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Relativistic and nonrelativistic elastodynamics with small shear strains

by

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ABSTRACT. – We present a new variational formulation for relativistic dynamics of isotropic hyperelastic solids. We introduce the shear strain tensor and study the geometry of characteristics in the cotangent bundle for the relativistic equations, under the assumption of small shear strains, and obtain a result on the stability of the double characteristic manifold. We then focus on the nonrelativistic limit of the above formulation, and compare it to the classical formulation of elastodynamics via displacements. We obtain a global existence result for small-amplitude elastic waves in materials under a constant isotropic deformation and a result on the formation of singularities for large data. © Elsevier, Paris


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1. FORMULATING RELATIVISTIC ELASTODYNAMICS

1.1. The Classical Formulation

Elastodynamics concerns itself with the time-evolution of elastic deformations in a solid body. Its classical formulation is based on the concept of displacements, i.e., elastic deformations of a solid away from a "totally relaxed" or "undeformed" reference state: Let $\phi : \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}^3$, $\phi = \phi(t, y)$ be a smooth deformation evolving in time of a material which in its undeformed state occupies the region $\mathcal{N} \subset \mathbb{R}^3$. Let $F = D_y \phi$ be the deformation gradient. The stored energy $W$ of an isotropic hyperelastic material depends on $F$ through the principal invariants $i_1, i_2, i_3$ of the (left Cauchy-Green) strain tensor $B = FF^T$. In the absence of body forces, the equations of motion, which can be derived from a Lagrangian, are

$$\frac{\partial^2 \phi^i}{\partial t^2} - \frac{\partial \mathcal{L}}{\partial (\partial_i \phi^i)} = 0. \quad (1.1)$$

This is a second-order, quasi-linear, and (under the right assumptions on $W$) hyperbolic system of partial differential equations.

Now let $\psi(t, y) := \phi(t, y) - y$ be the displacement field. When $\psi$ is infinitesimal, the equations of linear elasticity provide a good model for the dynamics of the body. These equations arise from the expansion around the trivial solution $\phi = y$ of equations (1.1) and ignoring the higher order terms:

$$L\psi := \frac{\partial^2 \psi}{\partial t^2} - c_2^2 \Delta \psi - (c_1^2 - c_2^2) \nabla (\nabla \cdot \psi) = 0, \quad (1.2)$$

where $c_1 > c_2 > 0$ are the propagation speeds of longitudinal and transverse waves, respectively:

$$
c_1^2 := 4(W_{11} + 4W_{12} + 2W_{13} + 4W_{22} + 4W_{23} + W_{33})$

$$
c_2^2 := -2(W_2 + W_3),$$

with $W_k$ and $W_{mn}$ denoting the first and second partial derivatives of $W$ with respect to the invariants $i_k$, evaluated at the unstressed reference state $i_1 = 3, i_2 = 3, i_3 = 1$.

1.2. Nonlinear and Relativistic Dynamics

Nonlinear elasticity comes in when finite displacements are considered. For a typical solid three-dimensional body whose linear dimensions are comparable, say a metallic cube, the displacements cannot get too large without ceasing to be elastic displacements. This is because most solid
materials, rubber being a notable exception, have a small elastic limit, accumulation of strains beyond which point will cause the material to crack, either immediately or eventually after going through a plastic phase, and in any case leaving the domain of applicability of the elasticity model. The hypothesis of small displacements is thus natural and reasonable when dealing with three-dimensional bodies with comparable linear dimensions. In the classical nonlinear theory of elasticity, large displacements are mostly considered when the elastic body in question has a special geometry that allows the displacements to accumulate and become large in a certain direction, while the strains are still small. This is the case for rods and plates, and more generally, for objects of modest size but with non-comparable linear dimensions.

One situation where it is natural to consider strains which are not uniformly small is in geophysics and astrophysics, where the solid bodies under study have comparable dimensions but are of a celestial scale in size, so that because of self-gravitation, the isotropic pressure in the interior of these bodies is enormous. Linear elasticity clearly does not apply to these bodies since they are very far from being relaxed. In studying the elastic response of such media to a disturbance, e.g. quakes on Earth or in a white dwarf star, one has to bear in mind that the shear forces produced by any feasible disturbance will be very small compared to the hydrostatic pressure in the background. Therefore the correct perturbative analysis involves linearizations not around a state of no stress, but rather around a state of very large isotropic pressure which is a nontrivial static or stationary solution of the equations of motion.

Astrophysical applications of elastodynamics necessitate the inclusion of relativistic effects in the theory. While special-relativistic effects in mechanics are commonly thought of as becoming apparent only when material velocities reach an appreciable fraction of the speed of light, there is another way for these effects to show which is perhaps more relevant to the dynamics of a solid body: It is when the material density becomes so high that the speed of sound waves travelling in the body begins to approach the speed of light. This is indeed the situation in a typical neutron star, with a radius of only about 10 km and a mass equal to 1.4 solar masses. In this case of course, the gravitational field is also very strong and general-relativistic effects must be taken into account.

In a truly relativistic theory of elastodynamics, the classical notion of displacement has to be put aside, since there is no invariant way of defining either a displacement vector or an undeformed state. This is because there is no reference frame in which gravity can be switched off. Starting with
Souriau [15], there have been many attempts, successful and otherwise, to present a Lagrangian formulation of relativistic elastodynamics which does not refer to displacements and is free of any assumptions on the existence of a globally relaxed state. Recently, Kijowski & Magli [10] proposed their formulation, which is based on mappings from the spacetime into an abstract material manifold. Independently, D. Christodoulou succeeded in generalizing his formulation of relativistic fluid mechanics [4] to the completely general case of the adiabatic dynamics of perfectly elastic, aeolotropic media in presence of dislocations and electromagnetic fields [5]. When reduced to the special case of isotropic solids, Christodoulou’s formulation, described below, agrees with that of Kijowski & Magli up to the definition of the strain tensor, but is more general since it includes some thermodynamics as well.

1.3. The Variational Formulation

The dynamics of a body are described by a mapping $f$ from the 4-dimensional spacetime manifold $(\mathcal{M}, g)$ into an abstract 3-dimensional Riemannian manifold $(\mathcal{N}, m)$, called the material manifold. The differential of the mapping is required to have maximal rank, with the 1-dimensional null spaces being timelike at every point. The inverse image of a point $y \in \mathcal{N}$ under $f$ is thus a timelike curve in $\mathcal{M}$ which is the world-line of the particle labeled $y$ in the material. The material velocity $u$ is the unit future-directed tangent vector field of these curves. The orthogonal complement of $u_x$ in $T_x \mathcal{M}$ is a hyperplane $\Sigma_x$, the simultaneous space at $x$, on which the metric of spacetime $g$ induces a Riemannian metric $\gamma$,

$$\gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu.$$  

The (relativistic) strain $h$ is defined to be the pullback of the material metric $m$ under the mapping $f$,

$$h_{\mu\nu} := (f^* m)_{\mu\nu} = \partial_\mu f^a \partial_\nu f^b m_{ab},$$  

and is thus a symmetric 2-tensor living on $\Sigma_x$. The velocity $u_x$ is an eigenvector of $h$ with eigenvalue zero. The other three eigenvalues, $h_1, h_2, h_3$ have to be positive since $m$ is positive definite and $D_x f$ is an isomorphism between $\Sigma_x$ and $T_{f(x)} \mathcal{N}$, which is taken to be orientation-preserving. The Lagrangian density $\mathcal{L}$ of a homogeneous, isotropic, perfectly elastic solid in general must be a function of $h$ which is invariant under the

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1 See Maugin [12] for a critique of some of these formulations.
orthogonal group $O(3)$ acting on $\Sigma_x$, i.e. a function of the three invariants $q_1, q_2, q_3$ of $h$:

\[ q_1 := h_1 + h_2 + h_3 = \text{tr}_v h = \text{tr}_g h \]
\[ q_2 := h_1 h_2 + h_2 h_3 + h_3 h_1 = \frac{1}{2}[(\text{tr} h)^2 - \text{tr}(h^2)] \]
\[ q_3 := h_1 h_2 h_3 = \det_v h. \]

Thus for a solid we have

\[ \mathcal{L} = \mathcal{L}(s, q_1, q_2, q_3), \]

where $s : \mathcal{M} \to \mathbb{R}^+$ is the entropy per particle, defined as a function on $\mathcal{M}$ which is independent of the spacetime metric $g$. Fluid dynamics can be thought of as a subcase, where $\mathcal{L}$ depends only on $q_3$ and $s$.

The energy-momentum-stress tensor $T$ of the material can now be computed from the formula

\[ T_{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \mathcal{L} g_{\mu\nu}. \]

Letting $H_x^+$ be the hyperboloid of unit future-directed timelike vectors at $x$, we define $\rho$, the (proper) energy density at $x$ by

\[ \rho(x) = \inf_{v \in H_x^+} T(v, v). \]

Then it is easy to see that in our case we have $\rho = \mathcal{L}$, the velocity $u$ is the eigenvector of $T$ corresponding to eigenvalue $\rho$, and that

\[ T_{\mu\nu} = \rho u_\mu u_\nu + S_{\mu\nu}, \quad (1.4) \]

where $S$ is the stress tensor,

\[ S_{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial q_1} h_{\mu\nu} + 2 \frac{\partial \mathcal{L}}{\partial q_2} (q_1 h_{\mu\nu} - h^2_{\mu\nu}) + (2 \frac{\partial \mathcal{L}}{\partial q_3} q_3 - \mathcal{L}) \gamma_{\mu\nu}. \quad (1.5) \]

The stress tensor satisfies $S_{\mu\nu} u^\mu = 0$ and thus is a symmetric bilinear form living on $\Sigma_x$. The eigenvalues of $S$ relative to $\gamma$ are called the principal pressures $p_1, p_2, p_3$, which unlike the case of a perfect fluid, do not have to be positive or equal. The positivity condition for the energy momentum tensor implies that $\rho \geq \max\{|p_1|, |p_2|, |p_3|\}$. 

