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Nonexistence of minimal blow-up solutions of equations $iu_t = -\Delta u - k(x)|u|^{4/N}u$ in \mathbb{R}^N

by

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ABSTRACT. – In this paper, we prove the existence of blow-up solutions of Equation of the form $iu_t = -\Delta u - k(x)|u|^{4/N}u$ in \mathbb{R}^N under some conditions on $k(x)$. We then consider the problem to find minimal blow-up solutions in L^2 .

RÉSUMÉ. – On démontre l'existence de solutions explosives pour des équations de la forme $iu_t = -\Delta u - k(x)|u|^{4/N}u$ dans \mathbb{R}^N , sous certaines conditions sur $k(x)$. On considère ensuite le problème de trouver des solutions singulières minimales dans L^2 .

Mots clés : Schrödinger, critique, explosion, minimal, stabilité.

1. INTRODUCTION

In the present paper, we consider the nonhomogeneous nonlinear Schrödinger equation with critical exponent

$$(1.1) \quad i \frac{\partial u}{\partial t} = -\Delta u - k(x)|u|^{4/N}u$$

and

$$(1.2) \quad u(0, \cdot) = \phi(\cdot),$$

where Δ is the Laplace operator on \mathbb{R}^N , $u : [0, T) \times \mathbb{R}^N \rightarrow \mathbb{C}$ and $\phi \in H^1(\mathbb{R}^N)$.

We assume in this paper that k is a given C^1 function such that there are $k_1 > 0$, $k_2 > 0$ and $c > 0$ such that

$$(H.1) \quad \forall x \in \mathbb{R}^N, \quad k_1 \leq k(x) \leq k_2,$$

$$(H.2) \quad \forall x \in \mathbb{R}^N, \quad |\nabla k(x)| + |x \cdot \nabla k(x)| \leq c,$$

$$(H.3) \quad \text{there is } x_0 \in \mathbb{R}^N, \quad k(x_0) = k_2.$$

We say that $u(\cdot)$ is a solution of Eq. (1.1)-(1.2) on $[0, T)$ if $\forall t \in [0, T)$,

$$(1.3) \quad u(t) = S(t)\phi + i \int_0^t S(t-s) \{k(x) |u(s)|^{4/N} u(s)\} ds,$$

where $S(\cdot)$ is the group with infinitesimal generator $i\Delta$ and, for each t , $u(t)$ denotes the function $x \rightarrow u(t, x)$.

It is easy to prove as in the homogeneous case:

$$(1.4) \quad k(x) \equiv k_0,$$

that Eq. (1.1)-(1.2) has a unique solution $u(t)$ in $H^1(\mathbb{R}^N)$ and there exists $T > 0$ such that, $\forall t \in [0, T)$, $u(t) \in H^1(\mathbb{R}^N)$ and either

$$T = +\infty,$$

or

$$T < +\infty \quad \text{and} \quad \lim_{t \rightarrow T^-} \|u(t)\|_{H^1} = +\infty,$$

where $\|\cdot\|_{H^1}$ is the usual norm on H^1 , and H^1 is $H^1(\mathbb{R}^N)$ (see Ginibre and Velo [2], Kato [6]).

Furthermore, we have $\forall t \in [0, T)$,

$$(1.5) \quad \int_{\mathbb{R}^N} |u(t, x)|^2 dx = \int_{\mathbb{R}^N} |\phi(x)|^2 dx,$$

$$(1.6) \quad \begin{aligned} E(u(t)) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(t, x)|^2 dx \\ &\quad - \frac{1}{\frac{4}{N} + 2} \int_{\mathbb{R}^N} k(x) |u(t, x)|^{\frac{4}{N} + 2} dx \\ &= E(\phi). \end{aligned}$$

In this paper we are interested in the study of singular solutions of Eq. (1.1)-(1.2). In the case where

$$(1.7) \quad k(x) \not\equiv k_0,$$

there are no results available.

Let us first recall some results in the case where $k(x) \equiv k_0$. For such a nonlinearity, there is another identity which is the following.

Let $\phi \in \Sigma = H^1 \cap \{|x|\phi \in L^2\}$ then $\forall t < T, u(t) \in \Sigma$, and

$$(1.8) \quad \frac{d}{dt} \int |x|^2 |u(t, x)|^2 dx = 4 \operatorname{Im} \int x \cdot \overline{\nabla} u u dx,$$

and

$$(1.9) \quad \frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 dx = 16 E(\phi).$$

From this identity, it follows easily that if

$$(1.10) \quad E(\phi) < 0$$

then

$$T < +\infty$$

(see Zakharov, Sobolev, Synach [15] and Glassey [5]). Moreover blow-up solutions have three important properties.

(i) They are bounded from below in L^2 (Weinstein [18]). That is, let Q_{k_0} be the unique radially symmetric solution of

$$(1.11) \quad \Delta u + k_0 |u|^{\frac{4}{N}} u = u$$

(see for existence Strauss, Berestycki, Lions, Peletier [1], [16], and for uniqueness Kwong [7]). If $u(t)$ is a blow-up solution then

$$\|\phi\|_{L^2} \geq \|Q_{k_0}\|_{L^2}.$$

(ii) The set of minimal blow-up solutions is known (Merle [10], [11]).

Let $u(t)$ be a blow-up solution with minimal mass in L^2 , ($\|\phi\|_{L^2} = \|Q_{k_0}\|_{L^2}$). There are then constants $\theta \in S^1, \omega > 0, x_0 \in \mathbb{R}^N, x_1 \in \mathbb{R}^N$,

$T > 0$ such that

$$(1.12) \quad u(t, x) = \left(\frac{\omega}{T-t} \right)^{\frac{N}{2}} \exp \left\{ i \left(\theta + \frac{|x-x_1|^2}{4(-T+t)} - \frac{\omega^2}{(-T+t)} \right) \right\} \\ \times Q_{k_0} \left(\frac{\omega(x-x_1)}{T-t} - \omega x_0 \right).$$

In [3], [4], we point out the importance of such solutions as limits of “stable” (from the numerical point of view) blow-up solutions for more complex equations which have (1.1) as a limit case (*see* Landam, Papanicolaou, C. and P. L. Sulem, Wang for numerical simulations [8], [14]).

(iii) At the blow-up time, there is a concentration phenomenon (Merle, Tsutsumi [12], Weinstein [19], Merle [9], Proposition A.3 in [4]). Indeed, let $u(t)$ be a blow-up solution of Eq. (1.1) and T its blow-up time. There is then $x(t)$ for $t > T$ such that

$$\forall R > 0, \quad \liminf_{t \rightarrow T} \|u(t)\|_{L^2(B(x(t), R))}^2 \geq \|Q_{k_0}\|_{L^2}^2,$$

where $B(x, R)$ is the ball of radius R and center x .

We first have the following result about existence of blow-up solutions.

THEOREM 1 (Existence and lower L^2 -bound of blow-up solutions, concentration at the blow-up time). – (i) *Lower L^2 -bound: Assume that k satisfies (H.1)-(H.2). Let $\phi \in H^1$ be such that*

$$\|\phi\|_{L^2} < \|Q_{k_2}\|_{L^2}.$$

Then $u(t)$ is globally defined in time.

(ii) *Existence of blow-up solutions: Let k satisfy (H.1)-(H.3). Assume in addition that k satisfies (H.4) or (H.4)' where*

$$(H.4) \quad \text{There is a } \rho_0 > 0 \text{ such that} \\ (x-x_0) \cdot \nabla k(x) < 0 \quad \text{for } 0 < |x-x_0| < \rho_0$$

and

$$(H.4)' \quad \forall x, \quad (x-x_0) \cdot \nabla k(x) \leq 0,$$

and x_0 is such that $k(x_0) = k_2$. Then there is $\varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$, there is $\phi_\varepsilon \in H^1$ such that

$$- \|\phi_\varepsilon\|_{L^2} = \|Q_{k_2}\|_{L^2} + \varepsilon,$$

– $u_\varepsilon(t)$ blows up in finite time where $u_\varepsilon(t)$ is the solution of Eq. (1.1) with initial data ϕ_ε . In addition, $\varepsilon_0 = +\infty$ when k satisfies (H.4)'.

(iii) Concentration at the blow-up time: Let k satisfy (H.1)-(H.2), let $u(t)$ be a blow-up solution of Eq. (1.1) and let T be its blow-up time. There is then $x(t)$ for $t < T$ such that

$$\forall R > 0, \quad \liminf_{t \rightarrow T} \|u(t)\|_{L^2(B(x(t), R))}^2 \geq \|Q_{k_2}\|_{L^2}^2.$$

Remark. – In part (ii), assumption (H.4) or (H.4)' can be weakened (see section 3) and x_0 can be a local maximum. However, it is still an open problem to show existence of blow-up solutions in the case where there is no local maximum of k .

Let us now consider k satisfying (H.1)-(H.3). The main question is whether there is or not L^2 -minimal blow-up solution: Is there a $\phi \in H^1$ such that

- $\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}$,
- $u(t)$ blows-up in finite time where $u(t)$ is the solution of (1.1)-(1.2).

These results related to L^2 -minimal blow-up solutions have a physical interest.

– In the case of existence of such a solution, we have a solution which blows up with minimal mass and is in some sense the limit point of numerically stable blow-up solution (see [8], [14]).

– In the case of nonexistence of such a solution, we obtain the existence of a space singularity which is in some sense, stable in time with respect to Eq. (1.1). We will call this kind of phenomenon a black hole (see Theorem 3).

THEOREM 2 (L^2 -minimal blow-up solutions). – Consider k satisfying (H.1)-(H.2) and (H.5) where

$$(H.5) \quad \begin{aligned} & \text{there are } \delta_0 > 0 \text{ and } R_0 > 0 \\ & \text{such that for } |x| > R_0, \\ & \quad k(x) \leq k_2 - \delta_0 \\ & \text{and } M = \{x; k(x) = k_2\} \text{ is finite.} \end{aligned}$$

(i) *Characterization:* Assume that $\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}$ and $u(t)$ blows-up in finite time. There is then $x_0 \in M$ such that

- $|u(t, x)|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}$ in the distribution sense,
- $|x - x_0|^2 |u(t, x)|^2 \rightarrow 0$ in L^1 , as $t \rightarrow T$.

(ii) *Nonexistence result: Assume in addition that for $x_0 \in M$, we have the following property:*

(H.6) *there is ρ_0 and $\alpha_0 \in (0, 1)$ such that*

$$\nabla k(x) \cdot (x - x_0) \leq -|x - x_0|^{1+\alpha_0}, \quad \text{for } |x - x_0| \leq \rho_0.$$

There is then no blow-up solutions such that

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}.$$

Remark. – In the case where $k(x) \equiv k_2$ globally or $k(x) \equiv k_2$ for x near x_0 , we are able to show the existence of minimal blow-up solution. Therefore, the existence of minimal blow-up solutions depends strongly on the form of the function $k(x)$ near the points where k achieves its maximum. However, we do not know exactly the case of limiting behavior near x_0 (where x_0 is such that $k(x_0) = k_2$) of k (between flatness near x_0 and assumption (H.5)) where there is nonexistence of minimal L^2 blow-up solutions.

We can in addition remark that in the elliptic situation in the case where $k(x) \not\equiv k_2$ there is no solution of the equation

$$\Delta v + k(x) |v|^{\frac{4}{N}} v = \omega v$$

where $\omega > 0$ such that

$$\|v\|_{L^2} = \|Q_{k_2}\|_{L^2}.$$

THEOREM 3 (Stability in time of singularity). – *Assume that x_0 is such that $k(x_0) = k_2$ and x_0 is a strict local maximum. Moreover, assume that there is no blow-up solution of Eq. (1.1)-(1.2) such that*

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}.$$

Consider now a sequence $\phi_n \in H^1$ such that

- $\|\phi_n\|_{L^2}^2 \rightarrow \|Q_{k_2}\|_{L^2}^2$,
- $|\phi_n(x)|^2 \rightharpoonup \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}$ in the distribution sense,
- there is a $c > 0$ such that

$$E_{\varepsilon_n}(\phi_n) \leq c$$

where $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$, $\varepsilon_n > 0$, $q \in \left(\frac{4}{N} + 1, 1 + \frac{4}{N-2}\right)$,

$E_\varepsilon(u) = E(u) + \frac{\varepsilon}{q+1} \int |u|^{q+1}$. Then $u_n(t)$, the solution of equation

$$(1.13) \quad iu_t = -\Delta u - k(x) |u|^{\frac{4}{N}} u + \varepsilon_n |u|^{q-1} u,$$

$$(1.14) \quad u(0) = \phi_n,$$

is such that

- $u_n(t)$ is defined for all time,
- for all time $t > 0$,

$$(1.15) \quad |u_n(t, x)|^2 \rightharpoonup \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}$$

in the distribution sense as $n \rightarrow +\infty$,

and

$$(1.16) \quad \|u_n(t)\|_{L^2} \rightarrow \|Q_{k_2}\|_{L^2}, \quad \text{as } n \rightarrow +\infty.$$

Remark. - In this case, we say that $\|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}$ is a singularity stable in time.

The plan of the paper is the following:

- In section two, we establish some conservation laws for solutions of (1.1) and derive some concentration properties at the blow-up time.
- In section three, we prove some blow-up results.
- Sections four and five are devoted to minimal blow-up solutions.
- Finally, in section six, we study the existence of black holes.

2. CONCENTRATION PROPERTIES OF BLOW-UP SOLUTIONS

In the first subsection, we give various identities satisfied by solutions of Eq. (1.1). We assume that $\phi \in \Sigma = H^1 \cap \{u; xu \in L^2\}$.

2.A. Conservation laws

Let us consider $u(t, x)$ solution of Eq. (1.1) and T its blow-up time.

PROPOSITION 2.1. - We have $\forall t \in [0, T)$,

$$(2.1) \quad (i) \quad \int |u(t, x)|^2 dx = \int |\phi(x)|^2 dx,$$

$$(2.2) \quad (ii) \quad E(u(t)) = E(\phi) \text{ where}$$

$$E(u) = \frac{1}{2} \int |\nabla u(x)|^2 dx - \frac{1}{\frac{4}{N} + 2} \int k(x)|u(t, x)|^{\frac{4}{N}+2} dx,$$

$$(2.3) \text{ (iii)} \quad \frac{d}{dt} \int |x|^2 |u(t, x)|^2 dx = 4 \operatorname{Im} \int \bar{u} \nabla u \cdot x,$$

$$(2.4) \quad \frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 dx \\ = 4 \left\{ 4E(\phi) + \frac{1}{\frac{2}{N} + 1} \int x \cdot \nabla k(x) |u(t, x)|^{\frac{4}{N}+2} dx \right\}.$$

Proof. – (i) and (ii) follow from direct calculation.

