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## **Planar Permutation Graphs**

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## Planar Permutation Graphs <sup>(1)</sup>

by

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### INTRODUCTION

One of the best known graphs in all of graph theory is the Petersen graph, shown in Figure 1, named after the Swedish mathematician. Petersen [3] proved that every cubic bridgeless graph contains a 1-factor. He also showed that not every such graph is 1-factorable by exhibiting a counterexample which has become classic.

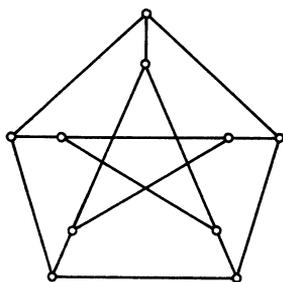


FIG. 1. — The Petersen graph.

This graph consists of two disjoint cycles of length 5 (a pentagon and a pentagram) joined by 5 additional lines. This is made clear in Figure 2 *a* as we see how the two cycles are linked. This graph is then redrawn in Figure 2 *b* to produce a labeling of the familiar Petersen graph shown in Figure 1.

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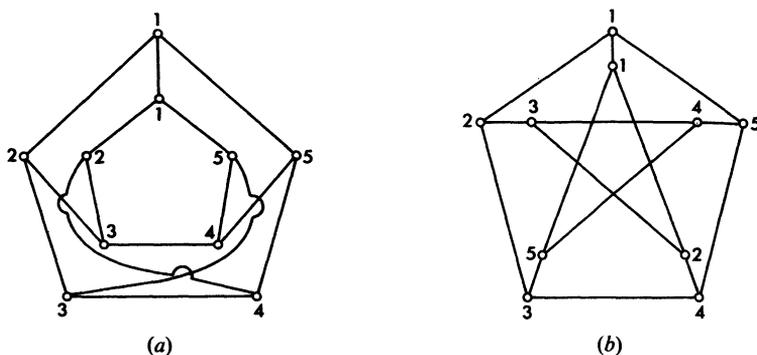


FIG. 2. — The first permutation graph.

The points of each of the two copies of  $C_5$  (the cycle of length 5) are labeled cyclically 1 through 5, with the points of the exterior cycle joined to the points of the interior cycle according to the rule

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}.$$

Thus the numbers on the top row of the permutation  $\alpha$  correspond to the exterior cycle and those in the second row to the interior cycle with a point  $i$  on the exterior cycle joined to a point  $j$  on the interior cycle if  $\alpha(i) = j$ . Therefore, the Petersen graph can be regarded as two disjoint copies of  $C_5$  joined according to this permutation  $\alpha$ . Looking at the Petersen graph from this viewpoint, we are led to the following, more general concept.

Consider two identical disjoint copies of a labeled graph  $G$  with  $p$  points. The  $\alpha$ -permutation graph  $P_\alpha(G)$  consists of these two copies of  $G$  along with  $p$  additional lines joining these graphs according to a given permutation  $\alpha$  on  $N_p = \{1, 2, \dots, p\}$ . A graph  $H$  is a permutation graph if there exists a labeled graph  $G$ , having  $p$  points, and a permutation  $\alpha$  on the set  $N_p$  such that  $H = P_\alpha(G)$ . We note that the graph  $P_\alpha(G)$  depends not only on the choice of the permutation  $\alpha$  but on the particular labeling of  $G$  as well. In fact, there are four permutation graphs which can be obtained from  $C_5$ : the Petersen graph which is known to be nonplanar (see [1]), the pentagonal prism (Figure 3a) which is planar, and the two nonplanar graphs in Figure 3b. Certainly, more than one permutation may result in the same permutation graph; indeed, there are 10 permutations which produce the Petersen graph as there are for the pentagonal prism, and each of the graphs in Figure 3b can be obtained from 50 permu-

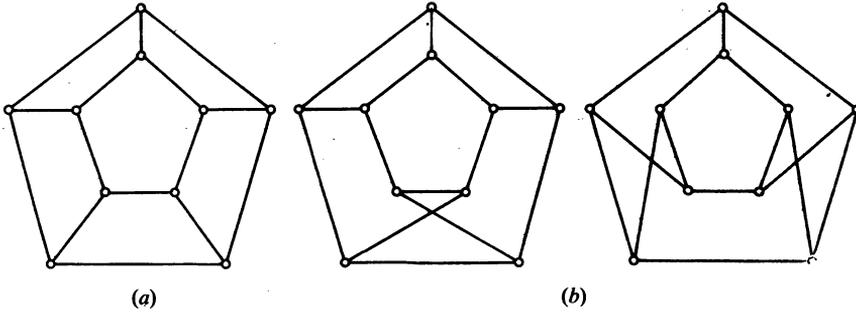


FIG. 3. — The other permutation graphs of  $C_5$ .

tations. For example, if the points of  $C_5$  are labeled cyclically, then the prism results from either the identity permutation or the cyclic permutation (1 2 3 4 5).

