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Remarks on relativistic Schrödinger operators and their extensions

by

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ABSTRACT. – The operator $H = f(-\Delta) + V(x)$ is considered in \mathbb{R}^n , where f is a general real increasing function on \mathbb{R}_+ , and $V(x)$ is a real potential with singularities. In particular, the relativistic Schrödinger operator $\sqrt{-\Delta + 1} + V(x)$ is included. Limiting absorption principles are proved for H , including the threshold. The smoothness of the extended resolvents is applied to long-time decay for the evolution group $\exp(-itH)$.

Key words: Relativistic Schrödinger operators, Limiting absorption principle, threshold regularity, long-time decay.

RÉSUMÉ. – Soit $H = f(-\Delta) + V(x)$ sur \mathbb{R}^n , où f est une fonction réelle croissante sur \mathbb{R}_+ , et $V(x)$ est un potentiel réel présentant des singularités. L'opérateur de Schrödinger relativiste $\sqrt{-\Delta + 1} + V(x)$ est un cas particulier dans cette classe. Nous prouvons le principe d'absorption limite sur H jusqu'au seuil. La régularité des résolvantes étendues est appliquée à la décroissance à temps long du groupe d'évolution $\exp(-itH)$.

1. INTRODUCTION

In a recent paper by T. Umeda [9] some spectral properties of the relativistic Schrödinger operator are studied. This operator is defined by

$$K = \sqrt{-\Delta + 1} + V(x), \quad x \in \mathbb{R}^n, \quad (1.1)$$

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where $V(x)$ is a short-range sufficiently small real potential. The main tool in the spectral study is a “Limiting Absorption Principle” (henceforth LAP), which states that in a suitable operator topology, the resolvent operator can be extended continuously up to the absolutely continuous spectrum (usually excluding “thresholds”).

Agmon [1] proved the LAP for short-range perturbations of elliptic (and principal-type) operators. Indeed, Agmon’s method, with minor modifications, can be used in the proof of the LAP for K .

An abstract theory for the LAP for partial differential operators was developed in [4], and applied in various studies, such as the spectral theory of the acoustic propagator [3] and global regularity and decay of time-dependent equations [2], [5], [6]. We refer the reader to references in [4], [9] for a variety of methods and results concerning the LAP.

In this paper we use the general method of [4] in order to deal with a whole class of operators (including K) of the form,

$$H = H_0 + V(x), \quad x \in \mathbb{R}^n, \quad (1.2)$$

$$H_0 = f(-\Delta), \quad (1.3)$$

where $f(\theta)$, $\theta \in \mathbb{R}_+ = (0, \infty)$, is a real-valued nonnegative continuous function.

We shall prove several results, which will require increasingly restrictive hypotheses on f . We use the terminology “ g is locally Hölder continuous in an open interval $\Omega \subseteq \mathbb{R}_+$ ” to mean that for every compact $S \subset \subset \Omega$, there exist constants $C > 0$, $0 < \alpha \leq 1$, such that

$$|g(\theta_1) - g(\theta_2)| \leq C |\theta_1 - \theta_2|^\alpha, \quad \theta_1, \theta_2 \in S. \quad (1.4)$$

Our basic hypothesis on f , which will be assumed throughout the paper is the following.

ASSUMPTION 1.1. – $f(\theta)$ is continuously differentiable in \mathbb{R}_+ , and its derivative $f'(\theta)$ is positive and locally Hölder continuous.

Clearly, this assumption renders H_0 to be a self-adjoint operator in $L^2(\mathbb{R}^n)$ (we identify $-\Delta$ with its natural self-adjoint realization in L^2). The spectrum $\sigma(H_0)$ is absolutely continuous and satisfies $\sigma(H_0) = [f(0), f(\infty))$ (we use $f(\infty) = \lim_{\theta \rightarrow \infty} f(\theta)$).

Next we list our additional assumptions, which will be used in our refinements of the basic LAP theorem (Theorem 2A).

ASSUMPTION 1.2. – $f'(\theta)$ satisfies a uniform Hölder condition near $\theta = 0$, namely, there exist constants $C > 0$, $0 < \alpha \leq 1$, such that

$$|f'(\theta_1) - f'(\theta_2)| \leq C |\theta_1 - \theta_2|^\alpha, \quad 0 < \theta_1, \theta_2 \leq 1, \quad (1.5)$$

and in addition $f'(0) > 0$.

