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2-Magnon scattering in the Heisenberg model

by

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ABSTRACT. – We prove asymptotic completeness for 2-magnon scattering in the Heisenberg model. The proof is based on a Mourre-estimate. The results equally apply to the scattering of two interacting particles on a lattice.

RÉSUMÉ. – On démontre la complétude asymptotique pour la diffusion à deux magnons dans le modèle de Heisenberg. La démonstration est basée sur une inégalité de Mourre. Les résultats sont également applicables à la diffusion de deux particules interagissant sur un réseau.

1. INTRODUCTION

The spin-$\frac{1}{2}$ Heisenberg model is formally given by the Hamiltonian

$$H = -\frac{1}{2} \sum_{\substack{\mathbf{x},\mathbf{y} \in \mathbb{Z}^\nu \\ |\mathbf{x}-\mathbf{y}|=1}} \boldsymbol{\sigma}^{(\mathbf{x})} \cdot \boldsymbol{\sigma}^{(\mathbf{y})}. \tag{1}$$

It describes a system of quantum-mechanical spins, one at each lattice site $\mathbf{x} \in \mathbb{Z}^\nu$, where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. In one of its ground states all spins point down, *i.e.*,

$$\phi_0 = \bigotimes_{\mathbf{x} \in \mathbb{Z}^\nu} \phi^\downarrow(\mathbf{x}), \quad \sigma_-^{(\mathbf{x})} \phi^\downarrow(\mathbf{x}) = 0 \ \ (\mathbf{x} \in \mathbb{Z}^\nu),$$

with $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)$. The Hilbert space $\mathcal{H}$ spanned by the states with all but finitely spins pointing down is the incomplete tensor product [14], [15]

$$\mathcal{H} = \bigotimes_{\mathbf{x} \in \mathbb{Z}^\nu}^{\phi_0} \mathbb{C}^2_{\mathbf{x}}$$

with respect to the ground state vector ϕ_0. There, the Hamiltonian is

$$H = -\frac{1}{2} \sum_{\substack{\mathbf{x},\mathbf{y}\in\mathbb{Z}^\nu \\ |\mathbf{x}-\mathbf{y}|=1}} \left(\boldsymbol{\sigma}^{(\mathbf{x})} \cdot \boldsymbol{\sigma}^{(\mathbf{y})} - 1\right)$$

$$= -\frac{1}{2} \sum_{\substack{\mathbf{x},\mathbf{y}\in\mathbb{Z}^\nu \\ |\mathbf{x}-\mathbf{y}|=1}} \left(\sigma_3^{(\mathbf{x})}\sigma_3^{(\mathbf{y})} - 1 + 4\sigma_+^{(\mathbf{x})}\sigma_-^{(\mathbf{y})}\right) \tag{2}$$

and differs from (1) by the subtraction of an infinite constant. Since H commutes with the *magnon* number $\mathbf{N} = 1/2 \sum_{\mathbf{x}\in\mathbb{Z}^\nu}(\sigma_3^{(\mathbf{x})} + 1)$, it leaves the n-magnon subspace $\mathcal{H}_n$ of $\mathcal{H} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ invariant. The restriction H_n of H to $\mathcal{H}_n$ is a bounded operator.

Pairs of magnons may exhibit bound states, as shown by [1] ($\nu = 1$) resp. by Hanus [6] and Wortis [17] ($\nu \leq 3$). The existence of scattering states, *i.e.*, of states whose asymptotic incoming and outgoing configurations are characterized by noninteracting wave-packets of magnons, has been proven by Watts [16] for $n = 2$ and by Hepp [7] for arbitrary n. The scattering states are described in terms of states in a Hilbert space which differs from the physical one. The so-called ideal spin waves were introduced in this context by Dyson [5] (see however [2]). Asymptotic completeness for $n = 2$ and arbitrary ν has been established - using a time-independent method - by Perez [11]. Here we give an alternate proof which depends on a Mourre estimate. More general scattering problems, including two-body scattering of interacting particles on a lattice, will be dealt with similarly.

The scattering of magnons emerges from the comparison of the dynamics of H_n on $\mathcal{H}_n$ with the one of $H_1^{(n)} = H_1 \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \cdots + \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes H_1$ on the n-particle space $\otimes^n \mathcal{H}_1 = \otimes^n \ell^2(\mathbb{Z}^\nu) =: \mathcal{F}_n$. Here and henceforth, $\mathcal{H}_1$ is identified with $\ell^2(\mathbb{Z}^\nu)$ through

$$\ell^2(\mathbb{Z}^\nu) \ni f \mapsto \sum_{\mathbf{x}\in\mathbb{Z}^\nu} f(\mathbf{x})\sigma_+^{(\mathbf{x})}\phi_0 \in \mathcal{H}_1.$$

The space $\mathcal{H}_n$ is identified with the subspace $\{f \in \mathcal{F}_n \,|\, f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ totally symmetric in $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and $= 0$ if $\mathbf{x}_i = \mathbf{x}_j$ for some $i \neq j\}$ by means of the isometric embedding $I : \mathcal{H}_n \to \mathcal{F}_n$ given by

$$I^* : f \mapsto \frac{1}{\sqrt{n!}} \sum_{\mathbf{x}_1,\ldots,\mathbf{x}_n\in\mathbb{Z}^\nu} f(\mathbf{x}_1, \ldots, \mathbf{x}_n)\sigma_+^{(\mathbf{x}_1)} \cdots \sigma_+^{(\mathbf{x}_n)}\phi_0.$$

Then, as mentioned above, it is a result due to Hepp and Watts that the *wave operators*

$$\Omega_\pm := \mathrm{s} - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}H_n t} I^* \mathrm{e}^{-\mathrm{i}H_1^{(n)}t}$$

exist, proving the existence of scattering states.

Let us now focus on the case $n = 2$. Roughly speaking, asymptotic completeness means that the large time behaviour of two magnons is either that of two free magnons or that of a bound pair. We will decompose the Hilbert space as a *direct integral*,

$$\mathcal{H}_2 \cong \int_{[0,2\pi)^\nu}^{\oplus} \mathcal{H}_K \, \mathrm{d}K, \tag{3}$$

where K is the total quasi-momentum. As this is a conserved quantity, the Hamiltonian H_2 has a *direct integral decomposition*

$$H_2 \cong \int_{[0,2\pi)^\nu}^{\oplus} H_2(K) \mathrm{d}K.$$

A similar situation occurs in the case of two non-relativistic particles, where the total momentum P plays the role of K. However, in contrast to the 2-magnon case, the P-dependence of $H_2(P) = P^2/2 + H_{\mathrm{rel}}$ is trivial, since it only amounts to an additive fiber-dependent constant. Now let $E_{\mathrm{pp/cont}} \cong \int_{[0,2\pi)^\nu}^{\oplus} E_{\mathrm{pp/cont}}(H_2(K)) \, \mathrm{d}K$ w.r.t. the isomorphism (3).

