J.-P. NICOLAS

Scattering of linear Dirac fields by a spherically symmetric Black-Hole


<http://www.numdam.org/item?id=AIHPA_1995__62_2_145_0>
Scattering of linear Dirac fields by a spherically symmetric Black-Hole

by

J.-P. NICOLAS

Mathematical Institute 24-29 St-Giles', Oxford OX1 3LB, England

or

CeReMaB, Université Bordeaux-I, Unité de Recherche Associée au CNRS n° 226, 351, cours de la Libération, 33405 Talence Cedex, France.

ABSTRACT. – We study the linear Dirac system outside a spherical Black-Hole. In the case of massless fields, we prove the existence and asymptotic completeness of classical wave operators at the horizon of the Black-Hole and at infinity.


1. INTRODUCTION

We develop a time-dependent scattering theory for the linear Dirac system on Schwarzschild-type metrics. The first time-dependent scattering results on the Schwarzschild metric were obtained by J. Dimock [8]. Using the short range at infinity of the interaction between gravity and a massless scalar field, he proved the existence and asymptotic completeness of classical wave-operators for the wave equation. The case of the Maxwell...
system in which the interaction is pseudo long-range has been worked out by A. Bachelot [2], and for the Regge-Wheeler equation, a complete scattering theory has been developed by A. Bachelot and A. Motet-Bachelot [3]. Our purpose in this work is to study the classical wave operators and their asymptotic completeness for the linear massless Dirac system on a general “Schwarzschild-type” metric which covers all the usual cases of spherical black-holes. The main tools are Cook’s method for the existence and the results obtained in [3] for the asymptotic completeness.

Let us consider the manifold $\mathbb{R}_+ \times [0, +\infty), \mathbb{R} \times S^2_{\phi, \psi}$ endowed with the pseudo-riemannian metric

$$g_{\mu \nu} \, dx^\mu \, dx^\nu = F(r) \, e^{2\phi(r)} \, dt^2 - \left[ F(r)^{-1} \, dr^2 + r^2 \, d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \right]$$ (1)

where $F, \phi \in C^\infty$ on $[0, +\infty)$. We assume the existence of three values $r_\nu$ of $r$, $0 \leq r_- < r_0 < r_+ \leq +\infty$, which are the only possible zeros of $F$, such that

$$F(r_\nu) = 0, \quad F'(r_\nu) = 2 \kappa_\nu, \quad \kappa_\nu \neq 0, \quad \text{if } 0 < r_\nu < +\infty,$$

$$F(r) > 0 \quad \text{for } r \in ]r_0, r_+[, \quad F(r) < 0 \quad \text{for } r \in ]r_-, r_0[.$$ When they are finite and non zero, $r_-, r_0$ and $r_+$ are the radii of the spheres called: horizon of the black-hole ($r_0$), Cauchy horizon ($r_-$) and cosmological horizon ($r_+$). $\kappa_\nu$ is the surface gravity at the horizon $\{r = r_\nu\}$. If $r_+$ is infinite, we assume moreover that

$$F(r) = 1 - \frac{r_1}{r} + O(r^{-2}), \quad r_1 > 0,$$

$$\delta(r) = \delta(+\infty) + o(r^{-1}), \quad r \to +\infty,$$

$$F'(r), \quad \delta'(r) = O(r^{-2}), \quad r \to +\infty.$$ All these properties are satisfied by usual spherical black-holes (see [13]).

NOTATIONS. – Let $(M, g)$ be a Riemannian manifold, $C^\infty_0(M)$ denotes the set of $C^\infty$ functions with compact support in $M$, $H^k(M, g)$, $k \in \mathbb{N}$ is the Sobolev space, completion of $C^\infty_0(M)$ for the norm

$$\| f \|^2_{H^k(M)} = \sum_{j=0}^k \int_M \langle \nabla^j f, \nabla^j f \rangle \, d\mu,$$

where $\nabla^j, d\mu$ and $\langle, \rangle$ are respectively the covariant derivatives, the measure of volume and the hermitian product associated with the metric $g$. We write $L^2(M, g) = H^0(M, g)$.

If $E$ is a distribution space on $M$, $E_{\text{comp}}$ represents the subspace of elements of $E$ with compact support in $M$.
The 2-dimensional euclidian sphere $S_2^2$ is endowed with its usual metric
\[ d\omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi. \]

2. THE COVARIANT GENERALIZATION OF THE LINEAR DIRAC SYSTEM ON SCHWARZSCHILD-TYPE METRICS

The covariant generalization of the Dirac system on the metric $g$ has the form
\[ (i \gamma^\mu \nabla_\mu - m) \Phi = 0, \quad m \geq 0 \quad (2) \]
for a particle with mass $m$, where $\Phi$ is a Dirac 4-spinor, the $\gamma^\mu$ are the contravariant Dirac matrices on curved space-time and $\nabla_\mu$ is the covariant derivation of spinor fields. We make the following choices of flat space-time Dirac matrices
\[ \gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \quad \gamma_{\tilde{\alpha}} = \begin{pmatrix} 0 & \sigma_\alpha \\ -\sigma_\alpha & 0 \end{pmatrix} \quad \alpha = 1, 2, 3 \quad (3) \]
where
\[ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
\[ \sigma_2 = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4) \]
are the Pauli matrices, and of local Lorentz frame
\[ e_{\tilde{\alpha}}^\mu = \begin{cases} |g^{\mu\nu}|^{\frac{1}{2}} & \text{if } \tilde{\alpha} = \mu, \\ 0 & \text{if } \tilde{\alpha} \neq \mu. \end{cases} \quad (5) \]
We recall that flat space-time Dirac matrices are a set of $4 \times 4$ matrices $\{\gamma_{\tilde{\alpha}}\}_{0 \leq \tilde{\alpha} \leq 3}$ such that
\[ \{\gamma_{\tilde{\alpha}}, \gamma_{\tilde{\beta}}\} = \gamma_{\tilde{\alpha}} \gamma_{\tilde{\beta}} + \gamma_{\tilde{\beta}} \gamma_{\tilde{\alpha}} = 2 \eta_{\tilde{\alpha}\tilde{\beta}} 1 \quad (\tilde{\alpha}, \tilde{\beta} = 0, 1, 2, 3) \quad (6) \]
where
\[ \eta_{\tilde{\alpha}\tilde{\beta}} = \text{diag} (1, -1, -1, -1) \quad (7) \]
is the Minkowski metric. The indices with a tilde refer to flat space-time and can be raised or lowered using $\eta_{\tilde{\alpha}\tilde{\beta}}$, whereas the indices without tilde refer to curved space-time and are raised or lowered using the metric $g$.

With these definitions, the $\gamma^\mu$ and $\nabla_\mu$ are then defined by (see for example [5], [7])
\[ \gamma^\mu = \gamma_{\tilde{\alpha}} e^{\tilde{\alpha}\mu} \quad (8) \]
and
\[ \nabla_\mu = \partial_\mu + \frac{1}{2} G_{[\alpha\beta]} \omega^{\alpha\beta}_\mu \]  
(9)

where
\[ G_{[\alpha\beta]} = \frac{1}{4} [\gamma_\alpha, \gamma_\beta] \equiv \frac{1}{4} (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha) \]  
(10)

are the generators of the spinor representation of the proper Lorentz group and
\[ \omega^{\alpha\beta}_\mu = \frac{1}{2} e^{\gamma_\nu} (e^{\beta}_{\nu,\mu} - e^{\beta}_{\mu,\nu}) - \frac{1}{2} e^{\delta_\nu} (e^{\alpha}_{\nu,\mu} - e^{\alpha}_{\mu,\nu}) \]
\[ + \frac{1}{2} e^{\gamma_\nu} e^{\beta_\sigma} (e^{\gamma}_{\nu,\sigma} - e^{\gamma}_{\sigma,\nu}) e^{\gamma}_\mu = -\omega^{\beta_\alpha}_\mu \]  
(11)

are the coefficients of the spin connection, \( \cdot_\mu \) standing for the derivation with respect to the \( \mu \)-th variable. We compute the \textit{a priori} non zero components:

\[ \omega^{\hat{t}\hat{r}}_t = \frac{1}{2} e^{\hat{t}\hat{r}} [\partial_t (e^{\hat{r}}_t) - \partial_t (e^{\hat{r}}_t)] - \frac{1}{2} e^{\hat{r}\hat{r}} [\partial_t (e^{\hat{r}}_t) - \partial_t (e^{\hat{r}}_t)] \]
\[ + \frac{1}{2} e^{\hat{t}\hat{t}} e^{\hat{r}\hat{r}} [\partial_r (e^{\hat{r}}_t) - \partial_t (e^{\hat{r}}_t)] e^{\hat{t}\hat{t}} \]
\[ = \frac{1}{2} e^{\hat{r}\hat{r}} \partial_r (e^{\hat{r}}_t) (1 + e^{\hat{t}\hat{t}} e^{\hat{t}\hat{t}}) = \frac{1}{2} (-F^{1/2} \partial_r (F^{1/2} e^\delta) \]
\[ \times (1 + F^{-1/2} e^{-\delta} F^{1/2} e^\delta) = -\left( \frac{F'}{2} + F'\delta' \right) e^\delta, \]

\[ \omega^{\hat{r}\hat{r}}_r = \frac{1}{2} e^{\hat{t}\hat{r}} [\partial_r (e^{\hat{r}}_t) - \partial_t (e^{\hat{r}}_t)] - \frac{1}{2} e^{\hat{r}\hat{r}} [\partial_r (e^{\hat{r}}_t) - \partial_r (e^{\hat{r}}_t)] \]
\[ + \frac{1}{2} e^{\hat{t}\hat{t}} e^{\hat{r}\hat{r}} [\partial_r (e^{\hat{r}}_t) - \partial_t (e^{\hat{r}}_t)] e^{\hat{t}\hat{t}} = 0, \]

\[ \omega^{\hat{t}\hat{\theta}}_t = \frac{1}{2} e^{\hat{t}\hat{\theta}} [\partial_t (e^{\hat{\theta}}_t) - \partial_t (e^{\hat{\theta}}_t)] - \frac{1}{2} e^{\hat{\theta}\hat{\theta}} [\partial_t (e^{\hat{\theta}}_t) - \partial_t (e^{\hat{\theta}}_t)] \]
\[ + \frac{1}{2} e^{\hat{t}\hat{t}} e^{\hat{\theta}\hat{\theta}} [\partial_t (e^{\hat{\theta}}_t) - \partial_t (e^{\hat{\theta}}_t)] e^{\hat{t}\hat{t}} = 0, \]

\[ \omega^{\hat{\theta}\hat{\theta}}_\theta = \frac{1}{2} e^{\hat{t}\hat{\theta}} [\partial_\theta (e^{\hat{\theta}}_t) - \partial_t (e^{\hat{\theta}}_t)] - \frac{1}{2} e^{\hat{\theta}\hat{\theta}} [\partial_\theta (e^{\hat{\theta}}_t) - \partial_\theta (e^{\hat{\theta}}_t)] \]
\[ + \frac{1}{2} e^{\hat{t}\hat{t}} e^{\hat{\theta}\hat{\theta}} [\partial_\theta (e^{\hat{\theta}}_t) - \partial_t (e^{\hat{\theta}}_t)] e^{\hat{t}\hat{t}} = 0, \]
\[ \omega_{\xi_\varphi} = \frac{1}{2} e^{it} \left[ \partial_t (e^{\xi_\varphi}) - \partial_t (e^{\xi_\varphi}) \right] - \frac{1}{2} e^{\xi_\varphi} \left[ \partial_t (e^{\xi_\varphi}) - \partial_t (e^{\xi_\varphi}) \right] \\
+ \frac{1}{2} e^{it} e^{\xi_\varphi} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_t (e^{\xi_\varphi}) \right] e_{t\varphi} = 0, \]

\[ \omega_{\xi_\varphi} = \frac{1}{2} e^{it} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_t (e^{\xi_\varphi}) \right] - \frac{1}{2} e^{\xi_\varphi} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_\varphi (e^{\xi_\varphi}) \right] \\
+ \frac{1}{2} e^{it} e^{\xi_\varphi} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_t (e^{\xi_\varphi}) \right] e_{\varphi\varphi} = 0, \]