(iii) Let us show that

$$(2.5) \quad \frac{d}{dt} \int \operatorname{Im} \bar{u} \nabla u \cdot x = 4E(\phi) \\ + \frac{1}{\frac{2}{N} + 1} \int x \cdot \nabla k(x) |u(t, x)|^{\frac{4}{N}+2} dx.$$

$$(2.6) \quad \frac{d}{dt} \operatorname{Im} \int \bar{u} \nabla u \cdot x = \operatorname{Im} \left\{ \int x \bar{u} \nabla \frac{\partial u}{\partial t} + \int x \frac{\partial \bar{u}}{\partial t} \nabla u \right\} \\ = \operatorname{Im} \left\{ 2 \int x \frac{\partial \bar{u}}{\partial t} \nabla u - N \int \bar{u} \frac{\partial \bar{u}}{\partial t} \right\}.$$

On the one hand,

$$(2.7) \quad N \operatorname{Im} \int \bar{u} \frac{\partial u}{\partial t} = -N \operatorname{Re} \int \bar{u} (\Delta u + k(x) |u|^{\frac{4}{N}} u) \\ = -N \int k(x) |u|^{\frac{4}{N}+2} + N \int |\nabla u|^2.$$

On the other hand

$$(2.8) \quad 2 \operatorname{Im} \int x \frac{\partial \bar{u}}{\partial t} \nabla u \\ = -2 \operatorname{Re} \left\{ \int x \Delta u \nabla \bar{u} + \int x k(x) |u|^{\frac{4}{N}} u \nabla \bar{u} \right\}$$

$$\begin{aligned} &= -(N-2) \int |\nabla u|^2 - \frac{1}{\frac{N}{2} + 1} \int xk(x) \nabla |u|^{\frac{4}{N}+2} \\ &= -(N-2) \int |\nabla u|^2 - \frac{N}{\frac{N}{2} + 1} \int k(x) |u|^{\frac{4}{N}+2} \\ &\quad + \frac{1}{\frac{N}{2} + 1} \int x \cdot \nabla k |u|^{\frac{4}{N}+2}; \end{aligned}$$

(for more detailed calculations see [11], p. 214).

From (2.6)-(2.8), (2.5) follows.

As in the case $k(x) \equiv k_0$, let us derive some consequences of these conservation laws.

COROLLARY 2.2.

(i)
$$\frac{d}{dt} \int \vec{x} |u(t, x)|^2 dx = 2 \operatorname{Im} \int \bar{u} \nabla u,$$

(ii)
$$\frac{d^2}{dt^2} \int \vec{x} |u(t, x)|^2 dx = \frac{2}{\frac{N}{2} + 1} \int \nabla k |u(t, x)|^{\frac{4}{N}+2} dx.$$

Proof. – We have for all $x_0 \in \mathbb{R}^N$:

$$\frac{d}{dt} \int |\vec{x} + \vec{x}_0|^2 |u(t, x)|^2 dx = 4 \operatorname{Im} \int \bar{u} \nabla u \cdot (\vec{x} + \vec{x}_0).$$

Therefore,

$$\begin{aligned} &\frac{d}{dt} \left\{ |x_0|^2 \int |u(t, x)|^2 dx + \int |x|^2 |u(t, x)|^2 dx \right. \\ &\quad \left. + 2 \vec{x}_0 \cdot \int \vec{x} |u(t, x)|^2 dx \right\} \\ &= 4 \operatorname{Im} \int \bar{u} \nabla u \cdot x + 4 \vec{x}_0 \operatorname{Im} \int \bar{u} \nabla u, \end{aligned}$$

and from Proposition 2.1,

(2.9)
$$\vec{x}_0 \cdot 2 \frac{d}{dt} \int \vec{x} |u(t, x)|^2 dx = \vec{x}_0 \cdot 4 \operatorname{Im} \int \bar{u} \nabla u.$$

(i) follows from the fact that (2.9) is true for all $x_0 \in \mathbb{R}^N$. Proof of part (ii) is similar.

Let us write an energy type identity from Proposition 2.1 derived in the case $k(x) \equiv k_2$ by Anosov and rediscovered by Ginibre and Velo.

COROLLARY 2.3. – *We have*

$$\begin{aligned} \tilde{E}_t(u(t)) &= \tilde{E}_0(u(0)) - \int_0^t \frac{s}{\frac{4}{N} + 2} \int x \cdot \nabla k |u(s, x)|^{\frac{4}{N}+2} dx ds \\ &= \frac{1}{8} \int |x|^2 |\phi(x)|^2 dx \\ &\quad - \int_0^t \frac{s}{\frac{4}{N} + 2} \int x \cdot \nabla k |u(s, x)|^{\frac{4}{N}+2} dx ds, \end{aligned}$$

where

$$\tilde{E}_t(u) = \frac{1}{2} \int |\nabla v|^2 - \frac{1}{\frac{4}{N} + 2} \int k(tx) |v|^{\frac{4}{N}+2}$$

with

$$v = |t|^{\frac{N}{2}} e^{-i\frac{|x|^2 t}{4}} u(xt).$$

Proof. – Let $u \in \Sigma$,

$$\begin{aligned} (2.10) \quad \tilde{E}_t(u) &= \frac{1}{2} \int |t|^N \left| \left(-\frac{ixt}{2} + t\nabla \right) u(xt) \right|^2 \\ &\quad - \frac{1}{\frac{4}{N} + 2} t^2 \int k(tx) |t|^N |u(xt)|^{\frac{4}{N}+2} dx \\ &= \frac{1}{2} \int \left| \left(-\frac{iy}{2} + t\nabla \right) u(y) \right|^2 dy \\ &\quad - \frac{t^2}{\frac{4}{N} + 2} \int k(y) |u(y)|^{\frac{4}{N}+2} dy \\ &= \frac{1}{2} \left\{ \frac{1}{4} \int |x|^2 |u(x)|^2 dx \right. \\ &\quad \left. - t \operatorname{Im} \int x \cdot \nabla u \bar{u} \right\} + t^2 E(u). \end{aligned}$$

Let us now consider $\tilde{E}_t(u(t))$

$$\begin{aligned} \frac{d}{dt} \tilde{E}_t(u(t)) &= \frac{1}{2} \left\{ \frac{1}{4} \frac{d}{dt} \int |x|^2 |u(t, x)|^2 dx - \operatorname{Im} \int x \cdot \nabla u(t) \bar{u}(t) \right\} \\ &\quad - \frac{t}{2} \frac{d}{dt} \int x \cdot \nabla u(t) \bar{u}(t) + 2t E(\phi). \end{aligned}$$

From Proposition 2.1, we have

$$\begin{aligned} \frac{d}{dt} \tilde{E}_t(u(t)) &= \frac{1}{2} \left\{ \operatorname{Im} \int x \cdot \nabla u(t) \bar{u}(t) - \operatorname{Im} \int x \cdot \nabla u(t) \bar{u}(t) \right\} \\ &\quad - \frac{t}{2} \left\{ 4E(\phi) + \frac{1}{\frac{N}{2} + 1} \int x \cdot \nabla k |u(t, x)|^{\frac{4}{N}+2} dx \right\} \\ &\quad + 2t E(\phi) \\ &= -\frac{t}{\frac{N}{2} + 2} \int x \cdot \nabla k |u(t, x)|^{\frac{4}{N}+2} dx, \end{aligned}$$

which concludes the proof of Corollary 2.3 and Section 2.A.

2.B. Concentration properties of blow-up solutions of Eq. (1.1)

In this section, we consider a blow-up solution of Eq. (1.1), $u(t)$. Let T be its blow-up time. Assume that

$$\begin{aligned} -0 < k_1 &\equiv \inf_{x \in \mathbb{R}^N} k(x) \leq k_2 \equiv \sup_{x \in \mathbb{R}^N} k(x) < +\infty, \\ -k &\in C^1, \\ -|\nabla k| &\leq c_0. \end{aligned}$$

We claim the following

PROPOSITION 2.4. – *There is $x(t) \in \mathbb{R}^N$ such that for all $R > 0$,*

$$\liminf_{t \rightarrow T} \|u(t)\|_{L^2(B(x(t), R))} \geq \|Q_{k_2}\|_{L^2},$$

where Q_{k_2} is the unique positive radially symmetric solution of

$$v = \Delta v + k_2 |v|^{\frac{4}{N}} v.$$

Remark. – From scaling argument, we have $Q_{k_2} = \frac{1}{k_2^{\frac{N}{4}}} Q$ where Q is the unique radially symmetric solution of (II, 1). In particular

$$\|Q_{k_2}\|_{L^2} = \frac{\|Q\|_{L^2}}{k_2^{\frac{N}{4}}}.$$

In fact, we have a slightly more precise result.

PROPOSITION 2.5. – *There is $x(t) \in \mathbb{R}^N$ such that for all $R > 0$,*

$$\liminf_{t \rightarrow T} \left\{ \frac{\|u(t)\|_{L^2(B(x(t), R))}}{\|Q_{k(x(t))}\|_{L^2}} \right\} \geq 1.$$

Remark.

$$\|Q_{k(x(t))}\|_{L^2} = \frac{\|Q\|_{L^2}}{[k(x(t))]^{\frac{N}{4}}} \geq \frac{\|Q\|_{L^2}}{k_2^{\frac{N}{4}}}.$$

Proof of Proposition 2.5 follows exactly the proof of Proposition 2.4 and will be omitted (it uses the fact that $\forall R > 0$,

$$\sup_{|x-y| \leq R} \left| k\left(\frac{x}{\lambda(t)}\right) - k\left(\frac{y}{\lambda(t)}\right) \right| \leq c_0 \frac{|x-y|}{\lambda(t)} \leq \frac{R c_0}{\lambda(t)} \xrightarrow{t \rightarrow T} 0,$$

where $\lambda(t) = \|\nabla u(t)\|_{L^2}$.

Sketch of proof of Proposition 2.4. – It is a consequence of similar results in [18], [12], [9], [4]. Indeed, we have

$$\begin{aligned} (2.11) \quad E_{k_2}(u(t)) &= \frac{1}{2} \int |\nabla u(t, x)|^2 \\ &\quad - \int \frac{k_2}{\frac{4}{N} + 2} |u(t, x)|^{\frac{4}{N}+2} dx \\ &\leq \frac{1}{2} \int |\nabla u(t, x)|^2 \\ &\quad - \int \frac{k(x)}{\frac{4}{N} + 2} |u(t, x)|^{\frac{4}{N}+2} dx \\ &\leq E(u(t)) = E(\phi) \end{aligned}$$

and

$$(2.12) \quad \|u(t)\|_{L^2} = \|\phi\|_{L^2}.$$

Let us argue by contradiction. Assume there are $R_0 > 0$, $\delta_0 > 0$ and a sequence $t_n \rightarrow T$ such that

$$\sup_{x \in \mathbb{R}^N} \left\{ \int_{|x-y| < R_0} |u(t_n, x)|^2 dy \right\} \leq \|Q_{k_2}\|_{L^2}^2 - \delta_0.$$

Then from results of [12], [19], [4], we have the existence of constants $c_1 > 0$ and $c_2 > 0$ such that

$$(2.13) \quad \forall t_n, \quad -c_1 + c_2 \int |\nabla u(t_n, x)|^2 dx \leq E_{k_2}(u(t_n))$$

(see from example Proposition A.3 in [4]).

From (2.11), we deduce that $\int |\nabla u(t_n, x)|^2 dx \leq c$ which contradicts that $t_n \rightarrow T$. This concludes the proof of Proposition 2.4 and Theorem 1. (iii).

As a direct consequence of Proposition 2.4 and (2.12), we obtain

COROLLARY 2.6. (Lower bound for blow-up solutions). – Assume

$$\|\phi\|_{L^2} < \|Q_{k_2}\|_{L^2} = \frac{\|Q\|_{L^2}}{k_2^{\frac{N}{4}}}.$$

Then the solution $u(t)$ is globally defined in time.

In fact, from the proof of Proposition 2.4, we have a useful corollary (see also [19]):

COROLLARY 2.7. – Let $u_n \in H^1$ be such that $\|u_n\|_{L^2} \rightarrow \|Q_{k_2}\|_{L^2}$, $\lambda_n = \|\nabla u_n\|_{L^2} \rightarrow +\infty$ as $n \rightarrow +\infty$ and $E(u_n) \leq c$ for a $c > 0$. There are sequences $x_n \in \mathbb{R}^N$, $\theta_n \in S^1$ such that

$$|u_n(x - x_n)|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0},$$

and

$$\lambda_n^{-\frac{N}{2}} e^{i\theta_n} u_n \left(\frac{x - x_n}{\lambda_n} \right) \rightarrow Q_{k_2} \text{ in } H^1.$$

3. BLOW-UP THEOREMS FOR SOLUTIONS OF EQ. (1.1)

In the homogeneous case

$$(3.1) \quad k(x) \equiv k_0;$$

blow-up theorems are obtained using the virial identity

$$(3.2) \quad \frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 dx = 16 E(\phi).$$

(see [5], [15]). If $E(\phi) < 0$, then using the fact

$$(3.3) \quad \forall t, \quad \int |x|^2 |u(t, x)|^2 dx > 0$$

and (3.2), we obtain a contradiction.

In the case where

$$k(x) \equiv k$$

such an identity is not true anymore (see (2.4)) and we have $\forall x_0 \in \mathbb{R}^N$,

$$(3.4) \quad \frac{d^2}{dt^2} \int |x - x_0|^2 |u(t, x)|^2 dx = 16 E(\phi) + \frac{4}{\frac{2}{N} + 1} \\ \times \int (x - x_0) \cdot \nabla k |u(t, x)|^{\frac{4}{N} + 2} dx.$$

Under some global or local conditions on the sign of

$$(x - x_0) \cdot \nabla k(x)$$

we are able to obtain some blow-up theorems for solutions of Eq. (1.1).

