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# Wavelet transform associated to an induced representation of $SL(n+2, \mathbf{R})$

by

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ABSTRACT. – Let $G = SL(n+2, \mathbf{R})$ and $P = MAN$ a parabolic subgroup of $G$ such that $N$ is isomorphic to the Heisenberg group $H_n$. Let $1 \otimes e^\lambda \otimes 1$ be a representation of $P$ and $\pi_\lambda = \mathrm{Ind}_P^G(1 \otimes e^\lambda \otimes 1)$ the induced representation of $G$ acting on $L^2(H_n)$. In this paper we shall obtain a condition on $\lambda$ and $\psi \in \mathcal{S}'(H_n)$ for which the matrix coefficients $\langle f, \pi_\lambda(g)\psi\rangle_{L^2(H_n)}$ are square-integrale on a subgroup $\overline{N} A_1 \simeq H_n \times \mathbf{R}$ of $G$ and $\|f\|^2_{L^2(H_n)} = c \int\int_{\overline{N} A_1} |\langle f, \pi_\lambda(\overline{n} a_1)\psi\rangle_{L^2(H_n)}|^2 \, d\overline{n} \, da_1$ for all $f \in \mathcal{S}(H_n)$.

RÉSUMÉ. – Soit $G = SL(n+2, \mathbf{R})$ et $P = MAN$ un sous-groupe parabolique de $G$ tel que $N$ soit isomorphe à $H_n$ le groupe de Heisenberg. Soit $1 \otimes e^\lambda \otimes 1$ une représentation de $P$ et $\pi_\lambda = \mathrm{Ind}_P^G(1 \otimes e^\lambda \otimes 1)$ la représentation induite de $G$ qui opère sur $L^2(H_n)$. Dans cet article on obtient une condition sur $\lambda$ et $\psi \in \mathcal{S}'(H_n)$ pour que les coefficients de matrice $\langle f, \pi_\lambda(g)\psi\rangle_{L^2(H_n)}$ soient de carré-intégrables sur un sous-groupe $\overline{N} A_1 \simeq H_n \times \mathbf{R}$ de $G$ et $\|f\|^2_{L^2(H_n)} = c \int\int_{\overline{N} A_1} |\langle f, \pi_\lambda(\overline{n} a_1)\psi\rangle_{L^2(H_n)}|^2 \, d\overline{n} \, da_1$ pour toute $f \in \mathcal{S}(H_n)$.

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## 1. INTRODUCTION

Let $G$ be a locally compact group and $(\pi, \mathcal{H})$ a representation of $G$ where $\mathcal{H}$ is a Hilbert space equipped with the norm $||.||_{\mathcal{H}}$ and the inner product $\langle . , . \rangle_{\mathcal{H}}$. For a subset $S$ of $G$ with a measure $ds$ on $S$ we say that $\psi \in \mathcal{H}$ is *$S$-admissible* for $\pi$ if there exists a positive constant $c_\psi$ such that

$$\int_S |\langle f, \pi(s)\psi\rangle_{\mathcal{H}}|^2 \, ds = c_\psi \, ||f||^2_{\mathcal{H}} \quad \text{for all } f \in \mathcal{H}. \tag{1}$$

Then $\psi \in \mathcal{H}$ is $S$-admissible for $\pi$ if and only, if, as a functional on $\mathcal{H}$,

$$f = c_\psi^{-1} \int_S \langle f, \pi(s)\psi\rangle_{\mathcal{H}} \, \pi(s)\psi \, ds \quad \text{for all } f \in \mathcal{H}. \tag{2}$$

Clearly, (2) implies (1). Conversely, we suppose (1) and define $T_f : \mathcal{H} \to \mathbf{C}$ by $T_f(h) = c_\psi^{-1} \int_S \langle f, \pi(s)\psi\rangle_{\mathcal{H}} \langle \pi(s)\psi, h\rangle_{\mathcal{H}} \, ds \, (h \in \mathcal{H})$. Then the Schwarz inequality yields that $|T_f(h)| \leq ||f||_{\mathcal{H}} \, ||h||_{\mathcal{H}}$, so $T_f$ is a bounded linear functional on $\mathcal{H}$. Therefore, it follows from Riesz Representation Theorem that there exists $f_0 \in \mathcal{H}$ such that $T_f(h) = \langle f_0, h\rangle_{\mathcal{H}}$ and $||T_f|| = ||f_0||_{\mathcal{H}} \leq ||f||_{\mathcal{H}}$. Especially, $T_f(f) = \langle f_0, f\rangle_{\mathcal{H}} = ||f||^2_{\mathcal{H}}$. Thereby, $||f - f_0||^2_{\mathcal{H}} = ||f||^2_{\mathcal{H}} - 2\Re\langle f_0, f\rangle_{\mathcal{H}} + ||f_0||^2_{\mathcal{H}} \leq 0$ and thus, $f = f_0$. This proves (2). We call $\langle f, \pi(s)\psi\rangle$ the *wavelet transform* of $f$ associated to $(G, \pi, S, \psi)$ and (3) the inversion formula of the transform. Here we put $T_f(s) = \langle f, \pi(s)\psi\rangle_{\mathcal{H}}$ $(s \in S)$. If $\pi$ is unitary, it satisfies a partial covariance: $T_{\pi(s_1)\pi(s_2)^{-1} f}(s_1) = T_f(s_2)$ $(s_1, s_2 \in S)$ and furthermore, if $S$ is a subgroup of $G$, it is the covariance property: $T_{\pi(s_2) f}(s_1) = T_f(s_2^{-1} s_1)$.