\[ \omega_{\xi_\varphi} = \frac{1}{2} e^{ir} \left[ \partial_r (e^{\xi_\varphi}) - \partial_r (e^{\xi_\varphi}) \right] - \frac{1}{2} e^{\xi_\varphi} \left[ \partial_r (e^{\xi_\varphi}) - \partial_r (e^{\xi_\varphi}) \right] \\
+ \frac{1}{2} e^{ir} e^{\xi_\varphi} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_r (e^{\xi_\varphi}) \right] e_{r\varphi} = 0, \]

\[ \omega_{\xi_\varphi} = \frac{1}{2} e^{ir} \left[ \partial_\theta (e^{\xi_\varphi}) - \partial_r (e^{\xi_\varphi}) \right] - \frac{1}{2} e^{\xi_\varphi} \left[ \partial_\theta (e^{\xi_\varphi}) - \partial_\theta (e^{\xi_\varphi}) \right] \\
+ \frac{1}{2} e^{ir} e^{\xi_\varphi} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_r (e^{\xi_\varphi}) \right] e_{r\theta} = F^{1/2}, \]

\[ \omega_{\xi_\varphi} = \frac{1}{2} e^{ir} \left[ \partial_r (e^{\xi_\varphi}) - \partial_r (e^{\xi_\varphi}) \right] - \frac{1}{2} e^{\xi_\varphi} \left[ \partial_r (e^{\xi_\varphi}) - \partial_r (e^{\xi_\varphi}) \right] \\
+ \frac{1}{2} e^{ir} e^{\xi_\varphi} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_r (e^{\xi_\varphi}) \right] e_{r\varphi} = 0, \]

\[ \omega_{\xi_\varphi} = \frac{1}{2} e^{ir} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_r (e^{\xi_\varphi}) \right] - \frac{1}{2} e^{\xi_\varphi} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_\varphi (e^{\xi_\varphi}) \right] \\
+ \frac{1}{2} e^{ir} e^{\xi_\varphi} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_r (e^{\xi_\varphi}) \right] e_{\varphi\varphi} = \frac{F^{1/2}}{2} \sin \theta, \]

\[ \omega_{\xi_\varphi} = \frac{1}{2} e^{\xi_\varphi} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_\theta (e^{\xi_\varphi}) \right] - \frac{1}{2} e^{\xi_\varphi} \left[ \partial_\theta (e^{\xi_\varphi}) - \partial_\varphi (e^{\xi_\varphi}) \right] \\
+ \frac{1}{2} e^{\xi_\varphi} e^{\xi_\varphi} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_\theta (e^{\xi_\varphi}) \right] e_{\theta\varphi} = 0, \]

\[ \omega_{\xi_\varphi} = \frac{1}{2} e^{\xi_\varphi} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_\theta (e^{\xi_\varphi}) \right] - \frac{1}{2} e^{\xi_\varphi} \left[ \partial_\theta (e^{\xi_\varphi}) - \partial_\varphi (e^{\xi_\varphi}) \right] \\
+ \frac{1}{2} e^{\xi_\varphi} e^{\xi_\varphi} \left[ \partial_\varphi (e^{\xi_\varphi}) - \partial_\theta (e^{\xi_\varphi}) \right] e_{\theta\varphi} = \cos \theta \]
and we obtain the following expression for the linear massive Dirac equation outside a spherical black-hole:

\[
\begin{align*}
\gamma^0 \partial_t + Fe^{\delta} \gamma^1 \left( \partial_r + \frac{1}{r} \frac{F'}{4F} + \delta' \right) + \frac{F^{1/2} e^{\delta}}{r} \gamma^2 \left( \partial_\theta + \frac{1}{2} \cot \theta \right) \\
+ \frac{F^{1/2} e^{\delta}}{r \sin \theta} \gamma^3 \partial_\varphi + i F^{1/2} e^{\delta} m \right) \Phi = 0.
\end{align*}
\] (12)

We introduce the frame with respect to which we shall express the equation, \( \mathcal{R}' = \left( \frac{1}{r \sin \theta} \partial_\varphi, - \frac{1}{r} \partial_\theta, F^{1/2} \partial_r \right) \), image of \( \mathcal{R} = \left( F^{1/2} \partial_r, \frac{1}{r} \partial_\theta, \frac{1}{r \sin \theta} \partial_\varphi \right) \) by the spatial rotation \( f \) with Euler angles (see for example [15]) \((\varphi, \theta, \psi) = (0, \pi/2, \pi)\), and the Regge-Wheeler variable \( r_* \) defined by

\[
\frac{dr}{dr_*} = F e^{\delta}, \quad r \in ]r_0, r_+ [.
\] (13)

The spinor

\[
\Psi = T_{(f^{-1})} r F^{1/4} e^{\delta/2} \Phi,
\] (14)

where \( T_{(f^{-1})} \) is the spin transformation associated with the rotation \( f^{-1} \), satisfies

\[
\partial_t \Psi = i H \Psi,
\]

\[
H = i \left[ \gamma^0 \gamma^3 \partial_{r_*} - \frac{F^{1/2} e^{\delta}}{r} \gamma^0 \gamma^2 \left( \partial_\theta + \frac{1}{2} \cot \theta \right) \\
+ \frac{F^{1/2} e^{\delta}}{r \sin \theta} \gamma^3 \partial_\varphi + i \gamma^0 F^{1/2} e^{\delta} m \right]
\] (15)

on the domain \( \mathbb{R}_t \times \mathbb{R}_{r_*} \times S_\omega^2 \) representing the exterior of the black-hole in the variables \((t, r_*, \omega)\).

We recall (see [7]) that, given a spatial rotation \( f \) of angle \( \theta \) around a unit vector \( n = (n_1, n_2, n_3) \), its associated spin transformation \( T_f \) is

\[
T_f = \text{Exp} \left\{ [n_1 G_{[\bar{2}, \bar{3}]} + n_2 G_{[\bar{3}, \bar{1}]} + n_3 G_{[\bar{1}, \bar{2}]}] \theta \right\}
\] (16)

where \( \text{Exp} \) is the exponential mapping.

3. GLOBAL CAUCHY PROBLEM

We introduce the Hilbert space

\[
\mathcal{H} = \{ L^2 (\mathbb{R}_{r_*} \times S_\omega^2, \, dr_*^2 + d\omega^2) \}^4.
\] (17)

Annales de l’Institut Henri Poincaré - Physique théorique
THEOREM 3.1. − Given \( \Psi_0 \in \mathcal{H} \), equation (15) has a unique solution \( \Psi \) such that
\[
\Psi \in C(\mathbb{R}_t; \mathcal{H}), \quad \Psi|_{t=0} = \Psi_0.
\] (18)

Moreover, for any \( t \in \mathbb{R} \)
\[
\| \Psi(t) \|_{\mathcal{H}} = \| \Psi_0 \|_{\mathcal{H}}.
\] (19)

Proof. − We show that the operator
\[
\tilde{H} = H + \gamma_0 F^{1/2} e^\delta m
\] (20)
is self-adjoint with dense domain on \( \mathcal{H} \). We decompose \( \mathcal{H} \) using generalized spherical functions of weights 1/2 and \(-1/2\). Let
\[
\mathcal{I} = \{(l, m, n); 2l, 2m, 2n \in \mathbb{Z}; l - |m|, l - |n| \in \mathbb{N}\}
\] (21)
and for any half-integer \( m \)
\[
\mathcal{I}_m = \{(l, n); (l, m, n) \in \mathcal{I}\}.
\] (22)

For \( (l, m, n) \in \mathcal{I} \), we define the function \( T_{mn}^l \) of \( (\varphi_1, \theta, \varphi_2) \), \( \varphi_1, \varphi_2 \in [0, 2\pi], \theta \in [0, \pi] \), by
\[
T_{mn}^l (\varphi_1, \theta, \varphi_2) = e^{-im\varphi_2} u_{mn}^l (\theta) e^{-in\varphi_1}
\] (23)
where \( u_{mn}^l \) satisfies the following ordinary differential equations
\[
\frac{d^2 u_{mn}^l}{d\theta^2} + \cotg \theta \frac{du_{mn}^l}{d\theta} + \left[ l(l+1) - \frac{n^2 - 2mn \cos \theta + m^2}{\sin^2 \theta} \right] u_{mn}^l = 0,
\] (24)
\[
\frac{du_{mn}^l}{d\theta} - \frac{n - m \cos \theta}{\sin \theta} u_{mn}^l = -i [(l + m)(l - m + 1)]^{1/2} u_{m-1,n}^l,
\] (25)
\[
\frac{du_{mn}^l}{d\theta} + \frac{n - m \cos \theta}{\sin \theta} u_{mn}^l = -i [(l + m + 1)(l - m)]^{1/2} u_{m+1,n}^l
\] (26)
and the normalization condition
\[
\int_0^{\pi} |u_{mn}^l (\theta)|^2 \sin \theta d\theta = \frac{1}{4 \pi^2}.
\] (27)

We know from [12], that \( \{T_{mn}^l\}_{(l,m,n) \in \mathcal{I}_1} \) is a Hilbert basis of
\[
L^2([0, 2\pi]_\varphi_1 \times [0, \pi]_{\theta} \times [0, 2\pi]_{\varphi_2}; \sin^2 \theta d\varphi_1^2 + d\theta^2 + d\varphi_2^2).
\] (28)
Thus, for any half-integer \( m \),
\[
\{ T_{mn}^l (\varphi, \theta, 0) = e^{-im\varphi} u_{mn}^l (\theta) \} \forall (l,n) \in I_m
\]
is a Hilbert basis of \( L^2 (S^2_\omega, d\omega^2) \). In particular,
\[
\mathcal{H} = \bigoplus_{(l,n) \in I_{\frac{1}{2}}} \mathcal{H}_{ln}
\]
(29)
where
\[
\mathcal{H}_{ln} = \{ t(f_1 T_{-\frac{1}{2}, n}^l, f_2 T_{\frac{1}{2}, n}^l, f_3 T_{-\frac{1}{2}, n}^l, f_4 T_{\frac{1}{2}, n}^l) ;
\]
\[
f_i \in L^2 (R, d\omega^2), \; i = 1, 2, 3, 4 \},
\]
or equivalently,
\[
\mathcal{H}_{ln} = [L^2 (R, d\omega^2)]^4 \otimes F_{ln}; \; F_{ln} = \{ T_{-\frac{1}{2}, n}^l, T_{\frac{1}{2}, n}^l, T_{-\frac{1}{2}, n}^l, T_{\frac{1}{2}, n}^l \}
\]
(31)
where the \( T_{\pm\frac{1}{2}, n} \) are seen as functions of only \( \varphi, \theta \). Let
\[
\Psi = t(f_1, f_2, f_3, f_4) \otimes F_{ln} \in \mathcal{H}_{ln}.
\]

Denoting \( \alpha = F^{1/2} e^{\delta} \), the four components of \( \tilde{H} \Psi \) are
\[
i \partial_r f_3 T_{-\frac{1}{2}, n}^l - \frac{\alpha}{r} f_4 \left( \partial_\theta + \frac{1}{2} \cot \theta \right) T_{\frac{1}{2}, n}^l + i \frac{\alpha}{r \sin \theta} f_4 \partial_\varphi T_{\frac{1}{2}, n}^l,
\]
\[
-i \partial_r f_4 T_{\frac{1}{2}, n}^l + \frac{\alpha}{r} f_3 \left( \partial_\theta + \frac{1}{2} \cot \theta \right) T_{-\frac{1}{2}, n}^l + i \frac{\alpha}{r \sin \theta} f_3 \partial_\varphi T_{-\frac{1}{2}, n}^l,
\]
\[
i \partial_r f_1 T_{-\frac{1}{2}, n}^l - \frac{\alpha}{r} f_2 \left( \partial_\theta + \frac{1}{2} \cot \theta \right) T_{\frac{1}{2}, n}^l + i \frac{\alpha}{r \sin \theta} f_2 \partial_\varphi T_{\frac{1}{2}, n}^l,
\]
\[
-i \partial_r f_2 T_{\frac{1}{2}, n}^l + \frac{\alpha}{r} f_1 \left( \partial_\theta + \frac{1}{2} \cot \theta \right) T_{-\frac{1}{2}, n}^l + i \frac{\alpha}{r \sin \theta} f_1 \partial_\varphi T_{-\frac{1}{2}, n}^l.
\]

Relations (25) and (26) yield
\[
\left( \partial_\theta + \frac{1}{2} \cot \theta \right) T_{\frac{1}{2}, n}^l = \frac{n}{\sin \theta} T_{\frac{1}{2}, n}^l - i \left( l + \frac{1}{2} \right) T_{-\frac{1}{2}, n}^l,
\]
(32)
\[
\left( \partial_\theta + \frac{1}{2} \cot \theta \right) T_{-\frac{1}{2}, n}^l = \frac{-n}{\sin \theta} T_{-\frac{1}{2}, n}^l - i \left( l + \frac{1}{2} \right) T_{\frac{1}{2}, n}^l.
\]
(33)

