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Blow-up solutions and strong instability of standing waves for the generalized Davey-Stewartson system in $\mathbb{R}^2$

by

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ABSTRACT. – We study the instability of standing wave $e^{iwt} \varphi_{\omega}(x)$ for the equation

$$iu_t + \Delta u + a|u|^{p-1}u + E_1(|u|^2)u = 0$$

in $\mathbb{R}^2$, where $\varphi_{\omega}$ is a ground state. We prove that if $a(p - 3) > 0$, then there exist blow-up solutions of (*) arbitrarily close to the standing wave.

RÉSUMÉ. – Nous étudions l’instabilité de l’onde stationnaire $e^{iwt} \varphi_{\omega}(x)$ pour l’équation

$$iu_t + \Delta u + a|u|^{p-1}u + E_1(|u|^2)u = 0$$

dans $\mathbb{R}^2$, où $\varphi_{\omega}$ est un état fondamental. Nous prouvons que si $a(p - 3) > 0$, il existe solutions de (*) explosant en temps fini, arbitrairement voisine de l’onde stationnaire.

1. INTRODUCTION AND RESULT

We consider the instability of standing waves for the following equation:

$$iu_t + \Delta u + a|u|^{p-1}u + E_1(|u|^2)u = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n, \quad (1.1)$$

where $a \in \mathbb{R}$, $1 < p < 2^* - 1$, $n = 2$ or $3$, and $E_1$ is the singular integral operator with symbol $\sigma_1(\xi) = \xi_1^2/|\xi|^2$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. Equation
(1.1), for \( n = 2 \) and \( p = 3 \), describes the evolution of weakly nonlinear water waves that travel predominantly in one direction (see [3], [4] and [2]). By a standing wave, we mean a solution of (1.1) with the form

\[ u_\omega(t, x) = e^{ixt} \varphi_\omega(x), \]

where \( \omega > 0 \) and \( \varphi_\omega \) is a ground state (least action solution) of the problem:

\[
\begin{align*}
-\Delta \psi + \omega \psi - a|\psi|^{p-1}\psi - E_1(|\psi|^2) \psi &= 0, \\
\psi &\in H^1(\mathbb{R}^n), \\
\psi &\neq 0.
\end{align*}
\]

Here the action \( S_\omega \) of (1.2) is defined by

\[ S_\omega(v) = \frac{1}{2} \| \nabla v \|^2 + \frac{\omega}{2} \| v \|^2 - \frac{a}{p+1} \| v \|_{p+1}^{p+1} - \frac{1}{4} B_1(|v|^2), \]

where \( B_1(|v|^2) = \int |v|^2 E_1(|v|^2) \, dx \). We denote by \( G_\omega \) the set of all ground states for (1.2).

**Definition 1.1.** For \( \Omega \subset H^1(\mathbb{R}^n) \), we say that the set \( \Omega \) is stable if for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( u_0 \in H^1(\mathbb{R}^n) \) satisfies

\[ \inf_{\varphi \in \Omega} \| u_0 - \varphi \|_{H^1} < \delta, \]

then the solution \( u(t) \) of (1.1) with \( u(0) = u_0 \) satisfies

\[ \sup_{0 \leq t < \infty} \inf_{\varphi \in \Omega} \| u(t) - \varphi \|_{H^1} < \epsilon. \]

Otherwise, \( \Omega \) is said to be unstable. Moreover, for \( \varphi_\omega \in G_\omega \), we say that the standing wave \( u_\omega(t) = e^{ixt} \varphi_\omega \) is unstable if \( \{ e^{i\theta} \varphi_\omega(\cdot + y) : \theta \in \mathbb{R}, y \in \mathbb{R}^n \} \) is unstable. Furthermore, we say that \( u_\omega \) is strongly unstable if for any \( \epsilon > 0 \) there exists \( u_0 \in H^1(\mathbb{R}^n) \) such that \( \| u_0 - \varphi_\omega \|_{H^1} < \epsilon \) and the solution \( u(t) \) of (1.1) with \( u(0) = u_0 \) blows up in a finite time.