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The equations of motion for a solid body are
\[ \nabla^\nu T_{\mu \nu} = 0, \quad (1.6) \]
which express the conservation laws of energy and momentum. (1.6) is a direct consequence of the spacetime metric $g$ satisfying the Einstein equations,
\[ G_{\mu \nu} := R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = \frac{2\kappa}{c^4} T_{\mu \nu}, \quad (1.7) \]
where $R_{\mu \nu}$ is the Ricci curvature of the spacetime metric $g_{\mu \nu}$, $R$ is its scalar curvature, $\kappa$ is $4\pi$ times Newton’s constant of gravitation and $c$ the speed of light. This is because $\nabla^\nu G_{\mu \nu} = 0$, in view of the twice-contracted Bianchi identities.

To equations (1.6) we need to append the equivalent of the equation of continuity for solids, which here takes the form
\[ \mathcal{L}_u h = 0, \quad (1.8) \]
where $\mathcal{L}$ denotes the Lie derivative. This is a consequence of (1.3). In a coordinate frame (1.8) reads
\[ u^\lambda \nabla_\lambda h_{\mu \nu} + h_{\lambda \nu} \nabla_\mu u^\lambda + h_{\mu \lambda} \nabla_\nu u^\lambda = 0. \quad (1.9) \]
Let
\[ n := \sqrt{q_3} \]
denote the number density of the material in physical space, i.e., the number of particles (or equivalently, the number of flow lines) per unit volume of $\Sigma_x$. Let $\omega$ denote the volume form of $(\mathcal{N}, m)$, and let $v$ be the volume 3-form induced by $g$ on $\Sigma_x$, i.e., $v_{\alpha \beta \gamma} = u^\lambda \epsilon_{\lambda \alpha \beta \gamma}$ where $\epsilon$ is the volume 4-form of $(\mathcal{M}, g)$. Then it is clear that $f^\ast \omega = n v$, and hence $d(n v) = d(f^\ast \omega) = f^\ast(d \omega) = 0$. We thus have the law of conservation of particle number
\[ \nabla_\mu(n u^\mu) = 0. \quad (1.10) \]
(This can also be derived from (1.8) and therefore is not a new equation.) If we regard the spacetime metric $g$ as a fixed background metric (the test-relativistic case), then (1.6) and (1.8) together form a complete system.

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\[ \text{(This can also be derived from (1.8) and therefore is not a new equation.)} \]

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\[ \text{If we regard the spacetime metric } g \text{ as a fixed background metric (the test-relativistic case), then (1.6) and (1.8) together form a complete system.} \]
of 10 equations, where the 10 unknowns are the entropy $s$, the velocity field $u$ which is subject to the normalizing condition

$$u_\mu u^\mu = -1,$$

and the strain tensor $h$ which is subject to the condition

$$h_{\mu\nu} u^\mu = 0.$$

If the effects of the dynamics of the solid on the gravitational field are not negligible, then $g$ is also unknown and we need to add (1.7) to the system of equations.

Finally, it can be shown (see [4]) that for $C^1$ solutions, the component of (1.6) in the direction of $u$ is equivalent to the adiabatic condition:

$$\nabla_u s = u^\mu \partial_\mu s = 0. \quad (1.13)$$

1.4. The shear strain tensor

Let $\sigma$ be the symmetric 2-covariant tensor defined by

$$\sigma_{\mu\nu} := d^{-1/3}_3 h_{\mu\nu} - \gamma_{\mu\nu}. \quad (1.14)$$

Thus $\sigma$ measures the deviation from isotropy. We call it the shear strain tensor. From the definition it is clear that $\sigma$ is like $\gamma$ and $h$ a tensor living on $\Sigma_x$. Let $\sigma_1, \sigma_2, \sigma_3$ be the eigenvalues of $\sigma$ restricted to $\Sigma_x$. Since

$$h_i = n^{2/3}(1 + \sigma_i),$$

we must have

$$(1 + \sigma_1)(1 + \sigma_2)(1 + \sigma_3) = 1,$$

and thus only two of the $\sigma_i$ are independent. Let $|\sigma| := \max\{|\sigma_1|, |\sigma_2|, |\sigma_3|\}$ and let

$$\tau = \text{tr}\sigma, \quad \kappa = \frac{1}{2}[(\text{tr}\sigma)^2 - \text{tr}\sigma^2], \quad \delta = \det\sigma,$$

be the invariants of $\sigma$. Then $\tau + \kappa + \delta = 0$ and thus we only need to consider $\tau$ and $\delta$. Moreover,

$$\tau = O(|\sigma|^2), \quad \delta = O(|\sigma|^3).$$
The invariants of $h$ and $\sigma$ are related in the following way:

$$q_1 = n^{2/3}(3 + \tau), \quad q_2 = n^{4/3}(3 + \tau - \delta), \quad q_3 = n^2.$$ 

Let

$$e := \frac{\rho}{n}$$

be the energy per particle. The constitutive equation of the material is the specification of $e$ as a smooth function of the thermodynamic variables:

$$e = e(s, n, \tau, \delta) \quad (1.15)$$

Substituting this in (1.4) and (1.5) we obtain

$$T_{\mu\nu} = \rho u_\mu u_\nu + p\gamma_{\mu\nu} + q\sigma_{\mu\nu} + w\sigma_{\mu\nu}^2, \quad (1.16)$$

where

$$p := n^2 \frac{de}{dn} - \frac{2}{3} n \tau \frac{de}{d\tau} - \frac{4}{3} n(\tau + \delta) \frac{de}{d\delta}, \quad q := 2n \frac{de}{d\tau} - 2n \tau \frac{de}{d\delta}, \quad w := 2n \frac{de}{d\delta}.$$ 

In astrophysical applications, e.g. the case of a white dwarf or a neutron star, the shear strains are much smaller than the hydrostatic compression, which is due to self-gravitation. To study small deviations from an isotropic background state, we can expand $e$ around $\sigma = 0$:

$$e(s, n, \tau, \delta) = e_0(s, n) + \tau e_2(s, n) + \delta e_3(s, n) + O(|\sigma|^4), \quad (1.17)$$

where $e_i$ are smooth functions on $\mathbb{R}^+ \times \mathbb{R}^+$. In particular, $e_0$ is the energy per particle of the background isotropic state, and $e_2$ can be thought of as the modulus of rigidity.

We make the following assumption regarding the constitutive equation:

For every fixed $s > 0$ and $n > 0$, $e$ has a unique local minimum at $\sigma = 0$. \quad (1.18)

This is already reflected in (1.17) by the absence of a first order term in $\sigma$, and (since $\tau > 0$ for $\sigma_i$ small) by the requirement that

$$e_2 > 0. \quad (1.19)$$

However, we do not need to impose this a priori since, as we shall see later, it is a necessary condition for the hyperbolicity of the system of equations.
2. THE GEOMETRY OF CHARACTERISTICS

2.1. The Characteristic Set

The symbol \( \sigma_\xi \) of the system (1.6, 1.8) at a given covector \( \xi \in T^*_x M \) is a linear operator on the space of variations \((\dot{s}, \dot{u}, \dot{h})\). It is obtained by replacing all terms of the form \( \nabla_\mu F^\alpha \) in the principal part of the equations with \( \xi_\mu \dot{F}^\alpha \), where \( F^\alpha \) is any one of the unknowns. The set of covectors \( \xi \) such that the null space of \( \sigma_\xi \) is nontrivial is by definition the characteristic set \( C_x^* \) in the cotangent bundle.

We note that the algebraic constraints (1.11, 1.12) on \( u \) and \( h \) imply the following constraints on the variations \( \dot{u}, \dot{h} \):

\[
\begin{align*}
    u_\mu \dot{u}^\mu &= 0, \\
    h_{\mu\nu} \dot{u}^\nu + \dot{h}_{\mu\nu} u^\nu &= 0.
\end{align*}
\]  
(2.1)

To calculate the components of the symbol, we choose as a basis for \( T_x M \) the orthonormal frame of vectors \( \{E_0, E_1, E_2, E_3\} \), with \( E_0 = u \) and \( E_i \) the eigenvectors of \( \sigma \). We take the dual basis for this frame.

\[
\begin{align*}
    g_{00} &= -1, \quad g_{0i} = 0, \quad g_{ij} = \gamma_{ij}, \quad \gamma_{0\nu} = 0, \quad \sigma_{0\nu} = 0, \quad \sigma_{ij} = 0 \quad \text{for} \quad i \neq j, \\
    u^0 &= 1, \quad u_0 = -1, \quad u^i = 0, \quad \dot{u}^0 = 0.
\end{align*}
\]

Hence the symbol of (1.9) is

\[
\xi_0 \dot{h}_{\mu\nu} + (h_{\mu\lambda} \xi_\nu + h_{\lambda\nu} \xi_\mu) \dot{u}^\lambda.
\]  
(2.2)

We will also need the symbol of the adiabatic condition (1.13):

\[
\begin{align*}
    u_\mu \xi_\mu \dot{s}.
\end{align*}
\]  
(2.3)

Let \( \xi_u \) denote the component of \( \xi \) in the direction of (the cotangent vector dual to) \( u \). Thus \( \xi_u = \xi_\mu u^\mu = \xi_0 \) in the frame we have chosen. Let \( \xi \in C_x^* \) and assume \( \xi_u = 0 \). Then from (2.2) we see that \( \dot{u}^i = 0 \). Moreover, (2.3) implies that \( \dot{s} \) is arbitrary. This means that the 3-dimensional spacelike plane \( \Pi_0 := \{\xi \in T_x^* M \mid \xi_u = 0\} \) is part of the characteristic set. The remaining equations, obtained by setting the symbol of (1.6) equal to zero, will be three equations for the six variations \( \dot{h}_{ij} \), and it is easy to see that the null space of \( \sigma_\xi \) for \( \xi \in \Pi_0 \) is four-dimensional.