We state some well-known examples of the wavelet transform in our scheme. When $S = G$, $ds$ a Haar measure of $G$, and $(\pi, \mathcal{H})$ a square-integrable representation of $G$, Duflo and Moore [DM] find a $G$-admissible vector $\psi \in \mathcal{H}$: for example, Gabor transform and Grossmann-Morlet transform correspond to a square-integrable representation of the Weyl-Heisenberg group and the one-dimensional affine group respectively (*cf.* [MW, § 3]), and a reproducing formula for a weighted Bergman space on a bounded symmetric domain relates to a holomorphic discrete series of a semisimple Lie group (*cf.* [B], [K]). For another example we refer to [VP]. Let $(G, H)$ be a semisimple symmetric pair and put $S = KA = \sigma(G/H)$, $\sigma : G/H \to G$ is a flat section, $ds = dkda$, and $(\pi, \mathcal{H})$ a square-integrable representation of $G$ mod $H$. Then we can find a $H$-invariant distribution vector $\psi \in \mathcal{H}_{-\infty}$ for which (1) and (2) hold. Recently this idea "square-integrability mod $H$" was generalized to some other pairs $(G, H)$ and non-flat sections $\sigma : G/H \to G$: Ali, Antoine, and Gazeau [AAG] obtain wavelet transforms associated to the Wigner representation of the Poincaré group $\mathcal{P}_+^\uparrow (1, 1)$, and Torrésani [T1, 2], Kalisa and Torrésani [KT] do to the Stone-von Neumann type representation of the affine Weyl-Heisenberg group $G_{aWH}$.

In this paper we shall investigate a transform associated to a principal series representation of $SL(n+2, \mathbf{R})\,(n \in \mathbf{N})$. To explain our goal we look at an example by taking $G = SL(2, \mathbf{R})$. The holomorphic discrete series $\pi_n\,(n \geq 1/2,\ n \in \mathbf{Z})$ is realized on $H_n^2$, here $H_n^2$ is the Hilbert space of holomorphic functions on the upper half plane $\mathbf{C}^+$ with inner product $\langle \phi, \psi \rangle_{H_n^2} = \Gamma(2n-1) \int_{\mathbf{C}^+} \phi(x+iy) \overline{\psi}(x+iy)\, y^{2n-2}\, dx\, dy$. Then each $\pi_n$ is square-integrable:

$$(3) \qquad \int_G |\langle \phi, \pi_n(g)\psi \rangle_{H_n^2}|^2\, ds = \frac{c}{2n-1} \|\phi\|_{H_n^2} \|\psi\|_{H_n^2} \quad \text{for all } \phi, \psi \in H_n^2,$$

where $c$ is independent of $n$ (*cf.* [Su, § 10 and Prop. 7.18 in Chap. V]) and, as stated above, the inversion formula follows as

$$\phi(x) = c_\psi^{-1} \int_G \langle \phi, \pi_n(g)\psi \rangle_{H_n^2} \pi_n(g)\psi(x)\, dg \quad (x \in G),$$

where $c_\psi = (2n-1)^{-1} c \|\psi\|_{H_n^2}$. Let $\hat{H}_n^2$ be the Hilbert space of functions on $\mathbf{R}^+$ with inner product $\langle \Phi, \Psi \rangle_{H_n^2} = 2^{2n-1} \int_{\mathbf{R}^+} \Phi(t)\, \hat{\Psi}(t)\, t^{-2n+1}\, dt$. Then the inverse Fourier-Laplace transform $\mathcal{F}$ gives an isometry of $H_n^2$ onto $\hat{H}_n^2$ and, if we put $\hat{\pi}_n = \mathcal{F} \circ \pi_n \circ \mathcal{F}^{-1}$, $(\pi_n, H_n^2)$ and $(\hat{\pi}_n, \hat{H}_n^2)$ are unitary equivalent (*cf.* [Sa, p. 20]). In particular, (3) and the inversion formula hold by replacing $\pi_n$ and $H_n^2$ with $\hat{\pi}_n$ and $\hat{H}_n^2$ respectively. Here we note that $\|\Psi\|_{\hat{H}_{n+1/2}^2} = (2n-1)^{-1} \|\Psi\|_{\hat{H}_n^2}$ provided that $\Psi$ is $K$-invariant, and therefore, by the integral formula under the Iwasawa decomposition $G = \overline{N} AK$ we can deduce that

$$(4) \qquad \int_{\overline{N} A} |\langle \Phi, \hat{\pi}_n(\overline{n}\, a)\Psi \rangle_{\hat{H}_n^2}|^2\, d\overline{n}\, da = c \|\Phi\|_{\hat{H}_n^2} \|\Psi\|_{\hat{H}_{n+1/2}^2} \quad \text{for all } \Phi \in \hat{H}_n^2.$$