THEOREM 3.1 (Global condition on $(x - x_0) \cdot \nabla k(x)$). – Assume there is $x_0 \in \mathbb{R}^N$ such that

$$(3.5) \quad \forall x \in \mathbb{R}^N, \quad (x - x_0) \cdot \nabla k(x) \leq 0$$

so that x_0 is global maximum of $k(x)$.

(i) Let $\phi \in \Sigma$ be such that $E(\phi) < 0$. Then the solution $u(t)$ of Eq. (1.1) blows up in finite time.

For all $\varepsilon > 0$, there is ϕ_ε such that

$$- \|\phi_\varepsilon\|_{L^2} = \|Q_{k_2}\|_{L^2} + \varepsilon,$$

– $u_\varepsilon(t)$ blows-up in finite time, where $u_\varepsilon(t)$ is the solution of Eq. (1.1) with initial data ϕ_ε .

THEOREM 3.2 (Local condition on $(x - x_0) \cdot \nabla k(x)$). – Assume there is $x_0 \in \mathbb{R}^N$ and $\rho_0 > 0$ such that

$$(3.6) \quad (x - x_0) \cdot \nabla k(x) < 0, \quad \text{for } 0 < |x - x_0| < \rho_0,$$

so that x_0 is a local strict maximum of $k(x)$

$$k(x_0) \geq k(x) \quad \text{for } 0 < |x - x_0| < \rho_0.$$

There is ε_0 such that for all $0 < \varepsilon < \varepsilon_0$, there exists $\phi_\varepsilon \in \Sigma$ such that

- $\|\phi_\varepsilon\|_{L^2} = \|\mathcal{Q}_{k(x_0)}\|_{L^2} + \varepsilon,$
- $u_\varepsilon(t)$ blows up in finite time where $u_\varepsilon(t)$ is the solution of Eq. (1.1) with initial data ϕ_ε .

Remark. – Theorem 3.2 implies Theorem 3.1 but the proof of Theorem 3.1 is completely elementary. Assumption (3.6) can be weakened and replaced by

$$(3.6)' \quad \begin{cases} (x - x_0) \cdot \nabla k(x) \leq 0 & \text{for } 0 < |x - x_0| < \rho_0, \\ (x - x_0) \cdot \nabla k(x) < 0 & \text{on } S, \end{cases}$$

where S is a closed hypersurface included in $B(x_0, \rho_0)$ with x_0 in its interior.

In Theorem 3.1 or 3.2, we have to assume that x_0 is a local maximum. An open problem left in this direction is to obtain blow-up theorem in the case where there is no local maximum of k in \mathbb{R}^N . For example, consider in \mathbb{R} a function $k(x)$ such that

- $k' < 0,$
- $\lim_{x \rightarrow +\infty} k(x) = k_1 > 0,$
- $\lim_{x \rightarrow -\infty} k(x) = k_2 > 0.$

Is there a blow-up solution of Eq. (1.1)?

Proof of Theorem 3.1. – The proof is completely elementary.

(i) Let $\phi \in \Sigma$ such that $E(\phi) < 0$. Consider $y(t) = \int |x - x_0|^2 |u(t, x)|^2 dx$ and assume by contradiction that $u(t)$ and $y(t)$ are defined for all time; we have $\forall t > 0, y''(t) \leq 16 E(\phi)$. Thus by integration

$$\forall t > 0, \quad y(t) \leq y(0) + ty'(0) + 8t^2 E(\phi) = z(t).$$

Since $E(\phi) < 0, z(t) < 0$ for t large which is contradiction. This concludes the proof of (i).

(ii) (3.5) implies directly that x_0 is a global maximum. Let $k_2 = k(x_0)$. For all $\varepsilon > 0$, consider for $\lambda > 0, w_{\varepsilon, \lambda} = (1 + \varepsilon) \frac{1}{\lambda^{N/2}} \mathcal{Q}_{k_2} \left(\frac{(x - x_0)}{\lambda} \right)$. $\forall \lambda > 0,$

$$(3.7) \quad \|w_{\varepsilon, \lambda}\|_{L^2} = (1 + \varepsilon) \|\mathcal{Q}_{k_2}\|_{L^2}.$$

In addition,

$$\begin{aligned} E(w_{\varepsilon, \lambda}) &= \frac{1}{2} \int |\nabla w_{\varepsilon, \lambda}|^2 - \frac{1}{\frac{4}{N} + 2} \int k(x) |w_{\varepsilon, \lambda}|^{\frac{4}{N} + 2} \\ &= E_{k_2}(w_{\varepsilon, \lambda}) + \frac{1}{\frac{4}{N} + 2} \int (k(x_0) - k(x)) |w_{\varepsilon, \lambda}|^{\frac{4}{N} + 2} \end{aligned}$$

where $E_{k_2}(w) = \frac{1}{2} \int |\nabla w|^2 - \frac{1}{\frac{4}{N} + 2} \int k_2 |w|^{\frac{4}{N} + 2}$.

On the one hand, by scaling arguments

$$(3.8) \quad \begin{aligned} E_{k_2}(w_{\varepsilon, \lambda}) &= (1 + \varepsilon)^2 \frac{1}{\lambda^2} E_{k_2}(Q_{k_2}) \\ &\quad + ((1 + \varepsilon)^2 - (1 + \varepsilon)^{\frac{4}{N} + 2}) \frac{1}{\lambda^2} \int Q_{k_2}^{\frac{4}{N} + 2}. \end{aligned}$$

Since $E_{k_2}(Q_{k_2}) = 0$ (Pohazaev identity),

$$(3.9) \quad \forall \lambda > 0, \quad E_{k_2}(w_{\varepsilon, \lambda}) \leq -\frac{c(\varepsilon)}{\lambda^2} \quad \text{where } c(\varepsilon) > 0.$$

Since $\forall x, Q_{k_2}(x) \leq c_0 e^{-c_1|x|}$ and $|\nabla k(x)| \leq c_0$, for $\lambda > 1$,

$$(3.10) \quad \begin{aligned} &\left| \int (k(x_0) - k(x)) |w_{\varepsilon, \lambda}(x)|^{\frac{4}{N} + 2} \right| \\ &\leq c + \int_{|x-x_0| \leq 1} |k(x_0) - k(x)| |w_{\varepsilon, \lambda}|^{\frac{4}{N} + 2} \\ &\leq c + c \int_{\mathbb{R}^N} \frac{|x|}{\lambda^{2+N}} e^{-c_1 \frac{|x|}{\lambda}} dx \\ &\quad c + \frac{c}{\lambda} \int |y| e^{-c_1|y|} dy \leq c \left(1 + \frac{1}{\lambda} \right). \end{aligned}$$

From (3.9)-(3.10) we derive that for $\lambda \geq \lambda(\varepsilon)$, $E(w_{\varepsilon, \lambda}) < 0$ and for $\varepsilon > 0$, $\phi_\varepsilon = w_{\varepsilon, \lambda(\varepsilon)}$ satisfies the conclusions of Theorem 3.1. This concludes the proof of Theorem 3.1.

Proof of Theorem 3.2. – We remark that we had showed in the proof of Theorem 3.1 (ii) the following lemma.

LEMMA 3.3. – $\forall \varepsilon \in (0, 1)$, for all $A(\varepsilon) > 0$, there is a $\phi_\varepsilon \in \Sigma$ such that

$$- \|\phi_\varepsilon\|_{L^2} = \|Q_{k(x_0)}\|_{L^2} + \varepsilon,$$

$$- E(\phi_\varepsilon) = -A(\varepsilon),$$

$$- \int |x|^2 |\phi_\varepsilon|^2 \leq C, \text{ (where } C \text{ is independent of } \varepsilon \text{ and } A(\varepsilon)\text{),}$$

$$- \forall x \in \mathbb{R}^N, \phi_\varepsilon(x) \in \mathbb{R},$$

$$- \|\nabla \phi_\varepsilon\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} +\infty \text{ and } |\phi_\varepsilon(x)|^2 \xrightarrow{\varepsilon \rightarrow 0} \|Q_{k(x_0)}\|_{L^2}^2 \delta_{x=x_0}.$$

Proof. – It follows from the proof of Theorem 3.1 (ii) and direct computations.

We claim now for $A(\varepsilon)$ sufficiently large as $\varepsilon \rightarrow 0$, the solution $u_\varepsilon(t)$ associated with ϕ_ε blows up in finite time. We now assume that $A(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} +\infty$. We argue by contradiction. We suppose that $u_\varepsilon(t)$ is globally defined in time. The two key arguments of the proof are

– On one hand, the use of the geometry of $k(x)$ near x_0 to control the evolution of the concentration point;

– On the other hand, the use of local virial identity as in [10], [11]. We proceed in three steps to obtain a contradiction.

Step 1. – Concentration properties of $u_\varepsilon(t)$.

PROPOSITION 3.4 (Concentration in L^2 of $u_\varepsilon(t)$). – For all $\varepsilon' > 0$, there is ε_0 such that, $\forall \varepsilon \in (0, \varepsilon_0), \forall t \geq 0$,

$$(3.11) \quad \left| \int_{|x-x_0| \leq \varepsilon'} |u_\varepsilon(t, x)|^2 dx - \int_{\mathbb{R}^N} Q_{k(x_0)}^2(x) dx \right| < \varepsilon',$$

and

$$(3.12) \quad \int_{|x-x_0| \geq \varepsilon'} |u_\varepsilon(t, x)|^2 dx \leq \varepsilon'.$$

Proof of Proposition 3.4. – One uses the fact that x_0 is a strict local maximum and some contraction lemma.

LEMMA 3.5. – Consider a sequence $t_\varepsilon \in \mathbb{R}$. We then have

$$(3.13) \quad \|\nabla u_\varepsilon(t_\varepsilon)\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} +\infty.$$

Proof. – Indeed, by contradiction, assume there is a $c > 0$ such that for a sequence $\varepsilon_n \rightarrow 0$

$$\|\nabla u_{\varepsilon_n}(t_{\varepsilon_n})\|_{L^2} \leq c.$$

Then by Sobolev imbeddings

$$E(\phi_{\varepsilon_n}) = |E(u_{\varepsilon_n}(t_{\varepsilon_n}))| \leq \frac{1}{2} \|\nabla u_{\varepsilon_n}(t_{\varepsilon_n})\|_{L^2}^2 + c \|u_{\varepsilon_n}(t_{\varepsilon_n})\|_{L^{\frac{4}{N}+2}}^{\frac{4}{N}+2} \leq c$$

which contradicts the fact

$$|E(\phi_{\varepsilon_n})| = A(\varepsilon_n) \rightarrow +\infty \quad \text{as } n \rightarrow +\infty.$$

Similarly with Proposition 2.5, we have the following lemma.

LEMMA 3.6. – *Let u_n be such that for constants a, b ,*

$$(3.13) \quad \|u_n\|_{L^2} \leq a,$$

$$(3.14) \quad E(u_n) \leq b,$$

$$(3.15) \quad \|\nabla u_n\|_{L^2} \rightarrow +\infty \quad \text{as } n \rightarrow +\infty.$$

There is x_n such that for all $R > 0$

$$\liminf_{n \rightarrow +\infty} \left\{ \frac{\|u_n\|_{L^2(B(x_n, R))}}{\|Q_{k(x_n)}\|_{L^2}} \right\} \geq 1.$$

Proof. – See Corollary 2.7.

Applying Lemma 3.6 with $u_\varepsilon(t)$ ($a = 2\|Q_{k(x_0)}\|_{L^2}$, $b = 0$), we obtain the conclusion.

Indeed, consider $\delta > 0$ such that

$$(3.16) \quad \forall x, \quad \|Q_{k(x)}\|_{L^2}^2 \geq 2\delta.$$

$$\left((3.16) \text{ is equivalent to, } \forall x, \quad \frac{\|Q\|_{L^2}^2}{k(x)^{\frac{N}{2}}} \geq 2\delta \text{ or equivalently } \delta \leq \frac{\|Q\|_{L^2}^2}{2k_2^{\frac{N}{2}}} \right).$$

Consider, for each $\varepsilon > 0$, \tilde{T}_ε such that

$$(3.17) \quad \forall t \in [0, \tilde{T}_\varepsilon), \quad \|u_\varepsilon(t, x)\|_{L^2(B(x_0, \frac{\rho_0}{4}))}^2 \geq \|Q_{k(x_0)}\|_{L^2}^2 - \delta,$$

$$(3.18) \quad \|u_\varepsilon(\tilde{T}_\varepsilon, x)\|_{L^2(B(x_0, \frac{\rho_0}{4}))}^2 = \|Q_{k(x_0)}\|_{L^2}^2 - \delta.$$

From Lemma 3.3, for ε small enough, $\tilde{T}_\varepsilon > 0$. Let us show that for $\varepsilon < \varepsilon_0$ (where $\varepsilon_0 > 0$)

$$(3.19) \quad \tilde{T}_\varepsilon = +\infty.$$

Indeed, by contradiction, assume that for $\varepsilon_n \rightarrow 0$

$$(3.20) \quad \tilde{T}_{\varepsilon_n} = \tilde{T}_n < +\infty.$$

Consider $u_n = u_{\varepsilon_n}(\tilde{T}_n, x)$. u_n satisfies (3.13)-(3.15), therefore from Lemma 3.6, there is x_n

$$(3.21) \quad \forall R, \quad \liminf_{n \rightarrow +\infty} \|u_n\|_{L^2(B(x_n, R))}^2 \geq \limsup_{n \rightarrow +\infty} \|Q_{k(x_n)}\|_{L^2}^2.$$

We chain for n large

$$(3.22) \quad |x_n - x_0| \leq \frac{\rho_0}{2}.$$

Indeed if not

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \|u_n\|_{L^2(B(x_0, \frac{\rho_0}{4}))}^2 &\geq \lim_{n \rightarrow +\infty} \|u_n\|_{L^2(B(x_n, \frac{\rho_0}{4}))}^2 \\ &\geq \limsup_{n \rightarrow +\infty} \|Q_{k(x_0)}\|_{L^2}^2 \geq 2\delta \end{aligned}$$

and from (3.18)

$$(3.23) \quad \liminf_{n \rightarrow +\infty} \|u_n\|_{L^2}^2 \geq \|Q_{k(x_n)}\|_{L^2}^2 + \delta.$$

