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Extreme value distribution for the M/G/1 and the G/M/1 queueing systems


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Extreme value distribution for the M/G/1 and the G/M/1 queueing systems

by

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SUMMARY. — For the supremum of the virtual delay time in a busy cycle and for the supremum of the actual waiting times of the customers served in a busy cycle the Laplace-Stieltjes transforms of the distribution functions have been found recently. Also for the supremum of the number of customers simultaneously present in the system during a busy cycle the generating function of the distribution is known. For every one of these variables the limit distribution of the maximum of these variables over a finite number of busy cycles is derived in the present paper. These limit distributions are obtained for the queueing systems M/G/1 and G/M/1 and for traffic intensities equal to one and less than one.

1. SOME RELATIONS FOR THE M/G/1 SYSTEM

For the M/G/1 queueing system denote by \( v_t \) the virtual waiting time at time \( t \), by \( x_t \) the number of customers in the system at time \( t \) and by \( w_n \) the actual waiting time of the \( n \)th arriving customer with \( w_1 = 0 \). Further \( \zeta \) will denote the duration of a busy cycle and \( n \) the number of customers served in a busy cycle. Define

\[
\begin{align*}
V_{\text{max}} & \triangleq \sup_{0 < t < \zeta} v_t, \\
W_{\text{max}} & \triangleq \sup_{1 \leq n \leq n} w_n, \\
X_{\text{max}} & \triangleq \sup_{0 < t < \zeta} x_t,
\end{align*}
\]
so that $\varepsilon_{\text{max}}$ is the supremum of the virtual waiting time in a busy cycle, $w_{\text{max}}$ is the supremum of all actual waiting times of a busy cycle and $x_{\text{max}}$ the maximum number of customers simultaneously present in a busy cycle.

Denoting by $B(t)$ the distribution function of the service times and by $\alpha$ the average interarrival time then with

$$\beta(\rho) = \int_0^\infty e^{-\rho t} dB(t), \; \Re \rho \geq 0, \; B(0+) = 0, \; \beta = \int_0^\infty t dB(t) < \infty,$$

we have

$$\Pr \{ \varepsilon_{\text{max}} < \nu \} = \frac{1}{2\pi i} \oint_{C_n} \frac{e^{\nu \eta} }{\beta(\eta) + \alpha \eta - 1} d\eta, \; \Re \eta > \delta; \quad \nu > 0,$$

$$= 0, \quad \nu < 0,$$

$$\Pr \{ w_{\text{max}} < w \} = \frac{1}{2\pi i} \oint_{C_n} \frac{e^{\nu \eta} }{\beta(\eta) + \alpha \eta - 1} \frac{d\eta}{1 - \alpha \eta} \frac{1}{\beta(\eta)}, \; w > 0,$$

$$= 0, \quad w < 0,$$

and for $x = 1, 2, \ldots,$

$$\Pr \{ x_{\text{max}} \leq x \} = \frac{1}{2\pi i} \oint_{D_\omega} \frac{d\omega}{\omega^{x+1}} \frac{1}{\beta(\omega)} \frac{1}{\beta(\omega) - \omega} \frac{1}{\beta(\omega) - (1 - \omega)} \frac{1}{\beta(\omega) - (1 - \omega)}.$$

Here we used the notation

$$\frac{1}{2\pi i} \oint_{C_n} \ldots d\eta = \lim_{b \to \infty} \int_{R - ib}^{R + ib} \ldots d\eta, \; \Re = \Re \eta,$$

and $D_\omega$ is a circle in the complex $\omega$-plane with center at $\omega = 0$ and radius $|\omega|$, the positive direction of integration being counter clockwise. By $\delta$ is denoted the larger zero of $\beta(\eta) + \alpha \eta - 1$ with $\Re \eta \geq 0$, while $\mu$ is the smaller
zero inside or on the unit circle of \( \beta \left\{ \frac{1}{\alpha} (1 - \omega) \right\} - \omega \). It is well known (cf. Takacs [1]) that if \( a \overset{\text{def}}{=} \beta / \alpha \leq 1 \) then \( \delta = 0 \), \( \mu = 1 \); the zeros \( \delta \) and \( \mu \) have multiplicity one if \( a \neq 1 \), if \( a = 1 \) they have multiplicity two. The relations (1.1) and (1.3) have been derived by Takacs [2] and by Cohen [3], [4], [5], while the relation (1.2) has been obtained by Cohen [6].