Now we consider the limiting case of $n = 1/2$: the limit of discrete series $(\pi_{1/2}, H^2_{1/2}) \simeq (\hat{\pi}_{1/2}, \hat{H}^2_{1/2})$. Obviously, (3) and (4) collapse when $n$ goes to 1/2, because $\|\Psi\|_{\hat{H}^2_{n+1/2}} = (2n-1)^{-1} \|\Psi\|_{\hat{H}^2_n} \to \infty$ when $n \to 1/2$. However, if we drop the $K$-invariance of $\Psi$ and assume that $\|\Psi\|_{\hat{H}^2_1} < \infty$ for $\Psi \in \hat{H}^2_{1/2}$, we can deduce that

$$\int_{\overline{N}A} |\langle \Phi, \hat{\pi}_{1/2}(\overline{n}a)\Psi\rangle_{\hat{H}^2_{1/2}}|^2 \, d\overline{n}\, da = c\, \|\Phi\|_{\hat{H}^2_{1/2}} \|\Psi\|_{\hat{H}^2_1} \quad \text{for all } \Phi,\ \psi \in \hat{H}^2_{1/2}. \tag{5}$$

Observe that the wavelet transform $\langle \Phi, \hat{\pi}_{1/2}(\overline{n}a)\Psi\rangle$ is nothing but the affine wavelet transform obtained by Grossmann and Morlet (*see* § 5). Moreover, recall that the limit of discrete series $\pi_{1/2} \simeq \hat{\pi}_{1/2}$ is unitary equivalent to an irreducible component of a reducible unitary principal series of $G$, that is denoted by $V^{1/2,\,1/2}$ in [Su, p. 246]. Therefore, in this context we can say that the affine wavelet transform corresponds to the limit of discrete series or a reducible unitary principal series of $SL(2, \mathbf{R})$.

Our aim is to generalize this correspondance and to find a transform which associates to a principal series of $SL(n+2, \mathbf{R})$. Let $P = MAN$ be a parabolic subgroup of $G$ such that $N \simeq H_n$, the Heisenberg group, and and $\pi_\lambda = \mathrm{Ind}_P^G(1 \otimes e^\lambda \otimes 1)$ the induced representation of $G$. We identify $\overline{N}$ with $\mathbf{R}^{2n+1}$ to define $L^2(\overline{N})$, $\mathcal{S}(\overline{N})$, and $\mathcal{S}'(\overline{N})$. Then we shall find a subgroup $\overline{N}A_1 \simeq H_n \times \mathbf{R}$ of $G$, $\psi \in \mathcal{S}'(\overline{N})$, and a $\lambda$ for which

$$\int\int_{\overline{N}\times A_1} |\langle \phi, \pi_\lambda(\overline{n}a_1)\psi\rangle_{L^2(\overline{N})}|^2 \, d\overline{n}\, da_1 = c_\psi \|\phi\|^2_{L^2(\overline{N})} \tag{6}$$

for all $\phi \in \mathcal{S}(\overline{N})$. Of course, since calculation is carried on a subgroup $\overline{N}A$ of $G$, the whole results can be stated without using the $SL(n+2, \mathbf{R})$-scheme. However, to emphasise the correspondance of our transform and a principal series of $G$, we dare to use the $SL(n+2, \mathbf{R})$-scheme. Most of results in this paper can be generalized to the analytic continuation of discrete series, inclusing the limiting case, and to a principal series of semisimple Lie groups. They will appear in forthcoming papers.

## 2. HEISENBERG GROUP

Before starting the representation theory of $G = SL(n+2, \mathbf{R})$, we recall the one of the Heisenberg group $H_n$, to which the subgroup $\overline{N}$ of $G$ is isomorphic (*see* § 3 below). We refer to the general references [F] and [G].

Let $H_n = \{X = (p, q, t); p, q \in \mathbf{R}^n, t \in \mathbf{R}\}$ denote the *polarized* Heisenberg group with the group law:

$$(p, q, t)(p', q', t') = (p+p', q+q', t+t'+pq'),$$

where $xy = \sum_{i=1}^{n} x_i y_i$ for $x = (x_i), y = (y_i) \in \mathbf{R}^n$. We observe that $\{(0, 0, t); t \in \mathbf{R}\}$ is the center of $H_n$ and the Lebesgue measure $dqdpdt$ is an bi-invariant measure $dX$ on $H_n$. The Schrödinger representation $(\rho_h, L^2(\mathbf{R}^n))$ of $H_n$ with parameter $h \in \mathbf{R}^n \backslash \{0\}$ is given by

$$\rho_h(p, q, t)\, x = e^{2\pi i h t + 2\pi i q x} f(x + hp) \quad (f \in L^2(\mathbf{R}^n)). \tag{7}$$