Annales de l'Institut Henri Poincaré - Physique théorique
and we also have

\[ \partial_\varphi T_{\pm \frac{1}{2}, n} (\varphi, \theta, 0) = -in T_{\pm \frac{1}{2}, n} (\varphi, \theta, 0). \]  

(34)

Thus, the four components of \( \tilde{H} \Psi \) are

\[ \left( i \partial_r, f_3 + \frac{i \alpha}{r} \left( l + \frac{1}{2} \right) f_4 \right) T_{- \frac{1}{2}, n}, \]

\[ \left( -i \partial_r, f_4 - \frac{i \alpha}{r} \left( l + \frac{1}{2} \right) f_3 \right) T_{\frac{1}{2}, n}, \]

\[ \left( i \partial_r, f_1 + \frac{i \alpha}{r} \left( l + \frac{1}{2} \right) f_2 \right) T_{- \frac{1}{2}, n}, \]

\[ \left( -i \partial_r, f_2 - \frac{i \alpha}{r} \left( l + \frac{1}{2} \right) f_1 \right) T_{\frac{1}{2}, n}. \]

We see that on \( \mathcal{H}_{\text{in}}, \tilde{H} \) has the form

\[ \tilde{H} \big|_{\mathcal{H}_{\text{in}}} = \left( i \partial_r, L + \frac{\alpha}{r} \left( l + \frac{1}{2} \right) M \right)_{\text{r+}} \otimes 1_{\theta, \varphi} \]  

(35)

where the matrices \( L \) et \( M \), defined by

\[
L = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\end{bmatrix} \quad M = \begin{bmatrix}
0 & 0 & 0 & i \\
0 & 0 & -i & 0 \\
0 & i & 0 & 0 \\
-i & 0 & 0 & 0 \\
\end{bmatrix}
\]  

(36)

are hermitian and \( L \) is invertible. Since the function \( \alpha r^{-1} \) belongs to \( L^\infty (\mathbb{R}_{r+}) \), \( \tilde{H} \big|_{\mathcal{H}_{\text{in}}} \) is self-adjoint with domain

\[ D_{\text{in}} = [D (i \partial_{r+})]^4 \otimes F_{\text{in}} \simeq [H^1 (\mathbb{R}_{r+}; dr_{\ast}^2)]^4 \otimes F_{\text{in}} \]  

(37)

dense in \( \mathcal{H}_{\text{in}} \). On \( D_{\text{in}} \), we choose the following norm

\[
\| \Psi \|_{D_{\text{in}}}^2 = \| \Psi \|_{(L^2 (\mathbb{R}))^4}^2 + \left\| \left( i \partial_r, L + \frac{\alpha}{r} \left( l + \frac{1}{2} \right) M \right) \Psi \right\|_{(L^2 (\mathbb{R}))^4}^2
\]  

(38)

and we introduce the dense subspace of \( \mathcal{H} \)

\[ D (H) = \{ \Psi = \sum_{(l, n) \in \mathcal{I}_{\frac{1}{2}}} \Psi_{ln}; \Psi_{ln} \in D_{\text{in}}, \} \]

\[ \sum_{(l, n) \in \mathcal{I}_{\frac{1}{2}}} \| \Psi_{ln} \|_{D_{\text{in}}}^2 < +\infty \}. \]  

(39)

\[ \hat{H} \text{ is self-adjoint on } \mathcal{H} \text{ with domain } D(H), \gamma^0\alpha m \text{ is self-adjoint and bounded on } \mathcal{H}, \text{ therefore, } H \text{ is self-adjoint on } \mathcal{H} \text{ with dense domain } D(H). \text{ Theorem 3.1 follows from Stone's theorem.} \] Q.E.D.

4. WAVE OPERATORS AT THE HORIZON

When \( r \to r_0 \), the operator \( H \) has the formal limit
\[
H_0 = i\gamma^0 \gamma^3 \partial_r
\]
which is a self-adjoint operator on \( \mathcal{H} \) with dense domain
\[
D(H_0) = \{ H^1(\mathbb{R}^3; dr^2); L^2(\mathbb{S}^2; d\omega^2) \}^4.
\]
The spectrum of \( H_0 \) is purely absolutely continuous. We define the subspaces of incoming and outgoing waves associated with \( H_0 \):
\[
\mathcal{H}_0^\pm = \{ \Psi = \phi(u^1, u^2, u^3, u^4), \ u^3 = \mp u^1, \ u^4 = \pm u^2 \}.
\]
\( \mathcal{H}_0^\pm \) as well as the \( \mathcal{H}_{in} \) remain stable under \( H_0 \) and we have
\[
\mathcal{H} = \mathcal{H}_0^+ \oplus \mathcal{H}_0^-,
\]
\( \forall \Psi_0 \in \mathcal{H}_0^\pm, (e^{iH_0 t} \Psi_0)(r_*, \omega) = \Psi_0(r_\pm t, \omega) \).

Since we want to compare \( H \) with \( H_0 \) in the neighbourhood of the horizon, we introduce the cut-off function
\[
\chi_0 \in C^\infty(\mathbb{R}^3), \ 0 \leq \chi_0 \leq 1, \quad \exists a, b \in \mathbb{R}, \ a < b \text{ such that }
\]
\[
\text{for } r_* < a \quad \chi_0(r_*) = 1; \quad \text{for } r_* > b \quad \chi_0(r_*) = 0
\]
together with the identifying operator
\[
\mathcal{J}_0 : \mathcal{H} \to \mathcal{H}, \quad \Psi \mapsto \chi_0 \Psi.
\]

We consider the classical wave operators
\[
W_0^\pm \Psi_0 = \lim_{t \to \pm\infty} e^{-iHt} \mathcal{J}_0 e^{iH_0 t} \Psi_0 \text{ in } \mathcal{H}.
\]

**Theorem 4.1.** - The operator \( W_0^+ \) (resp. \( W_0^- \)) is well-defined from \( \mathcal{H}_0^+ \) (resp. \( \mathcal{H}_0^- \)) to \( \mathcal{H} \), is independent of the choice of \( \chi_0 \) satisfying (44), moreover
\[
\forall \Psi_0 \in \mathcal{H}_0^\pm, \| W_0^+ \Psi_0 \|_\mathcal{H} = \| \Psi_0 \|_\mathcal{H}.
\]

**Proof.** - We apply Cook's method. \( \mathcal{J}_0 \) being a bounded operator, it suffices to prove that for
\[
\Psi_0 \in D_{in}^\pm, \quad D_{in}^\pm = \mathcal{H}_0^\pm \cap \mathcal{H}_{in} \cap [C^\infty(\mathbb{R}^3 \times \mathbb{S}^2)]^4, \quad (l, n) \in \mathcal{T}_{in}^\pm
\]
we have
\[ \| (H \mathcal{I}_0 - \mathcal{I}_0 H_0) e^{iH_0 t} \Psi_0 \|_{\mathcal{H}} \in L^1 (\pm t > 0). \] (49)

Let for \((l, n) \in \mathcal{I}_{\frac{1}{2}}\)
\[ \Psi_0 \in D_{in}^{\pm}, \quad \text{Supp } \Psi_0 \subset [-R, R]_r \times S^2_\omega, \quad R > 0, \] (50)
then
\[ H e^{iH_0 t} \Psi_0 = \left( i \partial_{r_*} + \frac{\alpha}{r} \left( l + \frac{1}{2} \right) M - \alpha m \gamma^0 \right) \Psi_0 (r_* + t), \]
and
\[ H_0 e^{iH_0 t} \Psi_0 = i \partial_{r_*} L \Psi_0 (r_* + t). \]

\(\Psi_0\) being compactly supported, for \(t\) large enough,
\[ \| (H \mathcal{I}_0 - \mathcal{I}_0 H_0) e^{iH_0 t} \Psi_0 \|_{\mathcal{H}} \]
\[ = \left\| \left( \frac{\alpha}{r} \left( l + \frac{1}{2} \right) M - \alpha m \gamma^0 \right) e^{iH_0 t} \Psi_0 \right\|_{\mathcal{H}} \]
\[ \leq \left\| \left( l + \frac{1}{2} \right) \frac{\alpha}{r} + \alpha m \right\|_{L^\infty (-R-t, R-t)} \| \Psi_0 \|_{\mathcal{H}}. \]
\(\alpha\) is rapidly decreasing in \(r_*\) when \(r \to r_0\), therefore
\[ \| (H \mathcal{I}_0 - \mathcal{I}_0 H_0) e^{iH_0 t} \Psi_0 \|_{\mathcal{H}} \in L^1 (t > 0) \]
and \(W_0^+\) is well-defined. The same proof can of course be applied to \(W_0^-\). Furthermore, if \(\Psi_0 \in \mathcal{H}^{\pm}_{0+}\), we get from (43) that the energy of \(e^{iH_0 t} \Psi_0\) in a domain of \(\mathbb{R}_r \times S^2_\omega\) bounded to the left in \(r_*\) vanishes when \(t\) tends to infinity, which gives (47). If now we consider two different cut-off functions \(\chi_o\) and \(\chi'_o\), and the associated identifying operators \(\mathcal{J}_0\) and \(\mathcal{J}'_0\), the difference \(\chi_o - \chi'_o\) is compactly supported, thus
\[ \| e^{-iH t} \mathcal{J}_0 e^{iH_0 t} \Psi_0 - e^{-iH t} \mathcal{J}'_0 e^{iH_0 t} \Psi_0 \|_{\mathcal{H}} \to 0, \quad t \to \pm \infty. \]
Q.E.D.

**Remark 4.1.** – In the case where \(r_+\) is finite, we construct in the same way classical wave operators at the cosmological horizon
\[ W_{1+}^\pm \Psi_0 = \lim_{t \to \pm \infty} e^{-iH t} \mathcal{J}_1 e^{iH_0 t} \Psi_0 \text{ in } \mathcal{H} \] (51)
where the identifying operator \( J_1 \) is defined by
\[
J_1: \quad \mathcal{H} \to \mathcal{H}, \quad \Psi \to \chi_1 \Psi,
\]
(52)
\( \chi_1 \) being a cut-off function
\[
\chi_1 \in C^\infty (\mathbb{R}_r), \quad 0 \leq \chi_1 \leq 1,
\]
(53)
\[ \exists a, b \in \mathbb{R}, \quad a < b \text{ such that } \]
for \( r_* < a \) \( \chi_1 (r_*) = 0; \) for \( r_* > b \) \( \chi_1 (r_*) = 1. \)

\( W_1^+ \) (resp. \( W_1^- \)) is an isometry from \( \mathcal{H}_0^- \) (resp. \( \mathcal{H}_0^+ \)) to \( \mathcal{H} \) and is independent of the choice of \( \chi_1 \) satisfying (53).