Let \( w \) and \( x \) be stochastic variables with distribution functions given by

\[
(1.4) \quad E \{ e^{-\rho w} \} = \frac{(1 - a) \alpha \rho}{\beta(\rho) + \alpha \rho - 1}, \quad \Re \rho \geq 0, \quad a < 1,
\]

\[
(1.5) \quad E \{ \omega^x \} = (1 - a) \frac{\beta \left\{ \frac{1}{\alpha} (1 - \omega) \right\}}{\beta \left\{ \frac{1}{\alpha} (1 - \omega) \right\} - \omega}, \quad |\omega| \leq 1, \quad a < 1,
\]

so that the distribution of \( w \) is the stationary distribution of the (virtual or actual) waiting time for the M/G/1 queue, and the distribution of \( x \) is the stationary distribution of the number of customers present in the M/G/1 queueing system.

Further let \( \sigma \) be a negative exponentially distributed variable with expectation \( \alpha \) and \( r \) a variable with distribution function \( B(t) \). Assume that \( w \) and \( \sigma \) are independent, and also that \( w \) and \( r \) are independent. It follows from (1.1), ..., (1.5) that for \( a < 1 \),

\[
\Pr \{ v_{\max} < v \} = \frac{\Pr \{ w + r < v \}}{\Pr \{ w < v \}}, \quad v > 0,
\]

\[
\Pr \{ w_{\max} < w \} = \frac{\Pr \{ w < w \}}{\Pr \{ w < w + \sigma \}}, \quad w > 0,
\]

\[
\Pr \{ x_{\max} \leq x \} = 1 - \frac{\Pr \{ x = x \}}{\Pr \{ x \leq x \}}, \quad x = 0, 1, \ldots
\]

From (1.1) for \( v > 0 \), \( \Re \eta > 0 \), \( a \leq 1 \),

\[
(1.6) \quad 1 - \Pr \{ v_{\max} < v \} = \frac{\alpha}{2 \pi i} \int_{C_n} e^{\eta v} \frac{\eta d\eta}{\beta(\eta) + \alpha \eta - 1};
\]

\[
\frac{1}{2 \pi i} \int_{C_n} e^{\eta v} \frac{d\eta}{\beta(\eta) + \alpha \eta - 1};
\]
from (1.2) for \( w > 0, \frac{1}{\alpha} > \Re \eta > 0, a \leq 1, \)

\[
1 - \Pr \{ w_{\text{max}} < w \} = \frac{1}{2\pi i} \int_{C_\eta} \frac{e^{\eta w}}{e^{\eta w} - \beta(\eta) + \alpha \eta - 1} \eta d\eta
\]

(1.7)

and from (1.3) for \( x = 2, 3, \ldots, |\omega| < 1, \)

\[
1 - \Pr \{ x_{\text{max}} \leq x \} = \frac{1}{2\pi i} \int_{D_\omega} \frac{1 - \omega}{e^{\eta w} - \beta(\eta) + \alpha \eta - 1} \omega d\omega
\]

(1.8)

Define

\[
H(t) \overset{\text{def}}{=} \frac{1}{\beta} \int_0^t \{ 1 - B(\tau) \} d\tau, \quad h(t) \overset{\text{def}}{=} \frac{1}{\beta} \{ 1 - B(t) \}, \quad t > 0, \\
= 0, \quad t < 0,
\]

so that \( H(t) \) is a distribution function having a bounded and monotone density function \( h(t) \). Define for \( a \leq 1 \)

\[
K(t, a) \overset{\text{def}}{=} \sum_{n=0}^{\infty} a^n H^n(t),
\]

