(0)} + \nabla S^{(1)}) \\ & \times \int_t^\infty \tau^{-2} \nabla^2 S^{(0)} \cdot \nabla S^{(0)} d\tau - 2\nabla S^{(0)} \cdot \nabla S^{(1)}, \end{aligned} \tag{3.16}$$

where $\nabla^2 S$ denotes the Hessian matrix. In the first norm on the right hand side of (3.9), the contribution of the slowest component $(1/2)\, t^{-2} |\nabla S^{(0)}|^2$ is cancelled by the first term on the right hand side of (3.16) multiplied by $(1/2)\, t^{-2}$. Instead, the third term on the right hand side of (3.16) gives the strongest singularity at the origin, which should be minimized by choosing $\theta = 1/2 + \varepsilon/4$ with $0 < \varepsilon < 4(\mu - 1/2)$ in Lemma 2. The restriction $\theta > 1/2$ ensures the integrability of the associated time integral. The contribution of the thrid term is therefore estimated by

$$t^{-2} \left\| \left| \int_t^\infty \tau^{-2} \nabla^2 S^{(0)} \cdot \nabla S^{(0)} d\tau \right|^2 \hat{\phi} \right\| \leq C(\varepsilon) t^{-2-\varepsilon} \| |\xi|^{-8-\varepsilon} \hat{\phi} \|. \tag{3.17}$$

This requires $m = 8 + \varepsilon > 8$ in the assumption $\phi \in X_m$ of the theorems. Note that we may assume that $8 < m < 10$ without loss of generality. Other terms on the right hand side of (3.9) with (3.4) are estimated similarly except for the norm involving $\nabla\hat{\phi}$, where an additional use of (2.2) with $m = 8 + \varepsilon$ is needed. We have thus proved that for all $m > 8$

$$\|(i\partial_t - H) W(t) \phi\| \leq C(m) t^{-1-\mu} \|\phi\|_{X_m}. \tag{3.18}$$

(iii) When $0 < \sigma < \mu = 1$, we estimate (3.9) by using (2.6) in a way similar to the case (i) to obtain

$$\|(i\partial_t - H) W(t) \phi\| \leq C t^{-2} (\log t)^2 \|\phi\|_{X_4}, \tag{3.19}$$

where the slowest contribution proportional to $t^{-2} (\log t)^2$ is given by $t^{-2} \| |\nabla S|^2 \hat{\phi} \|$ from the right hand side of (3.9).

(iv) When $\sigma = \mu = 1$, we estimate (3.9) by using (2.6) in the same way as in the case (ii) with some necessary modifications. For instance, we replace (3.17) by

$$t^{-2} \left\| \left| \int_t^\infty \tau^{-2} \nabla^2 S^{(0)} \cdot \nabla S^{(0)} \, d\tau \right|^2 \hat{\phi} \right\| \leq C(\varepsilon)\, t^{-2-\varepsilon} (\log t)^{2+\varepsilon} \| |\xi|^{-8-\varepsilon} \hat{\phi} \|$$

by using Lemma 2 with $\theta = 1/2 + \varepsilon/4$ for $0 < \varepsilon < 2$. With $m > 8$, we finally obtain

$$\|(i\partial_t - H)\, W(t)\, \phi\| \leq C(m)\, t^{-2} (\log t) \|\phi\|_{X_m}, \tag{3.20}$$

where the slowest contribution proportional to $t^{-2} \log t$ is given by the terms linear in S from the right hand side of (3.9).

4. PROOF OF THEOREMS 2 AND 3

In this section we prove Theorems 2 and 3. Let $\hat{\phi} \in C_0^\infty (\mathbb{R}^n \backslash \{0\})$. By Cook's argument, we have for $t > s \geq 10$,

$$\|e^{itH}\, W(t)\, \phi - e^{isH}\, W(s)\, \phi\| \leq \int_s^t \|(i\partial_\tau - H\, W(\tau)\, \phi\| \, d\tau. \tag{4.1}$$

By (3.14), (3.18), (3.19), (3.20) and (4.1), $e^{itH}\, W(t)\, \phi$ converges strongly to a limit $\psi \in L^2$ as $t \to \infty$ and

$$\|\psi - e^{itH}\, W(t)\, \phi\| \leq \int_t^\infty \|(i\partial_\tau - H)\, W(\tau)\, \phi\| d\tau \leq C K_{\mu,\sigma}(t) \|\phi\|_{X_{l(\mu,\sigma)}}, \tag{4.2}$$

where

$$(K_{\mu,\sigma}(t),\, l(\mu,\sigma)) = \begin{cases} (t^{1-2\mu},\, 4) & \text{if } 1/2 < \mu < 1,\ 0 < \sigma \leq 2\mu - 1, \\ (t^{-\mu},\, m) & \text{if } 1/2 < \mu < 1,\ 2\mu - 1 \leq \sigma \leq \mu, \\ (t^{-1} (\log t)^2,\, 4) & \text{if } 0 < \sigma < \mu = 1, \\ (t^{-1} \log t,\, m) & \text{if } \sigma = \mu = 1. \end{cases}$$