(Remark that under (1.5) $f'(\theta)$ can be extended continuously to $\theta = 0$).

ASSUMPTION 1.3. – $f'(\theta)$ satisfies a uniform Hölder condition near $\theta = +\infty$, namely, there exists a constant $M > 0$ such that (1.5) holds for $\theta_1, \theta_2 > M$.

ASSUMPTION 1.4. – $f(\theta) \in C^2(\mathbb{R}_+)$.

Remark that Assumption 1.4 implies Assumption 1.1 and Assumption 1.2 (resp. Assumption 1.3) follows from Assumption 1.4 if $f''(\theta)$ is assumed bounded in $(0, 1)$ (resp. (M, ∞)).

The regularity conditions imposed on f are valid for a large class of Schrödinger operators. The method used here avoids the smoothness assumptions (and the growth conditions at infinity) imposed on f by the use of *phase space* techniques (cf. [7], [8]).

We emphasize that all the above assumptions, as well as some growth conditions imposed on f in the following sections, are satisfied in the case of the relativistic Schrödinger operator $f(\theta) = \sqrt{1 + \theta}$.

In Section 2 we prove (Theorem 2A) that Assumption 1.1 suffices to yield a LAP for H_0 in the interior of $\sigma(H_0)$, thus generalizing the result of [9]. Observe, in particular, that f satisfies a rather limited smoothness condition. Then by adding Assumption 1.2 we show (Theorem 2B) that the LAP can be extended to all of \mathbb{R} , thus crossing the threshold $f(0)$.

Our method of proof is very different from that of [9], requiring minimal smoothness and yielding the Hölder continuity of the limiting values of the resolvent.

In Section 3 we use the results (and techniques) of Section 2 in order to derive a decay result (as $|t| \rightarrow \infty$) of solutions to the time-dependent equation $i\partial_t u = H_0 u$, for all initial conditions $u(t=0)$ in $L^2(\mathbb{R}^n)$, $n \geq 3$. The analogy to the case of the Klein-Gordon equation is clear, and we refer the reader to [2] for similar results concerning the latter (as well as the wave) equation.

Finally in Section 4 we study the case of $H = H_0 + V(x)$. In particular, we remove the uniform boundedness (and smallness) imposed on V in [9], at the price of having (possibly) a discrete sequence of imbedded eigenvalues.

NOTATION. – We denote by $L^{2,s} = L^{2,s}(\mathbb{R}^n)$ – ($s \in \mathbb{R}$) the weighted $-L^2$ spaces normed by

$$\|g\|_s^2 := \int_{\mathbb{R}^n} (1 + |x|^2)^s |g(x)|^2 dx. \quad (1.6)$$

We write $\|g\|$ for $\|g\|_0$ and denote by (\cdot, \cdot) the $L^2(\mathbb{R}^n)$ inner-product. Denoting by \hat{g} the Fourier transform of g , the Sobolev space H_s ($s \in \mathbb{R}$) can be defined as

$$H_s = \{\hat{g} \mid g \in L^{2,s}(\mathbb{R}^n)\},$$

with the norm given by

$$\|\hat{g}\|_{H_s} := \|g\|_s. \quad (1.7)$$

For any two Hilbert spaces X, Y , we denote by $B(X, Y)$ the space of bounded linear operators from X to Y , equipped with the norm $\|T\|_{B(X, Y)}$.

2. THE OPERATOR H_0 TWO LIMITING ABSORPTION PRINCIPLES

In this section we prove two theorems concerning the LAP for H_0 , as defined by (1.3).

Let $\mathbb{C}^\pm = \{z \mid \pm \text{Im } z > 0\}$ and let $R_0(z) = (H_0 - z)^{-1}$, $\text{Im } z \neq 0$, be the resolvent of H_0 . In our first theorem we assume only that f satisfies Assumption 1.1.