(1.9)

so that

\[
\int_0^\infty e^{-\eta t} d_{\eta} K(t, a) = \frac{\alpha \eta}{\beta(\eta) + \alpha \eta - 1}, \quad \Re \eta > 0.
\]

(1.10)

Obviously, \( K(t, 1) \) is the renewal function of a renewal process with \( H(t) \) as renewal distribution. Since \( H(t) \) has a density which is monotone and bounded \( K(t, a) \) has for \( a \leq 1 \) a bounded derivative \( k(t, a) \) (cf. Feller [7], p. 358) and

\[
k(t, a) = \frac{d}{dt} K(t, a), \quad t > 0,
\]

(1.11)

\[
\int_0^\infty e^{-\eta t} k(t, a) dt = \frac{\alpha \eta}{\beta(\eta) + \alpha \eta - 1} - 1, \quad \Re \eta > 0.
\]

(1.12)
Since for $a \leq 1$
\[
\int_{0}^{\infty} K(w + \tau, a)e^{-\tau/a} \frac{d\tau}{a} = e^{w/a} \int_{t = w}^{\infty} e^{-t/a} K(t, a) \frac{dt}{a},
\]
we have for $w > 0$, $a \leq 1$,
\[
\frac{d}{dw} \int_{0}^{\infty} K(w + \tau, a)e^{-\tau/a} \frac{d\tau}{a} = e^{w/a} \int_{t = w}^{\infty} e^{-t/a} k(t, a) \frac{dt}{a},
\]
It is easily seen that for $a \leq 1$, $0 < \Re \eta < \frac{1}{\alpha},$
\[
\int_{0}^{\infty} e^{-\eta t/dw} \int_{0}^{\infty} K(w, t, a)e^{-t/a} \frac{dt}{a} = \frac{1}{1 - \alpha \eta} \frac{\alpha \eta}{\beta(\eta) + \alpha \eta - 1},
\]
Further for $|\omega| < 1$, $x = 0, 1, \ldots,$
\[
\frac{1}{2\pi i} \int_{D \omega} \frac{d\omega}{\omega^{x+1}} \frac{1 - \omega}{\beta \left\{ \frac{1}{\alpha (1 - \omega)} \right\} - \omega} = \int_{0}^{\infty} \frac{(t/x)^x}{x!} e^{-t/2} d(t, a),
\]
From (1.6), (1.7) and (1.8) it follows easily by using the inversion formula for the Laplace-Stieltjes transform that for $a \leq 1,$
\[
1 - \Pr \{ v_{\max} < v \} = \alpha \frac{k(v, a)}{K(v, a)} = \alpha \frac{d}{dv} \log K(v, a), \quad v > 0,
\]
\[
= \alpha \frac{d}{dv} \log \Pr \{ w < v \} \quad \text{if} \quad a < 1;
\]
\[
1 - \Pr \{ w_{\max} < w \} = \frac{\int_{0}^{\infty} k(w + t, a)e^{-t/a} dt}{\int_{0}^{\infty} K(w + t, a)e^{-t/a} dt}
\]
\[
= \frac{\alpha}{d} \log \int_{0}^{\infty} K(w + t, a)e^{-t/a} dt,
\]
\[
= \frac{\alpha}{d} \log \Pr \{ w < w + \alpha \} \quad \text{if} \quad a < 1, \quad w > 0,
\]
From the relations

\[ E\{v_{\max}\} = \int_0^\infty \{1 - \Pr\{v_{\max} < v\}\} \, dv, \]

\[ E\{v_{\max}^2\} = 2 \int_0^\infty v \{1 - \Pr\{v_{\max} < v\}\} \, dv, \]