5. WAVE OPERATORS AT INFINITY (MASSLESS CASE)

In all this paragraph, we shall assume that \( r_+ = +\infty \); the metric (1) is then asymptotically flat in the neighbourhood of infinity and we choose to compare \( H \) to an operator \( H_\infty \) which is equivalent to the hamiltonian operator for the Dirac equation on the Minkowski space-time. We also make the hypothesis that \( m = 0 \) in order to avoid long range perturbations at infinity. Let us consider on the Minkowski metric
\[
ds_{\mathcal{M}}^2 = dt^2 - dx^2 - dy^2 - dz^2; \quad x, y, z \in \mathbb{R}
\]
(54)
the massless Dirac system
\[
\{ \gamma^\delta \partial_t + \gamma^1 \partial_x + \gamma^2 \partial_y + \gamma^3 \partial_z \} \Phi = 0.
\]
(55)
The associated hamiltonian operator, defined by
\[
H_{\mathcal{M}} = i \gamma^\delta \{ \gamma^1 \partial_x + \gamma^2 \partial_y + \gamma^3 \partial_z \},
\]
(56)
is self-adjoint with dense domain on \([L^2 (\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z)]^4\) and if \( \Phi \in C (\mathbb{R}_t; [L^2 (\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z)]^4) \) is a solution of (55), its energy in a compact domain goes to zero when \( t \) goes to \( \pm \infty \). In addition, for any \( \Phi_0 \in [L^2 (\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z)]^4 \) with a compact support contained into
\[
B (0, R) = \{(x, y, z); \ 0 \leq \rho < R, \ \rho = (x^2 + y^2 + z^2)^{1/2}\},
\]
(57)
the solution \( \Phi \) of (55) associated with the initial data \( \Phi_0 \) satisfies
\[
\Phi (t, x, y, z) = 0 \quad \text{for} \quad 0 \leq \rho \leq |t| - R.
\]
(58)
At the point of spherical coordinates \((\rho, \theta, \varphi)\), we apply the spatial rotation \(f\) with Euler angles \((\pi/2, \theta, \pi - \varphi)\). The local frame \((\partial_x, \partial_y, \partial_z)\) is thus transformed by \(f^{-1}\) into

\[
(\partial_{x^1}, \partial_{x^2}, \partial_{x^3}) = \left(\frac{1}{\rho \sin \theta} \partial_{\varphi}, \frac{-1}{\rho} \partial_{\theta}, \partial_{\rho}\right).
\]

(59)

The spinor

\[
\Psi = \rho T_f \Phi,
\]

(60)

where \(T_f\) is the spin transformation associated with \(f\) defined in (16), satisfies

\[
\partial_t \Psi = i H_\infty \Psi,
\]

\[
H_\infty = i \left[ \gamma^0 \gamma^3 \partial_\rho - \frac{1}{\rho} \gamma^0 \gamma^5 \left( \partial_\theta + \frac{1}{2} \cot \theta \right) + \frac{1}{\rho \sin \theta} \gamma^0 \gamma^5 \partial_{\varphi} \right].
\]

(61)

The operator \(H_\infty\) on

\[
\mathcal{H}_\infty = \left\{ L^2([0, +\infty] \times S^2; d\rho^2 + d\omega^2) \right\}^4
\]

(62)

is unitarily equivalent to \(H_M\) on

\[
\left\{ L^2(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z; dx^2 + dy^2 + dz^2) \right\}. \]

Therefore, \(H_\infty\) is self-adjoint with dense domain on \(\mathcal{H}_\infty\) and if \(\Psi \in C(\mathbb{R}_t, \mathcal{H}_\infty)\) satisfies (61), then its energy in a compact domain goes to zero when \(t\) goes to \(\pm \infty\). Moreover, for

\[
\Psi_0 \in \mathcal{H}_\infty; \quad \text{Supp (}\Psi_0\text{)} \subset B(0, R)
\]

\[
\Psi(t) = e^{iH_\infty t} \Psi_0 \text{ satisfies}
\]

\[
\Psi(t, \rho, \theta, \varphi) = 0 \quad \text{for} \quad 0 \leq \rho \leq |t| - R.
\]

(63)

In order to avoid artificial long-range interactions, we choose

\[
\rho = r_* \geq 0
\]

(64)

and we introduce the cut-off function

\[
\chi_\infty \in C^\infty([0, +\infty], 0 \leq \chi_\infty \leq 1,
\]

\[
\exists 0 < a < b < +\infty \text{ such that}
\]

\[
\chi_\infty(r_*) = 0, \quad \text{for} \quad r_* \geq b \chi_\infty(r_*) = 1
\]

for \(0 \leq r_* \leq a\)

together with the identifying operator

$$J_\infty : \mathcal{H}_\infty \to \mathcal{H}; \quad \text{for } \Psi \in \mathcal{H}_\infty \left\{ \begin{array}{l}
(\mathcal{J} \Psi)|_{\{r_-, \geq 0\}} = \chi_\infty \Psi, \\
(\mathcal{J} \Psi)|_{\{r_-, \leq 0\}} = 0.
\end{array} \right. \quad (66)$$

We define the classical wave operators

$$W_\infty^\pm \Psi_0 = \underset{t \to \pm \infty}{\operatorname{lim}} e^{-iHt} J_\infty e^{iH_\infty^\pm t} \Psi_0 \quad \text{in } \mathcal{H}. \quad (67)$$

**Theorem 5.1.** - The operators $W_\infty^\pm$ are well-defined from $\mathcal{H}_\infty$ to $\mathcal{H}$, are independent of the choice of $\chi_\infty$ and

$$\forall \Psi_0 \in \mathcal{H}_\infty, \quad ||W_\infty^\pm \Psi_0||_\mathcal{H} = ||\Psi_0||_\mathcal{H}_\infty. \quad (68)$$

**Proof.** - For $(l, n) \in I_\frac{1}{2}$, we introduce the subspaces of $\mathcal{H}_\infty$

$$D_{in}^\infty = \{ \Psi = \psi(f_1, f_2, f_3, f_4) \otimes F_{ln} \in \mathcal{H}_\infty; 1 \leq i \leq 4, f_i \in C_0^\infty(\mathbb{R}_r^+ \times \mathbb{R}^{1+n}) \} \quad (69)$$

the direct sum of which is dense in $\mathcal{H}_\infty$. For $\Psi_0 \in D_{in}^\infty$,

$$H_\infty |_{D_{in}^\infty} = \left( i \partial_r, L + \frac{1}{r_*} \left( l + \frac{1}{2} \right) M \right)_{r_*} \quad (70)$$

where the matrices $L$ and $M$ are defined by (36), and

$$J_\infty \Psi_0 \in \mathcal{H}_{ln}. \quad (71)$$

$J_\infty$ being a bounded operator, it suffices to prove that for

$$\Psi_0 \in D_{in}^\infty, \quad \text{Supp}(\Psi_0) \subset B(0, R), \quad (72)$$

we have

$$|| (H J_\infty - J_\infty H_\infty) e^{iH_\infty^t} \Psi_0 ||_\mathcal{H} \in L^1(\mathbb{R}_t). \quad (73)$$

(63) yields

$$e^{iH_\infty^t} \Psi_0 = 0 \quad \text{in } \{ (t, r_*, \theta, \varphi); 0 \leq r_* \leq |t| - R \}. \quad (74)$$

Thus, for $|t|$ large enough

$$|| (H J_\infty - J_\infty H_\infty) e^{iH_\infty^t} \Psi_0 ||_\mathcal{H} \leq \left( l + \frac{1}{2} \right) ||\Psi_0||_{\mathcal{H}_\infty} \left\| \frac{\alpha}{r} - \frac{1}{r_*} \right\|_{L^\infty(|t|+R_1, +\infty[r_*])}. \quad (75)$$

*Annales de l'Institut Henri Poincaré - Physique théorique*
We study the asymptotic behavior of

$$\frac{\alpha}{r} - \frac{1}{r_*} = \frac{1}{r_*} \left( F^{1/2} e^\delta \frac{F_*}{r} - 1 \right)$$

when $r_*$ goes to $+\infty$. The Regge-Wheeler variables $r_*$ is defined with respect to $r$ by

$$r_* = \frac{1}{2\kappa_0} \left\{ \log |r - r_0| - \int_{r_0}^{r} \left[ \frac{1}{r - r_0} - \frac{2\kappa_0}{F e^\delta} \right] dr \right\}$$

(75)

where $2\kappa_0 = F'(r_0)$. For $r$ larger than $r_0 + 1$, we have

$$r_* = C + \int_{r_0+1}^{r} F^{-1} e^{-\delta} \, dr$$

(76)

where

$$2\kappa_0 C = -\int_{r_0}^{r_0+1} \left[ \frac{1}{r - r_0} - \frac{2\kappa_0}{F e^\delta} \right] dr.$$  

(77)

$F$ and $\delta$ satisfy

$$\delta(r) = o(r^{-1}); \quad F(r) = 1 - \frac{r_1}{r} + O(r^{-2}) \quad r_1 > 0; \quad r \to +\infty$$

and therefore

$$F^{-1}(r) e^{-\delta(r)} = 1 + \frac{r_1}{r} + o(r^{-1}),$$

$$r_* = r + r_1 \log (r) + o(\log (r)),$$

$$F^{1/2}(r) e^{\delta(r)} = 1 - \frac{r_1}{2r} + o(r^{-1})$$

which implies

$$F^{1/2}(r) e^{\delta(r)} \frac{r_*}{r} - 1 = r_1 \frac{\log (r)}{r}$$

$$+ o \left( \frac{\log (r)}{r} \right) = O(r^{-1/2}) = O(r_*^{-1/2}).$$

The operators $W^{\pm}_{\infty}$ are thus well-defined. The fact that they are isometries and do not depend upon the choice of the cut-off function can be verified using exactly the same remarks as in the case of the horizon.

Q.E.D.
6. ASYMPTOTIC COMPLETENESS OF OPERATORS $W_0^\pm$ AND $W_\infty^\pm$ (MASSLESS CASE)

We assume again that $m = 0$ and $r_+ = +\infty$. We introduce the inverse wave operators at the horizon and at infinity, defined for $\Psi_0 \in \mathcal{H}$ by

$$
\tilde{W}_0^\pm \Psi_0 = \lim_{t \to \pm \infty} e^{-iH_0 t} \mathcal{J}_0^* e^{iH t} \Psi_0 \quad \text{in } \mathcal{H},
$$

$$
\tilde{W}_\infty^\pm \Psi_0 = \lim_{t \to \pm \infty} e^{-iH_\infty t} \mathcal{J}_\infty^* e^{iH t} \Psi_0 \quad \text{in } \mathcal{H}_\infty,
$$

where $\mathcal{J}_0^*$ and $\mathcal{J}_\infty^*$ are respectively the adjoints of $\mathcal{J}_0$ and $\mathcal{J}_\infty$. We also define the wave operators $W^+$ and $W^-$ by

$$
\Psi_0 \in \mathcal{H}_0^\pm, \quad \Psi_\infty \in \mathcal{H}_\infty, \quad W^\pm (\Psi_0, \Psi_\infty) = W_0^\pm \Psi_0 + W_\infty^\pm \Psi_\infty
$$

as well as the inverse wave operators $\tilde{W}^+, \tilde{W}^-$.

Eventually, we define the scattering operator

$$
S = \tilde{W}^+ \tilde{W}^-.
$$

**Theorem 6.1.** - Operators $\tilde{W}_0^\pm$ (resp. $\tilde{W}_\infty^\pm$) are well defined from $\mathcal{H}$ into $\mathcal{H}_0^\pm$ (resp. from $\mathcal{H}_0^\pm$ into $\mathcal{H}_\infty$), are independent of the choice of $\chi_0$ (resp. $\chi_\infty$) and their norm is lower or equal to 1. Moreover

$W^\pm$ is an isometry of $\mathcal{H}_0^\pm \times \mathcal{H}_\infty$ onto $\mathcal{H}$.

$\tilde{W}_0^\pm$ is an isometry of $\mathcal{H}$ onto $\mathcal{H}_0^\pm \times \mathcal{H}_\infty$.

$S$ is an isometry of $\mathcal{H}_0^- \times \mathcal{H}_\infty$ onto $\mathcal{H}_0^+ \times \mathcal{H}_\infty$.

**Proof.** - For any solution $\Psi$ of (15) in $\mathcal{C}(\mathcal{R}; \mathcal{H}_{lm})$, $(l, n) \in \mathcal{I}_{\frac{1}{2}}$, we construct asymptotic profiles at the horizon and at infinity. The idea is that each component of $\Psi$ satisfies an equation of the form

$$
(\partial_t^2 - \partial_{r_*}^2 + V (r_*)) f = 0
$$

where the potential $V$ has the following properties

$$
V = V_+ - V_-; \quad V_+, V_- \geq 0,
$$

$$
V_+ (r_*) \leq C (1 + |r_*|)^{-1-\varepsilon}, \quad \varepsilon > 0,
$$

$$
V_- (r_*) \leq C (1 + |r_*|)^{-2-\varepsilon}, \quad \varepsilon > 0.
$$

We then apply the scattering results of [3]. This suffices to define $\tilde{W}_0^\pm$, but to prove the existence of $\tilde{W}^\pm_\infty$, we need to recover a solution of $$(\partial_t - iH_\infty) \Psi = 0$$ from the asymptotic profile at infinity.
Firstly, we study some spectral properties of the operator $H$:

**Proposition 6.1.** The point spectrum of $H$ is empty.

A straightforward consequence of proposition 6.1 is

**Corollary 6.1.** For $k \in \mathbb{N}$, the direct sum of the sets

$$
\mathcal{E}_{in}^k = \{ H^k \Psi; \Psi = \psi(f_1, f_2, f_3, f_4) \otimes F_{ln} \in \mathcal{H}_{ln}, \quad 1 \leq i \leq 4, \quad f_i \in C_0^\infty(\mathbb{R}_{r_*}); \quad (l,n) \in \mathcal{T}_2 \}
$$

is dense in $\mathcal{H}$.