Now suppose \( \xi \in C_x^* \) and \( \xi_u \neq 0 \). Then \( \dot{s} = 0 \) and from (2.2) we have

\[
\begin{align*}
    \dot{h}_{00} &= \dot{u}^0 = 0, \\
    \dot{h}_{0j} &= -h_{kj} \dot{u}^k, \\
    -\xi_0 \dot{h}_{ij} &= (h_{kj} \xi_i + h_{ik} \xi_j) \dot{u}^k.
\end{align*}
\]  
(2.4)
In studying the symbol of (1.6) we can ignore the terms with \( \mu = 0 \) since we already know that this component of the equation is equivalent to the adiabatic condition. Consider first the term \( \nu = 0, \mu = i \). This gives us

\[
\xi^0 \left\{ -(\rho + p)\gamma_{ik} - q\sigma_{ik} - w\sigma^2_{ik} \right\} \dot{u}^k,
\]

while the terms \( \nu = j, \mu = i \) give

\[
\xi^j \left\{ \dot{\rho} \gamma_{ij} + \dot{q}\sigma_{ij} + q\sigma_{ij} + \dot{w}\sigma^2_{ij} + w\sigma^2_{ij} \right\},
\]

where \( \dot{\rho} = \frac{\partial\rho}{\partial n} \dot{n} + \frac{\partial\rho}{\partial \tau} \dot{\tau} + \frac{\partial\rho}{\partial \phi} \dot{\phi} \), etc. To calculate \( \dot{n} \), we take the symbol of (1.10), which is \( \xi_{\mu}(\dot{n}u^\mu + n\dot{u}^\mu) \). Thus we have

\[
-\xi_0 \dot{n} = n\xi_k \dot{u}^k.
\]

From this, (1.14) and (2.4) we can calculate \( \dot{\sigma} \) and \( \dot{\sigma}^2 \) in terms of \( \dot{u} \). Let us define \( \zeta, \eta \in \mathbb{R}^3 \) as follows:

\[
\zeta := (\sigma_1 \xi_1, \sigma_2 \xi_2, \sigma_3 \xi_3), \quad \eta := (\sigma_1^2 \xi_1, \sigma_2^2 \xi_2, \sigma_3^2 \xi_3).
\]

We then have

\[
\dot{\sigma}_{00} = 0, \quad \dot{\sigma}_{0j} = -\sigma_{kj} \dot{u}^k
\]

\[
-\xi_0 \sigma_{ij} = \left\{ -\frac{2}{3}(\gamma_{ij} + \sigma_{ij})\xi_k + (\gamma_{kj} + \sigma_{kj})\xi_i + (\gamma_{ik} + \sigma_{ik})\xi_j \right\} \dot{u}^k
\]

\[
\dot{\sigma}^2_{00} = 0, \quad \dot{\sigma}^2_{0j} = -\sigma^2_{kj} \dot{u}^k
\]

\[
-\xi_0 \sigma^2_{ij} = \left\{ -\frac{4}{3}(\sigma_{ij} + \sigma^2_{ij})\xi_k + (\sigma_{kj} + \sigma^2_{kj})\xi_i + (\sigma_{ik} + \sigma^2_{ik})\xi_j
\]

\[
+ (\gamma_{kj} + \sigma_{kj})\xi_i + (\gamma_{ik} + \sigma_{ik})\xi_j \right\} \dot{u}^k.
\]

These formulas in turn allow us to compute

\[
-\xi_0 \dot{\tau} = -\xi_0 \text{tr} \dot{\sigma} = (-\frac{2}{3}\tau \xi_k + 2\xi_k) \dot{u}^k
\]

\[
-\xi_0 \dot{\delta} = \xi_0 (\dot{\tau} + \dot{\kappa}) = (-\frac{4}{3}(\tau + \delta)\xi_k - 2\tau \xi_k + 2\eta_k) \dot{u}^k.
\]

Thus substituting in the above for all the variations in terms of \( \dot{u} \) and regrouping terms, we have that for a \( \xi \in C^*_x \) with \( \xi_0 \neq 0 \), a vector \( \dot{u} \in \text{Null}(\sigma_\xi) \) satisfies

\[
P_{ik} \dot{u}^k = 0,
\]

(2.5)
where \( P = P(x, \xi) \) is the following matrix-valued quadratic form in \( \xi \):

\[
\begin{align*}
P_{ik} &= a_1 \xi_i \xi_k + b_1 (\xi_i \xi_k + \xi_k \xi_i) + c_1 (\xi_i \eta_k + \xi_k \eta_i) \\
&
+ d_1 \xi_i \xi_k + f_1 (\xi_i \eta_k + \xi_k \eta_i) + \eta_i \eta_k \\
&
+ \gamma_{ik} \{-\rho + p \xi_0^2 + q |\xi|^2 + w(\xi \cdot \zeta) + \sigma_{ik} \{ -q \xi_0^2 + q |\xi|^2 \\
&
+ w(|\xi|^2 + \xi \cdot \zeta) \} + \sigma^2_{ik} \{ -w \xi_0^2 + w |\xi|^2 \},
\end{align*}
\]

where \( \xi \cdot \zeta := \xi_k \zeta^k, |\xi|^2 := \xi_k \xi^k \), and

\[
\begin{align*}
a := &\ n \frac{\partial p}{\partial n} - \frac{2}{3} \tau \frac{\partial p}{\partial \tau} - \frac{4}{3} (\rho + 2) \frac{\partial p}{\partial \delta} + \frac{1}{3} q, \\
b := &\ 2 \frac{\partial p}{\partial \tau} - 2 \tau \frac{\partial p}{\partial \delta} + q + w \\
c := &\ 2 \frac{\partial p}{\partial \delta} + w \\
d := &\ 2 \frac{\partial q}{\partial \tau} - 2 \tau \frac{\partial q}{\partial \delta} + w \\
f := &\ 2 \frac{\partial q}{\partial \delta} \\
l := &\ 2 \frac{\partial w}{\partial \delta} .
\end{align*}
\]

Using now that the frame we have chosen consists of eigenvectors of \( \sigma \), we can rewrite \( P \) in a more concise way:

\[
P_{ik} = C^{mn}_{ik}(\sigma) \xi_m \xi_n - D_i(\sigma) \gamma_{ik} \xi_0^2, \tag{2.6}
\]

where

\[
\begin{align*}
C^{mn}_{ik}(\sigma) := &\ \delta^{m}_{i} \delta^{n}_{k} \{ a + b(\sigma_i + \sigma_k) + c(\sigma_i^2 + \sigma_k^2) + d \sigma_i \sigma_k \\
&
+ f(\sigma_i^2 \sigma_k + \sigma_i \sigma_k^2) + l \sigma_i^2 \sigma_k^2 \} \\
&
+ \gamma^{mn} \gamma_{ik} \{ q + w \sigma^m + \sigma_i (q + w (1 + \sigma^m)) + \sigma_i^2 w \},
\end{align*}
\]

and

\[
D_i(\sigma) := \rho + p + \sigma_i q + \sigma_i^2 w .
\]

The system (2.5) has a nontrivial solution \( \dot{u} \) if and only if \( \det P = 0 \). Let \( Q(\xi) = \det P \). Then \( Q \) is a homogeneous polynomial of degree six in \( \xi_\mu \), which is actually cubic in the squares of the \( \xi_\mu \), with coefficients that depend on \( x \), and we have

\[
C_x^* = \{ \xi \in T_x^* M \mid \xi_0^* Q(\xi, x) = 0 \}.
\]

2.2. Hyperbolicity

Let \( \mathcal{P} \) denote the vector space of quadratic forms in \( \mathbb{R}^4 \) with real \( 3 \times 3 \) matrix values. \( P \in \mathcal{P} \) is hyperbolic with respect to \( E_0 = (1, 0, 0, 0) \) if
\( P(E_0) \neq 0 \) and the zeros of \( \lambda \mapsto \det P(\lambda E_0 + \xi) \) are real for every \( \xi \in \mathbb{R}^4 \).

It is clear that if \( P \) is hyperbolic then

\[
P\left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)
\]

is a linear, second-order, constant-coefficient differential operator acting on vector-valued functions \( f : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) and is hyperbolic in the sense of the characteristic speeds being real in every direction (see e.g. [8]).

For the system of partial differential equations (1.6–1.8) to be hyperbolic the quadratic form \( P_{ik}(x, \xi) \) in (2.6) must be hyperbolic for all \( x \). We can obtain necessary conditions for hyperbolicity by looking at the zero-order terms in \( P \): If we set \( \sigma_i = 0 \) in (2.6) we obtain

\[
P_{ik}^0 := C_{ik}^{mn}(0)\xi_m\xi_n - D_i(0)\gamma_{ik}\xi_0^2,
\]

(2.7)

where

\[
C_{ik}^{mn}(0) = \alpha^0 \delta_i^m \delta_k^n + \xi_0^0 \gamma_{mn} \gamma_{ik}, \quad D_i(0) = \rho^0 + p^0.
\]

Here \( \rho^0 = ne_0 \) and \( p^0 = n\frac{\partial \rho^0}{\partial n} - \rho^0 \) are respectively the energy density and hydrostatic pressure of the background isotropic state, \( q^0 = 2ne_2 \) and \( a^0 = n\frac{\partial p^0}{\partial n} + \frac{1}{3}q^0 \). The characteristic surface for \( P_0^0 \) is the set of \( \xi \in T_x^*\mathcal{M} \) satisfying

\[
Q^0(\xi) := \det P_0^0(\xi) = (c_2^2|\xi|^2 - \xi_0^2)^2(c_1^2|\xi|^2 - \xi_0^2) = 0,
\]

(2.8)

with

\[
c_2^2 = \frac{q^0}{\rho^0 + p^0} = \frac{2e_2}{\partial \rho^0/\partial n},
\]

\[
c_1^2 = \frac{a^0 + q^0}{\rho^0 + p^0} = \frac{\partial \rho^0/\partial n + 8e_2/3}{\partial \rho^0/\partial n} = \eta_0^2 + \frac{4}{3}c_1^2,
\]

where \( \eta_0^2 = \frac{d\rho^0}{d\rho_0} \) (at constant \( s \)). If real, \( \eta_0 \) is the sound speed in the background state. Since \( P = P^0 + O(|\sigma|) \), it is clear that a necessary condition for the hyperbolicity of \( P \) is that \( c_1 \) and \( c_2 \) be real. Thus we arrive at the following natural assumptions about the equation of state (1.17):

\[
\frac{\partial \rho^0}{\partial n} > 0, \quad \frac{\partial p^0}{\partial n} > 0, \quad e_2 > 0.
\]

(2.9)

For the above theory to be consistent with Relativity, we also need causality to be satisfied, i.e., the characteristic set \( C_x^* \) must lie outside the light cone.
\( \xi_0^2 = |\xi|^2 \) in the cotangent space \( T^*_x \mathcal{M} \). This implies that \( c_1 < 1 \) and \( c_2 < 1 \). Thus we have the additional condition
\[
e_2 < \frac{3}{8} \left( \frac{\partial p^0}{\partial n} - \frac{\partial p^0}{\partial n} \right).	ag{2.10}\]

From (2.8) we see that the nonplanar portion of the characteristic set \( C^*_x \), up to first order terms in \( \sigma \), consists of three spherical cones, two of which coincide. The coincident cones carry the shear waves (transverse waves) travelling with speed \( c_2 \), and the single cone carries the compression waves (longitudinal waves) travelling with speed \( c_1 > \sqrt{4/3} c_2 \). This is the classical picture of linearized elasticity, i.e., the characteristics of (1.2).