THEOREM 2A. – *For every $s > \frac{1}{2}$, the operator-valued function*

$$z \rightarrow R_0(z), \quad z \in \mathbb{C}^+, \quad (\text{resp. } z \in \mathbb{C}^-)$$

can be extended continuously to the real axis, avoiding the points $f(0), f(\infty)$, in the uniform operator topology of $B(L^{2,s}, L^{2,-s})$.

Furthermore, the limiting values

$$R_0^\pm(\lambda) = \lim_{\varepsilon \rightarrow 0^+} R_0(\lambda \pm i\varepsilon), \quad \lambda \in \mathbb{R}, \quad \lambda \neq f(0), f(\infty), \quad (2.1)$$

are locally Hölder continuous in $B(L^{2,s}, L^{2,-s})$.

Proof. – It suffices to prove the statement for $f(0) < \lambda < f(\infty)$.

Denote by $\{E_0(\lambda)\}$, $\{E_{H_0}(\lambda)\}$, the spectral families associated with $-\Delta$, H_0 respectively. In view of the monotonicity of f ,

$$E_{H_0}(\lambda) = E_0(f^{-1}(\lambda)), \quad f(0) < \lambda < f(\infty). \quad (2.2)$$

As was shown in [4] the operator-valued function $E_0(\mu)$, $\mu > 0$, is weakly differentiable in $B(L^{2,s}, L^{2,-s})$, and its weak derivative $A_0(\mu) \in B(L^{2,s}, L^{2,-s})$ satisfies

$$[A_0(\mu)\varphi, \psi] = \frac{d}{d\mu} (E_0(\mu)\varphi, \psi) = \frac{1}{2\sqrt{\mu}} \int_{|\xi|^2=\mu} \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)} d\sigma_\xi, \quad (2.3)$$

where $\varphi, \psi \in L^{2,s}$ and $[\cdot, \cdot]$ denotes the $(L^{2,-s}, L^{2,s})$ pairing. Hence, by (2.2) $E_{H_0}(\lambda)$ is also weakly differentiable in $B(L^{2,s}, L^{2,-s})$ and its weak derivative $A_{H_0}(\lambda) \in B(L^{2,s}, L^{2,-s})$ satisfies,

$$A_{H_0}(\lambda) = [f'(f^{-1}(\lambda))]^{-1} \cdot A_0(f^{-1}(\lambda)), \quad f(0) < \lambda < f(\infty). \quad (2.4)$$

It follows from [4] that $A_0(\mu)$ is locally Hölder continuous in the uniform operator topology of $B(L^{2,s}, L^{2,-s})$ (with exponent depending only on s, n). Combining this with Assumption 1.1 we conclude from (2.4) that $A_{H_0}(\lambda)$ is locally Hölder continuous in $B(L^{2,s}, L^{2,-s})$. As shown in [4], this yields the statements of the theorem. \square

COROLLARY 2.1. – (Regularity of $R_0^\pm(\lambda)$). Let $L_{H_0}^{2,-s}$ be the graph-norm space of H_0 in $L^{2,-s}$ (i.e., the completion of $D(H_0)$ with respect to the norm $\|\varphi\|_{-s} + \|H_0\varphi\|_{-s}$). Then the operator space $B(L^{2,s}, L^{2,-s})$ in the statements of Theorem 2A can be replaced by $B(L^{2,s}, L_{H_0}^{2,-s})$.

Proof. – Follows simply from $H_0 R_0(z) = I + z R_0(z)$. \square

EXAMPLE 2.2. – If $f(\theta) \geq C\theta^{\gamma/2}$ as $\theta \rightarrow +\infty$, for some $\gamma > 0$, then the norm of $L_{H_0}^{2,-s}$ is stronger than that of the weighted Sobolev space H_γ^{-s} (which is the graph-norm space of $(-\Delta)^{\gamma/2}$). Corollary 2.1 then implies that $B(L^{2,s}, L^{2,-s})$ can be replaced by $B(L^{2,s}, H_\gamma^{-s})$. For the (free) relativistic Schrödinger operator $f(\theta) = \sqrt{1+\theta}$ and $\gamma = 1$, which is the result obtained in [9].