**Proof of proposition 6.1.** Let

$$
\Psi_{ln} = \phi \otimes F_{ln} \in \mathcal{H}_{ln}; \quad \phi = \psi(f_1, f_2, f_3, f_4) \in [L^2(\mathbb{R}, dr_*^2)]^4
$$

such that

$$
H \Psi_{ln} = \lambda \Psi_{ln}; \quad \lambda \in \mathbb{R}.
$$

Equation (87) is equivalent to

$$
\begin{align*}
\beta_{1} (r_*^2) &= \left( l + \frac{1}{2} \right) \frac{E^{1/2} e^\delta}{r}.
\end{align*}
$$

We first consider the case $\lambda = 0$. Putting

$$
\begin{align*}
g_1 &= f_1 + f_2, \quad g_2 = f_2 - f_1, \\
g_3 &= f_3 + f_4, \quad g_4 = f_4 - f_3,
\end{align*}
$$

we see that $g_1$ and $g_3$ are solutions of

$$
g' = -\beta_l g,
$$

while $g_2$ and $g_4$ satisfy

$$
f' = \beta_l f.
$$

Thus $\lambda = 0$ is an eigenvalue for $H$ if and only if there exists $l = \frac{1}{2} + k$, $k \in \mathbb{N}$, such that both equations (90) and (91) have solutions in $L^2(\mathbb{R}_{r_*}; dr_*^2)$. $\beta_l$ being smooth on $\mathbb{R}$, any solution of (90) or (91) in $L^1_{loc}(\mathbb{R})$ is necessarily smooth. Moreover, $\beta_l$ decreases exponentially when $r_*$ goes to $-\infty$, thus

$$
\forall r_*^1 \in \mathbb{R} \quad \beta_l \in L^1([-\infty, r_*^1]).
$$

and both integral equation
\begin{align}
    f(r_*) &= 1 + \int_{-\infty}^{r_*} \beta_i \cdot f \, dr_*, \\
    g(r_*) &= 1 - \int_{-\infty}^{r_*} \beta_i \cdot g \, dr_*
\end{align}

have a unique solution in $L^\infty([-\infty, r^1_*)]$, which can be extended on $\mathbb{R}$ as a smooth but not square integrable function. Therefore, (90) and (91) have no non trivial solution in $L^2(\mathbb{R})$ and $\lambda = 0$ is not an eigenvalue for $H$.

If now we suppose $\lambda \neq 0$, the components of $\phi$ satisfy
\begin{align}
    f_1'' &= (\beta_i^2 - \lambda^2) f_1 - \beta_i' f_2, \\
    f_2'' &= (\beta_i^2 - \lambda^2) f_2 - \beta_i' f_1, \\
    f_3'' &= (\beta_i^2 - \lambda^2) f_3 - \beta_i' f_4, \\
    f_4'' &= (\beta_i^2 - \lambda^2) f_4 - \beta_i' f_3.
\end{align}

Functions $g_1 = f_1 + f_2$ and $g_3 = f_3 + f_4$ are eigenvectors in $L^2(\mathbb{R})$ for the operator
\begin{equation}
    L_1 = -\partial^2_{r_*} + \beta_i^2 (r_*) - \beta_i' (r_*)
\end{equation}
associated with the eigenvalue $\lambda^2 > 0$, whereas $g_2 = f_2 - f_1$ and $g_4 = f_4 - f_3$ are eigenvectors in $L^2(\mathbb{R})$ for the operator
\begin{equation}
    L_2 = -\partial^2_{r_*} + \beta_i^2 (r_*) + \beta_i' (r_*)
\end{equation}
associated with the eigenvalue $\lambda^2 > 0$. It is easily seen that potentials
\begin{align}
    V_1 (r_*) &= \beta_i^2 (r_*) - \beta_i' (r_*), \\
    V_2 (r_*) &= \beta_i^2 (r_*) + \beta_i' (r_*)
\end{align}
satisfy (84). Therefore, the operators $L_1$ and $L_2$ are of the same type as the second order operators studied in [3] and have no strictly positive eigenvalue.

Q.E.D.

**Proof of corollary 6.1.** – For $(l, n) \in \mathcal{I}_{\frac{1}{2}}$ and $k \in \mathbb{N}$, if
\[ \Psi = \phi \otimes F_{ln} \in \mathcal{H}_{ln}; \quad \phi \in \mathcal{C}_0^\infty (\mathbb{R}_{r_*}) \]
then $\Psi$ belongs to $D(H^k |_{\mathcal{H}_{ln}})$. $\mathcal{E}_{ln}^k$ is well-defined and is a subset of $\mathcal{H}_{ln}$.

To prove corollary 6.1 it suffices to establish that for $(l, n) \in \mathcal{I}_{\frac{1}{2}}$ and
$k \in \mathbb{N}$, $\mathcal{E}_{ln}^{k}$ is dense in $\mathcal{H}_{ln}$. Let

$$\Psi_0 = \phi_0 \otimes F_{ln} \in \mathcal{H}_{ln}$$

be orthogonal to $\mathcal{E}_{ln}^{k}$. Then, for $\phi \in [C_{0}^{\infty} (\mathbb{R}_{r^*})]^{4}$

$$\langle \phi_0 , H^{k} |_{\mathcal{H}_{ln}} \phi \rangle_{L^{2} (\mathbb{R}_{r^*})} = 0,$$

$H^{k} |_{\mathcal{H}_{ln}}$ being here considered as an operator on $[L^{2} (\mathbb{R}_{r^*})]^{4}$. We have

$$H^{k} |_{\mathcal{H}_{ln}} \phi_0 = 0 \quad \text{in} \quad [D' (\mathbb{R}_{r^*})]^{4} \quad (100)$$

where $D' (\mathbb{R}_{r^*})$ is the space of distributions on $\mathbb{R}_{r^*}$. From (100), we deduce that $\Psi_0$ belongs to $D (H^{k} |_{\mathcal{H}_{ln}})$ and

$$H^{k} \Psi_0 = 0 \quad \text{in} \quad \mathcal{H}_{ln} \quad (101)$$

We know by proposition 6.1 that (101) has no non-trivial solution in $\mathcal{H}_{ln}$. Thus $\mathcal{E}_{ln}^{k}$ is dense in $\mathcal{H}_{ln}$.

Q.E.D.

We also study the spectral properties of operators $L_{1}$, $L_{2}$. We recall their definition for $l - 1/2 \in \mathbb{N}$

$$i = 1, 2, \quad L_{i} = -\partial_{r^*}^{2} + V_{i} (r_{*});$$

$$V_{i} (r_{*}) = \beta_{i}^{2} (r_{*}) + (-1)^{i} \beta_{i}^{1} (r_{*}). \quad (102)$$

**Proposition 6.2.** For $l - 1/2 \in \mathbb{N}$, the spectrum of operators $L_{1}$ and $L_{2}$ is purely absolutely continuous.

**Proof.** We already know that potentials $V_{1}$ and $V_{2}$ satisfy (84), which, from [3] implies that the singular spectrum of $L_{1}$ and $L_{2}$ is empty, that their absolutely continuous spectrum is $[0, +\infty]$ and that their point spectrum contains at the most a finite number of negative or zero eigenvalues, all of them being simple. Furthermore, $V_{1}$ and $V_{2}$ decrease exponentially when $r_{*} \to -\infty$ and 0 is not an eigenvalue. We show that $L_{1}$ and $L_{2}$ do not have any strictly negative eigenvalue either by a method similar to the one used in [3]. We recall that for $l - 1/2 \in \mathbb{N}$, equations

$$1 \leq i \leq 2, \quad L_{i} f = 0 \quad (103)$$

both have on $\mathbb{R}_{r^*}$ a unique continuous strictly positive solution, given respectively by (93) and (94). We consider the general case of a potential

$$V \in L^{\infty} (\mathbb{R}_{r^*}) \cap L^{2} (\mathbb{R}_{r^*}) \quad (104)$$

such that there exists a function $g$, continuous and strictly positive on $\mathbb{R}_{r^*}$, satisfying

$$L_{V} g = 0; \quad L_{V} = -\partial_{r^*}^{2} + V. \quad (105)$$
Let $f \in L^2(\mathbb{R}_r^*)$ be such that
\[ L_V f = -\lambda f, \quad \lambda > 0, \] (106)
which implies
\[ f \in H^2(\mathbb{R}_r^*). \] (107)

We define the cut-off function
\[ \chi \in C_0^\infty(\mathbb{R}_r^*), \quad \text{for} \quad |r_*| \leq \frac{1}{2} \] (108)
\[ \chi(r_*) = 1, \quad \text{for} \quad |r_*| \geq 1 \quad \chi(r_*) = 0. \]

Putting for $n \geq 1$
\[ f_n(r_*) = \chi \left( \frac{r_*}{n} \right) f(r_*), \] (109)
we easily see that
\[ \int_{[-n, n]} (|f'_n|^2 + V |f_n|^2) \, dr_* = -\lambda \int_{[-\frac{n}{2}, \frac{n}{2}]} |f|^2 \, dr_* + o(1). \] (110)

Thus, for $n$ large enough
\[ \int_{[-n, n]} [|f'_n|^2 + V |f_n|^2] \, dr_* < 0. \]

The operator $-\partial^2_{r_*} + V$ on $L^2([-n, n])$ with domain \{ $y \in H^2([-n, n]); y(\pm n) = 0$ \} has a strictly negative eigenvalue $-\lambda_n$ associated with an eigenvector $u$
\[ \begin{cases} -u'' + Vu = -\lambda_n u; & -n < r_* < n, \\ u(-n) = u(n) = 0. \end{cases} \] (111)

Even if it means changing $u$ into $-u$, there exist $\alpha$ and $\beta$ such that
\[ -n \leq \alpha < \beta \leq n, \]
\[ u(\alpha) = u(\beta) = 0, \quad u'(\alpha) > 0, \quad u'(\beta) < 0, \] (112)
\[ u > 0 \quad \text{for} \quad \alpha < r_* < \beta. \]

We denote
\[ I = \int_\alpha^\beta (u' g - ug')' \, dr_. \]

On the one hand, we can write
\[ I = u'(\beta) g(\beta) - u'(\alpha) g(\alpha), \]
$g$ being strictly positive on $\mathbb{R}$, (112) yields

$$I < 0.$$  

On the other hand

$$(u' g - ug')' = u'' g - g'' u = -\lambda_n ug,$$

thus

$$I = \lambda_n \int_\alpha^\beta u gdr > 0.$$  

We end up with a contradiction, which means that $L_V$ has no strictly negative eigenvalue.

Q.E.D.

We now prove the existence of the inverse wave operators $\tilde{W}_0^\pm$ and $\tilde{W}_\infty^\pm$.

For $(l, n) \in \mathcal{I}_{\frac{1}{2}}$, we consider the orthogonal decomposition of $\mathcal{H}_{ln}$

$$\mathcal{H}_{ln} = \mathcal{H}_{ln}^+ \oplus \mathcal{H}_{ln}^-,$$

$$\mathcal{H}_{ln}^\pm = \{ \Psi = \Psi^0(f_1, f_2, f_3, f_4) \otimes F_{ln} \in \mathcal{H}_{ln}; f_2 = \mp f_1, f_4 = \pm f_3 \}. \quad (113)$$

Each $\mathcal{H}_{ln}^\pm$ is stable under $H$ and by corollary 6.1, for $(l, n) \in \mathcal{I}_{\frac{1}{2}}$, $k \in \mathbb{N}$, the sets

$$\mathcal{E}_{ln}^{k, \pm} = \mathcal{E}_{ln}^k \cap \mathcal{H}_{ln}^\pm = \{ H^k \Psi; \Psi = \Psi^0(f_1, \mp f_1, f_3, \pm f_3) \otimes F_{ln} \in \mathcal{H}_{ln}^\pm; f_1, f_3 \in C_0^\infty(\mathbb{R}_r) \}$$

are respectively dense in $\mathcal{H}_{ln}^+$ and $\mathcal{H}_{ln}^-$. For $\Psi_0 \in \mathcal{E}_{ln}^{2, \pm}$ we establish the existence of the strong limits (78) and (79) defining $\tilde{W}_0^\pm \Psi_0$ and $\tilde{W}_\infty^\pm \Psi_0$. The following lemma guarantees the existence of asymptotic profiles for $\Psi_0$. The details of its proof will be given after the proof of theorem 6.1.