THEOREM 1.1. – *The limits*

$$\Omega_H = \mathrm{s} - \lim_{t \to \infty} \mathrm{e}^{\mathrm{i}H_2 t} I^* \mathrm{e}^{-\mathrm{i}H_1^{(2)} t}, \tag{4}$$

$$\Omega_H^* = \mathrm{s} - \lim_{t \to \infty} \mathrm{e}^{\mathrm{i}H_1^{(2)} t} I \mathrm{e}^{-\mathrm{i}H_2 t} E_{\mathrm{cont}} \tag{5}$$

exist and are mutually adjoint.

Asymptotic completeness is the statement (5). It implies that for any $\psi \in \mathcal{H}_2$ there exist $\phi \in \mathcal{F}_2$ such that

$$\|\mathrm{e}^{-\mathrm{i}H_2 t}\psi - (I^* \mathrm{e}^{-\mathrm{i}H_1^{(2)} t}\phi + \mathrm{e}^{-\mathrm{i}H_2 t} E_{\mathrm{pp}}\psi)\| \to 0$$

as $t \to \infty$, reflecting the picture given before.

In what follows, we will prove a slightly more general result than Theorem 1.1. Two interacting particles on a lattice can be viewed as a single particle in configuration space moving in a potential. Coinciding particles correspond to a sub-lattice, and their interaction to a potential invariant under translations along that sub-lattice. We are thus led to consider the scattering of a particle off a (possibly nonlocal) potential which is invariant under translations along a sub-lattice. More precisely, let

$$\mathcal{L} = \mathbb{Z}^\nu \times \mathbb{Z}^\nu \ni (\mathbf{x}_1, \mathbf{x}_2) = x$$

be the configuration space of two particles, and $\mathcal{D} = \{x \in \mathcal{L} \,|\, \mathbf{x}_1 = \mathbf{x}_2\}$. We then consider bounded operators H_0 and V on $\ell^2(\mathcal{L})$, where

$$(H_0\psi)(x) = \sum_{|y-x|=1} [\psi(x) - \psi(y)], \tag{6}$$

and

$$(V\psi)(x) = \sum_{y\in\mathcal{L}} V(x,y)\psi(y)$$

satisfies the following properties:

(i) V is selfadjoint, *i.e.*, $V(x,y) = \overline{V(y,x)}$.

(ii) V is invariant under translations along $\mathcal{D}$, *i.e.*, $V(x+d, y+d) = V(x,y)$ for $d \in \mathcal{D}$.

(iii) V is of finite range across $\mathcal{D}$, *i.e.*, there are at most finitely many equivalence classes $[x] \in \mathcal{L}/\mathcal{D}$ such that $V(x,y) \neq 0$ for some $x \in [x], y \in \mathcal{L}$.

We will actually discuss the scattering for the pair $(H_0, H = H_0 + V)$ when $\mathcal{L} = \mathbb{Z}^N$ and $\mathcal{D} \subset \mathcal{L}$ is an arbitrary sub-lattice, and see that it covers the situation of Theorem 1.1.

2. PROOFS

A character of $\mathcal{L}$ is a group homomorphism $\chi : \mathcal{L} \to \mathrm{S}^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$. Let

$$\widehat{\mathcal{L}} = \{\chi \,|\, \chi \text{ is a character of } \mathcal{L}\}$$

be the dual group of $\mathcal{L}$, *i.e.*, the Brillouin zone, equipped with its Haar measure $\mathrm{d}\mu(\chi)$. Similarly, let

$$\widehat{\mathcal{D}} = \{\phi \,|\, \phi \text{ is a character of } \mathcal{D}\},$$

with Haar measure $\mathrm{d}\nu(\phi)$. Define the group homomorphism $\pi : \widehat{\mathcal{L}} \to \widehat{\mathcal{D}}$ by

$$\chi \mapsto \pi(\chi) = \chi \restriction \mathcal{D}.$$

Then $(\widehat{\mathcal{L}}, \widehat{\mathcal{D}}, \pi)$ is a principal fiber bundle with structure group $\ker \pi$. Each fiber

$$F_\phi = \pi^{-1}(\phi) = \{\chi \in \widehat{\mathcal{L}} \,|\, \chi \restriction \mathcal{D} = \phi\}$$

is a coset of $\ker \pi$ in $\widehat{\mathcal{L}}$ and carries the measure $\mu_\phi(\cdot) = \mu_0(\chi^{-1}\cdot)$, where μ_0 is the Haar measure on $F_{\mathrm{id}} = \ker \pi$, and μ_ϕ is independent of $\chi \in F_\phi$.

Corresponding to this fibration of $\widehat{\mathcal{L}}$, $\mathrm{L}^2(\widehat{\mathcal{L}}, \mathrm{d}\mu)$ becomes a Hilbert space of sections [4],

$$\mathrm{L}^2(\widehat{\mathcal{L}}, \mathrm{d}\mu) \cong \int_{\widehat{\mathcal{D}}}^{\oplus} \mathrm{d}\nu(\phi)\, \mathrm{L}^2(F_\phi, \mathrm{d}\mu_\phi), \tag{7}$$

the isomorphism being $\hat{\psi} \mapsto \{\hat{\psi} \restriction F_\phi\}_{\phi \in \widehat{\mathcal{D}}}$. The Fourier transform

$$U : \ \ell^2(\mathcal{L}) \to \mathrm{L}^2(\widehat{\mathcal{L}}), \quad \psi \mapsto U\psi =: \hat{\psi}$$

defined by

$$\hat{\psi}(\chi) = \sum_{x \in \mathcal{L}} \overline{\chi(x)} \psi(x),$$

and the translation $T_a : \ell^2(\mathcal{L}) \to \ell^2(\mathcal{L}), a \in \mathcal{L}$, given by

$$(T_a \psi)(x) = \psi(x + a),$$

are unitary operators. Both H_0 and V commute with T_d for $d \in \mathcal{D}$. As a result, UH_0U^{-1} and UVU^{-1} are decomposable w.r.t. (7), *i.e.*,

$$UH_0U^{-1} \cong \int_{\widehat{\mathcal{D}}}^{\oplus} H_0(\phi)\, \mathrm{d}\nu(\phi), \quad UVU^{-1} \cong \int_{\widehat{\mathcal{D}}}^{\oplus} V(\phi)\, \mathrm{d}\nu(\phi).$$

Indeed, this follows from [12, Thm XIII.84], since

$$(UT_dU^{-1}\hat{\psi})(\chi) = \chi(d)\hat{\psi}(\chi) = \phi(d)\hat{\psi}(\chi)$$

for $\chi \in F_\phi$ and the span of the functions $\hat{d} : \phi \mapsto \phi(d)$, or rather of the multiplication operators associated to them, is strongly dense in the algebra of decomposable operators whose fibers are multiples of the identity.

