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Floquet operators with singular spectrum, III

by

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ABSTRACT. – The quasienergy for the time-periodic Hamiltonian

$$|p|^\alpha + v(\theta, t)$$

on $L_2[0, 2\pi]$ has no absolutely continuous spectrum if $0 < \alpha < 1$ and $v(\theta, t)$ is C^∞, although the gap between successive eigenvalues of $|p|^\alpha$ is decreasing. © Elsevier, Paris

Key words: Singular spectrum, Floquet theory, quasienergy, quantum stability, gap theorem.

RÉSUMÉ. – L'opérateur de quasi-énergie correspondant au Hamiltonien dépendant du temps

$$|p|^\alpha + v(\theta, t)$$

sur $L_2[0, 2\pi]$ n'a pas de spectre absolument continu si $0 < \alpha < 1$ et $v(\theta, t)$ est C^∞, bien que l'écart entre valeurs propres de $|p|^\alpha$ soit décroissant. © Elsevier, Paris

1. INTRODUCTION

Let H be a positive discrete self-adjoint operator on a separable Hilbert space $\mathcal{H}$, with non-degenerate eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots,$$

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and define the *gap* between eigenvalues

$$\Delta\lambda_n = \lambda_{n+1} - \lambda_n.$$

If $V(t)$ is a bounded strongly continuous perturbation of H, 2π-periodic in time, then the behavior of the system under the time-dependent Hamiltonian

$$H(t) = H + V(t)$$

is governed by the *quasienergy*

$$K = D + H + V(t)$$

on $\mathcal{H} \otimes L_2[0, 2\pi]$, where $D = -i\frac{d}{dt}$ with periodic boundary condition $u(0) = u(2\pi)$ in t.

In [3], the author proved the following result.

Gap Theorem. *If $V(t)$ is strongly C^∞, and*

$$\Delta\lambda_n \geq cn^\alpha$$

for some $\alpha > 0$, then K has no absolutely continuous component.

This result was extended to degenerate eigenvalues by the author [4], Nenciu [6, 7] and Joye [5].

The question naturally arises as to how essential the increasing gap condition is to this result. Hagedorn, Loss, and Slawny [2] show by explicit computation that the forced harmonic oscillator

$$-\frac{1}{2}\frac{d^2}{dx^2} + \frac{\omega_0^2}{2}x^2 + fx\sin(\omega t) \tag{1.1}$$

has a quasienergy with absolutely continuous spectrum in the resonant case $\omega = \omega_0$. Here, of course, $\Delta\lambda_n = \omega_0$ is constant. On the other hand, numerical experiments with the operator

$$|p|^{\frac{1}{2}} + v(\theta, t) \tag{1.2}$$

where $p = -id/d\theta$ on L_2 of the circle, showed no evidence of absolutely continuous spectrum, although $\Delta\lambda_n \sim n^{-\frac{1}{2}}$ [1].

In fact, we shall prove the following theorem.

THEOREM B. – *Let $v(\theta, t)$ be C^∞ and 2π-periodic in θ and t, and satisfy*

$$\int_0^{2\pi} v(\theta, t)dt = 0. \tag{1.3}$$

If $0 < \alpha < 1$, then the quasienergy for

$$|p|^\alpha + v(\theta, t)$$

has no absolutely continuous component.

The proof is a variant of the operator gauge transformation method of [3,II]. Transformation of K by $e^{iG(t)}$ leads, up to first-order terms in G and V, to the operator

$$D + H + \{i[H, G(t)] + V(t) - \dot{G}(t)\} + \cdots$$

In [3,II], $G(t)$ was chosen so that the *first two terms* in the braces cancel, effectively replacing $V(t)$ by $\dot{G}(t)$. In the present paper, the *last two terms* are made to cancel, effectively replacing $V(t)$ by $i[H, G(t)]$. Iteration eventually leads to the case that $V(t)$ is trace class, and the result follows from scattering theory.

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2. MAIN THEOREM

Let H be a positive discrete Hamiltonian with eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots.$$

Assume that

$$|\lambda_n - \lambda_m| \leq C|n - m|(nm)^{-\gamma}, \tag{2.1}$$

where $\gamma > 0$.