and

\[ \Pr\{w < 0 +\} = 1 - a \text{ if } a < 1, \]

it is found that for \(a < 1\) (cf. (1.16) and (1.17))

\[ E\{v_{\max}\} = \frac{\beta}{a} \log \frac{1}{1 - a}, \]

\[ E\{v_{\max}^2\} = -2 \frac{\beta}{a} \int_0^\infty \log \{1 - \Pr\{w \geq v\}\} \, dv, \]

\[ E\{w_{\max}\} = \frac{\beta}{a} \log \frac{1}{1 - a}, \]

\[ E\{w_{\max}^2\} = -2 \frac{\beta}{a} \int_0^\infty \log \Pr\{w < w + \sigma\} \, dw. \]

Since

\[ E\{v_{\max}^2\} = 2 \frac{\beta}{a} \sum_{n=1}^\infty \int_0^\infty \frac{1}{n} \Pr\{w \geq w\} \, dw, \]

and

\[ \Pr\{w \geq w\} < a, \]

we have

\[ 2 \frac{\beta}{a} \int_0^\infty \Pr\{w \geq w\} \, dw < E\{v_{\max}^2\} < 2 \frac{\beta}{a} \sum_{n=1}^\infty \frac{a^{n-1}}{n} \int_0^\infty \Pr\{w \geq w\} \, dw \]

so that since

\[ \int_0^\infty \Pr\{w \geq w\} \, dw = \frac{1}{2} \frac{a \beta}{1 - a \beta^2}, \]
with $\beta_2$ the second moment of $B(t)$, we obtain

$$\frac{\beta_2}{1 - a} < E \{ \xi_{\max}^2 \} < \frac{\beta_2}{1 - a} \log \frac{1}{1 - a}.$$  

It is seen that the second moment of $\xi_{\max}$ is finite if $\beta_2 < \infty$, a similar conclusion holds for $w_{\max}$ and $x_{\max}$. It is noted that $E \{ w \}$ is finite if $\beta_2 < \infty$, while $E \{ w \}$ is finite if $a < 1$ and $\beta_2 < \infty$.

2. EXTREME VALUE DISTRIBUTIONS FOR M/G/1

Suppose the server is idle at time $t = 0$. Denote by $\xi_{\max}$, $w_{\max}$ and $x_{\max}$ the supremum of $\xi_j$, of $w_j$ and of $x_j$ in the $j$th busy cycle of the queueing system M/G/1, $j = 1, 2, \ldots$. Obviously, $\xi_{\max}$, $j = 1, 2, \ldots$, are independent, identically distributed variables with finite first moment if $a < 1$ and with finite second moment if $\beta_2 < \infty$. If $a < 1$ then the strong law of large numbers applies for the sequence $\xi_{\max}$, $j = 1, 2, \ldots$; whereas if $\beta_2 < \infty$ the central limit theorem applies also for this sequence. Similar statements hold for the other sequences $w_{\max}$, $j = 1, 2, \ldots$, and $x_{\max}$, $j = 1, 2, \ldots$

Define for $n = 1, 2, \ldots$,  

$$V_n \overset{\text{def}}{=} \max_{1 \leq j \leq n} \xi_{\max}, \quad W_n \overset{\text{def}}{=} \max_{1 \leq j \leq n} w_{\max}, \quad X_n \overset{\text{def}}{=} \max_{1 \leq j \leq n} x_{\max},$$

i.e., $V_n$ is the supremum of the virtual waiting time in $n$ busy cycles, $W_n$ that of the actual waiting times in $n$ busy cycles and $X_n$ the supremum of the number of customers present simultaneously in the system during $n$ busy cycles. For these variables we shall derive some limit theorems.