For the standing wave \( u_\omega(t) = e^{ixt} \varphi_\omega \) with \( \varphi_\omega \in G_\omega \) of (1.1), Cipolatti [2] proved that if \( a(p - 3) \geq 0 \), and \( n = 2 \) or \( 3 \), then \( u_\omega \) is unstable for any \( \omega \in (0, \infty) \), and that if \( n = 2, p = 3 \) and \( a > -1 \), then \( u_\omega \) is strongly unstable for any \( \omega \in (0, \infty) \). After that, the author [5] proved that if \( a > 0 \), \( p \geq 1 + 4/n \), and \( n = 2 \) or \( 3 \), then \( u_\omega \) is unstable for any \( \omega \in (0, \infty) \), and that if \( n = 3, a > 0 \) and \( 1 < p < 7/3 \), then there exists a positive constant \( \omega_0 = \omega_0(a, p) \) such that \( u_\omega \) is unstable for any \( \omega \in (\omega_0, \infty) \). Moreover, the author [6] proved that if \( n = 3, a > 0 \) and \( 7/3 < p < 5 \), or \( a < 0 \) and \( 1 < p < 3 \), then \( u_\omega \) is strongly unstable for any \( \omega \in (0, \infty) \). On the other hand, when \( n = 2 \) and \( a(p - 3) < 0 \), the author [6] showed the existence of stable standing waves of (1.1).

Our result in this paper is the following.
Theorem 1.2. – Assume that \( n = 2 \) and \( a (p-3) > 0 \), or \( n = 3, a > 0 \) and \( p = 7/3 \). Then, for any \( \omega \in (0, \infty) \), the standing wave \( u_\omega(t) = e^{i\omega t} \varphi_\omega \) with \( \varphi_\omega \in G_\omega \) is strongly unstable in the sense of Definition 1.1.

Remark 1.3. – As stated above, we showed in [6] that if \( n = 3, a > 0 \) and \( 7/3 < p < 5 \) or \( a < 0 \) and \( 1 < p < 3 \), then \( u_\omega \) is strongly unstable for any \( \omega \in (0, \infty) \), by extending the method of Berestycki and Cazenave [1] to an anisotropic case \((1.1) \) contains an anisotropic nonlinearity \( E_1(|u|^2)u \). Following Berestycki and Cazenave [1], we consider the same minimization problem as in [6] (see Proposition 2.1 below). In the case of Theorem 1.2, we need some devices to obtain that its minimizing sequence is bounded in \( H^1(\mathbb{R}^n) \), and is not vanishing in \( L^q(\mathbb{R}^n) \) for some \( 2 < q < 2^* \), although it is easy in the case of [6] (see Proposition 2.2 below, and Lemma 4.2 in [6]). In particular, in order to show that the minimizing sequence is not vanishing in \( L^{p+1}(\mathbb{R}^2) \) when \( n = 2, a > 0 \) and \( p > 3 \), we need an estimate for the critical value of minimization problem (see Lemma 2.3 below).

In what follows, we omit the integral variables with respect to the spatial variable \( x \), and we omit the integral region when it is the whole space \( \mathbb{R}^n \). We denote the norms of \( L^q(\mathbb{R}^n) \) and \( H^1(\mathbb{R}^n) \) by \( \cdot \|_{L^q} \) and \( \cdot \|_{H^1} \), respectively. We put \( \lambda^\lambda(x) = \lambda^{n/2} \nu(\lambda x), \lambda > 0 \).

2. PROOF OF THEOREM 1.2

In this section, we give the proof of Theorem 1.2. We prove the case when \( n = 2 \) and \( a (p-3) > 0 \) only. The case when \( n = 3, a > 0 \) and \( p = 7/3 \) can be proved analogously to the case when \( n = 2, a > 0 \) and \( p > 3 \). Thus, we assume that \( n = 2, a > 0 \) throughout this section. Moreover, since we fix the parameter \( \omega \), we drop the subscript \( \omega \). Thus, we write \( \varphi \) for \( \varphi_\omega \), \( S \) for \( S_\omega \), and so on. We put

\[
P(v) = |\nabla v|^2 - \frac{p-1}{p+1} a|v|^{p+1} - \frac{1}{2} B_1(|v|^2). \tag{2.1}
\]

We note that \( P(v) = \partial_\lambda S(v^\lambda)|_{\lambda=1} \). We first prove a key proposition to obtain Theorem 1.2.

Proposition 2.1. – Assume that \( n = 2 \) and \( a (p-3) > 0 \). Then, \( \varphi \) is a ground state of \((1.2) \) if and only if \( \varphi \in M \) and \( m = S(\varphi) \), where

\[
m = \inf \{ S(v) : v \in M \},
\]

\[
M = \{ v \in H^1(\mathbb{R}^2) : v \neq 0, P(v) = 0 \}. \tag{2.2}
\]
In order to obtain a minimizer for (2.2), we consider the following minimization problem (2.3), instead of (2.2):

\[ m_1 = \inf \{ S^1(v) : v \in H^1(\mathbb{R}^2), v \neq 0, P(v) \leq 0 \}, \quad (2.3) \]

where

\[ S^1(v) = S(v) - \frac{1}{2} P(v) = \frac{\omega}{2} |v|^2_2 + \gamma |v|^{p+1}_{p+1}, \quad \gamma = \frac{a(p-3)}{2(p+1)} > 0. \]