### PLANAR PERMUTATION GRAPHS

Although  $C_5$  is obviously planar, we have seen that some permutation graphs of  $C_5$  are planar and others nonplanar. We develop a criterion for a permutation graph of a cycle as well as any other 2-connected graph to be planar.

A graph  $G$  is *homeomorphic from*  $H$  if it is possible to insert points of degree two into the lines of  $H$  to produce  $G$  (A graph  $G_1$  is *homeomorphic with*  $G_2$  if there exists a graph  $G_3$  which is homeomorphic from both  $G_1$  and  $G_2$ ). It is convenient to state in the following form the well-known theorem of Kuratowski [2]. A graph is planar if and only if it contains no subgraph homeomorphic from the complete graph  $K_5$  or from the complete bigraph  $K_{3,3}$ .

Given that  $P_\alpha(G)$  is planar, it is certainly clear that  $G$  is also planar since it is a subgraph of  $P_\alpha(G)$ . Furthermore,  $G$  must have the added property that it can be embedded in the plane so that all its points bound some region of  $G$ . Without loss of generality, we may assume this region to be exterior. If  $G$  did not have this property, then no matter how the points of the two copies of  $G$  are joined in the forming of a permutation graph, at least one of the added lines must cross some line in one of the copies of  $G$  so that  $P_\alpha(G)$  would be nonplanar. A connected graph having at least 3 points which can be embedded in the plane so that all its points lie on the exterior region will be called *outerplanar*. A disconnected

graph is considered outerplanar if all its components are. Of course every outerplanar nonseparable graph is hamiltonian. It is easy to see that all graphs with less than 6 lines are outerplanar. While all graphs with 6 lines are planar, there are two connected graphs among them which fail to be outerplanar, namely, the complete graph  $K_4$  and the « theta-graph »  $K_{2,3}$ .

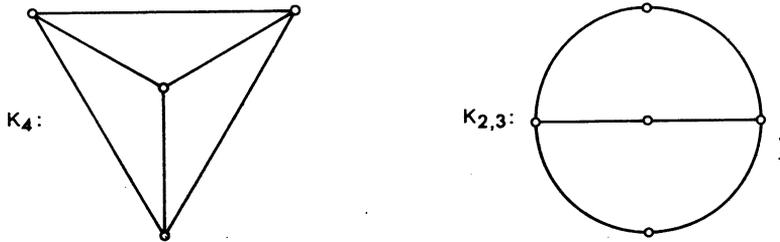


FIG. 4. — The prototypes of non-outerplanar graphs.

**THEOREM 1.** A graph  $G$  is outerplanar if and only if it contains no subgraph homeomorphic from  $K_4$  or  $K_{2,3}$ .

*Proof.* It is obvious that  $G$  is not outerplanar if it contains a subgraph homeomorphic from  $K_4$  or from  $K_{2,3}$ .

To prove the converse, let  $G$  contain no subgraph homeomorphic from  $K_4$  or from  $K_{2,3}$  but assume  $G$  is not outerplanar. If  $G$  is nonplanar, then, by Kuratowski's theorem, it contains a subgraph homeomorphic from  $K_5$  or from  $K_{3,3}$ , so it certainly contains one homeomorphic from  $K_4$  or from  $K_{2,3}$ . Hence,  $G$  is planar. Since  $G$  is not outerplanar, it must contain a block  $B$  with more than two points, which is not outerplanar. Embed  $B$  in the plane so that a maximum number of points lie on the exterior cycle  $Z$ . Since  $Z$  is not hamiltonian, there is at least one point which lies in the interior of  $Z$ . Let  $u$  be a point interior to  $Z$  which is adjacent to a point  $v_1$  on  $Z$ . Since  $B$  is a block,  $\deg u \geq 2$ . Hence, there is a path  $P$  from  $u$  to some other point  $v_2$  on  $Z$ . There are two cases to consider.