Define

$$\langle n \rangle = \begin{cases} |\, n \,| & \text{if } n \neq 0 \\ 1 & \text{if } n = 0. \end{cases}$$

We shall write operators as matrices in the representation in which H is diagonal. For $p > 1$ and $\alpha \geq 0$, define $\mathcal{X}(p, \alpha)$ to be the space of all infinite matrices

$$A = \{A_{nm} : \; n, m \geq 1\},$$

satisfying

$$|A_{nm}| \leq C(nm)^{-\alpha}\langle n-m\rangle^{-p}. \tag{2.2}$$

$\mathcal{X}(p,\alpha)$ is a Banach space under the norm

$$\|A\|_{p,\alpha} = \sup\{(nm)^{\alpha}\langle n-m\rangle^{p}|A_{n,m}| : \ n,m \geq 1\}.$$

For $\alpha = 0$, A defines a bounded operator on ℓ_2, since $\langle n\rangle^{-p}$ is summable. For $\alpha > 0$, every $A \in \mathcal{X}(0,\alpha)$ can be written as

$$A = \Lambda^{\alpha} A_0 \Lambda^{\alpha},$$

where Λ is the diagonal matrix with

$$\Lambda_{nm} = \frac{1}{n}\,\delta_{nm},$$

and $A_0 \in \mathcal{X}(p,0)$. The operators A in $\mathcal{X}(p,\alpha)$ are therefore *compact* for $\alpha > 0$, and, in fact,

$$\mathcal{X}(p,\alpha) \subset \mathcal{I}_q$$

for $2\alpha q > 1$, where $\mathcal{I}_q$ is the Shatten class. In particular, $A \in \mathcal{X}(p,\alpha)$ is *trace class* if $\alpha > \frac{1}{2}$.

Define $\mathcal{X}(\alpha)$ to be the space of all A such that $A \in \mathcal{X}(p,\alpha)$ for all $p > 1$. Again, $A \in \mathcal{X}(\alpha)$ is *trace class* if $\alpha > \frac{1}{2}$.

LEMMA 1. – *If $A \in \mathcal{X}(p,\alpha)$ and $B \in \mathcal{X}(p,\beta)$, then the product AB is in $\mathcal{X}(r,\alpha+\beta)$ if*

$$1 < r < min\{p - 1/2 - (\alpha+\beta)/2, p-\alpha, p-\beta\}.$$

Proof. – We note in preparation the two elementary inequalities

$$2j\langle m-j\rangle \geq m, \tag{2.3}$$

and

$$\langle n-m\rangle \leq 2\langle n-j\rangle\langle m-j\rangle, \tag{2.4}$$

which hold for $n,m,j \geq 1$. These follow from the triangle inequality and the fact that $a+b \leq 2ab$ if $a,b \geq 1$.

We have

$$\left|\sum_j A_{nj}B_{jm}\right| \le Cn^{-\alpha}m^{-\beta}\sum_j j^{-(\alpha+\beta)}\langle n-j\rangle^{-p}\langle j-m\rangle^{-p}$$
$$= C(nm)^{-(\alpha+\beta)}\langle n-m\rangle^{-r}\sum_j\left(\frac{m}{j}\right)^{\alpha}\left(\frac{n}{j}\right)^{\beta}$$
$$\times\left[\frac{\langle n-m\rangle}{\langle n-j\rangle\langle j-m\rangle}\right]^{r}[\langle n-j\rangle\langle j-m\rangle]^{r-p}$$
$$\le C2^{\alpha+\beta+r}(nm)^{-(\alpha+\beta)}\langle n-m\rangle^{-r}\sum_j\langle n-j\rangle^{\alpha+r-p}\langle j-m\rangle^{\beta+r-p}$$

Since the exponents in the sum are negative, it follows by Holder's inequality that the sum is uniformly bounded if

$$(p-r-\alpha)+(p-r-\beta)>1;$$

that is, if

$$r<p-1/2-(\alpha+\beta)/2.$$

COROLLARY 1. – *If $A\in\mathcal{X}(\alpha)$, and $B\in\mathcal{X}(\beta)$, then the product AB is in $\mathcal{X}(\alpha+\beta)$.*

LEMMA 2. – *If $A\in\mathcal{X}(p,\alpha)$ and H satisfies (2.1), then the commutator $[H,A]$ is in $\mathcal{X}(p-1,\alpha+\gamma)$.*

Proof. – We have

$$|(\lambda_n-\lambda_k)A_{nk}|\le C\langle n-k\rangle(nk)^{-\gamma}(nk)^{-\alpha}\langle n-k\rangle^{-p}.\quad\square$$

COROLLARY 2. – *If $A\in\mathcal{X}(\alpha)$ and H satisfies (2.1), then the commutator $[H,A]$ is in $\mathcal{X}(\alpha+\gamma)$.*

Let $V(t)$ be a 2π-periodic operator-valued function of t. We say that $V(t)$ is in a Banach space $\mathcal{X}$ *uniformly* iff $\|V(t)\|_{\mathcal{X}}$ is a bounded function of t. We say that $V(t)$ is in $\mathcal{X}(\alpha)$ *uniformly* iff $V(t)$ is in $\mathcal{X}(p,\alpha)$ *uniformly* for all $p>1$.