We then introduce E^0_{cont} and E_{cont} as

$$UE^0_{\mathrm{cont}}U^{-1} = \int_{\widehat{\mathcal{D}}}^{\oplus} E_{\mathrm{cont}}(H_0(\phi))\, \mathrm{d}\nu(\phi),$$

$$UE_{\mathrm{cont}}U^{-1} = \int_{\widehat{\mathcal{D}}}^{\oplus} E_{\mathrm{cont}}(H(\phi))\, \mathrm{d}\nu(\phi).$$

THEOREM 2.1 (Asymptotic completeness). – *The limits*

$$\Omega = \mathrm{s} - \lim_{t \to \infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} E^0_{\mathrm{cont}}, \tag{8}$$

$$\Omega^* = \mathrm{s} - \lim_{t \to \infty} \mathrm{e}^{\mathrm{i}tH_0} \mathrm{e}^{-\mathrm{i}tH} E_{\mathrm{cont}} \tag{9}$$

exist and are mutually adjoint.

We remark that it is not necessary to assume $\mathcal{L}/\mathcal{D}$ infinite, in which case $E^0_{\mathrm{cont}} = \mathbb{1}$ (see Lemma 2.4). However, if it is finite, then $E^0_{\mathrm{cont}} = E_{\mathrm{cont}} = 0$ and the theorem is trivial.

Proof of Theorem 1.1. – As in the first section, let

$$\mathcal{L} = \mathbb{Z}^\nu \times \mathbb{Z}^\nu \ni (\mathbf{x}_1, \mathbf{x}_2) = x$$

and

$$\mathcal{D} = \{x \in \mathcal{L} \,|\, \mathbf{x}_1 = \mathbf{x}_2\}.$$

Then

$$\mathcal{F}_2 = \ell^2(\mathbb{Z}^\nu) \otimes \ell^2(\mathbb{Z}^\nu) = \ell^2(\mathcal{L}).$$

Let H_0 and H be the bounded operators on $\ell^2(\mathcal{L})$ given by (6), resp. by

$$(H\psi)(x) = \begin{cases} \sum\limits_{\substack{|y-x|=1 \\ y\notin\mathcal{D}}} [\psi(x) - \psi(y)] & x \notin \mathcal{D} \\ 0 & x \in \mathcal{D}\,. \end{cases}$$

Clearly, $V = H - H_0$ satisfies properties (i-iii). Moreover, $2H_0 = H_1^{(2)}, 2I^*H = H_2 I^*$ and thus $I^* \mathrm{e}^{2\mathrm{i}Ht} = \mathrm{e}^{\mathrm{i}H_2 t} I^*$. Given Theorem 2.1, this and its adjoint imply Theorem 1.1 with $\Omega_H = I^*\Omega$. □

LEMMA 2.2. – (*a*) *$H_0(\phi)$ is multiplication with $E\lceil F_\phi$, where*

$$E(\chi) = 2\sum_{i=1}^{N} \big(1 - \mathrm{Re}\,\chi(e_i)\big). \tag{10}$$

(*b*) *$V(\phi) : \mathrm{L}^2(F_\phi) \to \mathrm{L}^2(F_\phi)$ is of finite rank for ν-a.e. ϕ.*

Proof. – (a) Follows immediately from $H_0 = \sum_{i=1}^{N} (2 - T_{e_i} - T_{-e_i})$.
(b) We will factorize U into two partial Fourier transforms, *i.e.*,

$$U = \Big(\int_{\widehat{\mathcal{D}}}^{\oplus} \mathrm{d}\nu(\phi) U_2(\phi)\Big) U_1, \tag{11}$$

where

$$U_1 : \ell^2(\mathcal{L}) \to \int_{\widehat{\mathcal{D}}}^{\oplus} \mathrm{d}\nu(\phi)\, \ell^2(\mathcal{L}/\mathcal{D}),$$
$$U_2(\phi) : \ell^2(\mathcal{L}/\mathcal{D}) \to \mathrm{L}^2(F_\phi).$$

The factorization requires the choice of a "gauge", *i.e.*, of an arbitrary measurable section $\phi \mapsto \chi_\phi \in F_\phi$. Then, (11) holds upon defining U_1 and U_2 as

$$(U_1\psi)(\phi, [x]) = \sum_{x\in[x]} \overline{\chi_\phi(x)}\psi(x), \tag{12}$$

$$(U_2(\phi)\psi)(\chi) = \sum_{[x]\in\mathcal{L}/\mathcal{D}} \overline{\chi(x)}\chi_\phi(x)\psi([x]) \qquad (\chi \in F_\phi). \tag{13}$$

We remark that (13) is independent of the choice $x \in [x]$, since $\bar{\chi}\chi_\phi \in \ker\pi$. We now set

$$V_\phi([x],[y]) = \sum_{x\in[x]} \overline{\chi_\phi(x)}V(x,y)\chi_\phi(y), \tag{14}$$

which is independent of the choice of $y \in [y]$ due to (ii), and is finite $\phi - \nu$-a.e. because V is bounded. Then

$$\begin{aligned}(U_1V\psi)(\phi,[x]) &= \sum_{x\in[x],y\in\mathcal{L}} \overline{\chi_\phi(x)}V(x,y)\psi(y)\\ &= \sum_{y\in\mathcal{L}} V_\phi([x],[y])\overline{\chi_\phi(y)}\psi(y)\\ &= \sum_{[y]\in\mathcal{L}/\mathcal{D}} V_\phi([x],[y])(U_1\psi)(\phi,[y]),\end{aligned}$$

i.e., $U_1VU_1^{-1}|_\phi = U_2(\phi)^{-1}V(\phi)U_2(\phi)$ has kernel (14) and is thus of finite rank by (iii). □

Part (b) already implies ([11], [12, Thm XI.8 or Sect. XI.14]) a weaker form of asymptotic completeness in which $E^{(0)}_{\text{cont}}$ is replaced by $E^{(0)}_{\text{ac}}$ in (8, 9).