Conditions (2.9) guarantee the hyperbolicity of \( P^0 \). However, the presence of a double root for \( Q^0 \) in (2.8) indicates that \( P \), considered as a perturbation of \( P^0 \), might still fail to be hyperbolic, since a double root can in general be perturbed into two roots or into none at all. In other words, we are looking at a perturbation of a non-strictly hyperbolic operator \( P^0 \). If this were a general, non-symmetric perturbation, there would be no reason for \( P \) to be hyperbolic.

However, from (2.6) it is clear that both \( P \) and \( P^0 \) are symmetric: \( P_{ik} = P_{ki} \). This is a consequence of the equations of motion being derivable from a Lagrangian. If we let \( D \) denote the diagonal matrix with entries \( D_i \) and define \( \tilde{P} = D^{-1/2} P D^{-1/2} \), then on the one hand for \( |\sigma| \) small \( \det \tilde{P} = 0 \) if and only if \( \det \tilde{P} = 0 \), and on the other hand
\[
\tilde{P}_{ik} = \tilde{C}^{mn}_{ik}(\sigma) \xi_m \xi_n - \gamma_{ik} \xi_0^2,
\]
so that for each fixed \( \xi \in S^2 \), the roots of the polynomial \( \tilde{Q}(\lambda) := \det \tilde{P}(\lambda, \tilde{\xi}) \) are the square roots of the eigenvalues of the symmetric bilinear form
\[
M_{ik}(\sigma) := \tilde{C}^{mn}_{ik}(\sigma) \xi_m \xi_n.
\]
Now,
\[
M_{ik}(0) = a^0 \xi_i \xi_k + q^0 \gamma_{ik},
\]
is positive definite provided \( q^0 > 0 \) and \( q^0 + a^0 > 0 \), which are guaranteed to hold by the assumptions (2.9). Thus for \( |\sigma| \) small enough, \( M_{ik}(\sigma) \) is positive definite as well, which implies that \( P \) is hyperbolic.

Having shown the hyperbolicity of \( P \), we can now move on to the question of the geometry of the characteristic surface \( Q = 0 \). Here the kind of perturbative analysis around \( P^0 \) done in the above is no longer
helpful, since the characteristic surface corresponding to $P^0$, although of very simple geometry, is highly degenerate in the sense that because of the coincident shear cones there is a double characteristic in every direction. We expect this picture to be very unstable with respect to perturbations. In the next section we introduce another operator, $P^1$, which corresponds to the case where $e = e_0 + \tau e_2$. We will obtain the characteristic surface for $P^1$ and then show that its shape is stable under further perturbations.

2.3. The slowness surface in John materials

Consider a hyperelastic solid material with the following constitutive equation:

$$e(s, n, \tau, \delta) = e_0(s, n) + \tau e_2(s, n),$$

where $e_0, e_2$ are smooth functions satisfying the hyperbolicity (2.9) and causality (2.10) conditions, but otherwise arbitrary. I propose to call materials with a constitutive equation of the above form John Materials, in honor of Fritz John’s (1910–1994) fundamental contributions to mathematical elasticity. The energy tensor of a John material is therefore of the form

$$T_{\mu\nu} = \rho u_{\mu} u_{\nu} + p \gamma_{\mu\nu} + q \sigma_{\mu\nu},$$

where

$$\rho = ne, \quad p = n^2 \frac{\partial e_0}{\partial n} + n\tau \left( n \frac{\partial e_2}{\partial n} - \frac{2}{3} e_2 \right), \quad q = 2ne_2.$$

Let $\xi \in C^*_x$ with $\xi_u \neq 0$ and let $\dot{u} \in \text{Null}(\sigma_\xi)$. Then $\dot{u}$ satisfies $P^1_{ik} \ddot{u}^k = 0$ where

$$P^1_{ik}(x, \xi) = \left( a + b(\sigma_i + \sigma_k) \right) \xi_i \xi_k + \left( q|\xi|^2 - (\rho + p)\xi_0^2 + \sigma_i q(|\xi|^2 - \xi_0^2) \right) \gamma_{ik},$$

with

$$a = n \frac{\partial p}{\partial n} + \frac{1}{3} q - \frac{2}{3} \tau \frac{\partial p}{\partial \tau}, \quad b = 2 \frac{\partial p}{\partial \tau} + q.$$

We note that $P^1|_{\sigma=0} = P^0$, hence $P^1$ is hyperbolic for $\sigma$ small enough. Let $Q^1 = \det P^1$. We will now show that the characteristic set

$$C^*_x(P^1) := \{ \xi \in \mathbb{R}^d \mid Q^1(x, \xi) = 0 \},$$

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contains only four double characteristics, by deriving the Kelvin form of the equation (see [6]). Let

\[ a_i := \frac{(a + b(\sigma_i + \sigma_j))(a + b(\sigma_i + \sigma_k))}{a + b(\sigma_j + \sigma_k)}, \quad i, j, k \text{ distinct} \]

\[ b_i := a + 2b\sigma_i - a_i = \frac{-b^2(\sigma_i - \sigma_j)(\sigma_i - \sigma_k)}{a + b(\sigma_j + \sigma_k)}, \quad i, j, k \text{ distinct} \]

\[ d_i := \rho + p + \sigma_i q \]

\[ f_i := q(1 + \sigma_i), \]

and we define

\[ \alpha_i := \sqrt{a_i} \xi_i, \quad \delta_i := b_i \xi_i^2 - d_i \xi_0^2 + f_i \xi_j \xi^j. \]

Then

\[ P^1_{ik} = \alpha_i \alpha_k + \gamma_{ik} \delta_i, \]

so that the equation for the characteristic surface is

\[ Q^4(\xi) = \alpha_i^2 \delta_2 \delta_3 + \alpha_2^2 \delta_1 \delta_3 + \alpha_3^2 \delta_1 \delta_2 + \delta_1 \delta_2 \delta_3 = 0. \quad (2.12) \]

Note that \( \delta_i = 0 \) is the equation for an elliptical cone in \( T^*_x M \), whose trace on the hyperplane \( \xi_0 = 1 \) is an ellipsoid of revolution with semiaxes \( r_i \) and \( s_i \),

\[ \frac{\xi_i^2}{s_i^2} + \sum_{j \neq i} \frac{\xi_j^2}{r_j^2} = 1, \]

where

\[ r_i^2 := \frac{d_i}{f_i} = \frac{\rho + p + \sigma_i}{1 + \sigma_i}, \quad s_i^2 := \frac{d_i}{f_i + b_i} = r_i^2 + O(|\sigma|^2) \]

Let us first assume that we are in the generic case where the principal shears \( \sigma_1, \sigma_2, \sigma_3 \) are distinct, and in particular, \( \sigma_1 < \sigma_2 < \sigma_3 \). Then since \( r_i \) is a monotone function of \( \sigma_i \), we have \( r_3 < r_2 < r_1 \) and \( s_3 < s_2 < s_1 \). Moreover, \( b_1 < 0, b_2 > 0 \) and \( b_3 < 0 \), so that \( s_1 > r_1, s_2 < r_2 \) and \( s_3 > r_3 \). Also

\[ s_2^2 - r_3^2 \geq r_2^2 - r_3^2 - C(\sigma_3 - \sigma_2)(\sigma_2 - \sigma_1) \geq C'(\sigma_3 - \sigma_2) > 0 \]

and similarly \( r_2 > s_3 \). We therefore conclude that the three ellipsoids \( \delta_i = 0 \) are mutually disjoint, so that for any \( \xi \), at most one of the \( \delta_i(\xi) \) can be zero.
Since $Q^1$ is a homogeneous polynomial, $C_x^\ast(P^1)$ is a conic subset of $T_x^\ast\mathcal{M} \cong \mathbb{R}^4$, and its geometry is therefore determined by a slice

$$S_x := C_x^\ast(P^1) \cap \{\xi_0 = 1\}.$$ 

$S_x$ is called the slowness surface at $x$. Now let $\xi \in S_x$. Then $Q^1(\xi) = 0$. Suppose that $\delta_i(\xi) = 0$ for some $1 \leq i \leq 3$. Then $\alpha_i \delta_j \delta_k = 0$ for $i, j, k$ distinct, and thus by the above we have $\alpha_i = 0$, which implies (since $a_i > 0$) that $\xi_i = 0$, and hence the three disjoint and mutually orthogonal circles

$$C_i := \{\xi \in \mathbb{R}^3 \mid \xi_i = 0, \xi_j^2 + \xi_k^2 = r_i^2, \ i, j, k \text{ distinct}\}$$

are part of the slowness surface.

If, on the other hand, $\delta_i \neq 0$ for $i = 1, 2, 3$, then from (2.12) we have that

$$1 + \sum_{k=1}^{3} \frac{\alpha_k^2(\xi)}{\delta_k(\xi)} = 0, \quad (2.13)$$

Let $\lambda := 1/|\xi|^2$ and $\tilde{\xi} := \xi/|\xi|$. Define

$$A_i(\tilde{\xi}) := \frac{a_i}{d_i} \tilde{\xi}^2, \quad B_i(\tilde{\xi}) := \frac{b_i \tilde{\xi}^2 + f_i}{d_i}.$$ 

Then (2.13) is equivalent to

$$\sum_{i=1}^{3} \frac{A_i}{\lambda - B_i} = 1. \quad (2.14)$$

For a given $\tilde{\xi} \in \mathbb{S}^2$, the values of $\lambda$ for which (2.14) is satisfied are the abscissae of the intersection points of the line $\varphi = 1$ with the curve

$$\varphi(\lambda) = \frac{A_1}{\lambda - B_1} + \frac{A_2}{\lambda - B_2} + \frac{A_3}{\lambda - B_3}. \quad (2.15)$$

We note that

$$A_i = \left(\frac{a^0}{p^0 + p^0} + O(|\sigma|)\right)\tilde{\xi}_i^2, \quad (2.16)$$

$$B_i = r_i + O(\Pi_{j \neq i}|\sigma_i - \sigma_j|)\tilde{\xi}_i^2, \quad (2.17)$$

and in particular $B_1 > B_2 > B_3$. It is now clear that if none of the $A_i$ are zero, then the curve $\varphi = \varphi(\lambda)$ has three vertical asymptotes at $\lambda = B_i$. 