LEMMA 3. – *Let H satisfy (2.1). Let $W\in\mathcal{X}(\gamma)$ and $V(t)$ be 2π-periodic, strongly continuous, and in $\mathcal{X}(\alpha)$ uniformly, where $\alpha\ge\gamma>0$. Then*

$$K=D+H+W+V(t)$$

is unitarily equivalent to

$$K_1 = D + H + W_1 + V_1(t) + T_1(t),$$

where $W_1 \in \mathcal{X}(\gamma)$, $V_1(t)$ *is* 2π*-periodic, strongly continuous and uniformly in* $\mathcal{X}(\alpha+\gamma)$, *and* $T_1(t)$ *is uniformly in trace class.*

Proof. – Let

$$V(t) = \bar{V} + \tilde{V}(t),$$

where

$$\int_0^{2\pi} \tilde{V}(t)dt = 0. \tag{2.5}$$

Define

$$G(t) = \int_0^t \tilde{V}(s)ds, \tag{2.6}$$

so that $G(t)$ is 2π-periodic, and

$$\dot{G}(t) = \tilde{V}(t).$$

Note that $\bar{V}$ is in $\mathcal{X}(\alpha)$ and $G(t)$ is in $\mathcal{X}(\alpha)$ uniformly.

If

$$adG(H) = [G, H],$$

then

$$\begin{aligned} e^{iG(t)}Ke^{-iG(t)} &= e^{iG(t)}(D + H + W + V(t))e^{-iG(t)} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!}[adG(t)]^n(D + H + W + V(t)) \\ &= D + H + W + V(t) \\ &\quad + \sum_{n=1}^{\infty} \frac{i^n}{n!}\{[adG(t)]^{n-1}([G(t), D] + [G(t), H]) \\ &\qquad\qquad + [adG(t)]^n(W + V(t))\}. \end{aligned}$$

But

$$[G(t), D] = i\dot{G}(t) = i\tilde{V}(t)$$

is in $\mathcal{X}(\alpha)$ uniformly by hypothesis, while $[G(t), H]$ is in $\mathcal{X}(\alpha+\gamma)$ uniformly by Corollary 2, and

$$[adG(t)]^n(W + V(t))$$

is in $\mathcal{X}(n\alpha + \gamma)$ uniformly by Corollary 1. It follows from Corollaries 1 and 2 that every term in (2.8) is in $\mathcal{X}(\alpha + \gamma)$, except for

$$D + H + W + V(t) + i^2\dot{G}(t) = D + H + W + \bar{V}.$$

Moreover, the terms of the series are all in trace class if $n\alpha > \frac{1}{2}$. Hence, (2.8) is equal to

$$D + H + W_1 + V_1(t) + T_1(t)$$

with $W_1 = W + \bar{V} \in \mathcal{X}(\gamma)$, $V_1(t) \in \mathcal{X}(\alpha + \gamma)$, and $T_1(t)$ in trace class uniformly. *Trace norm* convergence of the series presents no problem because of the factor $n!$. □

THEOREM B. – *Let H satisfy (2.1) for some $\gamma > 0$, and suppose that for some $\alpha > 0$, $W(t)$ is 2π-periodic, strongly continuous, and in $\mathcal{X}(\alpha)$ uniformly. Then*

$$K = D + H + W(t)$$

has no absolutely continuous component.

Proof. – If (2.1) holds for some positive γ, then it holds for any smaller positive number γ'. Since also $\mathcal{X}(\beta) \subset \mathcal{X}(\alpha)$ if $\alpha < \beta$, it follows that we may assume for simplicity that $\alpha = \gamma$. By Lemma 3, K is therefore unitarily equivalent to

$$K_1 = D + H + W_1 + V_1(t) + T_1(t),$$

with $W_1 \in \mathcal{X}(\gamma)$, and $V_1(t) \in \mathcal{X}(2\gamma)$, and $T_1(t)$ in trace class uniformly. From scattering theory, K_1, and hence also K, have the same absolutely continuous component as

$$\tilde{K}_1 = D + H + W_1 + V_1(t).$$

But $\tilde{K}_1$ satisfies the hypotheses of Theorem A with $\alpha = 2\gamma$. Continuing this process, we find that K has the same absolutely continuous component as an operator

$$\tilde{K}_N = D + H + W_N + V_N(t),$$

with $W_N \in \mathcal{X}(\gamma)$, $V_N(t) \in \mathcal{X}((N+1)\gamma)$. But if $(N+1)\gamma > \frac{1}{2}$, then $V_N(t)$ is trace class, so that $\tilde{K}_N$, and hence also K have the same absolutely continuous component as $D + H + W_N$ which is pure point. □

3. PROOF OF THEOREM B

Theorem B follows from Theorem A. The operator $H = |p|^{\alpha}$ has eigenvalues

$$0 = \lambda_1 < \lambda_2 = \lambda_3 < \lambda_4 = \lambda_5 < \cdots,$$

where

$$\lambda_{2j} = \lambda_{2j+1} = j^{\alpha}, \qquad j = 1, 2, \ldots$$

Matrices are taken in the basis 1, $e^{i\theta}$, $e^{-i\theta}$, $e^{2i\theta}, \ldots$ in which H is diagonal.