We will derive a Mourre estimate. As a preliminary, we identify $\langle\mathcal{L}\rangle = \mathbb{R}^N$, the $\mathbb{R}$-linear span of $\mathcal{L}$, with $\mathrm{T}^*_\chi\widehat{\mathcal{L}}$, the cotangent space of $\widehat{\mathcal{L}}$ at $\chi \in \widehat{\mathcal{L}}$, as follows. For $x \in \mathcal{L}$, let $\hat{x} : \widehat{\mathcal{L}} \to \mathrm{S}^1, \chi \mapsto \hat{x}(\chi) = \chi(x)$. The map

$$\omega : \mathcal{L} \to \mathrm{T}^*_\chi\widehat{\mathcal{L}}, \quad x \mapsto \omega(x) = -\mathrm{i}\frac{\mathrm{d}\hat{x}}{\hat{x}}\bigg|_\chi \tag{15}$$

is well-defined. Furthermore, it is $\mathbb{Z}$-linear and thus extends to a linear map from $\langle\mathcal{L}\rangle$ to $\mathrm{T}^*_\chi\widehat{\mathcal{L}}$. Indeed, $\widehat{x+y} = \hat{x}\cdot\hat{y}$, so that $\mathrm{d}(\widehat{x+y}) = \hat{y}\,\mathrm{d}\hat{x} + \hat{x}\,\mathrm{d}\hat{y}$ and hence $\omega(x+y) = \omega(x) + \omega(y)$. The so extended ω has trivial kernel since the differentials $\mathrm{d}\widehat{e_i}$ are linearly independent. It thus is an isomorphism from $\langle\mathcal{L}\rangle$ onto $\mathrm{T}^*_\chi\widehat{\mathcal{L}}$.

Let $Q(x) = s(x,x) \geq 0$, where $s(x,y)$ a symmetric $\mathbb{Z}$-bilinear form on $\mathcal{L}$. We set

$$A = \mathrm{i}\left[H_0, \frac{1}{2}Q(x)\right].$$

$Q(x)$ uniquely extends to a quadratic form on $\langle\mathcal{L}\rangle$. We can take Q so that its null space is $\langle\mathcal{D}\rangle$. Then, UQU^{-1} and UAU^{-1} are decomposable.

LEMMA 2.3. – *Let* $Q_* = Q \circ \omega^{-1}$. *Then*

$$U\,\mathrm{i}[H_0, A]U^{-1} = Q_*(\mathrm{d}E) \geq 0,$$

and for $\chi \in F_\phi$, $\bar{x} \in \mathrm{T}^*_\chi\widehat{\mathcal{L}}$,

$$Q_*(\bar{x}) = 0 \iff \bar{x}\restriction \mathrm{T}_\chi F_\phi = 0. \tag{16}$$

We remark that $\mathrm{d}E(\chi)$ is the group velocity of waves with "quasi-momentum" χ.

Proof. – For an arbitrary function $g(x)$ on $\mathcal{L}$ we have $[T_e, g] = (D_e\, g)T_e$, where $(D_e\, g)(x) = g(x+e) - g(x)$. Since $(D_f D_e\, Q) = 2s(e,f)$ we get

$$\mathrm{i}[H_0, A] = \sum_{i,j=1}^{N} \mathrm{i}^2(T_{e_i} - T_{-e_i})s(e_i, e_j)(T_{e_j} - T_{-e_j}).$$

Using $\mathrm{i}U(T_e - T_{-e})U^{-1} = -2\,\mathrm{Im}\,\widehat{e}$ and $\mathrm{d}E = -2\sum_{j=1}^N \mathrm{Re}\,\mathrm{d}\widehat{e_j} = \omega\big(2\sum_{j=1}^N(\mathrm{Im}\,\widehat{e_j})e_j\big)$ we obtain

$$U\,\mathrm{i}[H_0, A]\,U^{-1} = Q\bigg(-2\sum_{j=1}^{N}(\mathrm{Im}\,\widehat{e_j})e_j\bigg) = Q_*(\mathrm{d}E).$$

$Q_*(\bar{x}) = 0$ is equivalent to $\bar{x} \in \omega(\langle\mathcal{D}\rangle)$, so that (16) is a consequence of

$$\omega(\langle\mathcal{D}\rangle) = \{\bar{x} \in \mathrm{T}^*_\chi\widehat{\mathcal{L}} \,|\, \bar{x}\restriction\mathrm{T}_\chi F_\phi = 0\}.$$

Here, the inclusion $\subset$ follows from $\hat{d}(\chi) = \phi(d)$ for all $\chi \in F_\phi$, implying $\omega(d)\restriction\mathrm{T}_\chi F_\phi = -\mathrm{i}\hat{d}^{-1}\,\mathrm{d}\hat{d}\restriction\mathrm{T}_\chi F_\phi = 0$. Equality then follows by equality of dimensions. □

For each $\phi \in \widehat{\mathcal{D}}$, let

$$\begin{aligned}\mathcal{T}_\phi &= \{E(\chi) \,|\, \chi \in F_\phi,\, Q_*(\mathrm{d}E)(\chi) = 0\}\\ \mathcal{E}_\phi &= \{\text{Eigenvalues of } H(\phi)\},\\ \mathcal{E}_\phi^0 &= \{\text{Eigenvalues of } H_0(\phi)\}.\end{aligned}$$

$\mathcal{T}_\phi$ is the set of "thresholds".

LEMMA 2.4. – *(a) $\mathcal{T}_\phi$ is closed and countable.*

(b) $\mathcal{E}_\phi^0 \subset \mathcal{T}_\phi$ for all ϕ. Moreover, $E^0_{\mathrm{cont}} = \begin{cases} \mathbb{1} & \mathcal{L}/\mathcal{D} \text{ is infinite} \\ 0 & \mathcal{L}/\mathcal{D} \text{ is finite.}\end{cases}$

Proof. – (a) $\mathcal{T}_\phi$ is clearly closed. By (16) it consists of the critical values of $E\lceil F_\phi$. It is thus countable (actually: finite) by Sard's Theorem for analytic functions [10].

(b) The set

$$Z = \{\chi \in \widehat{\mathcal{L}} \,|\, Q_*(\mathrm{d}E(\chi)) = 0\} = \{\chi \in \widehat{\mathcal{L}} \,|\, \omega^{-1}(\mathrm{d}E(\chi)) \in \langle\mathcal{D}\rangle\}$$

is a level set of a real-analytic function. Thus $\mu(Z) = 0$ or $Z = \widehat{\mathcal{L}}$. The two possibilities correspond to $\langle\mathcal{D}\rangle \neq \langle\mathcal{L}\rangle$, resp. to $\langle\mathcal{D}\rangle = \langle\mathcal{L}\rangle$ (or, equivalently, $\mathcal{L}/\mathcal{D}$ infinite, resp. finite). Indeed, the map $\widehat{\mathcal{L}} \to \langle\mathcal{L}\rangle$, $\chi \mapsto \omega^{-1}(\mathrm{d}E(\chi))$ has full rank μ-a.e. Using (16) we have $Z_\phi = \{\chi \in F_\phi \,|\, \mathrm{d}E(\chi)\lceil \mathrm{T}_\chi F_\phi = 0\}$. The alternative above carries over to almost all fibers simultaneously: Either $\mu_\phi(Z_\phi) = 0$ (ν-a.e.) or $Z_\phi = F_\phi$ ($\phi \in \widehat{\mathcal{D}}$). On the other hand, for each $\phi \in \widehat{\mathcal{D}}$ we have either $\mu_\phi(\{\chi \in F_\phi \,|\, E(\chi) = \lambda\}) = 0$ for all $\lambda \in \mathbb{R}$, or E is constant on some connected component of F_ϕ, the value being in $\mathcal{T}_\phi$. Clearly, for ν-almost all ϕ, the first pair of alternatives coincide with the latter, thus showing

$$E^0_{\mathrm{pp}}(\phi) = \begin{cases} 0 & \mathcal{L}/\mathcal{D} \text{ is infinite} \\ \mathbb{1} & \mathcal{L}/\mathcal{D} \text{ is finite}\end{cases} \qquad (\nu\text{ -a.e.}).$$