Since $\|u_n\|_{L^2} = \|u_{\varepsilon_n}(\tilde{T}_{\varepsilon_n}, x)\|_{L^2} = \|\phi_{\varepsilon_n}\|_{L^2} \rightarrow \|Q_{k(x_0)}\|_{L^2}$ as $n \rightarrow +\infty$, we obtain a contradiction. We then remark that

$$(3.24) \quad x_n \rightarrow x_0 \quad \text{as } n \rightarrow +\infty.$$

Indeed, we have from (3.21)

$$\begin{aligned}
 (3.25) \quad \liminf_{n \rightarrow +\infty} \|\phi_{\varepsilon_n}\|_{L^2}^2 &= \liminf_{n \rightarrow +\infty} \|u_n\|_{L^2}^2 \\
 &\geq \limsup_{n \rightarrow +\infty} \|Q_{k(x_n)}\|_{L^2}^2 \\
 &\geq \limsup_{n \rightarrow +\infty} \frac{\|Q\|_{L^2}^2}{[k(x_n)]^{\frac{N}{2}}} \\
 &\geq \|Q_{k(x_0)}\|_{L^2}^2 \limsup_{n \rightarrow +\infty} \left[\frac{k(x_0)}{k(x_n)} \right]^{\frac{N}{2}}.
 \end{aligned}$$

From Lemma 3.3,

$$\|Q_{k(x_0)}\|_{L^2}^2 \geq \|Q_{k(x_0)}\|_{L^2}^2 \limsup_{n \rightarrow +\infty} \left[\frac{k(x_0)}{k(x_n)} \right]^{\frac{N}{2}}$$

or

$$\liminf_{n \rightarrow +\infty} k(x_n) \geq k(x_0),$$

which is equivalent from (3.6) and (3.22) to

$$x_n \rightarrow x_0 \quad \text{as } n \rightarrow +\infty.$$

From (3.21)

$$\begin{aligned}
 (3.26) \quad \liminf_{n \rightarrow +\infty} \|u_{\varepsilon_n}(\tilde{T}_{\varepsilon_n})\|_{L^2(x_0, \frac{\rho_0}{4})}^2 &\geq \liminf_{n \rightarrow +\infty} \|u_{\varepsilon_n}(\tilde{T}_{\varepsilon_n})\|_{L^2(x_n, \frac{\rho_0}{8})}^2 \\
 &\geq \liminf_{n \rightarrow +\infty} \|u_n\|_{L^2(x_n, \frac{\rho_0}{8})}^2 \\
 &\geq \|Q_{k(x_0)}\|_{L^2}^2
 \end{aligned}$$

which is a contradiction with (3.18). Therefore there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, $\tilde{T}_\varepsilon = +\infty$.

Let us conclude the proof of Proposition 3.4 by contradiction. We claim that (3.11), (3.12) follow from Lemma 3.3 and the conservation of mass. Assume there is t_{ε_n} and $\varepsilon_n \rightarrow 0$, $\varepsilon' > 0$ such that

$$(3.27) \quad \left| \|Q_{k(x_0)}\|_{L^2}^2 - \int_{|x-x_0| \leq \varepsilon'} |u_n|^2 \right| \geq \varepsilon',$$

where $u_n = u_{\varepsilon_n}(t_{\varepsilon_n}, x)$.

As before, there is x_n such that

$$(3.28) \quad \forall R, \quad \left[\liminf_{n \rightarrow +\infty} \|u_n\|_{L^2(B(x_0, R))}^2 \left[\frac{k(x_n)}{k(x_0)} \right]^{\frac{N}{2}} \right] \geq \|Q_{k(x_0)}\|_{L^2}^2.$$

We have from (3.17) and (3.19) by the same arguments than before that for n large $|x_n - x_0| \leq \frac{\rho_0}{2}$ and then $x_n \rightarrow x_0$ as $n \rightarrow +\infty$.

In particular, from (3.28) as classical arguments,

$$(3.29) \quad \forall R, \quad \liminf_{n \rightarrow +\infty} \|u_n\|_{L^2(B(x_0, R))}^2 \geq \|Q_{k(x_0)}\|_{L^2}^2.$$

Since

$$\|\phi_{\varepsilon_n}\|_{L^2} = \|u_{\varepsilon_n}\|_{L^2} = \|u_n\|_{L^2} \xrightarrow{n \rightarrow +\infty} \|Q_{k(x_0)}\|_{L^2},$$

we have

$$\lim_{n \rightarrow +\infty} (\|u_n\|_{L^2(B(x_0, R))}^2 - \|Q_{k(x_0)}\|_{L^2}^2) = 0,$$

which is a contradiction with (3.27). Thus Proposition 3.4 is proved.

Remark. – In the case where x_0 is a global maximum, we do not need to prove (3.19).

Step 2. – Energy estimates outside the concentration point.

Using local virial identity, we are able to prove the following proposition.

PROPOSITION 3.7. – *There are constants $0 < B_0 < \frac{\rho_0}{4}$, $c_1 > 0$ and $c_2 > 0$ independent of ε such that $\forall t$,*

$$\begin{aligned} & - \left[8E(\phi_\varepsilon)t^2 + \int_0^t (t-s) \frac{4}{\left(\frac{2}{N} + 1\right)} \int_{|x-x_0| \leq 2B_0} (x-x_0) \right. \\ & \quad \left. \times \nabla k(x)|u_\varepsilon(s, x)|^{\frac{4}{N}+2} dx ds \right] \\ & \geq c_2 \int_0^t (t-s) \int_{|x-x_0| \geq \frac{\rho_0}{2}} |\nabla u_\varepsilon(s, x)|^2 dx ds - c_1. \end{aligned}$$

Proof of Proposition 3.7. – Let us give some lemmas. As in [10], [11] for $k(x) \equiv k_0$ we have local virial identity.

LEMMA 3.8 (Local virial identity). – Consider $\psi \in C^4(\mathbb{R}^N, \mathbb{R})$ with compact support.

$$\begin{aligned}
 \text{(i)} \quad & \frac{d}{dt} \int \psi(x) |u(t, x)|^2 = 2 \operatorname{Im} \int \nabla \psi \nabla u \bar{u}, \\
 \text{(ii)} \quad & \frac{d^2}{dt^2} \int \psi(x) |u(t, x)|^2 = 2 \left\{ -\frac{2}{N \left(\frac{2}{N} + 1 \right)} \int \Delta \psi k |u|^{\frac{4}{N}+2} \right. \\
 & + 2 \sum_{i,j} \int \partial_i \partial_j \psi \partial_i u \overline{\partial_j u} - \frac{1}{2} \int |u|^2 \Delta^2 \psi \\
 & \left. + \frac{1}{\frac{2}{N} + 1} \int \nabla \psi \cdot \nabla k |u|^{\frac{4}{N}+2} dx \right\}.
 \end{aligned}$$

Proof. – It follows from similar calculation as in [10].

LEMMA 3.9. – Let $\rho(x) \in C^1(\mathbb{R}^N, \mathbb{R})$ such that $\rho \in L^\infty$ and $\nabla \rho \in L^\infty$. There is a $c_\rho > 0$ such that

$$\begin{aligned}
 \forall u \in H^1, \quad & \int |u(x)|^{\frac{4}{N}+2} \rho^2(x) dx \leq c_\rho \left(\int u^2 \right)^{\frac{2}{N}} \\
 & \times \left\{ \int |\nabla u|^2 \rho^2 + \int \nabla \rho^2 u^2 \right\}.
 \end{aligned}$$

Proof. – See [10] p. 434.

We claim now that applying Lemma 3.8 to a suitable function $\psi(x)$, we obtain Proposition 3.7. Indeed, consider ψ such that

$$(3.27) \quad \psi \in C^4(\mathbb{R}^N, \mathbb{R}) \quad \text{and} \quad \psi(x) = \psi(|x|),$$

$$\begin{aligned}
 (3.28) \quad & \psi(x) < |x|^2 \quad \text{for } |x| > \beta_0, \\
 & \psi(x) \equiv |x|^2 \quad \text{for } |x| \leq \beta_0, \\
 & \frac{1}{2} |x|^2 \leq x \cdot \nabla \psi \leq 3|x|^2 \quad \text{for } |x| \leq 2\beta_0, \\
 & \Delta \psi - 2N \geq 0 \quad \text{for } |x| \leq 2\beta_0,
 \end{aligned}$$

$$(3.29) \quad \psi(x) \equiv c \quad \text{for } |x| \geq \frac{\rho_0}{2},$$

there are a constant c_0 and a function g such that for

$$(3.30) \quad \beta_0 \leq |x| \leq \frac{\rho_0}{2}, \quad \forall a \in \mathbb{C}^N$$

$$\left(\sum_i |a_i|^2 - \sum_{i,j} \frac{\partial_i \partial_j \psi}{2} a_i \bar{a}_j \right) \geq g(x) \left(\sum_i |a_i|^2 \right),$$

where $g(x) \geq c_0$ for $|x| \geq 2\beta_0$ and $g(x) \geq 0, \forall x$.

The existence of such a ψ can be proved easily, and the proof is omitted.

We have then by Lemma 3.8 and Lemma 3.3, $\forall \varepsilon, \forall t > 0$,

$$(3.31) \quad \int \psi(x - x_0) |u_\varepsilon(t, x)|^2 dx$$

$$= \int \psi(x - x_0) |\phi_\varepsilon|^2$$

$$+ 2t \operatorname{Im} \int \nabla \psi \cdot \nabla \phi_\varepsilon \bar{\phi}_\varepsilon$$

$$+ 2 \int_0^t (t-s) \left\{ -\frac{2}{N \left(\frac{2}{N} + 1 \right)} \int \Delta \psi k |u_\varepsilon(s)|^{\frac{4}{N}+2} \right.$$

$$+ 2 \sum_{i,j} \int \partial_i \partial_j \psi \partial_i u_\varepsilon(s) \overline{\partial_j u_\varepsilon(s)} - \frac{1}{2} \int |u_\varepsilon(s)|^2 \Delta^2 \psi$$

$$\left. + \frac{1}{\frac{2}{N} + 2} \int \nabla k \nabla \psi (x - x_0) |x_\varepsilon(s)|^{\frac{4}{N}+2} \right\} ds.$$

From (3.31), the conservation of mass, Lemma 3.3 and (3.27)-(3.30), we obtain, $\forall \varepsilon, \forall t$

$$(3.32) \quad \left| \int_0^t (t-s) \left\{ -\frac{2}{N \left(\frac{2}{N} + 1 \right)} \int \Delta \psi k |u_\varepsilon(s)|^{\frac{4}{N}+2} \right. \right.$$

$$+ 2 \sum_{i,j} \int \partial_i \partial_j \psi \partial_i u_\varepsilon(s) \overline{\partial_j u_\varepsilon(s)}$$

$$\left. + \frac{1}{\frac{2}{N} + 2} \int \nabla k \nabla \psi (x - x_0) |u_\varepsilon(s)|^{\frac{4}{N}+2} \right\} ds \Big|$$

$$\leq c_1 + c_1 t^2.$$

Thus

$$\begin{aligned}
 (3.33) \quad & \left| \int_0^t (t-s) \left\{ 8 E(\phi_\varepsilon) + \int_{|x-x_0| \geq \beta_0} \frac{-2}{N \left(\frac{2}{N} + 1 \right)} \right. \right. \\
 & \quad \times (\Delta \psi - 2N) k |u_\varepsilon|^{\frac{4}{N}+2} \\
 & \quad \left. + 2 \left(\sum_{i,j} \partial_i \partial_j \psi \partial_i \mu_\varepsilon \partial_j \bar{u}_\varepsilon \right) - 4 \left(\sum_i |\partial_i u_\varepsilon|^2 \right) \right\} dx \\
 & \quad \left. + \frac{1}{\frac{2}{N} + 2} \int \nabla k \nabla \psi (x - x_0) |u_\varepsilon|^{\frac{4}{N}+2} dx \right\} ds \Big| \\
 & \leq c_1 + c_1 t^2,
 \end{aligned}$$

or equivalently

$$\begin{aligned}
 (3.34) \quad & \int_0^t 2 \int_{|x-x_0| \geq \beta_0} \left(\sum_i |\partial_i u|^2 \right) - \frac{1}{2} \left(\sum_{i,j} \partial_i \partial_j \psi \partial_i u \bar{\partial}_j u \right) \\
 & \leq c \left(|E(\phi_\varepsilon)| t^2 + 1 + \int_0^T \int_{\beta_0 \leq |x-x_0| \leq \frac{\rho_0}{2}} |u|^{\frac{4}{N}+2} dx ds \right) \\
 & \quad + \left| \int_0^T (T-t) \int_{|x-x_0| \leq 2\beta_0} \nabla k \cdot (x - x_0) |u|^{\frac{4}{N}+2} dx dt \right|.
 \end{aligned}$$

In addition, from (3.6) and a compactness argument in \mathbb{R}^N , we have

$$(3.35) \quad |(x - x_0) \cdot \nabla k| \geq c_0 > 0 \quad \text{for } \beta_0 \leq |x - x_0| \leq \frac{\rho_0}{2}.$$

Thus (3.34)-(3.35) yield Proposition 3.7.

Step 3. – Conclusion of the proof.