**Theorem 1.** — If $a = 1$ and $\beta_2$, the second moment of $B(t)$, is finite then the distributions of $\frac{1}{n\beta} V_n$, of $\frac{1}{n\beta} W_n$ and of $\frac{1}{n} X_n$ all converge for $n \to \infty$ to the distribution $G(x)$ with

$$G(x) = e^{-x^{-1}} \quad \text{for} \quad x > 0, \quad = 0 \quad \text{for} \quad x < 0.$$  

Proof. Since $\beta_2/2\beta$ is the first moment of $H(t)$, and since $h(t)$ is monotone we have from renewal theory (cf. Feller [7], p. 358)

$$\lim_{t \to \infty} \frac{K(t, 1)}{t} = \frac{2\beta}{\beta_2}, \quad \lim_{t \to \infty} k(t, 1) = \frac{2\beta}{\beta_2}.$$
Hence from (1.16) since \( a = 1 \)

\[
(2.2) \quad \lim_{v \to \infty} v \{ 1 - \Pr \{ x_{\max} < v \} \} = \alpha = \beta.
\]

From this relation and from

\[
\Pr \left\{ \frac{1}{n\beta} V_n < x \right\} = [\Pr \{ v_{\max} < n\beta x \}]^n = \left\{ 1 - \frac{\beta}{n\beta x} + o\left(\frac{1}{n}\right) \right\}^n, \quad x > 0,
\]

for \( n \to \infty \) it follows immediately that

\[
\lim_{n \to \infty} \Pr \left\{ \frac{1}{n\beta} V_n < x \right\} = e^{-x^{-1}}, \quad x > 0,
\]

\[
= 0, \quad x < 0,
\]

and the statement for \( V_n \) has been proved.

From (1.17) for \( a = 1 \)

\[
(2.4) \quad 1 - \Pr \{ w_{\max} < w \} = \alpha \int_{0}^{\infty} k(w + \tau, 1)e^{-\frac{\tau}{\alpha}} \frac{d\tau}{\alpha} \quad w > 0.
\]

For given \( \varepsilon > 0 \) a finite number \( W(\varepsilon) > 0 \) exists such that

\[
\left| k(w, 1) - \frac{2\beta}{\beta_2} \right| < \varepsilon \quad \text{for all} \quad w > W(\varepsilon),
\]

so that

\[
\left| k(w + t, 1) - \frac{2\beta}{\beta_2} \right| < \varepsilon \quad \text{for all} \quad w > W(\varepsilon), \quad t \geq 0.
\]

Consequently, since \( k(t, 1) \) is bounded

\[
\lim_{w \to \infty} \int_{0}^{\infty} k(t + w, 1)e^{-\frac{t}{\alpha}} \frac{dt}{\alpha} = \frac{2\beta}{\beta_2} \int_{0}^{\infty} e^{-\frac{t}{\alpha}} \frac{dt}{\alpha} = \frac{2\beta}{\beta_2}.
\]

Using (2.1) the same argumentation yields

\[
\lim_{w \to \infty} \int_{0}^{\infty} K(w + t, 1)e^{-\frac{t}{\alpha}} \frac{dt}{\alpha} = \lim_{w \to \infty} \int_{0}^{\infty} K(w + t, 1)e^{-\frac{t}{w + t}} \left\{ 1 + \frac{t}{w} \right\} \frac{dt}{\alpha} = \frac{2\beta}{\beta_2}.
\]

Hence from (2.4)

\[
\lim_{w \to \infty} w \{ 1 - \Pr \{ w_{\max} < w \} \} = \alpha = \beta,
\]