In general (even for $-\Delta$) the limiting values $R_0^\pm(\lambda)$ are unbounded (in $B(L^{2,s}, L^{2,-s})$) as $\lambda \rightarrow f(0)$. However, by strengthening the assumptions on f , taking $s > 1$ and restricting the dimension to $n \geq 3$ we can actually obtain the smooth behavior of $R_0^\pm(\lambda)$ near $\lambda = f(0)$.

THEOREM 2B. – *Let $n \geq 3$, $s > 1$, and let f satisfy Assumptions 1.1, 1.2, and assume in addition that $f(\infty) = \infty$. Then the operator-valued function,*

$$z \rightarrow R_0(z), \quad z \in \mathbb{C}^+ \quad (\text{resp. } z \in \mathbb{C}^-),$$

can be extended continuously to $\bar{\mathbb{C}}^+ = \mathbb{C}^+ \cup \mathbb{R}$ (resp. $\bar{\mathbb{C}}^- = \mathbb{C}^- \cup \mathbb{R}$) in the uniform operator topology of $B(L^{2,s}(\mathbb{R}^n), L^{2,-s}(\mathbb{R}^n))$. The limiting values $R_0^\pm(\lambda)$ are locally Hölder continuous in $\lambda \in \mathbb{R}$, with values in $B(L^{2,s}, L^{2,-s})$.

Proof. – In view of Theorem 2A, it suffices to deal with a neighborhood of $\lambda = f(0)$. We recall the following trace lemma, the proof of which can be found in [6, Appendix].

TRACE LEMMA. – *Let $\hat{\varphi} \in H_s(\mathbb{R}^n)$, $n \geq 3$, $\frac{1}{2} < s < \frac{3}{2}$. Then for any $r > 0$,*

$$\left(\int_{|\xi|=r} |\hat{\varphi}(\xi)|^2 d\sigma_r \right)^{\frac{1}{2}} \leq C \cdot \text{Min}(r^{s-\frac{1}{2}}, 1) \cdot \|\hat{\varphi}\|_{H_s}, \quad (2.5)$$

where C depends only on s, n .

Combining the trace lemma with (2.3)-(2.4) we obtain for $\lambda > f(0)$, $\varphi, \psi \in L^{2,s}(\mathbb{R}^n)$,

$$\begin{aligned} |[A_{H_0}(\lambda)\varphi, \psi]| &\leq C[f'(f^{-1}(\lambda))]^{-1} \\ &\cdot (f^{-1}(\lambda))^{-\frac{1}{2}} \cdot \text{Min}(f^{-1}(\lambda)^{s-\frac{1}{2}}, 1) \|\varphi\|_s \|\psi\|_s. \end{aligned} \quad (2.6)$$

In particular, $A_{H_0}(\lambda)$ extends to a locally Hölder continuous function on \mathbb{R} by setting $A_{H_0}(\lambda) \equiv 0$ for $\lambda \leq f(0)$. The conclusions now follow as in the case of the proof of Theorem 2A. \square

REMARK 2.3. – Note that if $s = 1$, then by (2.6), $A_{H_0}(\lambda)$ is bounded in $B(L^{2,1}, L^{2,-1})$ for λ near $f(0)$, but does not necessarily vanish as $\lambda \rightarrow f(0)$.

3. LARGE-TIME DECAY OF THE UNITARY GROUP $\exp(-itH_0)$

If $\varphi \in L^2(\mathbb{R}^n)$, then $u(t) = \exp(-itH_0)\varphi$ can be expressed as

$$u(t) = \int_{f(0)}^{f(\infty)} e^{-it\lambda} A_{H_0}(\lambda) \varphi d\lambda, \quad (3.1)$$

which must be interpreted in the appropriate weak sense.

We say that φ has “compact energy support” if $\hat{\varphi}(\xi)$ vanishes for ξ in a neighborhood of $\xi = 0$ and ∞ . This is equivalent, by (2.3)-(2.4), to the fact that $A_{H_0}(\lambda)\varphi$ is compactly supported in $(f(0), f(\infty))$.