□

Due to Lemma 2.3 we have the following *Mourre estimate* for H_0:

PROPOSITION 2.5. – *Let $\Delta \subset \mathbb{R}$ be open, $\overline{\Delta} \cap \mathcal{T}_\phi = \emptyset$. Then*

$$E_\Delta(H_0(\phi))\,\mathrm{i}[H_0(\phi), A(\phi)]\,E_\Delta(H_0(\phi)) \geq cE_\Delta(H_0(\phi))$$

for some $c > 0$.

Proof. – This follows immediately from $Q_*(\mathrm{d}E) \geq c > 0$ on every compact set $U \subset \{\chi \in F_\phi \,|\, Q_*(\mathrm{d}E(\chi)) \neq 0\}$ and from the continuity of $E(\chi)$. □

There is also a Mourre estimate for H:

THEOREM 2.6. – (*a*) *Let* $\Delta \subset \mathbb{R}$ *be open,* $\overline{\Delta} \cap \mathcal{T}_\phi = \emptyset$. *Then*

$$E_\Delta(H(\phi))\,\mathrm{i}[H(\phi), A(\phi)]\,E_\Delta(H(\phi)) \geq cE_\Delta(H(\phi)) + C(\phi) \tag{17}$$

for some $c > 0$ *and* $C(\phi)$ *compact.*

(*b*) *Non-threshold eigenvalues of* $H(\phi)$ *have finite multiplicity and can only accumulate at* $\mathcal{T}_\phi$.

Proof. – (a) Due to property (iii) of V, AV and VA are bounded. Furthermore $A(\phi)V(\phi)$ and $V(\phi)A(\phi)$ have finite rank since $V(\phi)$ has. It follows that

$$\mathrm{i}[H(\phi), A(\phi)] = \mathrm{i}[H_0(\phi), A(\phi)] + \text{compact}.$$

Let $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$. Then $(H(\phi)-z)^{-1}-(H_0(\phi)-z)^{-1} = -(H(\phi)-z)^{-1}V(\phi)(H_0(\phi) - z)^{-1}$ is compact and so is $f(H(\phi)) - f(H_0(\phi))$ for all $f \in \mathrm{C}_0(\mathbb{R})$. Choosing f such that $f = 1$ on Δ and $\operatorname{supp} f \cap \mathcal{T}_\phi = \emptyset$, we get

$$\bar{f}(H(\phi))\,\mathrm{i}[H(\phi), A(\phi)]\,f(H(\phi)) \geq c\,\bar{f}f(H(\phi)) + \text{compact}$$

for some $c > 0$ by Proposition 2.5. Multiplying from both sides with $E_\Delta(H(\phi))$ proves (a).

(b) Assume $H(\phi)\psi_n = \lambda_n\psi_n$ with $\|\psi_n\| = 1$ and $\lambda_n \to \lambda \notin \mathcal{T}_\phi$. Choose $\Delta \ni \lambda$ open such that $\overline{\Delta} \cap \mathcal{T}_\phi = \emptyset$. From $(\psi_n, \mathrm{i}[H(\phi), A(\phi)]\psi_n) = 0$ and (a) we conclude that $0 \geq c + (\psi_n, C(\phi)\psi_n)$ for all n. But this is impossible since $\psi_n \xrightarrow{\text{w}} 0$ and $C(\phi)$ is compact, hence $(\psi_n, C(\phi)\psi_n) \to 0$ as $n \to \infty$. □

COROLLARY 2.7. – *Let* $\lambda \notin (\mathcal{E}_\phi \cup \mathcal{T}_\phi)$. *Then there is an open* $\Delta \ni \lambda$ *such that*

$$E_\Delta(H(\phi))\,\mathrm{i}[H(\phi), A(\phi)]\,E_\Delta(H(\phi)) \geq cE_\Delta(H(\phi)) \tag{18}$$

for some $c > 0$.

Proof. – In (17), $C(\phi)$ can be replaced by $E_\Delta(H(\phi))C(\phi)E_\Delta(H(\phi))$. Since $\lambda \notin \mathcal{E}_\phi$ we have $E_\Delta(H(\phi)) \xrightarrow{\text{s}} E_{\{\lambda\}}(H(\phi)) = 0$ and so $E_\Delta(H(\phi))C(\phi)E_\Delta(H(\phi)) \to 0$ in norm as $\Delta \to \{\lambda\}$. Hence we can omit $C(\phi)$ in (17) at expense of making c and Δ smaller. □

PROPOSITION 2.8. – *Let* $\Delta \subset \mathbb{R}$ *be such that* $\overline{\Delta} \cap (\mathcal{E}_\phi \cup \mathcal{T}_\phi) = \emptyset$. *Then for any* $\alpha > 1$

$$\int_1^\infty \mathrm{d}t \, \|\langle A(\phi)\rangle^{-\alpha/2} \mathrm{e}^{-\mathrm{i}H(\phi)t} E_\Delta(H(\phi))\psi\|^2 \leq \mathrm{const.}\, \|\psi\|^2 \qquad (19)$$

$$\int_1^\infty \mathrm{d}t \, \|\langle A(\phi)\rangle^{-\alpha/2} \mathrm{e}^{-\mathrm{i}H_0(\phi)t} E_\Delta(H_0(\phi))\psi\|^2 \leq \mathrm{const.}\, \|\psi\|^2, \qquad (20)$$

for all $\psi \in \mathrm{L}^2(F_\phi)$, *where* $\langle x\rangle \equiv (x^2+1)^{1/2}$.

This propagation estimate is based on the Mourre estimate and will be proven in the appendix by making use of a propagation observable. Note that it follows also from [3, Thm 4.9] and [12, Thm XIII.25 and Corollary], or from [12, Thm 2.9].