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which separate the intersection points with \( \varphi = 1 \), and in this case there can be no multiple roots for (2.14) (see Figure 1).

Thus if \( S_x \) has multiple points, they have to be on a coordinate plane, where one of the \( A_i \) vanishes. If \( A_i = 0 \), then the vertical line \( \lambda = B_i \) is no longer an asymptote but a branch of the curve \( \varphi = \varphi(\lambda) \). Thus the intersection points with the line \( \varphi = 1 \) are still separated, unless \( \lambda = B_i \) is between the other two asymptotes, in which case a multiple point is again possible (See Figure 1). To be more precise, on a coordinate plane \( \xi_i = 0 \), the characteristic equation (2.12) factors into a quadratic and a quartic:

\[
Q^1|_{\xi_i=0} = \delta_i(\alpha_j^2\delta_k + \alpha_k^2\delta_j + \delta_j\delta_k), \quad i, j, k \text{ distinct.}
\]

The quadratic curve \( \delta_i = 0 \) is the circle \( C_i \). The quartic curve

\[
Q_{jk} := (\alpha_j^2\delta_k + \alpha_k^2\delta_j + \delta_j\delta_k)|_{\xi_i=0} = 0, \quad (2.18)
\]
cannot have a multiple point, at least for small \( \sigma \), since

\[
Q_{jk}|_{\sigma=0} = (c_1^2(\xi_j^2 + \xi_k^2) - 1)(c_2^2(\xi_j^2 + \xi_k^2) - 1).
\]

Thus the only possibility for a multiple point is that the quartic curve \( Q_{jk} = 0 \) intersect the circle \( C_i \). Moreover, only one of the two branches of the quartic can intersect the circle, since at \( \sigma = 0 \), \( 1/c_1^2 < 1/c_2^2 = r_i^2 \), and thus there are at most four double characteristics in this case. In fact it is not hard to see that

\[
Q_{jk}|_{\xi_j^2+\xi_k^2=r_i^2} = aq(q-\rho-p)((\rho+p)(\sigma_j-\sigma_i)+q(\sigma_k-\sigma_j)\xi_j^2)+O(|\sigma|^2)\xi_j^4,
\]

so that the directions in which we can have a double root are close to the following four directions in the coordinate plane \( \xi_i = 0 \):

\[
(\pm \sqrt{\frac{\sigma_i-\sigma_j}{\sigma_k-\sigma_j}}, \pm \sqrt{\frac{\sigma_i-\sigma_k}{\sigma_j-\sigma_k}}, 0).
\]
For these directions to be real, it is necessary for \( \sigma_i \) to be the middle eigenvalue of \( \sigma \), i.e., \( i = 2 \).

In the exceptional case when two of the principal shears coincide, the slowness surface \( S_x \) is a surface of revolution. To see this, suppose that \( \sigma_2 = \sigma_3 = \sigma \neq 0 \), so that \( \sigma_1 = 1/(1 + \sigma)^2 - 1 \). We then have

\[
a_2 = a_3 = \tilde{a}, \quad b_2 = b_3 = 0, \quad d_2 = d_3 = d, \quad f_2 = f_3 = f, \quad \delta_2 = \delta_3 = \delta,
\]

where \( \tilde{a} := a + 2b\sigma \), \( d := \rho + p + \sigma q \), \( f := q(1 + \sigma) \) and \( \delta := f|\xi|^2 - d\xi_0^2 \).

Therefore,

\[
Q^1 = \delta(\alpha_1^2 \delta + (\alpha_2^2 + \alpha_3^2)\delta_1 + \delta\delta_1) := \delta \tilde{Q},
\]

where, introducing cylindrical coordinates \( R^2 = \xi_2^2 + \xi_3^2 \), \( Z = \xi_1 \) we have

\[
\frac{\partial \tilde{Q}}{\partial \xi_0} \bigg|_{\xi_0 = 1} (R, Z) = a_1 Z^2[(R^2 + Z^2)f - d] + [f_1 R^2 + (b_1 + f_1)Z^2 - d_1][(\tilde{a} + f)R^2 + fZ^2 - d],
\]

and thus the surface \( \tilde{Q} = 0 \) is cylindrically symmetric with respect to the \( \xi_1 \)-axis. Once again, this quartic surface cannot have any multiple points since at \( \sigma = 0 \) its sheets are well-separated, hence the only possible multiple points would be the intersections of the outer sheet of this surface with the sphere \( \delta = 0 \). But then from the above we must have \( R = 0 \) at the multiple points, so that in this case there are only two double characteristics, located at \((\pm \sqrt{d/f}, 0, 0)\).

The following theorem summarizes all that can be concluded from the above analysis regarding the geometry of the characteristic set of \( P^1 \):

**Theorem 2.1.**  
1. The characteristic set \( C_x^*(P^1) = \{\xi \mid Q^1(\xi) = 0\} \) is a conic set in \( T^*_xM \), the slice \( \xi_u = 1 \) of which is a 2-dimensional algebraic surface in \( \mathbb{R}^3 \) with three real sheets. The innermost sheet of this surface is strictly convex and is separated from the two outer ones.

2. In the generic case when the three principal shears are distinct, the two outer sheets touch at only four points. These points lie on a plane \( \xi \cdot E = 0 \), where \( E \) is the eigenvector corresponding to the middle eigenvalue. Moreover, in the coordinate frame given by the eigenvectors of \( \sigma \), the slowness surface possesses reflectional symmetry with respect to the three coordinate planes. The trace of the surface on a coordinate plane \( \xi_i = 0 \) is comprised of a circle \( C_i \) of radius \( r_i \) centered at the origin, together with a non-self-intersecting quartic.
curve which is a small perturbation of two concentric circles with radii $1/c_1$ and $1/c_2$ centered at the origin.

3. In the exceptional cases when two of the principal shears coincide, the slowness surface is a surface of revolution whose axis of symmetry is the eigendirection corresponding to the other eigenvalue. The two outer sheets, one of which is a sphere, touch only at two points on that axis.

Figure 2 shows the positive octant of the surface in the generic and the exceptional cases.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{slowness_surface.png}
\caption{The Slowness Surface.}
\end{figure}

### 2.4. Stability of the characteristic surface

We now return to the general case of a material with an arbitrary constitutive equation (1.17) which is subject only to conditions (2.9–2.10). The corresponding operator $P$, defined in (2.6), can be thought of as a (symmetric) perturbation of the operator $P^1$ introduced in the previous section: $P = P^1 + O(|\sigma|)$. We want to show that for small $|\sigma|$, the characteristic set of $P$ has the same qualitative shape as that of $P^1$, i.e., $P$ has only four non-degenerate double characteristics close to those of $P^1$.

The study of perturbations of non-strictly hyperbolic $3 \times 3$ systems was taken up by F. John in [8], where he studies the neighborhood, in the space of all second-order, linear, constant coefficient operators, of a specific non-strictly hyperbolic operator $J$ with four non-degenerate double characteristics:

$$J_{ik}(\lambda, \xi) := (\lambda^2 - \epsilon_i |\xi|^2)\delta_{ik} - (1 - \epsilon_i)\xi_i\xi_k, \quad \text{for } 0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < 1,$$

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which he calls the “modified elasticity operator.” He shows that for a general (nonsymmetric) perturbation of $J$ to be hyperbolic, it has to satisfy certain identities, since the space of all hyperbolic operators in that neighborhood has a positive codimension. From his proof it follows that any hyperbolic perturbation of $J$ must necessarily be non-strictly hyperbolic, and in fact has exactly four non-degenerate double characteristics close to those of $J$. We note that the slowness surface of John’s elasticity operator $J$ looks just as in Figure 2, which justifies naming the material after him.

Expanding on John’s work, the following general result on the stability of non-degenerate double characteristics was later obtained by Hörmander [7], and also independently by Bernardi and Nishitani [1]. We recall that a nonzero vector $\xi^0 \in \mathbb{R}^4$ is called a characteristic of $P \in \mathcal{P}$ if $\det P(\xi^0) = 0$. $\xi^0$ is simple if the zero of $\det P(\xi)$ at $\xi = \xi^0$ is of order one, and double if it is of order two. It is easy to see that at a double characteristic, the rank of the Hessian of $\det P$ is at most 3. If this rank is 3 then the double characteristic is called non-degenerate.

**Theorem (Hörmander).** — Let $P \in \mathcal{P}$ be hyperbolic and have a non-degenerate double characteristic at $\xi^0$, with $\dim \ker P(\xi^0) = 2$. Let $\bar{P} \in \mathcal{P}$ be close enough to $P$. If $\bar{P}$ is hyperbolic, then it also has a non-degenerate double characteristic, which is close to $\xi^0$.

Thus in our case, we need to examine the Hessian of $Q^1$ at the multiple points. It will turn out that in the generic case, the rank of the Hessian is indeed 3, and thus we can conclude that $P$ has four double characteristics close to those of $P^1$. In the exceptional case however, the rank of the Hessian is one, and thus the addition of higher-order terms to $P^1$ can change the shape of the characteristics.