Clearly, if φ has compact energy support, then for every $t \in \mathbb{R}$ and every $\beta \in \mathbb{R}$, $u(t) \in H_\beta(\mathbb{R}^n)$. In fact, the norm $\|u(t)\|_{H_\beta}$ decays, as $|t| \rightarrow \infty$, in the following sense,

$$\int_{\mathbb{R}} \|(1+|x|^2)^{-\frac{s}{2}} u\|_{H_\beta}^2 dt \leq C \|\varphi\|^2, \quad (3.2)$$

where $s > \frac{1}{2}$ and C depends only on s, n, β and $\text{supp } \hat{\varphi}$. The proof of (3.2) is similar to proofs of analogous statements in [5] and is omitted.

In order to prove a global decay result, avoiding the “compact energy support” hypothesis, we need to strengthen somewhat the requirements imposed on f in Theorem 2B.

THEOREM 3A. – *Let $n \geq 3$ and let f satisfy Assumptions 1.1, 1.2. Assume in addition that for some constant $L > 0$,*

$$f'(\mu) \geq \frac{L}{\sqrt{\mu}} \quad \text{as } \mu \rightarrow +\infty. \quad (3.3)$$

Then there exists a constant $C > 0$, depending only on n, f , such that for every $\varphi \in L^2(\mathbb{R}^n)$, the L^2 -valued function $u(t) = \exp(-itH_0)\varphi$ satisfies

$$\int_{\mathbb{R}} \|(1+|x|^2)^{-\frac{1}{2}} u(t)\|^2 dt \leq C \|\varphi\|^2. \quad (3.4)$$

Proof. – It is seen from (2.6) and (3.3) that

$$\sup \{ \|A_{H_0}(\lambda)\|_{B(L^{2,1}, L^{2,-1})}, \lambda > f(0) \} < \infty. \quad (3.5)$$

By an obvious density argument, it suffices to establish (3.4) for $\varphi \in C_0^\infty(\mathbb{R}^n)$. Let $w(x, t) \in C_0^\infty(\mathbb{R}^{n+1})$ and use (3.1) to write

$$\begin{aligned} \int_{\mathbb{R}} (u(t), w(\cdot, t)) dt &= \int_{\mathbb{R}} \int_{f(0)}^{\infty} e^{-it\lambda} [A_{H_0}(\lambda)\varphi, w(\cdot, t)] d\lambda dt \\ &= \int_{f(0)}^{\infty} \left[A_{H_0}(\lambda)\varphi, \int_{\mathbb{R}} e^{+it\lambda} w(\cdot, t) dt \right] d\lambda. \end{aligned} \quad (3.6)$$

Denoting by $\tilde{w}(\cdot, \lambda) = \int_{\mathbb{R}} e^{+it\lambda} w(\cdot, t) dt$ the Fourier transform of $w(\cdot, t)$ with respect to t , we rewrite (3.6) as,

$$\int_{\mathbb{R}} (u(t), w(\cdot, t)) dt = \int_{f(0)}^{\infty} [A_{H_0}(\lambda)\varphi, \tilde{w}(\cdot, \lambda)] d\lambda. \quad (3.7)$$

Since $A_{H_0}(\lambda) = \frac{d}{d\lambda} E_{H_0}(\lambda)$, the bilinear form $[A_{H_0}(\lambda)\cdot, \cdot]$ is nonnegative and, for any

$$\psi \in L^{2,s}(\mathbb{R}^n), \quad \int_{f(0)}^{\infty} [A_{H_0}(\lambda) \psi, \psi] d\lambda = \|\psi\|^2.$$

Using these considerations and the Cauchy-Schwartz inequality in (3.7) we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}} (u(t), w(\cdot, t)) dt \right| &\leq \left(\int_{f(0)}^{\infty} [A_{H_0}(\lambda) \varphi, \varphi] d\lambda \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_{f(0)}^{\infty} [A_{H_0}(\lambda) \tilde{w}(\cdot, \lambda), \tilde{w}(\cdot, \lambda)] d\lambda \right)^{\frac{1}{2}} \\ &\leq C \|\varphi\| \cdot \left(\int_{f(0)}^{\infty} \|\tilde{w}(\cdot, \lambda)\|_1^2 d\lambda \right)^{\frac{1}{2}}, \end{aligned} \quad (3.8)$$