From Proposition 2.1, we have $\forall \varepsilon, \forall t > 0$,

$$\begin{aligned}
 & \frac{d}{dt} \int |x|^2 |u_\varepsilon(t, x)|^2 dx = 4 \operatorname{Im} \int \bar{u}_\varepsilon \nabla u_\varepsilon x, \\
 & \frac{d^2}{dt^2} \int |x|^2 |u_\varepsilon(t, x)|^2 dx \\
 & = 4 \left\{ 4 E(\phi_\varepsilon) + \frac{1}{\frac{2}{N} + 1} \int (x - x_0) \nabla k |u_\varepsilon|^{\frac{4}{N}+2} \right\}.
 \end{aligned}$$

We integrate twice these identities and using Lemma 3.3 we obtain for t ,

$$(3.36) \quad \begin{aligned} y_\varepsilon(t) &= \int |x|^2 |u_\varepsilon(t, x)|^2 dx = 8E(\phi_\varepsilon)t^2 \\ &\quad + \int |x|^2 |\phi_\varepsilon|^2 + \int_0^t (t-s) \frac{4}{\frac{2}{N} + 1} \\ &\quad \times \int (x-x_0) \nabla k |u_\varepsilon(s)|^{\frac{4}{N}+2} dx ds \end{aligned}$$

$$(3.37) \quad \begin{aligned} &= 8E(\phi_\varepsilon)t^2 + \int |x|^2 |\phi_\varepsilon|^2 \\ &\quad + \int_0^t (t-s) \frac{4}{\frac{2}{N} + 1} \\ &\quad \times \int_{|x-x_0| \leq \rho_0} (x-x_0) \nabla k |u_\varepsilon(s)|^{\frac{4}{N}+2} dx ds \\ &\quad + \int_0^t (t-s) \frac{4}{\frac{2}{N} + 1} \\ &\quad \times \int_{|x-x_0| \geq \rho_0} (x-x_0) \nabla k |u_\varepsilon|^{\frac{4}{N}+2} dx ds. \end{aligned}$$

Let us estimate the last term.

LEMMA 3.10. – *There is a constant $c(\varepsilon)$ depending only on ε such that*

$$(i) \quad c(\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

$$(ii) \quad \forall t, \quad \left| \int_0^t (t-s) \frac{4}{\frac{2}{N} + 1} \times \int_{|x-x_0| \geq \rho_0} |x-x_0| \nabla k |u_\varepsilon|^{\frac{4}{N}+2} dx ds \right| \leq c(\varepsilon) \left| c_1 + 8E(\phi_\varepsilon)t^2 + \int_0^t (t-s) \frac{4}{\frac{2}{N} + 1} \times \int_{|x-x_0| \leq \frac{\rho_0}{2}} (x-x_0) \nabla k |u_\varepsilon|^{\frac{4}{N}+2} dx ds \right|.$$

Proof. – We have

$$\begin{aligned}
 (3.38) \quad \beta_\varepsilon(t) &= \left| \int_0^t (t-s) \frac{4}{\frac{2}{N} + 1} \int_{|x-x_0| \geq \rho_0} \right. \\
 &\quad \left. \times (x-x_0) \nabla k |u_\varepsilon|^{\frac{4}{N}+2} dx ds \right| \\
 &\leq c \int_0^t (t-s) \int_{|x-x_0| \geq \rho_0} |u_\varepsilon|^{\frac{4}{N}+2} dx ds \\
 &\leq c \int_0^t (t-s) \int \rho^2(x) |u_\varepsilon|^{\frac{4}{N}+2} dx ds,
 \end{aligned}$$

where ρ is a C^∞ function such that

- $0 \leq \rho \leq 1$
- $\rho = 1$ for $|x - x_0| \geq \rho_0$
- $\rho = 0$ for $|x - x_0| \leq \frac{\rho_0}{2}$.

Therefore from Lemma 3.9,

$$\begin{aligned}
 (3.39) \quad \beta_\varepsilon(t) &\leq c \int_0^t (t-s) \left\{ \left(\int_{|x-x_0| \geq \rho_0} |u_\varepsilon(s, x)|^2 dx \right)^{\frac{2}{N}} \right. \\
 &\quad \left(\int |\nabla u_\varepsilon(s, x)|^2 \rho^2(x) dx \right. \\
 &\quad \left. + \int \nabla \rho^2(x) |u_\varepsilon(s, x)|^2 dx \right) ds \\
 &\leq c \int_0^t (t-s) \left\{ \left(\int_{|x-x_0| \geq \frac{\rho_0}{2}} |u_\varepsilon(s, x)|^2 dx \right)^{\frac{2}{N}} \right. \\
 &\quad \left. \left(\int_{|x-x_0| \geq \frac{\rho_0}{2}} |u_\varepsilon(s, x)|^2 + |\nabla u_\varepsilon(s, x)|^2 dx \right) \right\} ds.
 \end{aligned}$$

From Step 1, we have

$$\begin{aligned}
 (3.40) \quad \beta_\varepsilon(t) &\leq c(\varepsilon) \left\{ c_1 + t^2 + \int_0^t (t-s) \right. \\
 &\quad \left. \times \int_{|x-x_0| \geq \frac{\rho_0}{2}} |\nabla u_\varepsilon(s, x)|^2 dx ds \right\}
 \end{aligned}$$

where

$$c(\varepsilon) = \left(\sup_{t \in \mathbb{R}} \int_{|x-x_0| \geq \frac{\rho_0}{2}} |u_\varepsilon(t, x)|^2 \right)^{\frac{2}{N}} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

From Proposition 3.7 and (3.40), we conclude the proof of Lemma 3.10.

Let ε_0 be such that

$$c(\varepsilon) \leq \frac{1}{2}, \quad \text{for } \varepsilon \leq \varepsilon_0.$$

For $\varepsilon \leq \varepsilon_0, \forall t,$

$$\begin{aligned} (3.41) \quad y_\varepsilon(t) &\leq \frac{1}{8} 8 E(\phi_\varepsilon) t^2 + c_1 - \frac{1}{2} E(\phi_\varepsilon) t^2 \\ &\quad + \frac{3}{2} \int_0^t (t-s) \frac{4}{\frac{4}{N} + 1} \\ &\quad \times \int_{|x-x_0| \leq \frac{\rho_0}{2}} (x-x_0) \nabla k |u_\varepsilon|^{\frac{4}{N}+2} dx ds \\ &\quad + \int_0^t (t-s) \frac{4}{\frac{4}{N} + 1} \\ &\quad \times \int_{\frac{\rho_0}{2} \leq |x-x_0| \leq \rho_0} (x-x_0) \nabla k |u_\varepsilon|^{\frac{4}{N}+2} dx ds. \end{aligned}$$

Since $(x-x_0) \nabla k \leq 0$ on $|x-x_0| \leq \rho_0$, we have

$$(3.42) \quad \forall t, \quad y_\varepsilon(t) \leq c_1 + \frac{1}{2} E(\phi_\varepsilon) t^2.$$

Therefore, from the fact that $y_\varepsilon(1) \geq 0$, we obtain that for a $c > 0$,

$$\forall \varepsilon \leq \varepsilon_0, \quad E(\phi_\varepsilon) \geq -c.$$

This is a contradiction with Lemma 3.3 and the solution $u_\varepsilon(t)$ for $\varepsilon \leq \varepsilon_0$ blows up in finite time. This concludes the proof of the Theorem 3.2 and Section 3.

4. PROPERTIES OF L^2 -MINIMAL BLOW-UP SOLUTIONS ($\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}$)

In this section, we assume that k is C^1 and

$$(4.1) \quad 0 < k_1 \equiv \inf_{x \in \mathbb{R}^N} k(x) \leq k(x) \leq \sup_{x \in \mathbb{R}^N} k(x) \equiv k_2 < +\infty.$$

Moreover, we assume compactness and nondegeneracy conditions on $k(x)$, that is

$$(4.1)' \quad \text{There are } R_0 > 0, c_0 > 0 \text{ and } \delta_0 > 0 \text{ such that for } |x| \geq R_0, \\ k(x) \leq k_2 - \delta_0, \quad |\nabla k(x)| \leq c_0,$$

and

$$(4.1)'' \quad \text{there are } x_1, \dots, x_p \text{ such that } M \\ = \{x; k(x) = k_2\} = \{x_1, \dots, x_p\}.$$

In this section we are interested by qualitative properties satisfied by blow-up solutions such that

$$(4.2) \quad \|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2} = \frac{\|Q\|_{L^2}}{k_2^{N/4}}.$$

We had seen in Section 2 that if

$$(4.3) \quad \|\phi\|_{L^2} < \|Q_{k_2}\|_{L^2}$$

then $u(t)$ is globally defined.

Moreover under some compactness assumptions on $k(x)$ in Section 3, we had seen that for all $\varepsilon > 0$, there is a blow-up solution with initial data ϕ_ε such that

$$(4.4) \quad \|\phi_\varepsilon\|_{L^2} = \|Q_{k_2}\|_{L^2} + \varepsilon.$$

Therefore, if $u(t)$, solution of Eq. (1.1) with initial data ϕ satisfying (4.2), blows-up in finite time $T < +\infty$, then $u(t)$ is a minimal blow-up solution in L^2 . Let $u(t)$ be such a solution.

In the case $k(x) \equiv k$, in [10], the following result has been proved: there is $x_0 \in \mathbb{R}^N$ such that

$$|u(t, x)|^2 \rightarrow \|Q_{k_0}\|_{L^2}^2 \delta_{x=x_0} \quad \text{as } t \rightarrow T, \\ |x - x_0|^2 |u|^2 \rightarrow 0 \quad \text{in } L^1 \quad \text{as } t \rightarrow T.$$

Using variational arguments we prove the following in the case where $k(x) \not\equiv k$.

PROPOSITION 4.1. – Assume that $\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}$ and $u(t)$ blows-up in finite time at $T < +\infty$. We then have the existence of x_0 such that

- $|u(t, x)|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}$ in the distribution sense as $t \rightarrow T$,
- $|x - x_0|^2 |u(t, x)|^2 \rightarrow 0$ in L^1 as $t \rightarrow T$,

and

$$\nabla k(x_0) = 0, \quad k(x_0) = k_2.$$

Remark. – It follows from Proposition 4.1 that we do not have ejection of mass in finite time with a minimal mass ($\|Q_{k_2}\|_{L^2}$). That is

$$|u(t, x - x(t))|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=0} \quad \text{and} \quad |x(t)| \rightarrow +\infty \quad \text{as } t \rightarrow T.$$

In the case where $k(x)$ does not satisfy (4.1)' and there is a sequence x_n such that

- $|x_n| \rightarrow +\infty$ as $n \rightarrow +\infty$,
- $k(x_n) \rightarrow k_2$ as $n \rightarrow +\infty$,

we still have the existence of $x(t)$ such that

$$|u(t, x + x(t))|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=0}.$$

But we do not know whether $x(t)$ is bounded or not.

Remark. – For a general initial data ($\|\phi\|_{L^2} > \|Q_{k_2}\|_{L^2}$, we don't know whether the concentration point of the solution in L^2 at the blow-up time is a critical point of $k(x)$ or not.

Proof of Proposition 4.1. – We establish the result in three steps. Let us consider $u(t)$ solution of Eq. (1.1) with initial data $\phi \in H^1$ such that

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}, \text{ and } u(t) \text{ blows up at } T < +\infty.$$

Step 1. – Variational estimates.

We show that there is $x(t)$ such that

$$|u(t, x - x(t))|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=0}$$

and

$$\forall \delta > 0, \quad \text{there is a } c_\delta > 0 \text{ such that } \forall t \in [0, T),$$

$$\int_{|x(t)-x|\geq\delta} |\nabla u(t, x)|^2 dx \leq c_\delta.$$

Step 2. – Localization of the concentration point.

There is $x_0 \in \mathbb{R}^N$ such that $x(t) \rightarrow x_0$ as $t \rightarrow T$. Moreover $k(x_0) = k_2$ and $\nabla k(x_0) = 0$.

Step 3. – Control of $u(t, x)$ for x large and conclusion.

We then show that

$$|x| |u(t, x)| \in L^2 \quad \text{for all } t \in [0, T]$$

and

$$|x - x_0|^2 |u(t, x)|^2 \rightarrow 0 \quad \text{in } L^1 \quad \text{as } t \rightarrow T.$$

Step 1. – Variational estimates: Concentration and compactness outside the concentration point.

We show that there is $x(t)$ such that

$$(4.5) \quad |u(t, x + x(t))|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=0}$$

and

$$(4.6) \quad \forall \delta > 0, \quad \text{there is a } c_\delta > 0 \text{ such that } \forall t \in [0, T),$$

$$\int_{|x(t)-x| \geq \delta} |\nabla u(t, x)|^2 dx \leq c_\delta.$$

We claim this result as a consequence of the concentration properties (Section 2.B) and a crucial compactness lemma.

LEMMA 4.2 ([10], p. 433). – *Let $u_n \in H^1(\mathbb{R}^N)$ and $R_0 > 0$ such that for a c_0 , we have $\forall n$:*

$$\begin{aligned} & - E_{k_2}(u_n) \leq c_0, \\ & - \int |u_n(x)|^2 dx \leq \int |Q_{k_2}(x)|^2 dx, \\ & - \int |\nabla u_n(x)|^2 dx \rightarrow +\infty \text{ as } n \rightarrow +\infty, \\ & - \int_{|x| \geq R_0} |u_n(x)|^2 dx \leq \varepsilon(N), \end{aligned}$$

where $\varepsilon(N) > 0$ is depending only on N . Then there is $A > 0$ depending only on R_0, c_0 such that

$$\forall n, \quad \int_{|x| \geq 4R_0} |\nabla u_n(x)|^2 dx \leq A.$$

Proof of (4.5)-(4.6). – Let be $x(t)$ defined in Section 2.B (Proposition 2.4). For all $R > 0$, we have

$$(4.7) \quad \liminf_{t \rightarrow T} \|u(t)\|_{L^2(B(x(t), R))} \geq \|Q_{k_2}\|_{L^2}.$$

Let

$$v(t, x) = |u(t, x + x(t))|^2.$$

$$(4.8) \quad \|v(t, x)\|_{L^1} = \|u(t, x)\|_{L^2}^2 = \|\phi\|_{L^2}^2 = \|Q_{k_2}\|_{L^2}^2,$$

and from (4.7)

$$(4.9) \quad \forall R > 0, \\ \liminf_{t \rightarrow T} \int_{|x| < R} v(t, x) = \liminf_{t \rightarrow T} \int_{|x| < R} |u(t, x + x(t))|^2 dx \\ \geq \|Q_{k_2}\|_{L^2}^2.$$

Therefore from (4.8)-(4.9)

$$(4.10) \quad v(t, x) \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=0} \quad \text{as } t \rightarrow T,$$

or equivalently

$$(4.11) \quad |u(t, x + x(t))|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=0} \quad \text{as } t \rightarrow T.$$

And, $\forall R > 0$

$$(4.12) \quad \int_{|x| > R} |u(t, x + x(t))|^2 dx \rightarrow 0 \quad \text{as } t \rightarrow T.$$

We now claim the following lemma.