We shall show that H satisfies (2.1), with $\gamma = (1 - \alpha)/2$. We have, for $j > k$,

$$\frac{j^{\alpha} - k^{\alpha}}{j - k} = \frac{\alpha}{\xi^{2\gamma}} \le \frac{2\alpha}{j^{2\gamma} + k^{2\gamma}} \le \alpha (jk)^{-\gamma}$$

by the mean value theorem and convexity of ξ^{α}. If $\lambda_n - \lambda_m = j^{\alpha} - k^{\alpha}$, then $n - m \ge (2j + 1) - 2k \ge j - k$, and so

$$\frac{\lambda_n - \lambda_m}{n - m} \le \frac{j^{\alpha} - k^{\alpha}}{j - k} \le \alpha (jk)^{-\gamma} \le \alpha 2^{-\gamma} (nm)^{-\gamma}.$$

By (1.3), we may write

$$v(\theta, t) = \dot{g}(\theta, t) = \frac{\partial}{\partial t} g(\theta, t)$$

for some $g(\theta, t)$ in C^{∞}. Since $v(\theta, t)$ is C^{∞} in θ, the operators $v(\theta, t)$ and $g(\theta, t)$ are in $\mathcal{X}(0, p)$ for all p. The operator K is therefore unitarily equivalent to

$$\begin{aligned} K_0 &= e^{ig(\theta,t)} (D + H + v(t, \theta)) e^{-ig(\theta,t)} &(3.1)\\ &= D - \dot{g}(\theta, t) + v(t, \theta) + e^{ig(\theta,t)} H e^{-ig(\theta,t)} \\ &= D + H + V(t), &(3.2)\end{aligned}$$

where

$$V(t) = e^{ig(\theta,t)} H e^{-ig(\theta,t)} - H.$$

The operator K_0 will satisfy the conditions of Theorem A with $\alpha = \gamma$, provided we show that $V(t)$ is uniformly in $\mathcal{X}(\gamma)$.

Write

$$W(s, t) = e^{isg(\theta,t)} H e^{-isg(\theta,t)} - H.$$

Then $W(0, t) = 0$ and

$$\frac{\partial W}{\partial s} = ie^{isg(\theta,t)}[g, H]e^{-isg(\theta,t)}. \tag{3.3}$$

Now g and $e^{\pm isg(\theta,t)}$ are C^∞ and hence in $\mathcal{X}(0)$, so $[g, H] \in \mathcal{X}(\gamma)$ by Corollary 2. By Corollary 1, the right side of (3.1) is in $\mathcal{X}(p, \gamma)$ uniformly in t and s. Regarding (3.1) as a differential equation in the Banach space $\mathcal{X}(p, \gamma)$, we find that $V(t) = W(1, t)$ is in $\mathcal{X}(p, \gamma)$ uniformly for all p. $\square$

Remark. – Actually, it is clear from the proof that differentiability in t is not actually required. Moreover, only a finite degree of differentiability in θ is required, depending on γ, although it did not seem worthwhile to quantify this.

REFERENCES

[1] J. BELLISSARD, Private communication, 1987.
[2] G. HAGEDORN, M. LOSS, and J. SAWNYl. Non-stochasticity of time-dependent Hamiltonians and the spectra of canonical transformations, *J. Phys. A*, Vol. **19**, 1986, pp. 521–531.
[3] J. S. HOWLAND, Floquet operators with singular spectrum, *Ann. Inst. H. Poincare*, Vol. **50**, 1989, I, pp. 309–323; II, pp. 325–334.
[4] J. S. HOWLAND, Quantum stability in "Schrödinger Operators: The Quantum Mechanical Many Body Problem," Lecture Notes in Physics, Vol. **403**, pp. 100–122. Springer Verlag, New York, 1992.
[5] A. JOYE, Absence of absolutely continuous spectrum of Floquet operators, *J. Stat. Phys.*, Vol. **75**, 1994, pp. 929–952.
[6] G. NENCIU, Floquet operators without absolutely continuous spectrum, *Ann. Inst. H. Poincare*, Vol. **59**, 1993, pp. 91–97.
[7] G. NENCIU, Adiabatic theory: Stability of systems with increasing gaps. Preprint 1995.

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