We recall that $Q^1(\xi)$ is a homogeneous polynomial of degree six in $\xi_\mu$, $\mu = 0, \ldots, 3$. Since the Hessian $\nabla^2 Q^1$ is also homogeneous, it is enough to find the rank of its restriction to $\xi_0 = 1$. Let

$$H_{\mu \nu}(\xi_1, \xi_2, \xi_3) := (\partial_\mu \partial_\nu Q^1)|_{\xi_0 = 1}.$$
Let us assume that we are in the generic case, where the principal shears $\sigma_i$ are distinct. By renumbering the axes we can arrange that $\sigma_3$ is the middle one, so that the double characteristics set is

$$D^*_x = \{ \xi \in C^*_x(P^1) \mid \xi_3 = 0, \xi_1^2 + \xi_2^2 = r_3^2, Q_{12} = 0 \},$$

where $Q_{12}$ is the quartic defined in (2.18). Since $Q^1$ is in fact cubic in the squares of the $\xi_\mu$, there are no terms in $H$ which are linear in $\xi_3$, so that

$$H(\xi_1, \xi_2, 0) = \begin{pmatrix} \hat{H}(\xi_1, \xi_2) & 0 \\ 0 & H_{33} \end{pmatrix}.$$

Thus to show that $H$ is rank three on $D^*_x$ it is enough to show that at the double characteristics, $H_{33} \neq 0$ and that the rank of $\hat{H}$ is at least two.

Let $\hat{H}$ be the $2 \times 2$ submatrix in the lower right corner of $H$. Then $\hat{H}_{ij} = \partial_i \partial_j (Q(1, \xi_1, \xi_2, 0)), i,j = 1,2$. It then suffices to show that $\hat{H}$ is nonsingular. Now,

$$Q(1, \xi_1, \xi_2, 0) = \delta_3(1, \xi_1, \xi_2, 0)Q_{12}(1, \xi_1, \xi_2, 0),$$

Therefore

$$\frac{\partial Q_{12}}{\partial \xi_1} \bigg|_{\xi_3 = 0, Q_{12} = 0} = 2f_3(\xi_1 \partial_1 Q_{12} + \xi_2 \partial_2 Q_{12}),$$

which is rank 2 if $\xi$ and $\nabla Q_{12}$ are linearly independent, i.e., if the intersection of the quartic curve $Q_{12} = 0$ with the circle $C_3$ is transverse. It is easy to compute that at an intersection point,

$$\xi_1 \partial_2 Q_{12} - \xi_2 \partial_1 Q_{12} = a^0 q^0 (p^0 + p^0 - q^0)(\sigma_1 - \sigma_2) + O(|\sigma|^3),$$

which is nonzero since we have assumed $\sigma_1 \neq \sigma_2$. Hence $\hat{H}$ is nonsingular.

To calculate $H_{33}$, we note

$$H_{33} = \partial_3^2 Q^1(1, \xi_1, \xi_2, \xi_3) \bigg|_{\xi_3 = 0, \xi_1^2 + \xi_2^2 = r_3^2},$$

so that, from the expression (2.12) for $Q^1$ we have

$$H_{33} = 2 \left[ a_3 \delta_1 \delta_2 + (b_3 + f_3)(\alpha_2^2 \delta_1 + \alpha_1^2 \delta_2 + \delta_1 \delta_2) \right] \xi_1^2 + \xi_2^2 = r_3^2, \xi_3 = 0$$

$$= \frac{2a_3}{f_3} \left[ f_3 b_1 \xi_1^2 + f_1 d_3 - d_1 f_3 \right] [b_2(d_3 - f_3 \xi_2^2) + f_2 d_3 - d_2 f_3]$$

$$= 2a^0 (p^0 + q^0 - q^0)^2 (\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3) + O(|\sigma|^3),$$

which is again nonzero since we are in the generic case. We have thus shown that the four double characteristics of $P^1$ are non-degenerate, so that
by the above theorem of Hörmander, any hyperbolic perturbation of \( P^1 \) must also have exactly four double characteristics which are close to those of \( P^1 \). We have thus established the following:

**Theorem 2.2.** At a point \( x \in \mathcal{M} \) where the three principal shears \( \sigma_i \) are distinct, the hyperbolic operator whose symbol is \( P \) defined in (2.6) has exactly four non-degenerate double characteristics which are \( O(|\sigma|) \)-close to the following four directions

\[
\left( \frac{q^0}{\rho^0 + p^0}, \pm \sqrt{\frac{\sigma_3 - \sigma_1}{\sigma_2 - \sigma_1}}, \pm \sqrt{\frac{\sigma_3 - \sigma_2}{\sigma_1 - \sigma_2}}, 0 \right).
\]

### 3. Nonrelativistic Elastodynamics

**3.1. Nonrelativistic dynamics and the relationship between the two formulations**

Corresponding to the system \((1.6, 1.8)\), there is a first-order system of equations for the nonrelativistic dynamics of a solid body, which can be obtained through the following well-known procedure: One begins with the Minkowski metric for the spacetime \( \mathcal{M} \):

\[
ds^2 = -(dx^0)^2 + \sum_{i=1}^{3}(dx^i)^2,
\]

and introduces the Newtonian time coordinate \( t := x^0/c \), the Newtonian velocity \( v^i := cu^i/u^0 \), and the internal energy \( \epsilon := \rho - nc^2 \), where \( c \) denotes the speed of light in vacuum. Taking the limit \( c \to \infty \) in \((1.6, 1.8)\), one then obtains:

\[
(\partial_t v^i + v^k \partial_k v^i) + \partial_j S^{ij} = 0 \tag{3.1}
\]

\[
\partial_t h_{ij} + S_{,i} h_{ij} = 0 \tag{3.2}
\]

\[
\partial_t s + v^k \partial_k s = 0 \tag{3.3}
\]

where \( S \) is the stress tensor

\[
S_{ij} := 2\epsilon_1 h_{ij} + 2\epsilon_2 (q_1 h_{ij} - h^2_{ij}) + (2q_3 \epsilon_3 - \epsilon)\delta_{ij},
\]

\footnote{Here we are assuming that the material is made up of only one kind of (macroscopic) particle, the rest mass of which is normalized to be one.}
with \( \varepsilon_k := \partial \varepsilon / \partial q_k \). The equation of continuity is a consequence of (3.2):

\[
\partial_t n + \partial_t (nv^i) = 0. \tag{3.4}
\]

Using it, the momentum equation (3.1) can also be written in the form

\[
\partial_t (nv^i) + \partial_j (nv^i v^j + S^i_j) = 0. \tag{3.5}
\]

The adiabatic condition (3.3) is satisfied by \( C^1 \) solutions. More generally, the energy equation is satisfied:

\[
\partial_t (\varepsilon + \frac{1}{2} n|v|^2) + \partial_t \{ (\varepsilon + \frac{1}{2} n|v|^2)v^i + S^i_j v^j \} = 0.
\]

The above nonrelativistic equations can also be put in the same context as the relativistic ones by letting \( f \) be a mapping from the Galilean spacetime \( \mathcal{G} \), with coordinates \((t, x^i)\), into the material manifold \( \mathcal{N} \), which is now taken to be a region the three-dimensional Euclidean space, and let \( f_t \) denote its restriction to the time slice \( \{t\} \times \mathbb{R}^3 \subset \mathcal{G} \). We define \( h \) to be the pullback of the Euclidean metric \( \delta \) under \( f_t \), i.e.

\[
h_{ij}(t, x) = \delta_{ab}(y) \frac{\partial y^a}{\partial x^i} \frac{\partial y^b}{\partial x^j}, \tag{3.6}
\]

where \( y(t, x) = f_t(x) \in \mathcal{N} \). The internal energy per unit deformed volume \( \varepsilon \) is then assumed to be a function of the three principal invariants \( q_1, q_2, q_3 \) of \( h \), as well as depending on the entropy \( s \).

We thus have two different formulations for nonrelativistic elastodynamics: The one given in the beginning of this paper, which leads to a second-order system for the displacement field, and the one described above which gives a first-order system for the velocity and strain fields (reminiscent of the Euler equations of fluid dynamics). The relationship between the two formulations becomes clear when we note that the deformation \( \phi : \mathbb{R} \times \mathcal{N} \rightarrow \mathbb{R}^3 \) is the inverse of the mapping \( f \):

\[
x = \phi(t, y) = f_t^{-1}(y).
\]

Thus, with \( F \) denoting the deformation gradient as before, \( F^a_i = \partial x^i / \partial y^a \), from (3.6) we have that

\[
h = F^{-1}(F^{-1})^T = (F^T F)^{-1}.
\]

Therefore, the matrix \( h \) is the inverse of the classical right Cauchy-Green strain matrix. Since \( FF^T \) and \( F^T F \) have the same eigenvalues, we conclude
that the eigenvalues of $h$ are the reciprocals of the eigenvalues of $B$, and hence the following relations hold between the principal invariants of $B$ and $h$:

\[
q_1 = \frac{i_2}{i_3}, \quad q_2 = \frac{i_1}{i_3}, \quad q_3 = \frac{1}{i_3}.
\] (3.7)

Moreover, by comparing the expressions for the energy integral in the two formulations we can find the relationship between $W$ and $\varepsilon$:

\[
\int_{\mathcal{N}} W(y) d^3y = \int_{f_t^{-1}(\mathcal{N})} (W \circ f_t)(x) |D_x f_t| d^3x = \int_{\mathbb{R}^3} (W \circ f_t)(x) n(x) d^3x.
\]

The integrand in the last integral above must be equal to the energy per unit deformed volume $\varepsilon(x)$, and hence $W \circ f_t = \varepsilon/n$. In other words

\[
W(i_1, i_2, i_3) = \frac{1}{\sqrt{q_3}} \varepsilon(q_1, q_2, q_3).
\] (3.8)

It should be stressed that in the general-relativistic case one cannot use a formulation based on displacements, and thus the variational formulation is the only one available, while for nonrelativistic dynamics, either of the two formulations can be used, and it is conceivable that for a given problem, one is more appropriate than the other. Later on in this section we will have the opportunity to use both of these formulations.