**Lemma 6.1.** Given $\Psi_0 \in \mathcal{E}_{ln}^{2, \pm}$, $(l, n) \in \mathcal{I}_{\frac{1}{2}}$, there exists $\Psi_1 \in [C(\mathbb{R}_t; H^1(\mathbb{R}_r)); C^1(\mathbb{R}_t; L^2(\mathbb{R}_r))])^4 \otimes F_{ln}$

such that

$$\partial_t \Psi_1 = i H_0 \Psi_1,$$  

and

$$\lim_{t \to \pm \infty} \| e^{i H t} \Psi_0 - \Psi_1(t) \|_{\mathcal{H}} = 0. \quad (117)$$
Any solution of (116) in $C(\mathbb{R}_t; \mathcal{H})$ and in particular $\Psi_1$ can be expressed in the form

$$\Psi_1(t) = e^{iH_0 t} \Psi_0^+ + e^{iH_0 t} \Psi_0^-$$

(118)

where

$$\Psi_0^+ \in \mathcal{H}_0^+, \quad \Psi_0^- \in \mathcal{H}_0^-.$$  

(119)

Thus, for a cut-off function $\chi_0$ satisfying (44), we have

$$\lim_{t \to +\infty} \| \mathcal{J}_0 \Psi_1(t) - e^{iH_0 t} \Psi_0^+ \|_{\mathcal{H}} = 0.$$  

(120)

That is to say that for $\Psi_0 \in \mathcal{E}_{ln}^{2\varepsilon}$, $(l, n) \in \mathcal{I}_{1/2}$, $\varepsilon = +, -$, there exists

$$\Psi_0^+ \in \mathcal{H}_0^+ \cap \mathcal{H}_{ln}^\varepsilon$$

(121)

such that

$$\lim_{t \to +\infty} \| \mathcal{J}_0 e^{iH t} \Psi_0 - e^{iH_0 t} \Psi_0^+ \|_{\mathcal{H}} = 0$$

(122)

and of course, we can similarly prove the existence of

$$\Psi_0^- \in \mathcal{H}_0^- \cap \mathcal{H}_{ln}^\varepsilon$$

(123)

such that

$$\lim_{t \to -\infty} \| \mathcal{J}_0 e^{iH t} \Psi_0 - e^{iH_0 t} \Psi_0^- \|_{\mathcal{H}} = 0.$$  

(124)

From (121) to (124), we conclude that $\tilde{W}_0^\pm \Psi_0$ is well-defined for $\Psi_0 \in \mathcal{E}_{ln}^{2\varepsilon}$, $(l, n) \in \mathcal{I}_{1/2}$, $\varepsilon = +, -$, and

$$\tilde{W}_0^\pm \Psi_0 \in \mathcal{H}_0^\pm, \quad \| \tilde{W}_0^\pm \Psi_0 \|_{\mathcal{H}_0} \leq \| \Psi_0 \|_{\mathcal{H}}.$$  

(125)

Then, corollary 6.1 yields that the operator $\tilde{W}_0^+$ (resp. $\tilde{W}_0^-$) is well-defined from $\mathcal{H}$ to $\mathcal{H}_0^+$ (resp. $\mathcal{H}_0^-$) and its norm is lower or equal to 1.

In order to prove the existence of $\tilde{W}_0^+$, we need to compare in the neighbourhood of the future infinity the outgoing part of $\Psi_1(t)$ with a solution of

$$(\partial_t - i H_{\infty}) \Psi = 0.$$  

(126)

**Lemma 6.2.** - The operator $W_0^\infty$

$$W_0^\infty \Psi_0 = s \lim_{t \to +\infty} e^{-iH_{\infty} t} \mathcal{J}_\infty^* e^{iH_0 t} \Psi_0$$

(127)

is well-defined from $\mathcal{H}_0^-$ to $\mathcal{H}_{\infty}$ and is independent of the choice of $\chi_\infty$ satisfying (65). Of course $W_0^\infty$ is defined as well from $\mathcal{H}_0^+$ to $\mathcal{H}_{\infty}$ and for $\Psi_0 \in \mathcal{H}_0^+$

$$W_0^\infty \Psi_0 = 0.$$
Lemma 6.2, and (118), (119) yield the existence of
\[ \Psi^+_{\infty} \in \mathcal{H}_{\infty} \] (128)
such that
\[ \lim_{t \to +\infty} \| \mathcal{J}^*_\infty \Psi_1(t) - e^{iH_{\infty}t} \Psi^+_{\infty} \|_{\mathcal{H}_{\infty}} = 0 \] (129)
and therefore
\[ \lim_{t \to +\infty} \| \mathcal{J}^*_\infty e^{iHt} \Psi_0 - e^{iH_{\infty}t} \Psi^+_{\infty} \|_{\mathcal{H}_{\infty}} = 0. \] (130)
which enables us to define \( \tilde{W}^+_{\infty} \) on \( \mathcal{E}^{2\pm}_{ln} \), \( (l, n) \in I_{\frac{1}{2}} \) and by density on \( \mathcal{H} \). The same thing can be done for \( \tilde{W}^-_{\infty} \). Let \( \chi_{\infty} \) and \( \chi'_{\infty} \) be two cut-off functions satisfying (65) and \( \mathcal{J}_{\infty} \) and \( \mathcal{J}'_{\infty} \) the associated identifying operators. For \( t \in \mathbb{R} \), \( \Psi_0 \in \mathcal{H} \)
\[ \| e^{-iH_{\infty}t} \mathcal{J}^*_\infty e^{iHt} \Psi_0 - e^{-iH_{\infty}t} \mathcal{J}'^*_\infty e^{iHt} \Psi_0 \|_{\mathcal{H}_{\infty}} \]
\[ \leq \| (\chi_{\infty} - \chi'_{\infty}) e^{iHt} \Psi_0 \|_{\mathcal{H}}, \]
and
\[ \lim_{t \to \pm\infty} \| e^{-iH_{\infty}t} \mathcal{J}^*_\infty e^{iHt} \Psi_0 - e^{-iH_{\infty}t} \mathcal{J}'^*_\infty e^{iHt} \Psi_0 \|_{\mathcal{H}_{\infty}} = 0. \]
Thus, the operators \( \tilde{W}^+_{\infty} \) are independent of the choice of \( \chi_{\infty} \) and by a similar argument, \( \tilde{W}^-_{\infty} \) are independent of the choice of \( \chi_0 \).

We still have to prove that \( W^\pm \) and \( \tilde{W}^\pm \) are bijective isometries, which yields that \( S \) is a bijective isometry by construction. Let \( \Psi \in \mathcal{H} \) and
\[ \Psi^\pm = \tilde{W}^\pm_0 \Psi, \quad \Psi^\pm_{\infty} = \tilde{W}^\pm_{\infty} \Psi. \] (131)
For \( \chi_0 \) satisfying (44) and \( \chi_{\infty} \) satisfying (65), we have
\[ \lim_{t \to \pm\infty} \| \mathcal{J}_0 (e^{iHt} \Psi - e^{iH_0t} \Psi^\pm_0) \|_{\mathcal{H}} = 0, \] (132)
\[ \lim_{t \to \pm\infty} \| \mathcal{J}^*_\infty \mathcal{J}^*_\infty e^{iHt} \Psi - \mathcal{J}^*_\infty e^{iH_{\infty}t} \Psi^\pm_{\infty} \|_{\mathcal{H}} = 0, \] (133)
\( \mathcal{J}_0 \mathcal{J}^*_\infty \) being simply the multiplication by \( \chi_{\infty} \). The local energy of \( e^{iHt} \Psi \) goes to 0 when \( t \) goes to \( \pm\infty \), therefore
\[ \lim_{t \to \pm\infty} \| (\chi_0 + \chi_{\infty} - 1) e^{iHt} \Psi \|_{\mathcal{H}} = 0. \] (134)
(132), (133) and (134) imply
\[ \lim_{t \to \pm\infty} \| e^{iHt} \Psi - \mathcal{J}_0 e^{iH_0t} \Psi^\pm_0 - \mathcal{J}^*_\infty e^{iH_{\infty}t} \Psi^\pm_{\infty} \|_{\mathcal{H}} = 0, \] (135)
which means

\[ W^\pm \tilde{W}^\pm = 1_{\mathcal{H}}. \]  \hfill (136)

If on the other hand we consider

\[ \Psi_0^\pm \in \mathcal{H}_0^\pm, \quad \Psi_\infty^\pm \in \mathcal{H}_\infty \]  \hfill (137)

and put

\[ \Psi = W^\pm (\Psi_0^\pm, \Psi_\infty^\pm), \]  \hfill (138)

we have (135) from which we get

\[ \lim_{t \to \pm \infty} \| \mathcal{J}_0^* (e^{iH_0 t} \Psi - \mathcal{J}_0 e^{iH_\infty t} \Psi_0^\pm - \mathcal{J}_\infty e^{iH_\infty t} \Psi_\infty^\pm) \|_{\mathcal{H}} = 0 \]  \hfill (139)

\[ \lim_{t \to \pm \infty} \| \mathcal{J}_\infty^* (e^{iH_0 t} \Psi - \mathcal{J}_\infty e^{iH_\infty t} \Psi_0^\pm - \mathcal{J}_\infty e^{iH_\infty t} \Psi_\infty^\pm) \|_{\mathcal{H}_\infty} = 0. \]  \hfill (140)

The local energy of \( e^{iH_0 t} \Psi_0^\pm \) and \( e^{iH_\infty t} \Psi_\infty^\pm \) goes to 0 when \( |t| \) goes to \( +\infty \), therefore (139) and (140) yield

\[ \lim_{t \to \pm \infty} \| \mathcal{J}_0^* e^{iH_0 t} \Psi - e^{iH_0 t} \Psi_0^\pm \|_{\mathcal{H}} = 0 \]  \hfill (141)

and

\[ \lim_{t \to \pm \infty} \| \mathcal{J}_\infty^* e^{iH_\infty t} \Psi - e^{iH_\infty t} \Psi_\infty^\pm \|_{\mathcal{H}_\infty} = 0, \]  \hfill (142)

thus

\[ \tilde{W}^\pm W^\pm = 1_{\mathcal{H}_0^\pm \times \mathcal{H}_\infty}. \]  \hfill (143)

(136) and (143) show that \( W^\pm \) and \( \tilde{W}^\pm \) are all bijections and if we choose \( \chi_0 \) and \( \chi_\infty \) such that their supports have no intersection, we deduce from (135)

\[ \| \Psi \|_{\mathcal{H}} = \| \Psi_0^\pm \|_{\mathcal{H}} + \| \Psi_\infty^\pm \|_{\mathcal{H}_\infty}. \]  \hfill (144)

Q.E.D.