so that, as above the statement for \( W_n \) follows.
For \( x = 1, 2, \ldots \),
\[
\int_{\mathcal{W}(e)}^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} \left| k(t, 1) - \frac{2\beta}{\beta_2} \right| \frac{dt}{\alpha} \leq \varepsilon \int_{\mathcal{W}(e)}^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} \frac{dt}{\alpha} \leq \varepsilon,
\]
and
\[
\lim_{x \to \infty} \int_{0}^{\mathcal{W}(e)} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} k(t, 1) \frac{dt}{\alpha} \leq \max_{0 \leq t \leq \mathcal{W}(e)} k(t, 1), \lim_{x \to \infty} \int_{0}^{\mathcal{W}(e)} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} dt = 0.
\]
It follows
\[
\lim_{x \to \infty} \int_{0}^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} \left( k(t, 1) - \frac{2\beta}{\beta_2} \right) \frac{dt}{\alpha} = 0,
\]
or
\[
\lim_{x \to \infty} \int_{0}^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} k(t, 1) \frac{dt}{\alpha} = \frac{2\beta}{\beta_2}.
\]
In the same way it is shown that
\[
\lim_{x \to \infty} \int_{0}^{\infty} \frac{(t/\alpha)^{x}}{x!} e^{-t/\alpha} k(t, 1) \frac{dt}{t} \frac{1}{\alpha} = \frac{2\beta}{\beta_2}.
\]
Hence from (1.12)
\[
\lim_{x \to \infty} x \{ 1 - \Pr \{ X^{(j)}_{\text{max}} \leq x \} \} = 1,
\]
the last relation leads as above to the statement for \( X_n \). The theorem is proved.

**Theorem 2.** — If \( a < 1, \rho_0 > 0 \) and \( -\rho_0 \) is the abcissa of convergence of \( \beta(\rho) \) and if \( \beta(-\rho_0 + 0) = \infty \) then for \( -\infty < x < \infty \),
\[
\lim_{n \to \infty} \Pr \left\{ \frac{1}{\beta} V_n < \frac{x + \log(nb_1)}{-\varepsilon \beta} \right\} = e^{-e^{-x}},
\]
\[
\lim_{n \to \infty} \Pr \left\{ \frac{1}{\beta} W_n < \frac{x + \log(nb_2)}{-\varepsilon \beta} \right\} = e^{-e^{-x}},
\]
\[
\lim_{n \to \infty} \Pr \left\{ X_n < \frac{x + \log(nb_3)}{\log(1 - \alpha \varepsilon)} \right\} = e^{-e^{-x}},
\]
with
\[
b_1 = \frac{\alpha - \beta}{\alpha + \beta'(\varepsilon) \alpha \varepsilon}, \quad b_2 = \frac{\alpha - \beta}{\alpha + \beta'(\varepsilon) (1 - \alpha \varepsilon)}, \quad b_3 = \frac{\alpha - \beta}{\alpha + \beta'(\varepsilon) \alpha \varepsilon (1 - \alpha \varepsilon)},
\]
\[
\beta'(\rho) = -\int_{0}^{\infty} te^{-\rho t} dB(t), \quad \text{Re} \rho > -\rho_0,
\]
and $\varepsilon$ is the zero of $\beta(\eta) + \alpha\eta - 1$, $\Re \eta < 0$ which is nearest to the imaginary axis $\Re \eta = 0$.

Proof. Since $\rho_0 > 0$ and $\alpha < 1$, the function $\beta(\eta) + \alpha\eta - 1$ has for $\Re \eta < 0$ a real zero. Denote by $\varepsilon$ its real zero nearest to the axis $\Re \eta = 0$. Clearly $\varepsilon > -\rho_0$. From

$$|\beta(\eta)| \leq \beta(\Re \eta) = 1 - \alpha \varepsilon < |1 - \alpha\eta| \quad \text{for} \quad \Re \eta = \varepsilon, \eta \neq \varepsilon,$$

it follows that $\varepsilon$ is the only zero with $\Re \eta = \varepsilon$. From

$$|\beta(\eta)| \leq \beta(\Re \eta) < |1 - \alpha\eta| \quad \text{for} \quad \Re \eta > \varepsilon$$