If \( P(v) < 0 \), then we have

\[ P(\lambda v) = \lambda^2 |\nabla v|^2 + \frac{p-1}{p+1} a \lambda^{p+1} |v|^{p+1}_{p+1} - \frac{1}{2} \lambda^4 B_1(|v|^2) > 0 \]

for sufficiently small \( \lambda > 0 \), so there exists a \( \lambda_0 \in (0, 1) \) such that \( P(\lambda_0 v) = 0 \). Moreover, since we get

\[ S^1(\lambda_0 v) = \frac{\omega}{2} \lambda_0^2 |v|^2_2 + \gamma \lambda_0^{p+1} |v|^{p+1}_{p+1} < S^1(v), \]

we obtain that

\[ m_1 = \inf \{ S^1(v) : v \in H^1(\mathbb{R}^2), v \neq 0, P(v) = 0 \} = m. \quad (2.4) \]

**Proposition 2.2.** – The minimization problem (2.3) is attained at some \( w \in M \).

Before giving the proof of Proposition 2.2, we prepare one lemma. We use Lemma 2.3 to show that a minimizing sequence for (2.3) is not vanishing in \( L^{p+1}(\mathbb{R}^2) \) when \( a > 0 \) and \( p > 3 \).

**Lemma 2.3.** – Let \( a > 0 \) and \( p > 3 \). Then, we have \( m_1 < \omega \mu_0/2 \), where

\[ \mu_0 = \inf \left\{ |v|^2_2 : v \in H^1(\mathbb{R}^2), v \neq 0, E_0(v) \equiv \frac{1}{2} |\nabla v|^2 - \frac{1}{4} B_1(|v|^2) \leq 0 \right\}. \]

**Proof.** – From Proposition 2.1 in [6], there exists a function \( Q \in H^1(\mathbb{R}^2) \) such that \( Q \neq 0 \), \( |Q|^2_2 = \mu_0 \) and \( E_0(Q) = 0 \). For \( 0 < \delta < 1 \) and \( \lambda > 0 \), we have by \( E_0(Q) = 0 \)

\[ P(\delta Q^\lambda) = \delta^2 \lambda^2 |\nabla Q|^2_2 - \frac{p-1}{p+1} a \delta^{p+1} \lambda^{p-1} |Q|^{p+1}_{p+1} - \frac{1}{2} \delta^4 \lambda^2 B_1(|Q|^2) \]

\[ = \delta^2 \lambda^2 (1 - \delta^2) |\nabla Q|^2_2 - \frac{p-1}{p+1} a \delta^{p+1} \lambda^{p-1} |Q|^{p+1}_{p+1}. \]
If we take $0 < \delta < 1$ and $\lambda > 0$ such that $P(\delta Q^\lambda) = 0$, then we have

$$\lambda = C(a, p, Q) \delta^{(1-p)/(p-3)} (1 - \delta)^{(1-p)/(p-3)}$$

and

$$S^1(\delta Q^\lambda) = \frac{\omega}{2} \delta^2 |Q|_2^2 + \gamma \delta^{p+1} \lambda^{p-1} |Q|_{p+1}^{p+1}$$

$$= \frac{\omega}{2} \delta^2 |Q|_2^2 + \frac{p-3}{2(p-1)} \delta^2 \lambda^2 (1 - \delta^2) |\nabla Q|_2^2.$$

Thus, if we take $\delta$ sufficiently close to 1, then we have $S^1(\delta Q^\lambda) < \omega |Q|_2^2/2$. Hence, from the definition of $m_1$, we obtain that $m_1 < \omega |Q|_2^2/2 = \omega \mu_0/2$.

**Remark 2.4.** - It is important to note that $m_1$ is strictly less than $\omega \mu_0/2$ in Lemma 2.3. This fact plays an essential role in the proof of Proposition 2.2.

**Proof of Proposition 2.2.** - Let $\{v_j\}$ be a minimizing sequence for (2.3). Since $\gamma > 0$, $\{v_j\}$ is bounded in $L^2(\mathbb{R}^2) \cap L^{p+1}(\mathbb{R}^2)$.