### 3.2. Characteristics in the nonrelativistic case

We can calculate the symbol of the nonrelativistic equations (3.1–3.3) in a similar way. By going into a Galilean frame moving with the body we can set $v^k = 0$ at the point that the symbol is calculated. Taking variations we then obtain, with $(\tau, \xi)$ in the cotangent bundle to the Galilean spacetime,

\[
\tau(\dot{n}^i) + \xi_j \dot{S}^{ij} = 0
\]

\[
\tau \dot{h}_{ij} + (h_{jk} \xi_i + h_{ik} \xi_j) \dot{v}^k = 0
\]

Moreover, the symbol of the continuity equation (3.4) is

\[
\tau \dot{n} + \xi_k \dot{n} v^k = 0.
\]

Once again, $\tau = 0$ is a characteristic, and if $\tau \neq 0$ then $\dot{s} = 0$ and we obtain that $\dot{n} = -n \xi_k \dot{v}^k / \tau$. We can now introduce the shear tensor $\sigma$ by setting
\[ h_{ij} = n^{2/3}(\delta_{ij} + \sigma_{ij}). \]

Once again it follows that \( S_{ij} = p\delta_{ij} + q\sigma_{ij} + w\sigma_{2}^{2}, \)
with \( p, q, w \) as before. Continuing the calculation as in the relativistic case,
we obtain that if \((\tau, \xi)\) belong to the characteristic set, then \( M_{ik}\dot{\nu}^{k} = 0 \)
for some nonzero \( \dot{\nu} \), where

\[ M_{ik} = C_{ik}^{mn} (\sigma)\xi_{m}\dot{\xi}_{n} - n\delta_{ik}\tau^{2} \]

where \( C_{ik}^{mn} \) is exactly as in the relativistic case. Thus the only difference is
that the factor \( D_{i}(\sigma) \) of the relativistic case is here replaced by \( n \). This does
not affect the above analysis concerning the geometry of characteristics, and
all the above conclusions are thus valid in the nonrelativistic case as well.

3.3. Elastic waves in pre-stressed materials

In order to show that the formulation developed in the above for
relativistic dynamics is not without application to the nonrelativistic case,
we present here two results on the propagation of elastic waves in materials
which are under a constant isotropic deformation. The results are easy
extensions of two theorems by Sideris, one on the global existence of
small-amplitude waves in hyperelastic materials [14], and the other on the
formation of singularities for compressible Euler equations [13].

3.3.1. Small-Data Global Existence

In [14], Sideris considers the problem of global existence of nonlinear,
isentropic elastic waves in an isotropic, homogeneous, hyperelastic material
which is initially stress-free and is filling the whole space. He shows that if
the stored energy function \( W \) satisfies certain conditions —called the null
conditions, then the Cauchy problem for the displacement admits globally
regular solutions, provided the initial data is sufficiently small in some
high Sobolev norm. The null conditions are algebraic relations between the
derivatives of \( W \) evaluated at the stress-free reference state. Previously,
John had shown [9] that if these conditions do not hold, then at least in the
spherically symmetric case singularities will develop from arbitrary small
data after a very long time.

Here we provide an extension of Sideris’s result to the case where the
solid material under study is not initially stress-free but instead subjected
to an initial constant isotropic deformation \( y \mapsto \lambda y \) for some fixed \( \lambda > 0 \).
The proof consists of repeating Sideris’s argument while keeping track
of the factors of \( \lambda \), and thus will be largely omitted. It suffices to say
that, following his method, analogous null conditions can be derived in
which \( \lambda \) appears as a parameter. If we accept that global existence of
nonlinear waves should not depend on the exact amount of pre-stress, then
these conditions have to hold for all \( \lambda \) in some interval, which makes them differential equations in \( \lambda \). By making suitable assumptions about the stored energy function, we can solve these equations to obtain the general class of materials for which the global existence result holds. In this way we hope to shed some light on the null conditions of Sideris.

We use the classical formulation of elastodynamics, as in [14], and start with the system (1.1). We consider small displacements off a pre-stressed state by letting

\[
u(t, y) := \phi(t, y) - \lambda y
\]
denote the perturbation. Here \( \lambda < 1 \) corresponds to initial compression and \( \lambda > 1 \) to the initial extension of the material. Let \( G := D_y u = F - \lambda I \) and \( C := B - \lambda^2 I = \lambda(G + G^T) + GG^T \). Let \( j_1, j_2, j_3 \) denote the principal invariants of \( C \). We have

\[
\begin{align*}
\nu_1 &= 3\lambda^2 + j_1 \\
\nu_2 &= 3\lambda^4 + 2\lambda^2 j_1 + j_2 \\
\nu_3 &= \lambda^6 + \lambda^4 j_1 + \lambda^2 j_2 + j_3.
\end{align*}
\]

We can thus regard \( W \) as a function of the \( j_k \) instead of the \( \nu_k \), defining

\[
\Sigma(j_1, j_2, j_3) := W(\nu_1, \nu_2, \nu_3).
\]

Hence

\[
\frac{\partial W}{\partial F} = \frac{\partial W}{\partial G} = \frac{\partial \Sigma}{\partial j_k} \frac{\partial j_k}{\partial G}.
\]

Moreover, up to fourth-order terms in \( G \),

\[
\begin{align*}
j_1 &= 2\lambda \text{tr} G + \text{tr} GG^T \\
j_2 &= 2\lambda^2(\text{tr} G)^2 - \lambda^2 \text{tr} G^2 - \lambda^2 \text{tr} GG^T + 2\lambda \text{tr} G \text{tr} GG^T - 2\lambda \text{tr} G^2 G^T + \ldots \\
j_3 &= \frac{4}{3}\lambda^3(\text{tr} G)^3 - 2\lambda^3 \text{tr} G(\text{tr} G^2 + \text{tr} GG^T) + \frac{2}{3}\lambda^3 \text{tr} G^3 + 2\lambda^3 \text{tr} G^2 G^T + \ldots
\end{align*}
\]

so that, upon Taylor-expanding \( \Sigma \) in powers of \( j_k \) around \( j_k = 0 \) and substituting in (1.1), keeping only terms of up to second order in \( u \) and its derivatives \(^5\), we obtain the following equation for the displacement \( u \):

\[
Lu = N(u, u),
\]

\(^5\) It was shown by Klainerman [11] that cubic terms in the nonlinearity do not affect the small-data global existence in elastodynamics.
where \( N \) denotes the quadratic terms, and \( L \) is the linear elasticity operator defined in (1.2), with the two propagation speeds

\[
c_1^2 = 2(\Sigma_1 + 2\lambda^2 \Sigma_{11}), \quad c_2^2 = 2(\Sigma_1 - \lambda^2 \Sigma_2).
\]

(3.10)

Here and later on \( \Sigma_k, \Sigma_{mn}, \) etc. denote the partial derivatives of \( \Sigma \) with respect to the \( j_k \)'s, evaluated at the isotropic background state \( j_1 = j_2 = j_3 = 0 \).

In Sideris's work [14, §4], the quadratic nonlinearity \( N(u, u) \) is analyzed in order to identify the terms in it with bad decay. Those terms are then eliminated by putting conditions on \( \Sigma \). Following that procedure, it is not hard to see that in our case, the null conditions of Sideris have the form

\[
\Sigma_{11} = \lambda^2 \Sigma_{12}, \quad 3 \Sigma_{12} + 2 \Sigma_{111} = 0, \quad \forall \lambda > 0.
\]

(3.11)

We note that here, \( \Sigma_{11}, \Sigma_{12}, \Sigma_{111} \) are functions of the parameter \( \lambda \). The extension of Sideris's result is therefore the following:

**Theorem 3.1.** — Assume that the stored energy function of a hyperelastic solid material satisfies the null conditions (3.11). Then the corresponding Cauchy problem for (3.9) with initial data that are small enough \(^6\) has a unique global solution.

Our task is now to identify a class of materials where the null conditions are satisfied. In the above we have assumed the perturbation \( \alpha \) to be small, while no smallness assumption has been made about the background isotropic pressure. We are thus in a situation where the strains are not uniformly small. This motivates us to use the nonrelativistic version of the shear strain tensor \( \sigma \):

\[
\sigma = (\det h)^{-1/3} h - I = (\det B)^{1/3}(F^T F)^{-1} - I,
\]

so that \( \sigma = 0 \) if \( F \) is a multiple of the identity. The invariants of \( B \) and \( \sigma \) are related as follows:

\[
\tau = \tau_3^{-2/3} \tau_2 - 3, \quad \delta = \tau_3^{-2/3} \tau_2 - \tau_3^{-1/3} \tau_1.
\]

and we have \( n = \tau_3^{-1/2} \) as before. Note that \( j_1 = j_2 = j_3 = 0 \) corresponds now to \( \tau = \delta = 0 \), i.e. the isotropic background state. Moreover, by (3.8),

\[
\Sigma(j_1, j_2, j_3) = W(\tau_1, \tau_2, \tau_3) = e(n, \tau, \delta),
\]

\(^6\) See [14, Thm. 5.1.] for the precise smallness condition.
where \( e \) is again the energy per particle \(^7\). If we assume as before that \( e \) has an expansion of the form (1.17), then we can calculate explicitly the partial derivatives of \( \Sigma \) that appear in (3.10) and (3.11) in terms of the parameter \( \lambda \). The remainder term in the expansion (1.17) will not affect this calculation since only derivatives of up to the third order are needed:

\[
\begin{align*}
\Sigma_1 &= -\frac{1}{2} \lambda^{-5} e_0' (\lambda^{-3}) \\
\Sigma_2 &= -\frac{1}{2} \lambda^{-7} e_0'' (\lambda^{-3}) - \lambda^{-4} e_2 (\lambda^{-3}) \\
\Sigma_{11} &= \frac{1}{4} \lambda^{-10} e_0'' (\lambda^{-3}) + \frac{3}{4} \lambda^{-7} e_0' (\lambda^{-3}) + \frac{2}{3} \lambda^{-4} e_2 (\lambda^{-3}) \\
\Sigma_{12} &= \frac{1}{4} \lambda^{-12} e_0'' (\lambda^{-3}) + \frac{3}{4} \lambda^{-9} e_0' (\lambda^{-3}) + \frac{4}{3} \lambda^{-6} e_2 (\lambda^{-3}) + \frac{1}{2} \lambda^{-9} e_2' (\lambda^{-3}) \\
&\quad + \frac{1}{3} \lambda^{-6} e_3 (\lambda^{-3}) \\
\Sigma_{111} &= -\frac{1}{8} \lambda^{-15} e_0''' (\lambda^{-3}) - \frac{9}{8} \lambda^{-12} e_0'' (\lambda^{-3}) - \frac{15}{8} \lambda^{-9} e_0' (\lambda^{-3}) \\
&\quad - \frac{20}{9} \lambda^{-6} e_2 (\lambda^{-3}) - \lambda^{-9} e_2' (\lambda^{-3}) - \frac{4}{9} \lambda^{-6} e_3 (\lambda^{-3}).
\end{align*}
\]