**Case 1. Points  $v_1$  and  $v_2$  are consecutive on  $Z$ .**

In this case, some point of  $P$  different from  $v_2$  must have degree at least 3; otherwise, the path could be transferred outside of  $Z$  to produce a planar embedding of  $B$  having a longer exterior cycle. Thus, there is a path from

a point of  $P$ , say  $w$ , to a point  $v_3$  of  $Z$  not containing any other point of  $P$  (See Figure 5a). The lines of  $Z$  and the 3 paths from  $w$  to  $Z$  induce a subgraph of  $B$  homeomorphic from  $K_4$ .

**Case 2. Points  $v_1$  and  $v_2$  are not consecutive on  $Z$ .**

Clearly, the lines of  $Z$  and those of the path through  $u$  from  $v_1$  to  $v_2$  constructed in Case 1 induce a subgraph homeomorphic from  $K_{2,3}$  (see Figure 5b), completing the proof.

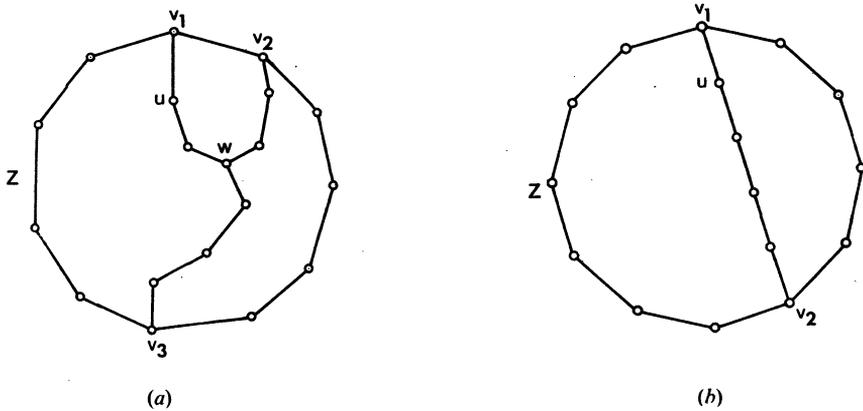


FIG. 5. — Homeomorphs of  $K_4$  and  $K_{2,3}$ .

We now return to the Petersen graph and ask which permutations applied to a given cycle or, more generally, to a given nonseparable outerplanar graph  $G$  result in a planar permutation graph. This may depend on how  $G$  is labeled. Since  $G$  is outerplanar, it can be embedded in the plane so that its exterior cycle  $Z$  is hamiltonian. If we label the points of  $Z$  cyclically, 1 through  $p$ , then we say that  $G$  is « cyclically labeled ». It is convenient to assume that every nonseparable outerplanar graph is cyclically labeled. One sees that every nonseparable outerplanar graph has  $2p$  cyclic labelings,  $p$  labelings of the points in cyclic clockwise order and  $p$  more counterclockwise.

Obviously, the number of ways of constructing a planar permutation graph from two disjoint copies of a nonseparable outerplanar graph with  $p$  points is the same as that of obtaining a planar permutation graph

from two copies of  $C_p$ , namely  $2p$ , the number of permutations is the dihedral group  $D_p$  of degree  $p$  generated by the two permutations:

$$\alpha_1 = \begin{pmatrix} 1 & 2 & \dots & p-1 & p \\ 2 & 3 & \dots & p & 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 & & & 2 & \dots & p \\ p & p-1 & \dots & 1 & & \end{pmatrix}$$

When  $\alpha \in D_p$ , we say  $\alpha$  is *dihedral*.

We now summarize these observations.

LEMMA. Given a nonseparable outerplanar graph  $G$ , the permutation graph  $P_\alpha(G)$  is planar if and only if  $\alpha$  is dihedral.

Combining Theorem 1 and the lemma, we arrive at a characterization of planar permutation graphs of nonseparable graphs.

THEOREM 2. The permutation graph  $P_\alpha(G)$  of a nonseparable graph  $G$  is planar if and only if  $G$  is outerplanar and  $\alpha$  is dihedral.

In general, the conclusion of Theorem 2 does not follow for connected outerplanar graphs with cutpoints, showing the necessity of the hypothesis that  $G$  is nonseparable. For example, consider the chain  $W_n$  with  $n$  points. It is easy to verify that all 24 permutation graphs of  $W_4$  are planar, not just those obtained from the 8 permutations in  $D_4$ . This is not so for  $C_5$  since  $P_\alpha(C_5)$  is not planar when

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{pmatrix},$$

for it contains a subgraph homeomorphic from  $K_{3,3}$ .

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