LEMMA 4.3. – (i) $\forall t \in [0, T)$,

$$\begin{aligned} & - \int (k_2 - k(x)) |u(t, x)|^{\frac{4}{N}+2} dx \leq \left(\frac{4}{N} + 2\right) E(\phi) \\ & - E_{k_2}(u(t)) \leq E(\phi). \end{aligned}$$

(ii) $\forall \delta > 0$, there is a $c_\delta > 0$ such that $\forall t \in [0, T)$

$$\int_{|x-x(t)| \geq \delta} |\nabla u(t, x)|^2 \leq c_\delta.$$

Proof. – (i) Indeed $\forall t \in [0, T)$,

$$E(u(t)) = E(\phi).$$

Therefore

$$(4.13) \quad \left\{ \frac{1}{2} \int |\nabla u(t, x)|^2 - \frac{1}{\frac{4}{N} + 2} \int k_2 |u(t, x)|^{\frac{4}{N}+2} dx \right\} \\ + \frac{1}{\frac{4}{N} + 2} \int (k_2 - k(x)) |u(t, x)|^{\frac{4}{N}+2} dx = E(\phi).$$

Since $\|u(t)\|_{L^2} = \|Q_{k_2}\|_{L^2}$, we have

$$(4.14) \quad E_{k_2}(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 - \frac{1}{\frac{4}{N} + 2} \\ \times \int k_2 |u(t, x)|^{\frac{4}{N}+2} dx \geq 0$$

and from (4.1),

$$(4.15) \quad \int (k_2 - k(x)) |u(t, x)|^{\frac{4}{N}+2} dx \geq 0.$$

From (4.13)-(4.15), we derive part (i) of the lemma.

(ii) Let $\delta > 0$. From (4.12), there is a $t_\delta < T$ such that

$$\forall t \in [t_\delta, T), \quad \int_{|x| > \frac{\delta}{4}} |u(t, x+x(t))|^2 dx < \varepsilon(N),$$

where $\varepsilon(N)$ is defined in Lemma 4.3.

From Lemma 4.2, we have the existence of $A_\delta > 0$ such that

$$\forall t \in [t_\delta, T), \quad \int_{|x| > \delta} |\nabla u(t, x+x(t))|^2 dx \leq A_\delta.$$

Since $\forall t \in [0, t_\delta]$,

$$\int_{|x|>\delta} |\nabla u(t, x + x(t))|^2 dx \leq \int |\nabla u(t, x)|^2 dx \leq c,$$

we have the conclusion. This concludes the proof of Lemma 4.3 and of (4.5)-(4.6).

Step 2. – Localisation of the concentration point.

In this step we use strongly the assumptions (4.1)'-(4.1)'. Since $\int \vec{x} |u(t, x)|^2 dx$ as $t \rightarrow T$ can not be controlled as in the case $k(x) \equiv k_0$, we cannot apply arguments such as in [10].

Let us show that there is x_0 such that

$$(4.16) \quad x(t) \rightarrow x_0 \text{ as } t \rightarrow T,$$

$$(4.17) \quad k(x_0) = k_2 \quad \text{and} \quad \nabla k(x_0) = 0.$$

Proof of (4.16)-(4.17).

LEMMA 4.4. – *There is a constant $c_0 > 0$ such that*

$$\forall t \in [0, T), \quad |x(t)| \leq c_0.$$

Proof. – Indeed, from Lemma 4.3 and (4.1)':

$$(4.17) \quad \forall t, \quad \int (k_2 - k(x)) |u(t, x)|^{\frac{4}{N}+2} dx \leq c,$$

and

$$\forall |x| \geq R_0, \quad k_2 - k(x) \geq \delta.$$

Therefore

$$\forall t, \quad \int_{|x| \geq R_0} \delta |u(t, x)|^{\frac{4}{N}+2} dx \leq c,$$

and

$$(4.18) \quad \forall t, \quad \int_{|x| \geq R_0} |u(t, x)|^{\frac{4}{N}+2} dx \leq \frac{c}{\delta}.$$

Moreover, from (4.11) and Hölder inequality we have

$$(4.19) \quad \int_{|x-x(t)| \leq 1} |u(t, x)|^{\frac{4}{N}+2} dx \rightarrow +\infty \quad \text{as } t \rightarrow T.$$

It follows from (4.18)-(4.19) that

$$\limsup_{t \rightarrow T} |x(t)| \leq R_0 + 1,$$

and the conclusion follows.

LEMMA 4.5. – *There is a x_0 such that*

$$x(t) \rightarrow x_0 \quad \text{as } t \rightarrow T \quad \text{and} \quad k(x_0) = k_2.$$

Remark. – It follows directly from $k(x_0) = k_2 = \max_{x \in \mathbb{R}^N} k(x)$ that $\nabla k(x_0) = 0$.

Proof. – (i) We first remark that

$$M(t) = \underset{i=1, \dots, p}{\text{Min}} \{|x(t) - x_i|\} \rightarrow 0 \quad \text{as } t \rightarrow T,$$

where x_1, \dots, x_p are defined by (4.1)''. Indeed, by contradiction, assume that there are $t_n \rightarrow T$ as $n \rightarrow +\infty$ and $\delta > 0$ such that

$$M(t_n) \geq \delta.$$

Compactness arguments in \mathbb{R}^N yield the existence of $\alpha > 0$ such that

$$(4.20) \quad \forall n, \quad \forall x \in B\left(x(t_n), \frac{\delta}{2}\right), \quad (k_2 - k(x)) \geq \alpha.$$

Therefore from Lemma 4.3,

$$\forall n, \quad \int_{|x-x(t_n)| < \frac{\delta}{2}} (k_2 - k(x)) |u(t_n, x)|^{\frac{4}{N}+2} dx \leq c_0$$

and

$$(4.21) \quad \forall n, \quad \int_{|x-x(t_n)| < \frac{\delta}{2}} |u(t_n, x)|^{\frac{4}{N}+2} dx \leq c.$$

(4.21) contradicts the fact that

$$\int_{|x-x(t_n)| < \frac{\delta}{2}} |u(t_n, x)|^{\frac{4}{N}+2} dx \xrightarrow{n \rightarrow +\infty} +\infty$$

(from (4.11)). Therefore

$$M(t) \rightarrow 0 \quad \text{as } t \rightarrow T.$$

(ii) Let us show now that there is $i \in \{1, \dots, p\}$ such that

$$(4.22) \quad x(t) \rightarrow x_i \quad \text{as } t \rightarrow T.$$

Let $\delta = \frac{1}{4} \min_{i \neq j} \{|x_i - x_j|\} > 0$ and $\psi \in C^\infty$ such that

$$\begin{aligned} -\psi(x) &\equiv 1 \quad \text{for } |x| < \delta, \\ -0 &\leq \psi(x) \leq 1, \\ -\psi(x) &\equiv 0 \quad \text{for } |x| > 2\delta. \end{aligned}$$

From Part (i) and Lemma 4.3 we have the existence of $c > 0$ such that

$$(4.23) \quad \forall t \in [0, T), \quad \forall i = 1, \dots, N, \quad \int_{\delta < |x - x_i| < 2\delta} |\nabla u(t, x)|^2 dx \leq c.$$

We remark that $\forall i = 1, \dots, p$, there is e_i such that

$$(4.24) \quad \int \psi(x - x_i) |u(t, x)|^2 dx \rightarrow e_i \quad \text{as } t \rightarrow T.$$

Indeed from direct calculations and (4.23),

$$\begin{aligned} &\left| \frac{d}{dt} \int \psi(x - x_i) |u(t, x)|^2 dx \right| \\ &= \left| 4 \operatorname{Im} \int \nabla \psi(x - x_i) u \overline{\nabla u} \right| \\ &= \left| 4 \operatorname{Im} \int_{\delta < |x - x_i| < 2\delta} \nabla \psi(x - x_i) u \overline{\nabla u} \right| \\ &\leq c \left(\int_{\delta < |x - x_i| < 2\delta} |\nabla u|^2 \right)^{\frac{1}{2}} \leq c \end{aligned}$$

and (4.24) follows.

Therefore, from (4.11)-(4.12) and (i), there is $i_0 \in \{1, \dots, p\}$ such that $e_{i_0} = \|Q_{k_2}\|_{L^2}$ and

$$x(t) \rightarrow x_{i_0} \quad \text{as } t \rightarrow T.$$

This concludes the proof of Lemma 4.5 and (4.16)-(4.17).

Step 3. – Control of the solution at infinity and conclusion.

Let us show that $\phi \in \Sigma$, that is

$$(4.25) \quad |x| |\phi(x)| \in L^2,$$

and

$$(4.26) \quad \int |x - x_0|^2 |u(t, x)|^2 dx \rightarrow 0 \quad \text{as } t \rightarrow T.$$

The proof will use the same type of argument than in [10].

We remark that from Lemmas 4.3 and 4.5, we have

$$(4.27) \quad \forall \delta > 0, \quad \text{there is a } c_\delta > 0 \text{ such that } \forall t \in [0, T),$$

$$\int_{|x-x_0| \geq \delta} |\nabla u(t, x)|^2 dx \leq c_\delta.$$

LEMMA 4.6.

$$(i) \quad \int |x|^2 |\phi(x)|^2 < +\infty.$$

There is a constant $c > 0$ such that

$$(ii) \quad \forall t \in [0, T), \quad \int |x - x_0|^2 |u(t, x)|^2 dx \leq c.$$

Proof. – Let us argue by contradiction. Suppose $\int |x|^2 |\phi(x)|^2 dx = +\infty$.

(i) Let us consider $\psi_A(x) = \tilde{\psi}_A(|x - x_0|)$ where

$$\begin{aligned} - \tilde{\psi}_A(0) &= 0, \\ - \tilde{\psi}'_A(r) &= 0 \quad \text{for } r \leq 1, \\ - \tilde{\psi}'_A(r) &= r - 1 \quad \text{for } 1 \leq r \leq A, \\ - \tilde{\psi}'_A(r) &= 2A - 1 - r \quad \text{for } A \leq r \leq 2A - 1 \\ - \tilde{\psi}'_A(r) &= 0 \quad \text{for } r \geq 2A - 1. \end{aligned}$$

By direct calculations, we have for a $c > 0$,

$$(4.28) \quad \forall x, \quad \forall A \geq 1, \quad |\nabla \psi_A|^2 \leq c\psi_A + c,$$

$$(4.29) \quad 1 + \psi_A(x) \geq \frac{|x - x_0|^2}{4}, \quad \forall 1 \leq |x - x_0| \leq A,$$

$$(4.30) \quad \psi_A(x) \equiv c_A, \quad \text{for } |x - x_0| \geq 2A - 1.$$

Let $Y_A(t) = \int \psi_A(x) |u(t, x)|^2 dx$. We have for a $c > 0$,

$$(4.31) \quad \forall A \geq 1, \quad \forall t, \quad |Y'_A(t)| \leq c\sqrt{Y_A(t) + 1},$$

$$(4.32) \quad Y_A(0) \rightarrow +\infty, \quad \text{as } A \rightarrow +\infty,$$

$$(4.33) \quad \forall A, \quad Y_A(t) \rightarrow 0 \quad \text{as } t \rightarrow T.$$

(4.33) follows from (4.11)-(4.12) and (4.30). (4.32) is a consequence of (4.29) and $\int |x|^2 |\phi(x)|^2 dx = +\infty$. (4.31) can be deduced from (4.28) and (4.27). Indeed,

$$\begin{aligned} Y'_A(t) &= 4 \operatorname{Im} \int \nabla \psi_A u \overline{\nabla u} \\ &\leq c \left(\int_{|x-x_0| \geq 1} |\nabla u|^2 \right)^{1/2} \left(\int |\nabla \psi_A|^2 |u|^2 \right)^{1/2} \\ &\leq c \left(\int \psi_A(x) |u|^2 + \int |u|^2 \right)^{1/2} \leq c(Y_A(t) + 1)^{1/2}. \end{aligned}$$

Integrating in time (4.31), we obtain

$$\forall A, \quad \forall t \in [0, T), \quad |\sqrt{Y_A(0) + 1} - \sqrt{Y_A(t) + 1}| \leq c.$$

Letting $t \rightarrow T$, we then have $\sqrt{Y_A(0) + 1} \leq c$, which contradicts (4.32). Therefore

$$\int |x|^2 |\phi(x)|^2 dx < +\infty.$$

(ii) Considering now

$$\psi(x) = \tilde{\psi}(|x - x_0|)$$

where $\tilde{\psi}(0) = 0$, $\tilde{\psi}'(r) = 0$ for $r \leq 1$, $\tilde{\psi}'(r) = r - 1$ for $r \geq 1$. We obtain

$$- |Y'(t)| \leq c \sqrt{Y(t) + 1},$$

$$- Y(0) < +\infty.$$

Therefore, there is a constant $c > 0$ such that

$$\forall t \in [0, T), \quad Y(t) \leq c,$$

and since $2 + \tilde{\psi}(r) \geq \frac{r^2}{4}$,

$$\forall t \in [0, T), \quad \int |x - x_0|^2 |u(t, x)|^2 dx \leq c.$$

LEMMA 4.7.

$$\lim_{t \rightarrow T} \int |x - x_0|^2 |u(t, x)|^2 dx = 0.$$

The proof is the same than the one in [10] (Step 2, p. 442). Let us recall the key parts of the proof. From (4.11)-(4.12), we have $\forall A \geq 0$,

$$(4.34) \quad \lim_{t \rightarrow T} \int_{|x - x_0| \leq A} |x - x_0|^2 |u(t, x)|^2 dx = 0.$$

The conclusion will follow from an uniform integrability property:

(4.35) $\forall \varepsilon > 0$, there is a A_ε such that

$$\forall t \in [0, T), \quad \int_{|x - x_0| \geq A_\varepsilon} |x - x_0|^2 |u(t, x)|^2 dx \leq \varepsilon.$$

Proof of (4.35). – Let us consider $\psi \in C^4(\mathbb{R}^N, \mathbb{R})$

$$- \psi(x) = \psi(|x - x_0|),$$

$$- \psi(x) = 0 \quad \text{for } |x - x_0| \leq 1,$$

$$- \frac{1}{2}|x|^2 \leq \psi(x) \leq |x|^2 \quad \text{for } |x - x_0| \geq 2,$$

– there is $c > 0$ such that $\forall x, \forall r \geq 0$,

$$|\nabla \psi(x)| \leq c|x - x_0| \text{ and } |\psi''(r)| + |\psi'''(r)| + |\psi''''(r)| \leq c,$$

and

$$\psi_A(x) = A^2 \psi\left(\frac{x}{A}\right).$$

Considering $\frac{d}{dt} \int \psi_A(x) |u(t, x)|^2 dx$, we obtain the existence of $\varepsilon(A) > 0$ such that

$$(4.36) \quad \sup_{t \in [0, T]} \int \psi_A(x) |u(t, x)|^2 dx \leq \varepsilon(A)$$

where $\varepsilon(A) \rightarrow 0$ as $A \rightarrow +\infty$ (see proof below). The fact that $\psi_A(x) \geq \frac{1}{2}|x - x_0|^2$ for $|x - x_0| \geq 2A$ implies (4.35) and the conclusion follows.