Then each $\rho_h$ is irreducible unitary and, by Stone-von Neumann Theorem, $\rho_h$ is, up to unitary equivalence, the only representation of $H_n$ with the central character $\pi(0, 0, t) = e^{2\pi i h t} I$ for $h \in \mathbf{R}^n \backslash \{0\}$. We define for $\phi \in L^1(H_n)$

$$\rho_h(\phi) = \int_{H_n} \phi(X)\, \rho_h(X)\, dX.$$

Then, for $\phi, \psi \in L^1(H_n)$

$$\rho_h(\phi * \psi) = \rho_h(\phi)\, \rho_h(\psi) \qquad \text{and} \qquad \rho_h(\phi)^* = \rho_h(\phi^{\sim}), \tag{8}$$

where $\phi * \psi(X) = \int_{H_n} \phi(Y)\, \psi(Y^{-1} X)\, dY$ and $\phi^{\sim}(X) = \overline{\phi}(X^{-1})$. Similarly for $\alpha(p, q) \in L^1(\mathbf{R}^n \times \mathbf{R}^n) = L^1(\mathbf{R}^{2n})$ we define

$$\rho_h^0(\alpha) = \int_{H_n} \alpha(p, q)\, \rho_h^0(p, q)\, dp\, dq,$$

where $\rho_h^0(p, q) = \rho_h(p, q, 0)$. It is clear that $\rho_h^0(\alpha)$ makes sense as an operator from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$ whenever $\alpha \in \mathcal{S}'(\mathbf{R}^{2n})$ (*see* [F, Theorem (1.30)]). For $\phi \in L^2(H_n)$ the Plancherel formula on $H_n$ is given as follows.

$$\|\phi\|^2_{L^2(H_n)} = \int_{\mathbf{R}} |h|^n \, \|\rho_h(\phi)\|^2_{HS}\, dh, \tag{9}$$

where $\|\,.\,\|_{HS}$ denotes the Hilbert-Schmidt norm.

## 3. INDUCED REPRESENTATION OF $SL(n+2,\ \mathbf{R})$

Let $G = SL(n+2,\ \mathbf{R})$ and $\mathfrak{g} = \mathfrak{sl}(n+2,\ \mathbf{R})$. According to the process in [H1, § 6], we shall define a parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ of $\mathfrak{g}$, the parabolic subgroup $P = MAN$ of $G$, and an induced representation $\pi_\lambda = \mathrm{Ind}_P^G (1 \otimes e^\lambda \otimes 1)$ of $G$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_0 + \mathfrak{n}_0$ be the Iwasawa decomposition of $\mathfrak{g}$ such that

$\mathfrak{k} = \mathfrak{so}(n+2)$,

$\mathfrak{a}_0 = \{\text{diagonal matrices in } \mathfrak{g}\}$,

$\mathfrak{n}_0 = \{\textit{lower} \text{ triangular matrices in } \mathfrak{g} \text{ with 0 on the diagonal}\}$.

Let $Z_\mathfrak{g}(\mathfrak{a}_0) = \mathfrak{m}_0 + \mathfrak{a}_0$. When $n = 0,\ 1$, we put $\mathfrak{m} = \mathfrak{m}_0$, $\mathfrak{a} = \mathfrak{a}_1 = \mathfrak{a}_0$, and $\mathfrak{n} = \mathfrak{n}_0$, that is, $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ is a minimal parabolic subalgebra of $\mathfrak{g}$. In the following, we may assume that $n \geq 2$. We define $e_i : \mathfrak{a}_0 \to \mathbf{R}$ $(1 \leq i \leq n+2)$ by $e_i(H) = h_i$ for $H = \mathrm{diag}(h_1,\ h_2,\ \ldots,\ h_{h+2}) \in \mathfrak{a}_0$, and put $\alpha_i = e_{i+1} - e_i$ $(1 \leq i \leq n+1)$. The set of roots of $(\mathfrak{g},\ \mathfrak{a}_0)$ positive for $\mathfrak{n}_0$ is given by $\Sigma = \{e_i - e_j;\ i > j\}$ and the subset consisting of simple roots is $\Sigma_0 = \{\alpha_i;\ 1 \leq i \leq n+1\}$. For $F = \{\alpha_i;\ 2 \leq i \leq n\}$ we set $\mathfrak{a} = \mathfrak{a}_F = \{H \in \mathfrak{a}_0;\ \alpha(H) = 0 \text{ for all } \alpha \in F\}$ and $\mathfrak{n} = \mathfrak{n}_F = \sum_{\alpha \in \Sigma \setminus \Sigma_F} \mathfrak{g}_\alpha$ where $\Sigma_F = \{\alpha \in \Sigma;\ \alpha|_{\mathfrak{a}_F} \equiv 0\}$ and $\mathfrak{g}_\alpha$ is the root space corresponding to $\alpha$. Explicitly, they are of the forms:

$$\mathfrak{a} = \left\{ \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 I_n & 0 \\ 0 & 0 & h_3 \end{pmatrix};\ h_1,\ h_2,\ h_3 \in \mathbf{R},\ h_1 + n\,h_2 + h_3 = 0 \right\},$$

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ t & q & 0 \end{pmatrix};\ p,\ q \in \mathbf{R}^n,\ t \in \mathbf{R} \right\},$$

where $I_n$ is the $n \times n$ unit matrix. Let $Z_\mathfrak{g}(\mathfrak{a}) = \mathfrak{m} + \mathfrak{a}$ and put $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$. The set of roots of $(\mathfrak{g},\ \mathfrak{a})$ positive for $\mathfrak{n}$ is given by $\Sigma(\mathfrak{a}) = \{\alpha^\sim;\ \alpha \in \Sigma\}$ where $\alpha^\sim = \alpha|_\mathfrak{a}$. We set $\mathfrak{a}_1 = \{H \in \mathfrak{a};\ \alpha_{n+1}^\sim(H) = 0\}$ and $\rho = \sum_{\alpha \in \Sigma(\mathfrak{a})} \alpha/2$, that is,