PROPOSITION 2.9. – *Let* $\Delta \subset \mathbb{R}$ *be open such that* $\overline{\Delta} \cap (\mathcal{E}_\phi \cup \mathcal{T}_\phi) = \emptyset$. *Then the wave operators*

$$\mathrm{s} - \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}H(\phi)t} \mathrm{e}^{-\mathrm{i}H_0(\phi)t} E_\Delta(H_0(\phi)), \qquad (21)$$

$$\mathrm{s} - \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}H_0(\phi)t} \mathrm{e}^{-\mathrm{i}H(\phi)t} E_\Delta(H(\phi) \qquad (22)$$

exist.

Proof. – We prove the existence of (21). The existence of (22) can be shown analogously. Let us omit writing the ϕ-dependence for convenience. First we claim that the limit (21) equals

$$\mathrm{s} - \lim_{t\to\infty} E_\Delta(H)\mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H_0t}E_\Delta(H_0) =: \Omega, \qquad (23)$$

provided this limit exists. To prove this, let $\psi = E_{\Delta'}(H_0)\psi$ for some compact $\Delta' \subset \Delta$ and let $f \in \mathrm{C}_0(\mathbb{R})$ with $\operatorname{supp} f \subset \Delta$, $f = 1$ on Δ'. Then $f(H) - f(H_0)$ is compact by the proof of Theorem 2.6. Since $\langle A\rangle^{-1}$ has trivial kernel, $\mathrm{Ran}E_\Delta(H_0) \subset \mathcal{H}_{\mathrm{ac}}(H_0)$ by Proposition 2.8 and [12, Thm XIII.23], implying $\mathrm{e}^{-\mathrm{i}H_0t}E_\Delta(H_0) \xrightarrow{\mathrm{w}} 0$ as $t \to \infty$. Hence

$$\begin{aligned}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H_0t}E_\Delta(H_0)\psi &= \lim_{t\to\infty} f(H)\mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H_0t}E_\Delta(H_0)\psi \\ &= \lim_{t\to\infty} E_\Delta(H)\mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H_0t}E_\Delta(H_0)\psi.\end{aligned}$$

The integral in

$$\Omega\psi = E_\Delta(H)E_\Delta(H_0)\psi + \lim_{t\to\infty}\int_0^t \mathrm{d}s E_\Delta(H)\mathrm{e}^{\mathrm{i}Hs}\mathrm{i}(H-H_0)\mathrm{e}^{-\mathrm{i}H_0s}E_\Delta(H_0)\psi$$

converges, because Proposition 2.8 and $\|\langle A\rangle V\langle A\rangle\| < \infty$ yield

$$\begin{aligned}
&\left\| \int_{t_1}^{t_2} \mathrm{d}s E_\Delta(H) \mathrm{e}^{\mathrm{i}Hs} V \mathrm{e}^{-\mathrm{i}H_0 s} E_\Delta(H_0)\psi \right\|^2 \\
&= \sup_{\|\varphi\|=1} \left| \int_{t_1}^{t_2} \mathrm{d}s \big(\varphi, E_\Delta(H) \mathrm{e}^{\mathrm{i}Hs} V \mathrm{e}^{-\mathrm{i}H_0 s} E_\Delta(H_0)\psi\big) \right|^2 \\
&\leq \|\langle A\rangle V\langle A\rangle\| \left(\sup_{\|\varphi\|=1} \int_{t_1}^{t_2} \mathrm{d}s \left\| \langle A\rangle^{-1} \mathrm{e}^{-\mathrm{i}Hs} E_\Delta(H)\varphi \right\|^2 \right) \\
&\quad \times \int_{t_1}^{t_2} \mathrm{d}s \left\| \langle A\rangle^{-1} \mathrm{e}^{-\mathrm{i}H_0 s} E_\Delta(H_0)\psi \right\|^2 \\
&\to 0
\end{aligned}$$

as $t_1, t_2 \to \infty$. This proves the existence of Ω. □

We can now finish the proof of asymptotic completeness:

Proof of Theorem 2.1. – By dominated convergence, it suffices to prove the claim on each fiber. So let $\psi \in E_{\mathrm{cont}}(H(\phi))$ and $\epsilon > 0$ fixed. By Theorem 2.6 b and Lemma 2.4 a there is $\Delta \subset \mathbb{R}$ open with $\overline{\Delta} \cap (\mathcal{E}_\phi \cup \mathcal{T}_\phi) = \emptyset$ such that $\|(1 - E_\Delta(H(\phi)))\psi\| \leq \epsilon$. The existence of (9) then follows from Proposition 2.9. The proof of (8) is identical. Then the mutual adjointness of Ω and Ω^* is immediate. □

A APPENDIX

To prove Proposition 2.8 we calculate commutator expansions using *almost analytic* extensions of functions defined on $\mathbb{R}$ [9], [8]. By this we understand an extension $\tilde{f}$ of f to the complex plane that satisfies the Cauchy-Riemann equation on the real axis: $\partial_{\bar{z}}\tilde{f} = 0$ for $z \in \mathbb{R}$. The extension $\tilde{f}$ can be chosen largely arbitrary, but the following one will do best for our purposes.

LEMMA A.1. – *Let* $f \in \mathrm{C}^{n+2}(\mathbb{R})$ *and* $\chi \in \mathrm{C}_0^\infty(\mathbb{R})$ *with* $\chi = 1$ *on some neighbourhood of* 0. *Assume* $\left\|f^{(k)}\right\|_{k-1} < \infty$ *for all* $k = 0, \ldots, n+2$, *where the norms* $\|\cdot\|_m$ *are defined by*

$$\|f\|_m = \int \mathrm{d}x \langle x\rangle^m |f(x)|, \quad \langle x\rangle \equiv (x^2+1)^{1/2}.$$

Then

$$\tilde{f}(z) = \chi\left(\frac{y}{\langle x\rangle}\right) \sum_{k=0}^{n+1} f^{(k)}(x) \frac{(\mathrm{i}y)^k}{k!}, \quad (z = x + \mathrm{i}y) \tag{A.1}$$

defines an almost analytic extension of f so that for any selfadjoint operator A and all $p = 0, \ldots, n$,

$$\frac{1}{p!} f^{(p)}(A) = \int \mathrm{d}\tilde{f}(z)\,(z-A)^{-p-1}, \quad \mathrm{d}\tilde{f}(z) \equiv -(2\pi)^{-1} \partial_{\bar{z}} \tilde{f}(z)\,\mathrm{d}x\,\mathrm{d}y, \tag{A.2}$$

the integral converging absolutely in norm sense due to the estimates

$$\int \mathrm{d}y\, |\partial_{\bar{z}} \tilde{f}(z)|\, |y|^{-p-1} \leq \text{const.} \sum_{k=0}^{n+2} \langle x \rangle^{k-p-1} |f^{(k)}(x)| \tag{A.3}$$

respectively

$$\int |\mathrm{d}\tilde{f}(z)|\, |\mathrm{Im}\, z|^{-p-1} \leq \text{const.} \sum_{k=0}^{n+2} \left\| f^{(k)} \right\|_{k-p-1}. \tag{A.4}$$