Proof of (4.36). – Let us define

$$Y_A(t) = \int \psi_A(x) |u(t, x)|^2 dx.$$

We have

$$(4.37) \quad \begin{aligned} |Y'_A(t)| &= \left| 2 \operatorname{Im} \int \nabla \psi_A(x) u \overline{\nabla u} \right| \\ &\leq 2 \left| \int_{|x| \geq A} \nabla \psi_A u \overline{\nabla u} \right| \\ &\leq 2 \left(\int_{|x| \geq A} |\nabla \psi_A|^2 |u|^2 \right)^{1/2} \\ &\quad \times \left(\int_{|x| \geq A} |\nabla u(t, x)|^2 dx \right)^{1/2}. \end{aligned}$$

We can remark that $\forall A \geq 1$,

$$\forall x, \quad |\nabla \psi_A|^2 \leq c \psi_A(x) + c.$$

Therefore from Lemma 4.6,

$$(4.38) \quad \begin{aligned} |Y'_A(t)| &\leq c \left(\int_{|x| \geq A} \psi_A(x) |u|^2 + \int_{|x| \geq A} |u|^2 \right)^{1/2} \\ &\quad \times \left(\int_{|x| \geq A} |\nabla u|^2 \right)^{1/2} \\ &\leq c \left(Y_A(t) + \frac{1}{A^2} \right)^{1/2} \left(\int_{|x| \geq A} |\nabla u|^2 \right)^{1/2}, \end{aligned}$$

or equivalently

$$(4.39) \quad |Y'_A(t)| \leq Y_A(t) + \frac{1}{A^2} + c \int_{|x| \geq A} |\nabla u|^2.$$

Since (4.27),

$$\int_0^T \int_{|x| \geq A} |\nabla u(t, x)|^2 dx dt \leq c,$$

the convergence dominated theorem yields

$$\lim_{A \rightarrow +\infty} \int_0^T \int_{|x| \geq A} |\nabla u(t, x)|^2 dx dt = 0.$$

Therefore by integration of (4.39),

$$\begin{aligned} \lim_{A \rightarrow +\infty} \left\{ \sup_{t \in [0, T]} Y_A(t) \right\} &\leq c \left\{ \lim_{A \rightarrow +\infty} \int_0^T \int_{|x| \geq A} |\nabla u(t, x)|^2 dx dt \right\} \\ &+ c \left\{ \lim_{A \rightarrow +\infty} Y_A(0) \right\} = 0, \end{aligned}$$

which concludes the proof of (4.36) and of Proposition 4.1.

5. NONEXISTENCE OF L^2 -MINIMAL BLOW-UP SOLUTIONS

In this section, we discuss nonexistence and existence of L^2 -minimal blow-up solutions.

Under some conditions on the function $k(x)$ at infinity, we saw in Section 4 that a blow-up solution such that

$$(5.1) \quad \|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}$$

concentrates at the blow-up time at a point x_0 such that

$$(5.2) \quad k(x_0) = k_2, \quad \nabla k(x_0) = 0.$$

In subsection 5.1, under some condition on the form of $k(x)$ for x near x_0 , we prove that such a solution does not exist. We briefly give the existence of such a solution in subsection 5.2 under some condition of flatness on $k(x)$ for x near x_0 .

5.1. Nonexistence of minimal blow-up solution

Let x_0 be such that $k(x_0) = k_2$ (in particular $\nabla k(x_0) = 0$). We assume for a $c_0 > 0$ that

$$\nabla k(x) \cdot (x - x_0) \leq -c_0 |x - x_0|^{1+\alpha_0} \quad \text{for } x \text{ near } x_0,$$

where $0 < \alpha_0 < 1$. It implies in particular

$$(5.3)x_0 \quad k(x_0) - k(x) \geq c |x - x_0|^{1+\alpha_0} \quad \text{for } x \text{ near } x_0;$$

(this condition does not allow $k(x)$ to be C^2 near x_0). We claim the following theorem.

THEOREM 5.1. – *Assume that $k(x)$ satisfies (5.3) $_{x_0}$. There is then no blow-up solution such that*

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}$$

and

$$|u(t, x)|^2 \rightharpoonup \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0} \text{ in the distribution sense as } t \rightarrow T$$

(where T is the blow-up time).

This theorem has the following corollary:

COROLLARY 5.2 (Nonexistence of L^2 -minimal blow-up solutions). – *Assume that k satisfies (4.1), (4.1)', (4.1)'' and all x_0 such that $k(x_0) = k_2$ satisfies (5.3) $_{x_0}$. There is no blow-up solutions such that*

$$\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}.$$

We remark that the corollary follows directly from Section 4 and Theorem 5.1. Let us prove Theorem 5.1.

Proof of Theorem 5.1. – We argue by contradiction. Assume there is a $\phi \in H^1$ such that

$$(5.4) \quad \|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2},$$

$u(t)$ blows-up in finite time T , and

$$(5.5) \quad |u(t, x)|^2 \rightharpoonup \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}.$$

A contradiction follows from asymptotic estimates on the solution and energy arguments.

LEMMA 5.3. (Energy estimates). – We have

$$(i) \quad E(\phi) \geq E_{k_2}(u(t)) \geq 0,$$

$$(ii) \quad E(\phi) \geq \frac{1}{\frac{4}{N} + 2} \int (k_2 - k(x)) |u(t, x)|^{\frac{4}{N}+2} dx \geq 0,$$

$$(iii) \quad E_{k_2}(u(t)) + \frac{1}{\frac{4}{N} + 2} \int (k_2 - k(x)) |u(t, x)|^{\frac{4}{N}+2} dx \leq E(\phi).$$

Proof. – Parts (i) and (ii) follow from $\|\phi\|_{L^2} = \|u(t)\|_{L^2} \leq \|Q_{k_2}\|_{L^2}$, Part iii), and the definition of k_2 . The conservation of the energy yields (iii).

We claim that

$$(5.6) \quad \int (k_2 - k(x)) |u(t, x)|^{\frac{4}{N}+2} dx \xrightarrow{t \rightarrow T} +\infty$$

which will be a contradiction with part (ii) of Lemma 5.3.

Proof of (5.6). – From (5.3) x_0 , (5.6) is implied by

$$(5.7) \quad \int_{|x-x_0| \leq \rho_0} |x-x_0|^{1+\alpha_0} |u(t, x)|^{\frac{4}{N}+2} dx \xrightarrow{t \rightarrow T} +\infty \quad \text{for a } \rho_0 > 0.$$

LEMMA 5.4. – We have the existence of $x(t) \rightarrow x_0$, $\theta(t) \in \mathbb{R}^2$, such that

$$(5.8) \quad \frac{1}{[\lambda(t)]^{\frac{N}{2}}} e^{i\theta(t)} u\left(t, x(t) + \frac{(x-x(t))}{\lambda(t)}\right) \rightarrow Q_{k_2}(x) \quad \text{in } H^1,$$

where $\lambda(t) = \|\nabla u(t)\|_{L^2} \rightarrow +\infty$.

Proof. – See Corollary 2.7.

Therefore for t near T

$$(5.9) \quad |x_0 - x(t)| < \frac{\rho_0}{2}$$

and

$$\begin{aligned}
 & \int_{|x-x_0| \leq \rho_0} |x - x_0|^{1+\alpha_0} |u(t, x)|^{\frac{4}{N}+2} dx \\
 & \geq \int_{|x(t)-x| \leq \frac{\rho_0}{2}} |(x - x(t) + (x(t) - x_0))|^{1+\alpha_0} \\
 & \quad \times |u(t, x - x(t) + x(t))|^{\frac{4}{N}+2} dx \\
 & \geq \int_{|y| \leq \frac{\rho_0}{2} \lambda(t)} \left| \frac{y}{\lambda(t)} + (x(t) - x_0) \right|^{1+\alpha_0} \\
 & \quad \times \left| u \left(t, \frac{y}{\lambda(t)} + x(t) \right) \right|^{\frac{4}{N}+2} dx \\
 & \geq \lambda(t)^2 \int_{|y| \leq 10} \frac{1}{\lambda(t)^{1+\alpha_0}} |y + (x(t) - x_0) \lambda(t)|^{1+\alpha_0} Q_{k_2}^{\frac{4}{N}+2}(y) dy \\
 & \geq c \left(\int_{|y| \leq 10} Q_{k_2}^{\frac{4}{N}+2}(y) dy \right) \frac{\lambda(t)^2}{\lambda(t)^{1+\alpha_0}} \\
 & \geq c \lambda(t)^{1-\alpha_0} \xrightarrow{t \rightarrow T} +\infty.
 \end{aligned}$$

This concludes the proof of (5.6). A contradiction follows and Theorem 5.1 is proved.

5.2. Existence of L^2 -blow-up solution and open problems

Using the same method as [9'], that is a fixed point and compactness argument near the solution of the homogeneous Schrödinger equation

$$u(t, x) = \frac{w^{\frac{N}{2}}}{t^{\frac{N}{2}}} e^{i \frac{|x-x_0|^2}{4t} - \frac{i w^2}{t}} Q_{k_2} \left(\frac{w(x-x_0)}{t} \right),$$

we are able to prove the following proposition.

PROPOSITION 5.4 (Existence L^2 -minimal blow-up solution under flatness condition). – Assume $k(x) \equiv k_2$ for x near x_0 . There is then a L^2 -minimal blow-up solution $u(t)$ such that

$$|u(t, x)|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0} \quad \text{as } t \rightarrow T$$

(where T is the blow-up time of $u(t)$).

Remark. – Section 5 leaves open the question of existence and nonexistence of L^2 -minimal blow-up solution in the case where k is a C^2 near x_0 and

$$c_1 \leq \left| \frac{D^2 k(x, x)}{|x - x_0|^{2+i}} \right| \leq c_2$$

for $i = 0, 1, \dots$

In addition, knowing which i (and eventually c_1, c_2) separates the cases of existence and nonexistence is an open question.

6. STABILITY OF SINGULARITY

In this section, we point out the relation between the nonexistence of minimal blow-up solutions and the existence of black holes. We define a black hole as a “space singularity stable in time with respect to initial data”. More precisely, assume that there is no minimal blow-up solution $Q(x_0) = k_2$ and x_0 is a strict local maximum. Then the singularity

$$(6.1) \quad |u|^2 = \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}$$

will be stable in time in some sense. That is,

THEOREM 6.1. – *Consider a sequence of initial data ϕ_n in H^1 such that*

$$(6.2) \quad \int |\phi_n|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2, \quad \text{as } n \rightarrow +\infty,$$

$$(6.3) \quad |\phi_n(x)|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}, \quad \text{as } n \rightarrow +\infty$$

in the distribution sense,

$$(6.4) \quad \text{there is a } c > 0 \text{ such that } E_{\varepsilon_n}(\phi_n) \leq c,$$

where $E_{\varepsilon_n}(u) = E(u) + \frac{\varepsilon_n}{q+1} \int |u|^{q+1}$, $\varepsilon_n > 0$, $\varepsilon_n \xrightarrow{n \rightarrow +\infty} 0$, and $\frac{N+2}{N-2} > q > \frac{4}{N} + 1$. Let $u_n(t)$ be the solution of

$$(6.5) \quad \begin{cases} iu_t = -\Delta u - k(x) |u|^{\frac{4}{N}} u + \varepsilon_n |u|^{q-1} u \\ u_n(0) = \phi_n. \end{cases}$$

$$(6.6) \quad \text{For all time } t > 0,$$

$$|u_n(t, x)|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0} \text{ in the distribution sense as } n \rightarrow +\infty.$$

Remark. – We have considered $u_n(t)$ solution of equation (6.5) to assure that $u_n(t)$ will be defined for all time. The same conclusions hold for solutions of equation (1.1) ($\varepsilon_n = 0$) on their maximum common time existence interval.

Remark. – In the case of nonexistence of minimal blow-up solution such that

$$E(\phi) \leq a,$$

if we assume $E(\phi_n) \rightarrow a$, same conclusion holds.

Remark. – It is an open problem to show that there is no black hole at a mass level different of $\|Q_{k_2}\|_{L^2}^2$. We conjecture there is none.

Proof of Theorem 6.1. – We do it in three steps.

Step 1. – Reduction.

We claim using concentration properties that Theorem 6.1 is implied by the following property

$$(6.7) \quad \forall t, \quad \liminf_{n \rightarrow +\infty} \left\{ \inf_{s \in [0, t]} \|\nabla u_n(s)\|_{L^2} \right\} = +\infty.$$

(6.7) implies (6.6). – Indeed, assume (6.7) and let us fix $t > 0$. From Corollary 2.7, there is a $x_n(s)$ such that

$$(6.8) \quad |u_n(s, x - x_n(s))|^2 \rightharpoonup \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}$$

in the distribution sense uniformly in s , that is: $\forall \delta_1 > 0, \forall \delta_2 > 0$, for n large

$$\sup_{s \in [0, t]} \int_{|x - x_n(s)| \geq \delta_2} |u_n(s, x)|^2 dx \leq \delta_1.$$

We remark that the energy identity

$$E_{\varepsilon_n}(u_n(t)) = E_{\varepsilon_n}(\phi_n) \leq c$$

$$\text{implies } E(u_n(t)) \leq c - \frac{\varepsilon_n}{q+1} \int |u_n(t)|^{q+1} \leq c.$$

Moreover, direct continuity arguments on the solution (with respect to the initial data) show that we can choose for a fixed n , $x_n(\cdot) : [0, t] \rightarrow \mathbb{R}^N$ continuous with respect to s .