$$\mathfrak{a}_1 = \left\{ H_s = \begin{pmatrix} -(n+1)\,s & 0 \\ 0 & s\,I_{n+1} \end{pmatrix};\ s \in \mathbf{R} \right\},$$

$$\rho = \frac{(n+1)\,(\alpha_1^\sim + \alpha_{n+1}^\sim)}{2}.$$

We denote by $M_0$, $A$, $A_1$, and $N$ the analytic subgroups of $G$ corresponding to $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{a}_1$, and $\mathfrak{n}$ respectively. We define $M = Z_K(\mathfrak{a})\,M_0$, where $Z_K(\mathfrak{a})$

is the centralizer of $\mathfrak{a}$ in $K = SO(n+2)$, and put $P = MAN$. We denote by $\theta$ the Cartan involution of $G$ given by $\theta(g) = {}^t g^{-1}\ (g \in G)$ and put $\overline{N} = \theta(H)$. Then it is easy to see that

$$\text{(10)} \qquad \overline{N} \simeq H_n \text{ and } \overline{N}P \text{ is open in } G \text{ whose complement has Haar measure } 0,$$

where the identification is given by

$$\begin{pmatrix} 1 & p & t \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \quad \mapsto \quad (p,\ q,\ t).$$

We put $d\overline{n} = dX$, $da = dA$, where $dA$ is the Lebesgue measure on $\mathfrak{a}$ normalized in such a way that $\int_A f(a)\,da = \int_{\mathfrak{a}} f(\exp A)\,dA$ ($f \in C_c(A)$), and $da_1 = da|_{A_1}$. We normalize the Haar measure $dm$ on $M$ and $d\overline{n} = d\theta(n)$ on $\overline{N}$ as the following integral formula holds:

$$\int_G f(g)\,dg = \int\int\int\int_{\overline{N}\times M\times A\times N} f(\overline{n}man)\,e^{2\rho(\log a)}\,d\overline{n}\,dm\,da\,dn$$

for $f \in C_c(G)$ (*see* [H2, § 19]).

Let $\mathfrak{a}_c^*$ denote the dual space of the complexification $\mathfrak{a}_c$ of $\mathfrak{a}$. For $\lambda \in \mathfrak{a}_c^*$ we define, out of the representation $1 \otimes e^\lambda \otimes 1\,(man) = e^{\lambda(\log a)}$ of $P = MAN$, a representation of $G$ denoted by

$$\pi_\lambda = \mathrm{Ind}_P^G(1 \otimes e^\lambda \otimes 1).$$

A dense subspace of the representation space $\mathcal{H}_\lambda$ is given by $\mathcal{H}_\lambda^0$ consisting of continuous functions $f$ on $G$ such that

$$\text{(11)} \qquad f(gman) = e^{-(\lambda+\rho)(\log a)}\,f(g) \quad (g \in G,\ man \in MAN)$$

with norm $\|f\|^2 = \int_K |f(k)|^2\,dk$. Moreover, $G$ acts on $\mathcal{H}_\lambda$ as $\pi_\lambda(g_0)\,f(g) = f(g_0^{-1}g)$ $(g_0 \in G)$ and $\pi_\lambda$ is unitary whenever $\lambda \in i\,\mathfrak{a}^*$. We observe that, by restricting $f \in \mathcal{H}_\lambda$ to $\overline{N}$, $\mathcal{H}_\lambda$ is identified with $L^2(\overline{N},\ e^{2\Re\lambda(H(\overline{n}))}\,d\overline{n})$, where $g \in G$ is decomposed under $G = KMAN$ as $g = kme^{H(g)}n$, and the action of $G$ is given by

$$\text{(12)} \qquad \pi_\lambda(g)\,F(\overline{n}) = e^{-(\lambda+\rho)\,\log a\,(g^{-1}\overline{n})}\,F(\overline{n}(g^{-1}\overline{n})),$$

where $g \in G$ is decomposed under $G = \overline{N} MAN$ as $g = \overline{n}(g)\, ma(g)\, n$. We define $\mathcal{S}(\overline{N}) = \mathcal{S}(H_n)$ by the Schwartz class $\mathcal{S}(\mathbf{R}^{2n+1})$. Then it follows from Lemma 8.5.23, 27, and Theorem 8.2.1 in [War] that $\mathcal{S}(\overline{N})$ is contained in $L^2(\overline{N},\, e^{\pm 2\Re\lambda(H(\overline{n}))}\, d\overline{n})$ and, from (12) that $\pi_\lambda(\overline{n}\, a)\,(\overline{n}\, a \in \overline{N} A)$ is an operator on $\mathcal{S}(\overline{N})$. We here define $\langle f,\, g\rangle_{L^2(\overline{N})} = \int_K f(k)\, \overline{g}(k)\, dk$ for $f,\, g \in \mathcal{H}_\lambda$. Since this form is nondegenerate and $G$-invariant on $\mathcal{H}_{-\overline{\lambda}} \times \mathcal{H}_\lambda$ (*cf.* [Wal, 8.3.11]), we see that $\pi_\lambda(\overline{n}\, a)$ is an operator on $\mathcal{S}'(\overline{N})$. Especially, $T_f(s) = \langle f, \pi_\lambda(s)\, \psi\rangle_{L^2(\overline{N})}$ $(s \in \overline{N} A)$ satisfies the covariance property: $T_{\pi_{-\overline{\lambda}}(s_2) f}(s_1) = T_f(s_2^{-1} s_1)$ $(s_1,\, s_2 \in S)$.