Proof of lemma 6.1. – Let \( \Psi_0 \in \mathcal{E}_{l,n}^{2\varepsilon}, (l, n) \in I_{1/2}, \varepsilon = +, - \). There exists

\[ \Psi'_0 = t(f_1, -\varepsilon f_1, f_3, \varepsilon f_3) \otimes F_{l,n} \in \mathcal{E}_{l,n}^{1\varepsilon} \]  \hfill (145)

such that

\[ \Psi_0 = iH \Psi'_0 \]  \hfill (146)

and

\[ \Psi''_0 = t(g_1, -\varepsilon g_1, g_3, \varepsilon g_3) \otimes F_{l,n} \in \mathcal{E}_{l,n}^{0\varepsilon} \]  \hfill (147)
such that

$$\Psi'_0 = -i H \Psi''_0.$$  \hfill (148)

We denote

$$\Psi = e^{iHt} \Phi; \quad \Psi = \phi \otimes F_{ln} = (\phi_1, -\epsilon \phi_3, \epsilon \phi_3) \otimes F_{ln}$$  \hfill (149)

and

$$\Psi = \partial_t \Psi = i H \Psi.$$  \hfill (150)

On the one hand, applying $\partial_t + i H$ to equation

$$(\partial_t - i H) \Psi = 0,$$

we obtain

$$\Psi = \partial_t - \partial_t^2 \Psi = 0$$

which, taking into account the fact that $\Psi$ takes its values in $\mathcal{H}_{ln}$ can also be written

$$(\partial_t^2 - \partial_t^2 + \beta_t^2 + \epsilon \beta_t') \phi_1 = 0,$$  \hfill (151)

$$\quad (\partial_t^2 - \partial_t^2 + \beta_t^2 - \epsilon \beta_t') \phi_3 = 0.$$  \hfill (152)

On the other hand

$$\phi_1 \mid_{t=0} = f_1; \quad \phi_3 \mid_{t=0} = f_3; \quad f_1, f_3 \in C_0^\infty (\mathbb{R}_r),$$  \hfill (153)

and since $\Psi_0 = H^2 \Psi''_0$

$$\partial_t \phi_1 \mid_{t=0} = (-\partial_t^2 + \beta_t^2 + \epsilon \beta_t') g_1, \quad g_1 \in C_0^\infty (\mathbb{R}_r),$$  \hfill (154)

$$\partial_t \phi_3 \mid_{t=0} = (-\partial_t^2 + \beta_t^2 - \epsilon \beta_t') g_3, \quad g_3 \in C_0^\infty (\mathbb{R}_r).$$  \hfill (155)

The scattering results obtained in [3] together with proposition 6.2 imply that for any solution

$$f \in C (\mathbb{R}_t; H^1 (\mathbb{R}_r) \cap C^1 (\mathbb{R}_t; L^2 (\mathbb{R}_r)))$$

of equation

$$(\partial_t^2 - \partial_t^2 + \beta_t^2 + \eta \beta_t') f = 0, \quad \eta = +, -$$
with initial data
\[ f \mid_{t=0} = \mu_1, \quad \partial_t f \mid_{t=0} = (-\partial^2_{r_+} + \beta^2_t + \eta \beta^t_t) \mu_2 \]
such that
\[ i = 1, 2 \quad \mu_i \in L^2(\mathbb{R}_r); \quad (-\partial^2_{r_+} + \beta^2_t + \eta \beta^t_t) \mu_i \in L^2(\mathbb{R}_r), \]
there exists a solution
\[ f_1 \in C(\mathbb{R}_t; H^1(\mathbb{R}_r)) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}_r)) \]
of
\[ (\partial^2_t - \partial^2_{r_+}) f_1 = 0 \]
such that
\[ \lim_{t \to +\infty} \left\| f (t) - f_1 (t) \right\|_{H^1(\mathbb{R}_r)} + \left\| \partial_t f (t) - \partial_t f_1 (t) \right\|_{L^2(\mathbb{R}_r)} = 0. \]
\( \tilde{\Psi} \) is the solution of (15) with initial data
\[ \Psi_0' \in [C_0^\infty(\mathbb{R}_r)]^4 \otimes F_{\text{in}} \]
therefore in particular,
\[ \phi_1, \phi_2 \in C(\mathbb{R}_t; H^1(\mathbb{R}_r)) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}_r)) \]
and (151) to (155) yield the existence of
\[ \tilde{\Psi}_1 \in [C(\mathbb{R}_t; H^1(\mathbb{R}_r)) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}_r))]^4 \otimes F_{\text{in}} \]
such that
\[ (\partial^2_t - \partial^2_{r_+}) \tilde{\Psi}_1 = 0 \]
and
\[ \lim_{t \to +\infty} \left\| e^{iHt} \tilde{\Psi}_0 - \tilde{\Psi}_1 \right\|_{\mathcal{H}} = 0, \quad \lim_{t \to +\infty} \left\| \partial_r (e^{iHt} \tilde{\Psi}_0 - \tilde{\Psi}_1) \right\|_{\mathcal{H}} = 0, \]
\[ \lim_{t \to +\infty} \left\| \partial_t (e^{iHt} \tilde{\Psi}_0 - \tilde{\Psi}_1) \right\|_{\mathcal{H}} = 0, \]
from which we deduce
\[ \lim_{t \to +\infty} \left\| e^{iHt} \Psi_0 - \partial_t \tilde{\Psi}_1 \right\|_{\mathcal{H}} = 0. \]
\( \Psi_0 \) being an element of \( \mathcal{E}_{in}^{2e} \subset \mathcal{E}^{1e}_{in} \), we can apply the previous construction to \( \Psi_0 \). We find that there exists

\[
\Psi_1 \in [C(\mathbb{R}_t; H^1(\mathbb{R}_r)) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}_r))]^4 \otimes F_{in}
\]
solution of

\[
(\partial^2_t - \partial^2_r) \Psi_1 = 0
\]
such that

\[
\lim_{t \to +\infty} \| e^{iHt} \Psi_0 - \Psi_1 \|_{\mathcal{H}} = 0,
\]

\[
\lim_{t \to +\infty} \| \partial_r (e^{iHt} \Psi_0 - \Psi_1) \|_{\mathcal{H}} = 0,
\]

\[
\lim_{t \to +\infty} \| \partial_t (e^{iHt} \Psi_0 - \Psi_1) \|_{\mathcal{H}} = 0.
\]

From (159) and (160) we deduce

\[
\lim_{t \to +\infty} \| (\partial_t - iH_0) (e^{iHt} \Psi_0 - \Psi_1) \|_{\mathcal{H}} = 0.
\]

\( e^{iHt} \Psi_0 \) being a solution of (15) in \( C(\mathbb{R}_t; \mathcal{H}_{in}) \), we have

\[
(\partial_t - iH) e^{iHt} \Psi_0 = (\partial_t - iH_0 - i\beta \gamma M) e^{iHt} \Psi_0 = 0
\]

and by (158)

\[
\lim_{t \to +\infty} \| i\beta \gamma M (e^{iHt} \Psi_0 - \partial_t \tilde{\Psi}_1) \|_{\mathcal{H}} = 0.
\]

\( \partial_t \tilde{\Psi}_1 \) is identically zero in

\[
\{(t, r_*, \omega); |r_*| \leq |t| - R, \omega \in S^2\},
\]

which is not true in general for \( \tilde{\Psi}_1 \), therefore

\[
\lim_{t \to +\infty} \| i\beta \gamma M \partial_t \tilde{\Psi}_1 \|_{\mathcal{H}} = 0
\]

and

\[
\lim_{t \to +\infty} \| i\beta \gamma M e^{iHt} \Psi_0 \|_{\mathcal{H}} = 0.
\]

(161), (162) and (163) give

\[
\lim_{t \to +\infty} \| (\partial_t - iH_0) \Psi_1 \|_{\mathcal{H}} = 0
\]
and \((\partial_t - iH_0)\psi_1\) being an element of \(\mathcal{C}(\mathbb{R}_t; \mathcal{H})\) and satisfying
\[
(\partial_t + iH_0) [(\partial_t - iH_0)\psi_1] = 0
\]
we must have
\[
(\partial_t - iH_0)\psi_1 = 0.
\]

Q.E.D.

**Proof of lemma 6.2.** - Let
\[
\psi_0 \in \mathcal{H}_0 \cap \mathcal{E}_{ln}^{0\varepsilon}, \quad (l, n) \in \mathcal{I}_{\frac{1}{2}}, \quad \varepsilon = +, -
\]
with
\[
\text{Supp} (\psi_0) \subset [-R, R] \times S_{\theta, \varphi}, \quad R > 0.
\]
\(\psi_0\) can be written
\[
\psi_0 = \psi(f_0, -\varepsilon f_0, f_0, \varepsilon f_0) \otimes F_{ln}, \quad f_0 \in C_0^\infty (\mathbb{R}_r),
\]
\[
\text{Supp} f_0 \subset [-R, R]
\]
and
\[
e^{iH_0t}\psi_0 = \psi(f, -\varepsilon f, f, \varepsilon f) \otimes F_{ln}, \quad f(t, r*) = f_0(r* - t).
\]
\(f\) is the solution of
\[
(\partial_t^2 - \partial_{r*}^2) f = 0
\]
associated with the initial data
\[
f|_{t=0} = f_0, \quad \partial_t f|_{t=0} = -\partial_{r*} f_0.
\]
Instead of applying Cook’s method to operators \(H_\infty\) and \(H_0\), which would give an apparently long-range perturbation at infinity, we work on the second order scalar equations and establish the existence of \(g_\eta\) solution of
\[
(\partial_t^2 - \partial_{r*}^2 + V_\eta (r*)) g_\eta = 0
\]
\[
V_\eta (r*) = \chi_\infty (r*) \frac{1}{r^*_2} \left( l + \frac{1}{2} \right)^2 + \eta \left( l + \frac{1}{2} \right), \quad \eta = +, -
\]
where \( \chi_\infty \) is a cut-off function satisfying (65); the solution \( g_\eta \) being such that
\[
\lim_{t \to +\infty} \| \partial_t (g_\eta - f) \|_{L^2(\mathbb{R})} = 0, \quad \lim_{t \to +\infty} \| \partial_r (g_\eta - f) \|_{L^2(\mathbb{R})} = 0, \quad (171)
\]
\[
\lim_{t \to +\infty} \left\| \frac{l + \frac{1}{2}}{\frac{2}{r}} (g_\eta - f) \right\|_{L^2(\mathbb{R})} = 0. \quad (172)
\]
In the case where \( l = 1/2 \) and \( \eta = - \), equations (168) and (170) are the same and it suffices to take \( g_- = f \). Let us now assume
\[
(l + \frac{1}{2})^2 + \eta (l + \frac{1}{2}) > 0. \quad (173)
\]
We write equations (168) and (170) in their hamiltonian form
\[
\partial_t \begin{pmatrix} f \\ \partial_t f \end{pmatrix} = - \begin{pmatrix} 0 & -1 \\ -\partial_r^2 & 0 \end{pmatrix} \begin{pmatrix} f \\ \partial_t f \end{pmatrix} = -A_0 \begin{pmatrix} f \\ \partial_t f \end{pmatrix}, \quad (174)
\]
\[
\partial_t \begin{pmatrix} g \\ \partial_t g \end{pmatrix} = - \begin{pmatrix} 0 & -1 \\ -\partial_r^2 + V_\eta & 0 \end{pmatrix} \begin{pmatrix} g \\ \partial_t g \end{pmatrix} = -A_\eta \begin{pmatrix} g \\ \partial_t g \end{pmatrix}. \quad (175)
\]
The operator \( iA_0 \) is skew-adjoint with dense domain on
\[
H_0 = B L^2_1(\mathbb{R}_r) \times L^2(\mathbb{R}_r) \quad (176)
\]
completion of \( [C_0^\infty(\mathbb{R}_r)]^2 \) for the norm
\[
\| \langle f_1, f_2 \rangle \|_{H_0}^2 = \int_{\mathbb{R}} \{ |\partial_r f_1|^2 + |f_2|^2 \} \, dr_\ast \quad (177)
\]
and \( iA_\eta \) is skew-adjoint with dense domain (cf. [3]) on
\[
H = H_1 \times L^2(\mathbb{R}_r) \quad (178)
\]
completion of \( [C_0^\infty(\mathbb{R}_r)]^2 \) for the norm
\[
\| \langle g_1, g_2 \rangle \|_H^2 = \int_{\mathbb{R}} \{ |\partial_r g_1|^2 + |g_2|^2 + V_\eta |g_1|^2 \} \, dr_\ast. \quad (179)
\]
Under assumption (173), the norm (179) is equivalent to
\[
\| \langle g_1, g_2 \rangle \|_H^2 = \| \langle g_1, g_2 \rangle \|_{H_0}^2 + \left\| \frac{l + \frac{1}{2}}{r_\ast} \chi_\infty g_1 \right\|_{L^2(\mathbb{R}_r)}^2. \quad (180)
\]
Moreover, any solution \( t(g, \partial_t g) \in C(\mathbb{R}_t; \mathbb{H}) \) of (170) satisfies the following energy estimate: for \( r_*^1 < r_*^2 \) and \( t \in \mathbb{R} \)

\[
\int_{r_*^1 < r_* < r_*^2} \{ |\partial_{r_*} g(t)|^2 + |\partial_t g(t)|^2 + V_\eta(r_*) |g(t)|^2 \} \, dr_* \leq \int_{r_*^1 - |t| < r_* < r_*^2 + |t|} \{ |\partial_{r_*} g(0)|^2 + |\partial_t g(0)|^2 + V_\eta(r_*) |g(0)|^2 \} \, dr_*
\]

which is very easily obtained by multiplying (170) by \( \partial_t g \) and integrating by parts on the domain

\[
\Omega_{t, r_*^1, r_*^2} = \{(\tau, r_*); \tau \in (0, t), r_*^1 - |t - \tau| < r_* < r_*^2 + |t - \tau| \}.
\]

and for \( r_* \) large enough

Thus

\[
\text{and}
\]

\[
\text{and for } r_* \text{ large enough}
\]

\[
V_\eta(r_*) = C r_*^{-2}, \quad C > 0,
\]