Setting \( n = \lambda^{-3} \), the null conditions (3.11) thus take the form of two ordinary differential equations for three unknown functions:

\[
\begin{align*}
3n e_2'(n) + 4e_2(n) + 2e_3(n) &= 0 \\
9n^3 e_0'''(n) + 54n^2 e_0''(n) + 54ne_0'(n) + 18ne_2'(n) + 16e_2(n) - 4e_3(n) &= 0.
\end{align*}
\]

Thus \( e_2 \) and \( e_3 \) can be found in terms of \( e_0 \):

\[
\begin{align*}
e_2(n) &= cn^{-1} - \frac{3}{8} n^2 e_0''(n) - \frac{9}{8} ne_0'(n) \\
e_3(n) &= -\frac{c}{2} n^{-1} + \frac{9}{16} n^3 e_0'''(n) + \frac{57}{16} n^2 e_0''(n) + \frac{63}{16} ne_0'(n),
\end{align*}
\]

with \( c \) an arbitrary constant. The arbitrary function \( e_0 \) is subject to further conditions to ensure the hyperbolicity of equations, or equivalently, the linear stability: The propagation speeds \( c_1 \) and \( c_2 \), have to be real, with \( c_1 > c_2 > 0 \). This implies that there must exist a constant \( C > 0 \) such that

\[
\begin{align*}
(n^3 e_0'(n))' &< C \\
3n^{4/3}(n^5/3 e_0'(n))' &> -C
\end{align*}
\]

\(^7\) Since we are in the isentropic case the dependence on the entropy is suppressed throughout this section.
In addition, it is reasonable to assume that the stress-free state \( n = 1, \tau = \delta = 0 \) is a minimum of the energy, so that

\[
e_0(1) = 0, \quad e'_0(1) = 0. \quad (3.16)
\]

We thus arrive at the following:

**Proposition 1.** — If the stored energy function \( \Sigma \) of a homogeneous, isotropic, hyperelastic solid satisfies (1.18), then the null conditions (3.11) hold for \( \Sigma \) if and only if the coefficients \( e_i \) in the expansion (1.17) satisfy (3.12–3.15).

**Remark 2.** — It should be noted that conditions (3.14–3.15) are overly restrictive. In particular, they do not allow for \( e_0(n) \to \infty \) as \( n \to \infty \), contrary to what is often assumed about stored energy functions. In fact, by integrating (3.14) it is easy to see that the following is a necessary condition for \( e_0 \):

\[
e_0(n) \leq C \left( 1 - \frac{1}{n} \right)^2.
\]

**Remark 2.** — As a simple example of materials satisfying the null conditions, we may consider the class of Hadamard materials, where \( W \) has the following form

\[W(n_1, n_2, n_3) = a n_1 + b n_2 + f(n_3).\]

It is then easy to see that \( W \) satisfies the null conditions (3.11) if and only if

\[f(n_3) = c n_3 + d_1 \sqrt{n_3} + d_2,
\]

where

\[d_1 := -2(a + 2b + c), \quad d_2 := b + c - a.\]

The corresponding coefficients in the expansion (1.17) are as follows:

\[
e_3(n) = -an^{-2/3}
\]
\[
e_2(n) = an^{-2/3} + bn^{-4/3}
\]
\[
e_0(x) = 3an^{-2/3} + 3bn^{-4/3} + cn^{-2} + d_1 n^{-1} + d_2.
\]

The two speeds of propagation (measured with respect to coordinates in the undeformed state) are thus

\[
c_2^2 = 2n^{2/3} e_2(n) = 2a + 2bn^{-2/3},
\]
\[
c_1^2 = n^{2/3} [n^2 e_0'(n) + \frac{8}{3} e_2(n)] = 2a + 4bn^{-2/3} + 2cn^{-2},
\]
and hence the hyperbolicity requirement is

\[ a > 0, \quad b > 0, \quad c > 0. \]

3.3.2. Large-Data Blowup

In this section we provide an example of a Cauchy problem for the nonrelativistic system (3.1–3.3), whose \( C^1 \) solutions cannot have infinite lifespan. This is an extension to elastodynamics of Sideris’s large-data blowup result [13] for compressible Euler equations, and the proof follows his original argument very closely. We consider solid materials with a constitutive equation of the form

\[ e = e_0 + \tau e_2 + \delta e_3, \tag{3.17} \]

where \( e_i \) satisfy the following conditions:

- (M1) \( \partial e_2/\partial n \equiv 0 \), i.e., \( e_2 = e_2(s) \).
- (M2) \( e_2(s) > 0 \) for all \( s > 0 \).
- (M3) \( \partial e_3/\partial n \equiv 0 \), i.e., \( e_3 = e_3(s) \).

Let \( S \) denote the corresponding stress tensor. We define the “pressure”:

\[ \bar{p} := \frac{1}{3} \text{tr} S = n^2 \frac{\partial e}{\partial n} = n^2 \frac{\partial e_0}{\partial n}. \]

and assume that

- (M4) \( \bar{p}(n, s) \) is an increasing, convex function of \( n \), for each \( s > 0 \).
- (M5) \( \bar{p}(n, s) \) is a non-decreasing function of \( s \), for each \( n > 0 \).

We propose initial data that represent a compactly-supported perturbation of some quiet background isotropic state, i.e., we assume that

- (D1) There exists \( R_0 > 0 \) such that \( v(0, x) \equiv 0 \), \( h(0, x) \equiv n_0^{2/3} I \), and \( s(0, x) = s_0 \), for \( |x| > R_0 \), where \( n_0 \) and \( s_0 \) are positive constants.

By a Domain of Dependence argument, it follows that as long as the solution remains \( C^1 \),

\[ v(t, x) \equiv 0, \quad h(t, x) \equiv n_0^{2/3} I, \quad s(t, x) \equiv s_0, \quad \text{for } |x| > R(t), \tag{3.18} \]

where

\[ R(t) := R_0 + c_1^0 t, \]

and \( c_1^0 \) is the speed of propagation of longitudinal waves in the quiet background, i.e.

\[ c_1^0 := \sqrt{\eta_0^2 + \frac{8}{3} e_2(s_0)}, \quad \text{where } \eta_0^2 := \frac{\partial \bar{p}}{\partial n}(n_0, s_0). \]
Let
\[ Q(t) := \int_{\mathbb{R}^3} x^i v_i(t, x) n(t, x) \, d^3x \] (3.19)
be the total radial momentum, which is finite by (3.18). Let \( p_0 \) denote the background pressure,
\[ p_0 := \bar{p}(n_0, s_0), \]
and let \( S_0 \) denote the background stress tensor,
\[ S_0^{ij} := p_0 \delta^{ij}. \]

By (3.5) we then have
\[
\frac{dQ}{dt} = -\int_{|x| \leq R(t)} x_i \partial_j (n v^i v^j + S^{ij}) \, d^3x = -\int_{\mathbb{R}^3} x_i \partial_j (n v^i v^j + S^{ij} - S_0^{ij}) \, d^3x
\]
\[ = \int_{\mathbb{R}^3} \delta_{ij} (n v^i v^j + S^{ij} - S_0^{ij}) \, d^3x = \int_{\mathbb{R}^3} n|v|^2 + 3(\bar{p} - p_0), \]

Let
\[ M(t) := \int_{\mathbb{R}^3} (n(t, x) - n_0) \, d^3x. \]
Then by (3.4) we have \( dM/dt \equiv 0 \), so that \( M \) is conserved: \( M(t) = M(0) = M_0 \). Our further assumptions on the initial data are that

(D2) \( M_0 \geq 0 \).

(D3) \( s(0, x) \geq s_0 \) for all \( x \).

The adiabatic condition (3.3) shows that the entropy \( s \) remains constant along the flow lines, and thus (D3) implies that \( s(t, x) \geq s_0 \) for \( t > 0 \) as well. Hence, by (M4) and (M5),
\[
\bar{p}(n, s) - p_0 = \bar{p}(n, s) - \bar{p}(n, s_0) + \bar{p}(n, s_0) - \bar{p}(n_0, s_0) \geq \eta_0^2 (n - n_0).
\]

It then follows that
\[
\int_{\mathbb{R}^3} (\bar{p} - p_0) \, d^3x \geq \eta_0^2 M_0 \geq 0,
\]
so that we have
\[
\frac{dQ}{dt} \geq \int_{\mathbb{R}^3} n|v|^2 \, d^3x.
\]
Therefore $Q$ is an increasing function of $t$. In particular $Q(t) \geq 0$ for all $t > 0$ if $Q_0 := Q(0) \geq 0$. On the other hand

$$|Q|^2 \leq \int n|x|^2d^3x \int n|v|^2d^3x \leq R^2(t)[M_0 + \frac{4\pi n_0}{3}R^3(t)] \int n|v|^2d^3x,$$

so that we arrive at the following differential inequality:

$$\frac{dQ}{dt} \geq \frac{Q^2}{M_0 R^2(t) + \frac{4\pi}{3} n_0 R^5(t)}.$$

Integrating this, we have

$$\frac{1}{Q(t)} \leq \frac{1}{Q_0} - \frac{1}{c_1^0} \int_{R_0 + c_1^0 t}^{\infty} \frac{dR}{M_0 R^2 + \frac{4\pi}{3} n_0 R^5}.$$

This contradicts the positivity of $Q$ provided our last assumption on the initial data holds, i.e., if

$$(D4) \quad Q_0 > c_1^0 \left( \int_{R_0}^{\infty} \frac{dR}{M_0 R^2 + \frac{4\pi}{3} n_0 R^5} \right)^{-1}.$$ The contradiction implies that there exists a certain $T^* < \infty$ by which time a $C^1$ solution has to develop a singularity. In particular, the domain-of-dependence may break down at an earlier time, perhaps because a shock discontinuity forms. We have thus proved:

**Theorem 3.2.** – Suppose that the constitutive equation of a hyperelastic solid is of the form (3.17) and satisfies (M1–M5). Then the corresponding Cauchy problem for (3.1)–(3.3) with initial data satisfying (D1–D4) cannot have a global-in-time $C^1$ solution.

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**References**


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