## 4. MAIN THEOREM

Let $P = MAN$ and $A_1 \subset A$ be the subgroups of $G = SL(n+2,\, \mathbf{R})$ introduced in paragraph 3. We suppose that $\alpha(p,\, q) \in \mathcal{S}'(\mathbf{R}^{2n})$ and $\beta(t) \in L^1(\mathbf{R})$ satisfy the following condition: there exists $\gamma : \mathbf{R} \to \mathbf{R}$ such that

$$(13) \qquad \begin{cases} \text{(i)} & \rho_h^0(\alpha)\, \rho_h^0(\alpha)^* = \gamma(h)\, I \quad \text{a.e. } h \in \mathbf{R}, \\ \text{(ii)} & \displaystyle\int_0^\infty |\mathcal{F}^{-1} \beta(\xi)|^2\, \gamma(\xi)\, \frac{d\xi}{|\xi|} \\ & \displaystyle = \int_{-\infty}^0 |\mathcal{F}^{-1} \beta(\xi)|^2\, \gamma(\xi)\, \frac{d\xi}{|\xi|} = c_{\alpha,\beta} < \infty, \end{cases}$$

where $\rho_h^0(\alpha)$ is an operator from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$ (*see* § 2), $I$ is the identity operator, and $\mathcal{F}^{-1}$ is the inverse Fourier transform on $\mathbf{R}$. When $G = SL(2,\, \mathbf{R})$, we ignore the function $\alpha$. we set

$$(14) \qquad \psi_{\alpha,\beta}(p,\, q,\, t) = \alpha(p,\, q)\, \beta(t).$$

THEOREM 4.1. – *Let $\psi_{\alpha,\beta}$ be as above and suppose $\lambda|_{\mathfrak{a}_1} = (n+1)\, \tilde{\alpha_1}/2$. Then, $\psi_{\alpha,\beta} \in \mathcal{S}'(\overline{N})$ is $\overline{N} A_1$-admissible for $\pi_\lambda$, that is, there exists a positive constant $c = c_{\psi_{\alpha,\beta}}$ such that*

$$\int\!\!\int_{\overline{N} \times A_1} |\langle f,\, \pi_\lambda(\overline{n}\, a_1)\, \psi_{\alpha,\beta}\rangle_{L^2(\overline{N})}|^2\, d\overline{n}\, da_1 = c\, \|f\|^2_{L^2(\overline{N})} \quad \textit{for all } f \in \mathcal{S}(\overline{N}).$$

*Proof.* – We first observe from (8) and (9) that for $f \in \mathcal{S}(\overline{N})$

$$
\begin{aligned}
&\int\int_{\overline{N}\times A_1} |\langle f,\, \pi_\lambda(\overline{n}\, a_1)\, \psi_{\alpha,\beta}\rangle_{L^2(\overline{N})}|^2\, d\overline{n}\, da_1 \\
&\quad = \int\int_{\overline{N}\times A_1} |(f * (\pi_\lambda(a_1)\, \psi_{\alpha,\beta})^\sim (\overline{n})|^2\, d\overline{n}\, da_1 \\
&\quad = \int_{A_1}\int_{\mathbf{R}} |h|^n\, \|\rho_h(f * (\pi_\lambda(a_1)\, \psi_{\alpha,\beta})^\sim)\|_{HS}^2\, dh\, da_1 \\
&\quad = \int_{\mathbf{R}} |h|^n\, \mathrm{Tr}\Big(\rho_h(f) \int_{A_1} \rho_h(\pi_\lambda(a_1)\, \psi_{\alpha,\beta})^* \\
&\qquad\qquad \times \rho_h(\pi_\lambda(a_1)\, \psi_{\alpha,\beta})\, da_1\, \rho_h(f)^*\Big)\, dh.
\end{aligned}
$$

Since $A_1 = \{a_s = \exp(H_s);\ s \in \mathbf{R}\}$, $(\lambda + \rho)|_{\mathfrak{a}_1} = (n+1)\,\alpha_1^\sim$ and $\alpha_1^\sim(\log a_s) = (n+2)\, s$, it follows from (10) and (11) that

$$
\begin{aligned}
\pi_\lambda(a_s)\, \psi_{\alpha,\beta}(\overline{n}) &= \psi_{\alpha,\beta}(a_s^{-1}\, \overline{n}\, a_s \cdot a_s^{-1}) \\
&= \psi_{\alpha,\beta}(e^{(n+2)s}\, p,\ q,\ e^{(n+2)s}\, t)\, e^{(n+1)(n+2)s}
\end{aligned}
$$

and thereby, from (7), (11), and (14) that

$$
\begin{aligned}
&\rho_h(\pi_\lambda(a_s)\, \psi_{\alpha,\beta}) \\
&\quad = \int_{\overline{N}} \pi_\lambda(a_s)\, \psi_{\alpha,\beta}(\overline{n})\, \rho_h(\overline{n})\, d\overline{n} \\
&\quad = \int\int\int_{\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}} \psi_{\alpha,\beta}(e^{(n+2)s}\, p,\ q,\ e^{(n+2)s}\, t) \\
&\qquad \times e^{2\pi i h t}\, \rho_h^0(p,\, q)\, dp\, dq\, dt\, e^{(n+1)(n+2)s} \\
&\quad = \mathcal{F}^{-1}\beta(e^{-(n+2)s}\, h) \int\int_{\mathbf{R}^n\times\mathbf{R}^n} \alpha(e^{(n+2)s}\, p,\ q)\, \rho_h^0(p,\, q)\, dp\, dq \\
&\qquad \times e^{n(n+2)s} \\
&\quad = \mathcal{F}^{-1}\beta(e^{-(n+2)s}\, h)\ \rho^0_{e^{-(n+2)s}\, h}(\alpha).
\end{aligned}
$$