By (3.2) and Lemma 2 with $\theta = \mu > \frac{1}{2}$,

$$\begin{aligned}
&\|U(t)\exp(-iS(t,-i\nabla))\phi - W(t)\phi\| \\
&\quad = \|(1 - M(-t))\exp(-iS(t,-i\nabla))\phi\| \\
&\quad \le t^{-1/2}\|x\exp(-iS(t,i\nabla))\phi\| \\
&\quad = t^{-1/2}\|\nabla(\exp(-iS(t,\cdot))\hat{\phi})\| \to 0 \quad \text{as } t \to \infty. \qquad (4.3)
\end{aligned}$$

By Lemma 2 with $\theta = \mu$ and the dominated convergence theorem with $\omega_0 \in L^1$,

$$\begin{aligned}
&\|\exp(-iS^{(0)}(t,-i\nabla))\phi - \exp(-iS(t,-i\nabla))\phi\| \\
&\quad = \|(\exp(iS(t,\cdot) - iS^{(0)}(t,\cdot)) - 1)\hat{\phi}\| \to 0 \quad \text{as } t \to \infty. \qquad (4.4)
\end{aligned}$$

From (4.2), (4.3), (4.4) we obtain

$$\begin{aligned}
&\|e^{itH}U(t)\exp(-iS^{(0)}(t,-i\nabla))\phi - \psi\| \\
&\quad \le \|e^{itH}U(t)(\exp(-iS^{(0)}(t,-i\nabla))\phi - \exp(-iS(t,-i\nabla))\phi)\| \\
&\qquad + \|e^{itH}U(t)\exp(-iS(t,-i\nabla))\phi - e^{itH}W(t)\phi\| \\
&\qquad + \|e^{itH}W(t)\phi - \psi\| \to 0 \quad \text{as } t \to \infty. \qquad (4.5)
\end{aligned}$$

This proves $\psi = \Omega\phi$ and (4.2) implies

$$\|e^{-itH}\Omega\phi - W(t)\phi\| \le CK_{\mu,\sigma}(t)\|\phi\|_{X_{l(\mu,\sigma)}}. \qquad (4.6)$$

By part (5) of Proposition 1 and the boundedness uniform in time of $e^{-itH}\Omega$ and $W(t)$, (4.6) still holds for all $\phi \in X_{l(\mu,\sigma)}$. On the other hand, (3.1) shows

$$\|W(t)\phi\|_{L^2_{-\sigma}} = \|D(t)\hat{\phi}\|_{L^2_{-\sigma}} = \|\langle t\xi\rangle^{-\sigma}\hat{\phi}\| \le t^{-\sigma}\||\xi|^{-\sigma}\hat{\phi}\|. \qquad (4.7)$$

Combining (4.6) and (4.7), we obtain Theorem 2 for $t \ge 10$, where we have chosen $(K_{\mu,\sigma}(t), l(\mu,\sigma)) = (t^{1-2\mu}, 4)$ in the limiting case $\sigma = 2\mu - 1 < 1$. This completes the proof of Theorem 2 since $e^{-itH}\Omega\phi$ is estimated by $\|\phi\|$ in the norm of $L^2_{-\sigma}$. By (4.7) and the dominated convergence theorem,

$$\lim_{t\to\infty} t^{\sigma}\|W(t)\phi\|_{L^2_{-\sigma}} = \||\xi|^{-\sigma}\hat{\phi}\|. \qquad (4.8)$$

Theorem 3 follows from (4.8) and (4.6), where we have chosen $(K_{\mu,\sigma}(t), l(\mu, \sigma)) = (t^{-\mu}, m)$ in the limiting case $\sigma = 2\mu - 1 < 1$ since $\mu > \sigma = 2\mu - 1$.

Q.E.D.

Concluding Remarks. – (1) The results in this paper concern the time-decay estimates for $e^{-itH}\,\Omega$ rather than e^{-itH} itself. Accordingly, if one looks at the behavior of e^{-itH}, then there arises the problem of characterizing the corresponding space ΩX_m of initial states. If Ω is complete (the difficulty could come from the singular part V_3), then the relation $\Omega\Omega^* = E_c(H)$ and an operational calculus yield $\Omega D(|H_0|^{-m/2}) = E_c(H) D(|H|^{-m/2})$, where $E_c(H)$ denotes the projection onto the continuous spectral subspace for H. Given the completeness of Ω, the only thing one should obtain is the mapping property of Ω in the weighted space L_2^2 and the best thing one could hope is that

$$\Omega X_m = L_2^2 \cap E_c(H)\, D(|H|^{-m/2}).$$

To prove the formula above, the methods in [1], [13], [14] seem useful although the author does not have a definite answer.

(2) The results in this paper break down for potentials with slower decay *i.e.* $\mu \leq 1/2$. This is natural since the existence of wave operators of Dollard type breaks down in the same range. In order to include the case $\mu \leq 1/2$, one needs to refine the phase function S by further iteration with optimal decay rate preserved. This gives higher singularity at the origin in the momentum space. We could give a full description on this subject but the whole procedure is somewhat involved and therefore the details are omitted.

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