thus

\[
\left\| \partial_t (e^{A_n t} e^{-A_0 t} \phi_0) \right\|_{\mathbb{H}} = O(t^{-2}); \quad t \to +\infty,
\]

and

\[
\left\| \partial_t (e^{A_n t} e^{-A_0 t} \phi_0) \right\|_{\mathbb{H}} \in L^1(t > 0).
\]
The limit (183) is therefore well-defined and if \( g_\eta \) is the solution of (170) such that
\[
\begin{pmatrix}
g_\eta(t) \\
\partial_t g_\eta(t)
\end{pmatrix}
= e^{-A_\eta t} \phi_\infty,
\] (186)
then
\[
\lim_{t \to +\infty} \| t^\varepsilon (g_\eta, \partial_t g_\eta) - t^\varepsilon (f, \partial_t f) \|_H = 0.
\] (187)
This last limit together with the equivalence of norms (179) and (180) gives (171) and (172). Moreover, for \( r_\ast < t - R \)
\[
g_\eta(t, r_\ast) = 0 \quad \text{and} \quad \partial_t g_\eta(t, r_\ast) = 0.
\] (188)
Indeed, for \( t \in \mathbb{R}, \varepsilon > 0 \) we choose \( \tau \in \mathbb{R} \) such that
\[
\| \phi_\infty - e^{iA_\eta \tau} e^{-iA_0 \tau} \phi_0 \|_H \leq \varepsilon, \quad \tau \geq t.
\] (189)
For \( t(f_1, f_2) \in H_\varepsilon \), we denote
\[
\mathcal{L}(t(f_1, f_2)) = |\partial_\ast f_1|^2 + V_\eta |f_2|^2 + |f_2|^2.
\] (190)
Let us consider
\[
\int_{r_\ast < t - R} \mathcal{L}(e^{-iA_\eta t} \phi_\infty) \, dr_\ast 
\leq \int_{r_\ast < t - R} \mathcal{L}[e^{-iA_\eta t} (\phi_\infty - e^{iA_\eta \tau} e^{-iA_0 \tau} \phi_0)] \, dr_\ast 
+ \int_{r_\ast < t - R} \mathcal{L}(e^{-iA_\eta (t-\tau)} e^{-iA_0 \tau} \phi_0) \, dr_\ast.
\] (181) and (189) yield
\[
\int_{r_\ast < t - R} \mathcal{L}(e^{-iA_\eta t} \phi_\infty) \, dr_\ast \leq \varepsilon^2 + \int_{r_\ast < \tau - R} \mathcal{L}(e^{-iA_0 \tau} \phi_0) \, dr_\ast,
\]
and this last integral is zero since
\[
\text{Supp} (e^{-iA_\eta \tau} \phi_0) \subset [\tau - R, \tau + R].
\]
(188) is therefore satisfied and for \( t \) large enough \( g_\eta \) is a solution of
\[
\left[ \partial_t^2 - \partial_\ast^2 + \frac{1}{r_\ast^2} \left( \left( l + \frac{1}{2} \right)^2 + \eta \left( l + \frac{1}{2} \right) \right) \right] g_\eta = 0.
\] (191)
Let us now introduce
\[
\Psi_\infty(t) = t^\varepsilon (g_- (t), -\varepsilon g_- (t), g_e (t), \varepsilon g_e (t)) \otimes F_{\varepsilon}.
\] (192)
There exists $t_0 > 0$ such that, for $t \geq t_0$, $g_\varepsilon$ and $g_{-\varepsilon}$ satisfy
\begin{align}
\left[ \partial_t^2 - \partial_{r_*}^2 + \frac{1}{r_*^2} \left( \left( l + \frac{1}{2} \right)^2 + \varepsilon \left( l + \frac{1}{2} \right) \right) \right] g_\varepsilon &= 0, \\
\left[ \partial_t^2 - \partial_{r_*}^2 + \frac{1}{r_*^2} \left( \left( l + \frac{1}{2} \right)^2 - \varepsilon \left( l + \frac{1}{2} \right) \right) \right] g_{-\varepsilon} &= 0, 
\end{align}
with
\begin{align}
g_\varepsilon, \ g_{-\varepsilon} &\in C ([t_0, +\infty[ ; H_1), \\
\partial_t g_\varepsilon, \ \partial_t g_{-\varepsilon} &\in C ([t_0, +\infty[ ; L^2 (\mathbb{R}_r)).
\end{align}
Moreover, for $t \geq t_0$
\begin{align}
\text{Supp} (g_\varepsilon (t), \ g_{-\varepsilon} (t), \ \partial_t g_\varepsilon (t), \ \partial_t g_{-\varepsilon} (t)) \\
\subset [t - R, +\infty [ \subset [0, +\infty [.
\end{align}
Thus, the quantities
\begin{align}
\partial_t \tilde{\Psi}_\infty, \ \partial_{r_*} \tilde{\Psi}_\infty, \ \left( l + \frac{1}{2} \right) r_*^{-1} \tilde{\Psi}_\infty
\end{align}
belong to $C ([t_0, +\infty[ ; \mathcal{H})$ and (171), (172) yield
\begin{align}
\lim_{t \to +\infty} \| \partial_t (\tilde{\Psi}_\infty (t) - e^{iH_0t} \Psi_0) \|_\mathcal{H} = 0, \\
\lim_{t \to +\infty} \| \partial_{r_*} (\tilde{\Psi}_\infty (t) - e^{iH_0t} \Psi_0) \|_\mathcal{H} = 0,
\end{align}
\begin{align}
\lim_{t \to +\infty} \left\| \left( l + \frac{1}{2} \right) r_*^{-1} (\tilde{\Psi}_\infty (t) - e^{iH_0t} \Psi_0) \right\|_\mathcal{H} = 0.
\end{align}
In particular, we have
\begin{align}
\lim_{t \to +\infty} \left\| \left( \partial_t + L \partial_{r_*} - i \left( l + \frac{1}{2} \right) r_*^{-1} M \right) \times (\tilde{\Psi}_\infty (t) - e^{iH_0t} \Psi_0) \right\|_\mathcal{H} = 0.
\end{align}
Since $e^{iH_0t} \Psi_0$ is a solution of
\begin{align}
(\partial_t + L \partial_{r_*}) e^{iH_0t} \Psi_0 = 0,
\end{align}
we have
\[ \left\| (\partial_t + L \partial r_\ast - i \left( l + \frac{1}{2} \right) r_\ast^{-1} M ) e^{iH_{0t} t} \Psi_0 \right\|_{\mathcal{H}} \]
\[ = \left( l + \frac{1}{2} \right) \left\| r_\ast^{-1} e^{iH_{0t} t} \Psi_0 \right\|_{\mathcal{H}} = O(t^{-1}), \quad t \to \infty \]
and therefore
\[ \lim_{t \to +\infty} \left\| (\partial_t + L \partial r_\ast - i \left( l + \frac{1}{2} \right) r_\ast^{-1} M ) \tilde{\Psi}_\infty(t) \right\|_{\mathcal{H}} = 0. \quad (200) \]

We introduce
\[ \Psi_\infty = \tilde{\Psi}_\infty |_{\{ r_\ast \geq 0 \}}. \quad (201) \]

The quantities
\[ \partial_t \Psi_\infty, \quad \partial r_\ast \Psi_\infty, \quad \left( l + \frac{1}{2} \right) r_\ast^{-1} \Psi_\infty \]
belong to \( \mathcal{C} ([t_0, +\infty [; \mathcal{H}_{\infty}^{\varepsilon \text{ln}}) \) where, for \( (l, n) \in \mathcal{I}_{\frac{1}{2}} \) and \( \varepsilon = +, - \)
\[ \mathcal{H}_{\infty}^{\varepsilon \text{ln}} = \{ t(f, -\varepsilon f, \varepsilon g, \varepsilon g) \otimes F_{ln} \in \mathcal{H}_{\infty} \}. \quad (202) \]

From (200), we get
\[ \lim_{t \to +\infty} \left\| (\partial_t + L \partial r_\ast - i \left( l + \frac{1}{2} \right) r_\ast^{-1} M ) \Psi_\infty(t) \right\|_{\mathcal{H}_{\infty}} = 0 \quad (203) \]
and, the function
\[ \left( \partial_t + L \partial r_\ast - i \left( l + \frac{1}{2} \right) r_\ast^{-1} M \right) \Psi_\infty \in \mathcal{C} ([t_0, +\infty [; \mathcal{H}_{\infty}^{\varepsilon \text{ln}}) \]
satisfies
\[ \left( \partial_t - L \partial r_\ast + i \left( l + \frac{1}{2} \right) r_\ast^{-1} M \right) \times \left[ \left( \partial_t + L \partial r_\ast - i \left( l + \frac{1}{2} \right) r_\ast^{-1} M \right) \Psi_\infty \right] = 0. \quad (204) \]

Therefore, we must have for \( t \geq t_0 \)
\[ \left( \partial_t + L \partial r_\ast - i \left( l + \frac{1}{2} \right) r_\ast^{-1} M \right) \Psi_\infty(t) = 0 \quad \text{in} \ \mathcal{H}_{\infty}. \]
\( H_1 \) being a distribution space, we can write in the sense of distributions for \( t \geq t_0 \)

\[
\partial_t \left( \partial_t + L \partial_{\tau^*} - i \left( l + \frac{1}{2} \right) r_*^{-1} M \right) \Psi_\infty (t)
= \left( \partial_t + L \partial_{\tau^*} - i \left( l + \frac{1}{2} \right) r_*^{-1} M \right) \partial_t \Psi_\infty (t) = 0 \quad \text{in } \mathcal{H}_\infty,
\]

which implies that \( \partial_t \Psi_\infty \) is a solution in \( C ([t_0, +\infty [; \mathcal{H}_\infty^{\text{eln}}) \) of

\[
(\partial_t - i H_\infty) \Psi = 0.
\]

This solution can be extended to \( C (\mathbb{R}_t; \mathcal{H}_\infty^{\text{eln}}) \) and we denote

\[
\Psi_\infty^0 = e^{-iH_\infty t_0} \partial_t \Psi_\infty (t_0)
\]

its initial data at \( t = 0 \). From (196), (197), we get

\[
\lim_{t \to +\infty} \| e^{iH_\infty t} \Psi_\infty^0 - \mathcal{J}_\infty^* \partial_t (e^{iH_0 t} \Psi_0) \|_{\mathcal{H}_\infty} = 0. \tag{206}
\]

The value of \( \partial_t (e^{iH_0 t} \Psi_0) \) at \( t = 0 \) is \( i H_0 \Psi_0 \). \( H_0 \) is a self-adjoint operator with dense domain on \( \mathcal{H} \), its point spectrum is empty and the spaces \( \mathcal{H}_0^\pm \), \( \mathcal{H}_\infty^\pm \) are invariant under \( H_0 \). Therefore the direct sum of the sets

\[
\{ H_0 \Psi_0; \Psi_0 \in \mathcal{H}_0^- \cap \mathcal{E}_{l_n}^0 \}; \quad (l, n) \in I^\pm, \quad \epsilon = +, - \tag{207}
\]

is dense in \( \mathcal{H}_0^- \). (206) shows that for an initial data \( H_0 \Psi_0 \) in a set of type (207), the limit

\[
\Psi_\infty^0 = s \lim_{t \to +\infty} e^{-iH_\infty t} \mathcal{J}_\infty^* e^{iH_0 t} H_0 \Psi_0 \tag{208}
\]

exists in \( \mathcal{H}_\infty \). The operator \( W^\infty_0 \) is consequently well-defined from \( \mathcal{H}_0^- \) into \( \mathcal{H}_\infty^- \). Since the local energy of the solution \( e^{iH_0 t} H_0 \Psi_0 \) goes to zero when \( |t| \) goes to \( +\infty \), the limit \( \Psi_\infty^0 \) is independent of the choice of \( \chi_\infty \) satisfying (65).

Q.E.D.

**7. CONCLUSION**

The scattering theory developed in this paper is only valid for the linear massless Dirac system. In the case of a massive field and when space-time is asymptotically flat, the mass of the field induces long-range perturbations at infinity and classical wave operators will probably not exist. However,
using the methods developed by J. Dollard and G. Velo [10] and by V. Enss and B. Thaller [11] about the relativistic Coulomb scattering of Dirac fields as well as the works of A. Bachelot [1] and J. Dimock and B. Kay [9] on the Klein-Gordon equation on the Schwarzschild metric, it must be possible to show the existence and asymptotic completeness of Dollard-modified wave operators at infinity.

REFERENCES


(Manuscript received April, 12, 1994.)