Let now A and H be selfadjoint. Multiple commutators are defined recursively by

$$\mathrm{ad}_A^{(k)}(H) = [\mathrm{ad}_A^{(k-1)}(H), A], \quad \mathrm{ad}_A^{(0)}(H) = H.$$

Then we have

PROPOSITION A.2. – *Let* $g \in C_0^\infty(\mathbb{R})$, $f \in C^{n+2}(\mathbb{R})$ *such that for some* $0 \leq p \leq n$ $\|f^{(k)}\|_{k-p-1} < \infty$ *for all* $k = 0, \ldots, p+2$, *and let* $\tilde{f}$ *be defined by* (A.1). *Suppose A and H are selfadjoint such that* $\mathrm{ad}_A^{(k)}(H)$ *is H-bounded for* $k \leq p$. *Then,* $\mathrm{ad}_A^{(k)}(g(H))$ *is bounded for* $k \leq p$ *and* $[g(H), f(A)]$ *can be expanded as*

$$[g(H), f(A)] = \sum_{k=1}^{p-1} \frac{1}{k!} f^{(k)}(A)\, \mathrm{ad}_A^{(k)}(g(H)) + R_p,$$
$$R_p = \int \mathrm{d}\tilde{f}(z)\,(z-A)^{-p}\, \mathrm{ad}_A^{(p)}(g(H))\,(z-A)^{-1}, \tag{A.5}$$

respectively as

$$[g(H), f(A)] = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k!}\, \mathrm{ad}_A^{(k)}(g(H))\, f^{(k)}(A) + \tilde{R}_p,$$
$$\tilde{R}_p = (-1)^{p-1} \int \mathrm{d}\tilde{f}(z)\,(z-A)^{-1}\, \mathrm{ad}_A^{(p)}(g(H))\,(z-A)^{-p}. \tag{A.6}$$

Note that the conditions on f in Proposition A.2 are weaker than the ones in Lemma A.1.

Proof of Proposition 2.8. – We prove (19). The proof of (20) identical. Let us for convenience omit the variable ϕ. We consider a propagation observable

$$\Phi_\lambda = E_\Delta(H)F_\alpha(A/\lambda)E_\Delta(H),$$

where

$$F_\alpha(x) = \int_{-\infty}^{x} \langle s\rangle^{-\alpha} \mathrm{d}s$$

with $1 < \alpha < 2$. In fact, this will suffice since $\langle x\rangle^{-\beta} \leq \langle x\rangle^{-\alpha}$ for $\alpha \leq \beta$. From $\|F_\alpha\|_\infty \leq 4(\alpha-1)^{-1}$ we conclude that Φ_λ is bounded and together with $\|(\frac{\mathrm{d}}{\mathrm{d}x})^k\langle x\rangle^{-\alpha}\|_{k-n} < \infty$ for $n \geq 0, k \geq 0$, we get

$$\left\|F_\alpha^{(k)}\right\|_{k-n-1} < \infty \qquad (n \geq 1, k \geq 0).$$

Moreover, $\mathrm{ad}_A^{(k)}(H)$ is bounded for all $k \geq 0$. Hence, we can use Proposition A.2 (with $n = 3, p = 2$) and get by taking the half-sum of (A.5) and (A.6)

$$\begin{aligned}
&\mathrm{i}[H, F_\alpha(A)]\\
&= \frac{1}{2}\left(\frac{1}{\langle A\rangle^\alpha}\mathrm{i}[H, A] + \mathrm{i}[H, A]\frac{1}{\langle A\rangle^\alpha}\right)\\
&\quad - \frac{\mathrm{i}}{2}\int \mathrm{d}\tilde{F}_\alpha(z)(z-A)^{-2}\mathrm{ad}_A^{(3)}(H)(z-A)^{-2}\\
&= \frac{1}{\langle A\rangle^{\alpha/2}}\left\{\mathrm{i}[H, A] - \frac{1}{2}\left[\left[\mathrm{i}[H, A], \frac{1}{\langle A\rangle^{\alpha/2}}\right], \langle A\rangle^{\alpha/2}\right]\right.\\
&\quad \left. - \frac{\mathrm{i}}{2}\int \mathrm{d}\tilde{F}_\alpha(z)\langle A\rangle^{\alpha/2}(z-A)^{-2}\mathrm{ad}_A^{(3)}(H)(z-A)^{-2}\langle A\rangle^{\alpha/2}\right\}\frac{1}{\langle A\rangle^{\alpha/2}}\\
&=: \frac{1}{\langle A\rangle^{\alpha/2}}\{\mathrm{i}[H, A] - R\}\frac{1}{\langle A\rangle^{\alpha/2}}, \qquad R = R^*.
\end{aligned} \tag{A.7}$$

LEMMA. – $\|R\| \leq$ const. $\|\mathrm{ad}_A^{(3)}(H)\|$, *the constant being independent of* A. *In particular, the constant is independent of* λ *as* A *is replaced by* A/λ.

Proof. – The functions $f(x) = \langle A\rangle^{-\alpha/2}$ and $g(x) = \langle A\rangle^{\alpha/2}$ satisfy $\|f^{(k)}\|_{k-2} < \infty$, $\|g^{(k)}\|_{k-2} < \infty$ for $k \geq 0$. Using (A.5) twice with $n = p = 1$ we obtain

$$\begin{aligned}
&\left[\left[\mathrm{i}[H, A], \frac{1}{\langle A\rangle^{\alpha/2}}\right], \langle A\rangle^{\alpha/2}\right]\\
&= \mathrm{i}\int \mathrm{d}\tilde{f}(z)\int \mathrm{d}\tilde{g}(\zeta)(\zeta-A)^{-1}(z-A)^{-1}\mathrm{ad}_A^{(3)}(H)(z-A)^{-1}(\zeta-A)^{-1}.
\end{aligned}$$

By (A.4) this is estimated in norm by

$$\|\mathrm{ad}_A^{(3)}(H)\| \int |\mathrm{d}\tilde{f}(z)|\,|\mathrm{Im}\,z|^{-2} \int |\mathrm{d}\tilde{g}(\zeta)|\,|\mathrm{Im}\,\zeta|^{-2}$$

$$\leq \text{const.}\,\|\mathrm{ad}_A^{(3)}(H)\| \sum_{k,l=0}^{3} \left\|f^{(k)}\right\|_{k-2} \left\|g^{(l)}\right\|_{l-2}.$$