where we have used (3.5) in the last step. The Plancherel theorem now yields,

$$\left| \int_{\mathbb{R}} (u(t), w(\cdot, t)) dt \right| \leq C \|\varphi\| \left(\int_{\mathbb{R}} \|w(\cdot, t)\|_1^2 dt \right)^{\frac{1}{2}}, \quad (3.9)$$

which can be rewritten as,

$$\begin{aligned} \left| \int_{\mathbb{R}} ((1+|x|^2)^{-\frac{1}{2}} u(t), w(\cdot, t)) dt \right| \\ \leq C \|\varphi\| \left(\int_{\mathbb{R}^{n+1}} |w(x, t)|^2 dx dt \right)^{\frac{1}{2}}, \end{aligned} \quad (3.10)$$

and which is equivalent to (3.4) by duality. \square

COROLLARY 3.1. – *The function $f(\theta) = \sqrt{1+\theta}$ satisfies all the hypotheses of Theorem 3A, so that (3.4) applies in the case of the relativistic (free) Schrödinger operator $H_0 = \sqrt{-\Delta + 1}$.*

REMARK 3.2. – (Asymptotic behavior of $R_0^\pm(\lambda)$ as $\lambda \rightarrow +\infty$). If f satisfies Assumption 1.3 and (3.3) is satisfied, then we deduce from (2.3)-(2.4) that $A_{H_0}(\lambda)$ is uniformly bounded and uniformly Hölder continuous for large λ . This implies, in view of the Privaloff-Korn theorem, that, for every $s > \frac{1}{2}$

$$\limsup_{\lambda \rightarrow +\infty} \|R_0^\pm(\lambda)\|_{B(L^{2,s}, L^{2,-s})} < \infty. \quad (3.11)$$

In particular, this holds in the case of $\sqrt{-\Delta + 1}$. However, as the example in [9] shows, it is not true that $R_0^\pm(\lambda) \rightarrow 0$ as $\lambda \rightarrow +\infty$ in the operator norm.

4. THE OPERATOR $H = H_0 + V(x)$

In order to allow (some) singular behavior of $V(x)$, we need to assume some regularity of $R_0^\pm(\lambda)$, as expressed by Corollary 2.1 and Example 2.2. Also, to exploit the general method of [4], more smoothness (locally) is required for f . Thus, we now assume that f satisfies Assumption 1.4 and, in addition, for some $\gamma > 0$,

$$f(\theta) > C\theta^{\gamma/2} \quad \text{as } \theta \rightarrow +\infty. \quad (4.1)$$

In view of Example 2.2, H_0 satisfies the LAP in $B(L^{2,s}, H_\gamma^{-s})$, for $f(0) < \lambda < \infty$, $s > \frac{1}{2}$.

From the general theory in [4, Section 3] we now obtain:

THEOREM 4A. – *Let $f \in C^2(\mathbb{R}_+)$ satisfy (4.1). Let $V(x)$ be a real potential such that, for some $\varepsilon > 0$, the multiplication operator $(1 + |x|)^{1+\varepsilon} V(x)$ is compact from $H_\gamma(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Then:*

(i) *The continuous spectrum $\sigma_c(H) = [f(0), \infty)$ is absolutely continuous, except possibly for a discrete sequence Λ_i of imbedded eigenvalues, which can accumulate only at $f(0), \infty$.*

(ii) *The resolvent $R(z) = (H - z)^{-1}$, $z \in \mathbb{C}^+$ (resp. $z \in \mathbb{C}^-$) can be extended continuously, with respect to the operator norm topology of $B(L^{2,s}, H_\gamma^{-s})$, $s > \frac{1}{2}$, to $\mathbb{C}^+ \cup (\sigma_c(H) \setminus \Lambda)$ (resp. $\mathbb{C}^- \cup (\sigma_c(H) \setminus \Lambda)$), where $\Lambda = \Lambda_i \cup \{f(0)\}$. Furthermore, the limiting values $R^\pm(\lambda)$, $\lambda \in \sigma_c(H) \setminus \Lambda$, are locally Hölder continuous in the same topology.*

REMARK 4.1. – Following the proof in [4, Section 5] we obtain here that if $\varphi \in L^{2,s}$ satisfies $\varphi = -VR_0^+(\lambda)\varphi$ for some $\lambda > f(0)$, then

$$|[(A_{H_0}(\mu) - A_{H_0}(\lambda))\varphi, \varphi]| \leq C|\mu - \lambda|^{1+\delta}, \quad \delta > 0,$$

for μ in a neighborhood of λ . It is here that the assumption $f \in C^2(\mathbb{R}_+)$ is required.