First, we show that $\{v_j\}$ is bounded in $H^1(\mathbb{R}^2)$. When $a > 0$ and $p > 3$, we see that $\{v_j\}$ is bounded in $L^4(\mathbb{R}^2)$, $B_1(|v_j|) \leq |v_j|_4$ and $P(v_j) \leq 0$, so that we have $\sup_j |\nabla v_j|_2^2 < \infty$. When $a < 0$ and $1 < p < 3$, we have from $P(v_j) \leq 0$

$$|\nabla v_j|_2^2 \leq |\nabla v_j|_2^2 + \frac{p-1}{p+1} |a| |v_j|_p^{p+1} \leq \frac{1}{2} B_1(|v_j|_2)$$

$$\leq \frac{1}{2} |v_j|_4^4 \leq C_1 |v_j|_p^{p+1} |\nabla v_j|_2^{3-p}$$

for some $C_1 > 0$. Here we have used the Gagliardo-Nirenberg inequality. Since $\{v_j\}$ is bounded in $L^{p+1}(\mathbb{R}^2)$, we have $|\nabla v_j|_2^2 \leq C_2 |\nabla v_j|_2^{3-p}$ for some $C_2 > 0$, so that we have $|\nabla v_j|_2^{p-1} \leq C_2$.

Next, we show that $\liminf_{j \to \infty} |v_j|_p^{p+1} > 0$ when $a > 0$ and $p > 3$. In fact, suppose that $|v_j|_p^{p+1} \to 0$. Then, since we have

$$B_1(|v_j|_2^2) \leq |v_j|_4^4 \leq |v_j|_2^{2(p-3)/(p-1)} |v_j|_p^{2(p+1)/(p-1)}$$

and $\{v_j\}$ is bounded in $L^2(\mathbb{R}^2)$, we have $B_1(|v_j|_2^2) \to 0$, and from $P(v_j) \leq 0$ we have $|\nabla v_j|_2 \to 0$. From the fact that $P(v_j) \leq 0$, Proposition 2.1 in [6] and the Gagliardo-Nirenberg inequality, we have

$$|\nabla v_j|_2^2 \leq \frac{p-1}{p+1} a |v_j|_p^{p+1} + \frac{1}{2} B_1(|v_j|_2^2)$$

$$\leq \frac{p-1}{p+1} a C_3 |v_j|_2^2 |\nabla v_j|_2^{p-1} + \frac{1}{\mu_0} |v_j|_2^2 |\nabla v_j|_2^2$$

$$\leq C_4 |\nabla v_j|_2^{p-1} + \frac{1}{\mu_0} |v_j|_2^2 |\nabla v_j|_2^2.$$
for some positive constants $C_3$ and $C_4$, so that we have
\[ 1 \leq C_4 |\nabla v_j|^p - \frac{3}{p^2} + \frac{1}{\mu_0} |v_j|^2. \]
It follows from $|\nabla v_j|_2 \to 0$ that $\mu_0 \leq \liminf_{j \to \infty} |v_j|^2$. Since $S^1(v_j) \to m_1$, we have $\omega_0 \mu_0 / 2 \leq m_1$. However, this contradicts Lemma 2.3. Therefore, we obtain that $\liminf_{j \to \infty} |v_j|^{p+1}_{p+1} > 0$ in the case of $a > 0$ and $p > 3$.

Next, we show that $\liminf_{j \to \infty} |v_j|^{4}_{4} > 0$ when $a < 0$ and $1 < p < 3$. In fact, suppose that $|v_j|^{4}_{4} \to 0$. Then, from $P(v_j) \leq 0$ we have $|\nabla v_j|_2 \to 0$. Again from $P(v_j) \leq 0$, we have
\[
|\nabla v_j|^2 + \frac{p-1}{p+1} |a| |v_j|^{p+1}_{p+1} \leq \frac{1}{2} B_1(|v_j|^2)
\]
\[
\leq \frac{1}{2} |v_j|^4 \leq \frac{p-1}{p+1} |a| |v_j|^{p+1}_{p+1} + C_5 |v_j|^5,
\]
so that we have
\[
|\nabla v_j|^2 \leq C_5 |v_j|^5 \leq C_6 |v_j|^2 |\nabla v_j|^2 \leq C_7 |\nabla v_j|^3
\]
for some positive constants $C_5$, $C_6$ and $C_7$. However, this contradicts $|\nabla v_j|_2 \to 0$. Therefore, we obtain that $\liminf_{j \to \infty} |v_j|^{4}_{4} > 0$ in the case of $a < 0$ and $1 < p < 3$.

From the above results, we can prove Proposition 2.2 in the same way as the proof of Lemma 4.2 in [6]. □

From (2.4) and Proposition 2.2, we obtain a minimizer of (2.2), that is, there exists a $w \in M$ such that $m = S(w)$.

**Lemma 2.5.** If $w \in M$ satisfies $m = S(w)$, then we have $S'(w) = 0$.

We can prove Lemma 2.5 similarly to the proof of Lemma 4.3 in [6]. Moreover, since we have $P(\psi) = 0$ for any solution $\psi$ of (1.2), Proposition 2.1 follows from Proposition 2.2 and Lemma 2.5. Finally, we can prove Theorem 1.2 from Proposition 2.1 in the same way as the proof of Theorem 1.2 in [6].

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