For the other contribution we get using (A.3) and (A.4)

$$\left\| \int \mathrm{d}\tilde{F}_\alpha(z) \frac{\langle A\rangle^{\alpha/2}}{z-A}(z-A)^{-1}\mathrm{ad}_A^{(3)}(H)(z-A)^{-1}\frac{\langle A\rangle^{\alpha/2}}{z-A} \right\|$$

$$\leq \int |\mathrm{d}\tilde{F}_\alpha(z)| \left\| \frac{\langle A\rangle}{z-A} \right\|^2 |\mathrm{Im}\,z|^{-2}\, \|\mathrm{ad}_A^{(3)}(H)\|$$

$$\leq \|\mathrm{ad}_A^{(3)}(H)\| \int |\mathrm{d}\tilde{F}_\alpha(z)| \big(\langle \mathrm{Re}\,z\rangle^2 |\mathrm{Im}\,z|^{-2} + 1 \big) |\mathrm{Im}\,z|^{-2}$$

$$\leq \text{const.}\,\|\mathrm{ad}_A^{(3)}(H)\| \sum_{k=0}^{5} \left\| F_\alpha^{(k)} \right\|_{k-2}$$

which is also of the claimed form. □

Now let Δ' be open such that $\overline{\Delta} \subset \Delta'$ and $\overline{\Delta'} \cap (\mathcal{T} \cup \mathcal{E}) = \emptyset$. Denote by c' the Mourre constant of Δ', so that (18) holds on Δ', and let $g \in C_0^\infty(\mathbb{R})$ with $0 \leq g \leq 1$, $\mathrm{supp}\, g \subset \Delta'$, $g = 1$ on Δ. Then

$$\begin{aligned}\left[g(H), \frac{1}{\langle A\rangle^{\alpha/2}} \right] &= -\frac{1}{\langle A\rangle^{\alpha/2}} [g(H), \langle A\rangle^{\alpha/2}] \frac{1}{\langle A\rangle^{\alpha/2}} \\ &=: \frac{1}{\langle A\rangle^{\alpha/2}} R_1 = -R_1^* \frac{1}{\langle A\rangle^{\alpha/2}} \end{aligned} \tag{A.8}$$

with

$$\|R_1\| \leq \text{const.}\, \|\, [H, A]\, \|, \tag{A.9}$$

the constant being again independent of A. In fact, for $f(x) = \langle x\rangle^{\alpha/2}$ we have

$$[g(H), f(A)]$$
$$= \int \mathrm{d}\tilde{g}(z) \int \mathrm{d}\tilde{f}(\zeta)(z-H)^{-1}(\zeta-A)^{-1}[H,A](\zeta-A)^{-1}(z-H)^{-1}$$

with $\|(\zeta - A)^{-1}\| \leq |\mathrm{Im}\,\zeta|$, which implies (A.8) using (A.4).

Let $R(\lambda)$ and $R_1(\lambda)$ be the above remainders we obtain upon replacing A by A/λ. Then $\|R(\lambda)\| \leq \text{const.}\, \lambda^{-3}$ due to the Lemma and $\|R_1(\lambda)\| \leq \text{const.}\, \lambda^{-1}$ by (A.9). By setting $g = g(H)$, $E_\Delta = E_\Delta(H)$ and using (A.7, A.8) we thus obtain

$$\begin{aligned}
\mathrm{i}[H, \lambda\Phi_\lambda] &= E_\Delta g\, \mathrm{i}\big[H, \lambda F_\alpha(A/\lambda)\big]\, g E_\Delta \\
&= E_\Delta g \frac{1}{\langle A/\lambda\rangle^{\alpha/2}} \{\mathrm{i}[H, A] - \lambda R(\lambda)\} \frac{1}{\langle A/\lambda\rangle^{\alpha/2}} g E_\Delta \\
&= E_\Delta \frac{1}{\langle A/\lambda\rangle^{\alpha/2}} g\, \mathrm{i}[H, A]\, g \frac{1}{\langle A/\lambda\rangle^{\alpha/2}} E_\Delta \\
&\quad + E_\Delta \frac{1}{\langle A/\lambda\rangle^{\alpha/2}} \left\{ \frac{1}{2}\big(R_1(\lambda)\mathrm{i}[H, A](1-g) + \text{h.c.}\big) - \lambda R(\lambda) \right\} \\
&\quad \times \frac{1}{\langle A/\lambda\rangle^{\alpha/2}} E_\Delta \\
&=: E_\Delta \frac{1}{\langle A/\lambda\rangle^{\alpha/2}} \{g\, \mathrm{i}[H, A]\, g - R_2(\lambda)\} \frac{1}{\langle A/\lambda\rangle^{\alpha/2}} E_\Delta \\
&\geq E_\Delta \frac{1}{\langle A/\lambda\rangle^{\alpha/2}} \{g c' g - R_2(\lambda)\} \frac{1}{\langle A/\lambda\rangle^{\alpha/2}} E_\Delta \\
&= E_\Delta \frac{1}{\langle A/\lambda\rangle^{\alpha/2}} \{c' - \tilde{R}(\lambda)\} \frac{1}{\langle A/\lambda\rangle^{\alpha/2}} E_\Delta \qquad \text{(A.10)}
\end{aligned}$$

where the last line follows by commuting back $g(H)$ and the resulting expressions similar to the ones in (A.10) are absorbed in $\tilde{R}(\lambda)$. Since $\|\tilde{R}(\lambda)\| \leq c\lambda^{-1}$ we conclude that

$$\mathrm{i}[H, \lambda\Phi_\lambda] \geq \frac{c'}{2} E_\Delta \frac{1}{\langle A/\lambda\rangle^{\alpha}} E_\Delta \geq \frac{c'}{2} E_\Delta \frac{1}{\langle A\rangle^{\alpha}} E_\Delta$$

for $\lambda \geq \max\{2c/c', 1\}$. The claim then follows using a standard argument:

$$\begin{aligned}
\int_1^{t_0} \mathrm{d}t\, \|\langle A\rangle^{-\alpha/2} E_\Delta \psi_t\|^2 &= \int_1^{t_0} \mathrm{d}t\, (\psi_t, E_\Delta \langle A\rangle^{-\alpha} E_\Delta \psi_t) \\
&\leq \frac{2}{c'} \int_1^{t_0} \mathrm{d}t\, (\psi_t, \mathrm{i}[H, \lambda\Phi_\lambda]\psi_t) \\
&= \frac{2}{c'} \int_1^{t_0} \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t} (\psi_t, \lambda\Phi_\lambda\psi_t) \\
&\leq \frac{4}{c'} \|\lambda\Phi_\lambda\|\, \|\psi\|^2,
\end{aligned}$$

where $\psi_t \equiv \mathrm{e}^{-\mathrm{i}Ht}\psi$. □

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