REMARK 4.2. – For $f(\theta) = \sqrt{1 + \theta}$ all the hypotheses of Theorem 4A are satisfied with $\gamma = 1$. Thus, the operator $K = \sqrt{-\Delta + 1} + V(x)$ satisfies the LAP for any potential $V(x)$ such that

$$|V(x)| \leq C(1 + |x|)^{-1-\varepsilon}. \quad (4.2)$$

We refer to [1] for more general criteria for compactness in $B(H_1, L^2)$.

As in [9], if C in (4.2) is sufficiently small, then there are no imbedded eigenvalues.

In order to extend the LAP for H across the threshold $\lambda = f(0)$, in analogy with Theorem 2B, one needs to ensure that $I + VR_0^\pm(f(0))$ are invertible in $L^{2,s}$ (compare [6]). Assuming that f satisfies the growth condition (4.1) we recall the following definition [6].

DEFINITION 4.3. – *The point $\lambda = f(0)$ is said to be a resonance of H if, for some $s > 1$, there exists $0 \neq \psi \in H_\gamma^{-s}$, such that $H\psi = 0$ in the sense of distributions.*

We can now formulate the following theorem, which supplements Theorem 4A.

THEOREM 4B. – *Let $n \geq 3$, $s > 1$. Let $f \in C^2(\mathbb{R}_+)$ and assume that f satisfies (4.1) and that $f''(\theta)$ is uniformly bounded in a neighborhood of $\theta = 0$. Assume further that $V(x)$ satisfies, for some C , $\delta > 0$,*

$$|V(x)| \leq C(1 + |x|^2)^{-1-\delta}, \quad (4.3)$$

and that $H = H_0 + V$ has no resonance at $f(0)$.

Then there exists $\eta > 0$ such that the limits

$$R^\pm(\lambda) = \lim_{\varepsilon \rightarrow 0^+} R(\lambda \pm i\varepsilon), \quad \lambda \in (f(0) - \eta, f(0) + \eta), \quad (4.4)$$

exist and are Hölder continuous in the operator norm topology of $B(L^{2,s}, H_\gamma^{-s})$. In particular, H has no eigenvalues in $(f(0) - \eta, f(0) + \eta)$.

Proof. – It follows from Theorem 2B that $R_0(z)$ can be extended continuously to $\overline{\mathbb{C}^+}$ (resp. $\overline{\mathbb{C}^-}$). By our non-resonance assumption the operators $I + VR_0^+(\lambda)$ (resp. $I + VR_0^-(\lambda)$) are invertible in $L^{2,s}$ for λ near $f(0)$ (note that (4.3) implies the compactness of V). The proof may now be concluded as the proof of Theorem 2 in [6]. \square

REMARK 4.4. – Note that by combining the assumptions of Theorem 4B and Remark 3.2 (i.e., (3.3) and Assumptions 1.1-1.3, or, more simply, (3.3) and the uniform boundedness of $f''(\theta)$, $\theta \in \mathbb{R}_+$), it follows that if $\Lambda_i = \phi$,

$$\sup \{ \|R^\pm(\lambda)\|_{B(L^{2,s}, L^{2,-s})}, \lambda \in \sigma_c(H) \} < \infty. \quad (4.5)$$

Since the weak derivative $A_H(\lambda) = \frac{d}{d\lambda} E_H(\lambda)$ of the spectral family associated with H , $E_H(\lambda)$, satisfies the equality,

$$A_H(\lambda) = \frac{1}{2\pi i} (R^+(\lambda) - R^-(\lambda)), \quad (4.6)$$

it follows that $\{A_H(\lambda), \lambda \in \sigma_c(H)\}$ is also uniformly bounded in $B(L^{2,s}, L^{2,-s})$. We now proceed as in the proof of Theorem 3A to obtain a large-time decay result for solutions of $iu_t = Hu$, where $u(t=0)$ belongs to the absolutely continuous subspace with respect to H .

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