Then, we can deduce from (13) (i) and (ii) that

$$\begin{aligned}
&\int_{A_1} \rho_h(\pi_\lambda(a_1)\psi_{\alpha,\beta})^* \rho_h(\pi_\lambda(a_1)\psi_{\alpha,\beta})\, da_1 \\
&= \int_{\mathbf{R}} |\mathcal{F}^{-1}\beta(e^{-(n+2)s}h)|^2 \rho^0_{e^{-(n+2)s}h}(\alpha)^* \rho^0_{e^{-(n+2)s}h}(\alpha)\, ds \\
&= \begin{cases} \displaystyle\int_0^\infty |\mathcal{F}^{-1}\beta(\xi)|^2 \gamma(\xi) I \frac{d\xi}{|\xi|} & \text{if } h > 0 \\ \displaystyle\int_{-\infty}^0 |\mathcal{F}^{-1}\beta(\xi)|^2 \gamma(\xi) I \frac{d\xi}{|\xi|} & \text{if } h < 0 \end{cases} \\
&= c_n\, c_{\alpha,\beta}\, I,
\end{aligned}$$

where $c_n = 1/(n+2)$ and thus,

$$\begin{aligned}
&\int\int_{\overline{N}\times A_1} |\langle f, \pi_\lambda(\overline{n}\, a_1)\psi_{\alpha,\beta}\rangle_{L^2(\overline{N})}|^2\, d\overline{n}\, da_1 \\
&= c_n\, c_{\alpha,\beta} \int_{\mathbf{R}} |h|^n \operatorname{Tr}(\rho_h(f)\,(\rho_h(f)^*)\, dh \\
&= c_n\, c_{\alpha,\beta}\, \|f\|^2_{L^2(\overline{N})}. \quad \square
\end{aligned}$$

## 5. EXAMPLES

We conclude with some examples of $\psi_{\alpha,\beta}(p, q, t) = \alpha(p, q)\beta(t)$ satisfying the condition (13) (i) and (ii). In the case of $SL(2, \mathbf{R})$, as said before, the function $\alpha(p, q)$ is ignored, so the condition (13) (ii) only lives out with $\gamma \equiv 1$:

$$\int_0^\infty |\mathcal{F}^{-1}\beta(\xi)|^2 \frac{d\xi}{|\xi|} = \int_{-\infty}^0 |\mathcal{F}^{-1}\beta(\xi)|^2 \frac{d\xi}{|\xi|} < \infty. \tag{15}$$

This condition is noting but the admissible condition for the affine wavelet transform on $L^2(\mathbf{R})$. Actually, when $G = SL(2, \mathbf{R})$, the wavelet transform in (2) is of the form:

$$\langle f, \pi_\lambda(\overline{n}_t\, a_s)\beta\rangle_{L^2(H_0)} = e^{2s} \int_{\mathbf{R}} f(t')\,\beta(e^{2s}(t'-t))\, dt'$$

and hence, if we set $a = e^{2s}$, it coincides with the affine wavelet transform on $L^2(\mathbf{R})$. This is quite natural because $\overline{N}\, A_1$ is isomorphic to the affine group "$ab + b$" (*cf.* [HW], 3.3).

We now suppose that $n \geq 1$, and we give some examples of $\alpha(p, q) \in \mathcal{S}'(\mathbf{R}^{2n})$ satisfying (13) (i) and obtain the function $\gamma : \mathbf{R} \to \mathbf{R}$.

(*a*) If $\alpha(p, q) = \delta(p - p_0)\, \delta(q - q_0)$ for some $p_0, q_0 \in \mathbf{R}^n$, it easily see that $\rho_h^0(\alpha) f(x) = e^{2\pi i q_0 x} f(x + h p_0)$ and $\|\rho_h^0(\alpha) f\|^2_{L^2(\mathbf{R}^n)} = \|f\|^2_{L^2(\mathbf{R}^n)}$. Therefore, $\alpha(p, q)$ satisfies (13) (i) with $\gamma \equiv 1$ and hence the condition on $\beta$ is the same as in (15).