We claim that

$$\lim_{n \rightarrow +\infty} \sup_{s \in [0, t]} |x_n(s) - x_0| = 0.$$

Indeed, by contradiction, assume there is $\delta > 0$ such that $\forall n$, there is $s_n \in [0, t]$ such that

$$(6.9) \quad \forall n, \quad |x_n(s_n) - x_0| \geq \delta.$$

We remark from (5.2) that

$$(6.10) \quad x_n(0) \rightarrow x_0 \quad \text{as } n \rightarrow +\infty.$$

Since $x_n(s)$ is a continuous function of s , there is a sequence $\tau_n \in [0, t]$ such that

$$(6.11) \quad |x_n(\tau_n) - x_0| = \delta.$$

From the fact that x_0 is a strict local maximum, taking δ small enough, there is $\varepsilon > 0$ such that

$$(6.12) \quad k(x_n(\tau_n)) \leq k_2 - \varepsilon_0.$$

By similar arguments than in the proof of Proposition 2.5, we have in addition

$$(6.13) \quad \liminf_{n \rightarrow +\infty} \frac{\|\phi_n\|_{L^2}}{\|Q_{k(x_n(\tau_n))}\|_{L^2}} \geq \liminf_{n \rightarrow +\infty} \frac{\|u_n(\tau_n)\|_{L^2}}{\|Q_{k(x_n(\tau_n))}\|_{L^2}} \\ \geq \liminf_{n \rightarrow +\infty} \frac{\|u_n(\tau_n)\|_{L^2(B(x_n(\tau_n)))}}{\|Q_{k(x_n(\tau_n))}\|_{L^2}} \geq 1.$$

Going to the limit in (6.13) as $n \rightarrow +\infty$, we obtain

$$(6.14) \quad \frac{\|Q\|_{L^2}}{k_2^{\frac{N}{2}}} = \|Q_{k_2}\|_{L^2} \geq \limsup_{n \rightarrow +\infty} \|Q_{k(x_n(\tau_n))}\|_{L^2} \\ \geq \limsup_{n \rightarrow +\infty} \frac{\|Q\|_{L^2}}{[k(x_n(\tau_n))]^{\frac{N}{4}}} \\ \geq \frac{\|Q\|_{L^2}}{(k_2 - \varepsilon_0)^{\frac{N}{4}}},$$

which is a contradiction. This concludes the proof of (6.8) and the fact that (6.7) implies (6.6).

Proof of (6.7). – We are now reduced to prove (6.7). Let us argue by contradiction; assume there is a sequence s_n such that

$$(6.15) \quad |s_n| \leq c \quad \text{and} \quad \|\nabla u_n(s_n)\|_{L^2} + \|u_n(s_n)\|_{L^2} \leq c.$$

There is then a $\delta_0 > 0$, by Sobolev imbedding such that

$$\int_{|x-x_0| \leq \delta_0} |u_n(s_n, x)|^2 \leq \frac{1}{2} \|Q_{k_2}\|_{L^2}^2.$$

The fact that x_0 is a strict local maximum implies that taking δ_0 sufficiently small, there is a $\varepsilon_0 > 0$ such that

$$(6.16) \quad k(x) \leq k_2 - \varepsilon_0 \quad \text{for} \quad |x - x_0| = \delta_0.$$

Consider now $t_n \in [0, s_n]$ such that

$$(6.17) \quad \int_{|x-x_0| \leq \delta_0} |u_n(t_n, x)|^2 dx = \frac{1}{2} \|Q_{k_2}\|_{L^2}^2,$$

$$\text{for } t \in [0, t_n], \quad \int_{|x-x_0| \leq \delta_0} |u_n(t, x)|^2 dx \geq \frac{1}{2} \|Q_{k_2}\|_{L^2}^2.$$

We have then t_n such that for a $c_0 > 0$, $\delta_0 > 0$ and

$$(6.18) \quad |t_n| \leq c_0,$$

$$(6.19) \quad \|\nabla u_n(t_n)\|_{L^2} \leq c_0,$$

$$(6.20) \quad \int_{|x-x_0| \leq \delta_0} |u_n(t_n, x)|^2 dx = \frac{1}{2} \|Q_{k_2}\|_{L^2}^2.$$

We just have to check (6.19). We argue by contradiction: assume for a subsequence also denoted t_n

$$(6.21) \quad \|\nabla u_n(t_n)\|_{L^2} \xrightarrow{n \rightarrow +\infty} +\infty.$$

Then by Corollary 2.7 and Proposition 2.5 (see (6.13)), we have

$$(6.22) \quad |u_n(t_n, x - x_n)|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0},$$

and

$$(6.23) \quad \liminf_{n \rightarrow +\infty} \left\{ \frac{\|\phi_n\|_{L^2}}{\|Q_{k(x_n)}\|_{L^2}} \right\} \geq \liminf_{n \rightarrow +\infty} \left\{ \frac{\|u_n(t_n)\|_{L^2(B(x_n, 1))}}{\|Q_{k(x_n)}\|_{L^2}} \right\} \geq 1.$$

Since $\|\phi_n\|_{L^2} \rightarrow \|Q_{k_2}\|_{L^2}$, we have from (6.17) and (6.22),

$$(6.24) \quad x_n \rightarrow \hat{x} \quad |x_0 - \hat{x}| = \delta_0.$$

(6.23) implies that

$$(6.25) \quad \|Q_{k_2}\|_{L^2} \geq \|Q_{k(\hat{x})}\|_{L^2},$$

that is

$$\frac{\|Q\|_{L^2}}{k_2^{\frac{N}{2}}} \geq \frac{\|Q\|_{L^2}}{[k(\hat{x})]^{\frac{N}{2}}} \quad \text{or} \quad k(\hat{x}) \geq k_2,$$

which is a contradiction with (6.16) and (6.24). Thus (6.19) is proved. Let us now obtain a contradiction with $u_n(t_n)$.

Step 2. – Compactness of $u_n(t_n)$ in L^2 .

LEMMA 6.2. – *There is a $\phi \in H^1$ such that*

$$(6.26) \quad u_n(t_n) \rightarrow \phi \text{ in } L^2 \quad \text{as } n \rightarrow +\infty$$

(eventually subtracting a subsequence).

Proof of Lemma 6.2. – From (6.19) and (6.20) and the fact that

$$(6.27) \quad \|u(t_n)\|_{L^2} = \|\phi_n\|_{L^2} \xrightarrow[n \rightarrow +\infty]{} \|Q_{k_2}\|_{L^2},$$

we have, by standard compactness arguments, (eventually subtracting a subsequence) the existence of $\phi \in H^1$ such that

$$(6.28) \quad u_n(t_n) \rightarrow \phi \text{ in } L^2_{\text{loc}} \quad \text{as } n \rightarrow +\infty.$$

In addition,

$$(6.29) \quad \|\nabla \phi\|_{H^1} \leq c,$$

$$(6.30) \quad \|\phi\|_{L^2} \leq \|Q_{k_2}\|_{L^2}.$$

We claim that in fact

$$(6.31) \quad \|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}.$$

Then (6.31) together with (6.27)-(6.28) give that

$$(6.32) \quad u_n(t_n) \rightarrow \phi \quad \text{in } L^2 \quad \text{as } n \rightarrow +\infty.$$

We show (6.31) by contradiction. We have to avoid in some sense dichotomy. Assume that

$$(6.33) \quad \|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2} - \delta \quad \text{where } \delta > 0.$$

We can remark from (6.37), (6.33) and (6.28) that

$$(6.34) \quad \frac{1}{2} \|Q_{k_2}\|_{L^2} \leq \|\phi\|_{L^2} \quad \text{or} \quad \delta < \frac{\|Q_{k_2}\|_{L^2}}{2}.$$

We then have the existence of R_0 and a sequence $R_n \rightarrow +\infty$ such that for n large

$$(6.35) \quad \|u_n(t_n)\|_{L^2(|x|>R_0)} \geq \|Q_{k_2}\|_{L^2} - \delta - \frac{\delta}{8};$$

and

$$(6.36) \quad \|u_n(t_n)\|_{L^2(|x|>R_n)} = \delta - \frac{\delta}{8}.$$

We consider now ψ such that

$$\psi \in C^\infty, \quad |\psi| \leq 1, \quad \psi \equiv 0 \quad \text{for } |x| \leq \frac{1}{2}, \quad \psi \equiv 1 \quad \text{for } |x| \geq 1.$$

Let us consider t'_n such that

$$(6.37) \quad \text{for } t \in [t'_n, t_n], \quad \int \psi\left(\frac{x}{R_n}\right) |u_n(t, x)|^2 dx \geq \frac{\delta}{2},$$

$$(6.38) \quad \int \psi\left(\frac{x}{R_n}\right) |u_n(t'_n, x)|^2 dx = \frac{\delta}{2}.$$

We have from (6.15), (6.2)-(6.3) that

$$(6.39) \quad 0 < t'_n < t_n \quad \text{and} \quad 0 \leq t_n - t'_n \leq c.$$

In addition, we have, for $c > 0$,

$$(6.40) \quad \forall t \in [t'_n, t_n], \quad \|\nabla u_n(t)\|_{L^2} \leq c.$$

Indeed by contradiction Lemma 5.6 (ii) implies that for x_n and $\tau_n \in [t'_n, t_n]$

$$(6.41) \quad |u_n(\tau_n, x - x_n)|^2 \rightarrow \|Q_{k_2}\|_{L^2}^2 \delta_{x=x_0}.$$

We have in addition

$$\|u_n\|_{L^2}^2 = \|\phi_n\|_{L^2}^2 \rightarrow \|Q_{k_2}\|_{L^2}^2.$$

For n large $|\|u_n\|_{L^2}^2 - \|Q_{k_2}\|_{L^2}^2| \leq \frac{1}{8} \|Q_{k_2}\|_{L^2}^2$ and from (6.17)

$$\int_{|x-x_0| \leq \delta_0} |u_n(\tau_n, x)|^2 dx \geq \frac{1}{2} \|Q_{k_2}\|_{L^2}^2,$$

we obtain using (6.40) that

$$(6.42) \quad |x - x_0| \leq 2\delta_0.$$

Then, from (6.41)-(6.42), we obtain for n large

$$\|u_n(\tau_n)\|_{L^2(|x-x_0| \geq 3\delta_0)} \leq \frac{\delta}{4}$$

or

$$\|u_n(\tau_n)\|_{L^2(|x| > R_n/2)} \leq \frac{\delta}{4},$$

which is a contradiction with (6.37). Therefore (6.40) is proved.

Let $y_n(s) = \int \psi\left(\frac{x}{R_n}\right) |u_n(t_n - s, x)|^2 dx$. We have

$$(6.43) \quad y_n(0) \geq \int_{|x| > R_n} |u_n(t_n, x)|^2 dx \geq \delta - \frac{\delta}{8},$$

$$(6.44) \quad y_n(t_n - t'_n) = \int \psi\left(\frac{x}{R_n}\right) |u_n(t'_n, x)|^2 dx = \frac{\delta}{2},$$

$$(6.45) \quad \begin{aligned} & \text{for } s \in [0, t_n - t'_n], \\ |y'_n(s)| & \leq \frac{c}{R_n} \left| \int \nabla \psi \left(\frac{x}{R_n} \right) \overline{\nabla u_n} u_n \right| \leq \frac{c}{R_n}. \end{aligned}$$

Integrating (6.45), we obtain from (6.39)

$$|y_n(t_n - t'_n) - y_n(0)| \leq |t_n - t'_n| \frac{c}{R_n} \leq \frac{c}{R_n} \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

which is a contradiction with (6.43)-(6.44). This concludes the proofs of (6.31) and of Lemma 6.2.

Remark. – In the case where

$$\begin{aligned} & - \varepsilon_n = 0, \\ & - \|\phi_n\|_{L^2} \leq \|Q_{k_2}\|_{L^2}, \\ & - \text{for a } c > 0, \quad \int |x|^2 |\phi_n(x)|^2 dx \leq c, \end{aligned}$$

there is a simpler proof of (6.31).

Step 3. – Conclusion of the proof.

We have then the existence of $\phi \in H^1$ such that

$$(6.46) \quad u_n(t_n) \rightarrow \phi \quad \text{in } L^2 \quad \text{as } n \rightarrow +\infty.$$

Since $\|\phi\|_{L^2} = \|Q_{k_2}\|_{L^2}$ and the fact that there is no minimal blow-up solutions, the solution of Eq. (1.1) with initial data ϕ , $u(t)$ is globally defined for all $t \in \mathbb{R}$ (using conjugation for $t > 0$ and for $t < 0$). Moreover, there is a $c > 0$ such that

$$\text{for } t \in [-c_0, 0], \quad \|\nabla u(t)\|_{L^2} \leq c,$$

(where c_0 is defined in (6.18)).

Continuity arguments with respect to the initial data in L^2 implies in fact that

$$(6.47) \quad u_n(t_n + t) \rightarrow u(t) \quad \text{in } C([-c_0, 0], L^2) \quad \text{as } n \rightarrow +\infty.$$

In the case $\varepsilon_n = 0$ it follows from a result of Cazenave and Weissler (Theorem 1.2 of [1']). In the general case, we can see from Kato [6] that

$$u_n(t_n) \rightarrow \phi$$

in standard Cauchy space where continuity with respect initial data is true from (6.19) and $\phi \in H^1$.

Since $|t_n| \leq c_0$, fom (6.45) we have

$$\int |u_n(t_n - t_n) - u(-t_n)|^2 \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

or equivalently

$$(6.48) \quad \int |\phi_n(x) - u(-t_n)|^2 \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

From (6.3), (6.48),

$$|u(-t_n, x)|^2 \rightarrow \|Q_{k_2}\|_{L^2} \delta_{x=x_0} \quad \text{as } n \rightarrow +\infty$$

in the distribution sense which is a contradiction with the fact

$$\|\nabla u(-t_n)\|_{L^2} \leq c.$$

This concludes the proof of (6.7) and of Theorem 6.1.

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