(*b*) Let $\alpha(p, q) = \alpha_0(p)\, e^{\pi i p q}$ where $|\alpha_0| \equiv 1$. Since $\int e^{2\pi i q x}\, dq = \delta(x)$, it follows that $\rho_h^0(\alpha) f(x) = 2^n \alpha_0(-2x) f((1 - 2h)x)$ and $\|\rho_h^0(\alpha) f\|^2_{L^2(\mathbf{R}^n)} = 2^{2n} |1 - 2h|^{-n} \|f\|^2_{L^2(\mathbf{R}^n)}$. Therefore, $\alpha(p, q)$ satisfies (13) (i) with $\gamma(h) = 2^{2n} |1 - 2h|^{-n}$ and (13) (ii) is of the form:

$$(16) \qquad 2^n \int_0^\infty |\mathcal{F}^{-1} \beta(\xi)|^2 |1 - 2\xi|^{-n} \frac{d\xi}{|\xi|} = 2^n \int_{-\infty}^0 |\mathcal{F}^{-1} \beta(\xi)|^2 |1 - 2\xi|^{-n} \frac{d\xi}{|\xi|} < \infty.$$

(*c*) If $\alpha(p, q) = \alpha_1(q)\, e^{\pi i p q}$ where $|\alpha_1| \equiv 1$, then $\rho_h^0(\alpha) f(x) = 2^n \mathcal{F}[\alpha_1' . \mathcal{F}^{-1} f]((1 - 2h)x)$ where $\alpha_1'(s) = \alpha_1(2hx)$, so the function $\gamma$ and the condition on $\beta$ are the same as in (*b*).

(*d*) If $\alpha(p, q) = 2^{-n/2} e^{in\pi/4} e^{-\pi i (p^2 + q^2)/2} e^{\pi i p q}$, then from the formula for the distribution Fourier transform of the Gaussian functions (*cf.* [F, Theorem 2 in Appendix A]) it follows that $\rho_h^0(\alpha) f(x) = h^{-n} e^{2\pi i (1 - 1/h) x^2} \mathcal{F}^{-1} f(x/h)$ and $\|\rho_h^0(\alpha) f\|^2_{L^2(\mathbf{R}^n)} = |h|^{-n} \|f\|^2_{L^2(\mathbf{R}^n)}$. Therefore, $\alpha(p, q)$ satisfies (13) (i) with $\gamma(h) = |h|^{-n}$ and hence (13) (ii) is given by

$$\int_0^\infty |\mathcal{F}^{-1} \beta(\xi)|^2 \frac{d\xi}{|\xi|^{n+1}} = \int_{-\infty}^0 |\mathcal{F}^{-1} \beta(\xi)|^2 \frac{d\xi}{|\xi|^{n+1}} < \infty.$$

(*e*) We now consider the Gaussian functions:

$$\alpha(p, q) = e^{-\pi i (pBp - 2pAq + qCq)} e^{\pi i p q},$$

where $A$, $B$, and $C$ denote $n \times n$ real matrices. We set $D = {}^t A + I/2$. If $C = 0$ and $D$ is invertible, it follows as in (*b*) that

$$\rho_h^0(\alpha) f(x) = \det{}^{-1} D \cdot e^{-\pi i x {}^t D^{-1} B D^{-1} x} f((I - h D^{-1}) x),$$
$$\gamma(h) = |\det D|^{-2} |\det(I - h D^{-1})|^{-1}.$$

On the other hand, if $C$ is invertible and symmetric, and $B =^t DC^{-1} D$, it follows as in (*d*) that

$$\rho_h^0(\alpha) f(x) = e^{-\pi i \#(C)/4} |\det C|^{-1/2} h^{-n} \times e^{2\pi i x C^{-1} \left(\frac{I}{2} - \frac{D}{h}\right) x} \mathcal{F}^{-1} f(C^{-1} Dx/h),$$

$$\gamma(h) = |h|^{-n} |\det D|^{-1},$$

where $\#(C)$ is the number of positive eigenvalues of $C$ minus the number of negative eigenvalues.

*Remark.* – (1) We note that the process to obtain $\rho_h^0(\alpha)$ is exactly same as the one used in the Weyl correspondence of pseudodifferential operators (*cf.* [F, Chap. 2]). In fact the above calculation of $\rho_h^0(\alpha)$ also follows from Proposition (2.28) in [F] by generalizing the results for $\rho_1$ to $\rho_h$ and by arranging the isomorphism from the Heisenberg group to the polarized one. Especially, in the case (*e*) the set of operators $\gamma(h)^{-1/2} \rho_h^0(\alpha)$ corresponds to the range of the metapletic representation of $Sp(n, \mathbf{R})$ (*see* [F, Chap. 4 and Chap. 5]).

(2) We suppose that $B = C = 0$ and $D$ is invertible in (*e*). Then $\overline{N} A_1$-admissible vectors $\psi_{\alpha, \beta}$ are $M_0 A_1'$-invariant, where $A_1'$ is the analytic subgroup of $G$ corresponding to $\mathfrak{a}_1' = \{H \in \mathfrak{a};\ (e_{n+2} - e_1)(H) = 0\}$. In general, if $\alpha(p, q)$ is a function of $pq$, then $\overline{N} A_1$-admissible vectors $\psi_{\alpha, \beta}$ are $M_0 A_1'$-invariant, and moreover, if $\alpha(p, q)$ is an even function of $pq$ and $\beta(t)$ is even, then $\psi_{\alpha, \beta}$ are $M_0 A_1'$-invariant.

(3) Examples in paragraph 5 don't cover the results in [KT]. In order to obtain their transforms in our $SL(n+2, \mathbf{R})$-scheme we need deeper analysis on $\psi$. It will be done in a forthcoming paper.

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