and from Rouche's theorem it is seen that $\beta(\eta) + \alpha\eta - 1$ has only one zero with $\Re \eta > \varepsilon$; this zero is $\eta = 0$. Hence $\varepsilon$ is the zero with $\Re \eta < 0$ nearest to the axis $\Re \eta = 0$. Moreover, $\varepsilon$ is a single zero, since

$$\beta'(\varepsilon) + \alpha = \beta'(\varepsilon) + \frac{1 - \beta(\varepsilon)}{\varepsilon} = -\sum_{n=1}^{\infty} \int_0^{\infty} \frac{(e^n - 1)}{n!} dB(t) < 0,$$

the series being convergent. If $\beta(\eta) + \alpha\eta - 1$ has a second zero $\varepsilon_1$ with $\Re \varepsilon_1 < 0$ then $-\rho_0 < \Re \varepsilon_1 < \varepsilon$. Let $C_\xi$ be a line parallel to the imaginary axis with $\Re \varepsilon_1 < \Re \xi < \varepsilon$ if $\varepsilon_1$ exists, otherwise $-\rho_0 < \Re \xi < \varepsilon$. The function $\beta(\eta) + \alpha\eta - 1$ is analytic for $\Re \eta > \Re \xi$ and has single zeros at $\eta = \varepsilon$ and $\eta = 0$. From Cauchy's theorem it follows for

$$\Re \eta > 0 > \varepsilon > \Re \xi > \Re \varepsilon_1 > -\rho_0$$

$$\frac{\alpha}{2\pi i} \int_{C_\eta} e^{\eta\beta(\eta) + \alpha\eta - 1} = \frac{\alpha e^{\eta\beta(\eta) + \alpha\eta - 1}}{\beta(\varepsilon) + \alpha\eta - 1} + \frac{\alpha}{2\pi i} \int_{C_\xi} e^{\xi\beta(\eta) + \alpha\eta - 1}, \quad v > 0,$$

$$\frac{1}{2\pi i} \int_{C_\eta} e^{\eta\beta(\eta) + \alpha\eta - 1} = \frac{1}{\alpha - \beta} + \frac{1}{\alpha + \beta'\varepsilon} e^{\varepsilon\beta(\eta) + \alpha\eta - 1}, \quad v > 0.$$

It is easily verified that

$$\lim_{v \to \infty} \frac{\alpha e^{\xi\beta(\eta) + \alpha\eta - 1}}{\beta(\varepsilon) + \alpha\eta - 1} = 0, \quad \lim_{v \to \infty} \frac{\alpha e^{\xi\beta(\eta) + \alpha\eta - 1}}{\beta(\varepsilon) + \alpha\eta - 1} = 0.$$

Hence from (1.6) we obtain

$$\lim_{v \to \infty} e^{-\varepsilon v} \{ 1 - \Pr \{ \xi_{\text{max}} < v \} \} = \frac{\alpha - \beta}{\alpha + \beta'\varepsilon} \alpha \varepsilon = b_1 > 0.$$
Therefore
\[ \Pr \left\{ \frac{1}{\beta} V_n < \frac{x + \log (nb_1)}{-\varepsilon \beta} \right\} = \left[ \Pr \left\{ \frac{x + \log (nb_1)}{b_{\max}} < \frac{x + \log (nb_1)}{-\varepsilon \beta} \right\} \right]^n, \]
so that for \( n \to \infty \)
\[ \Pr \left\{ \frac{1}{\beta} V_n < \frac{x + \log (nb_1)}{-\varepsilon \beta} \right\} = \left[ 1 - b_1 e^{-x - \log(nb_1)} + o \left( \frac{1}{n} \right) \right]^n, \]
i.e.
\[ \lim_{n \to \infty} \Pr \left\{ \frac{1}{\beta} V_n < \frac{x + \log (nb_1)}{-\varepsilon \beta} \right\} = e^{-e^{-x}}, \quad -\infty < x < \infty. \]
This proves the statement for \( V_n \), that for \( W_n \) is proved in the same way.
The statement for \( X_n \) is also analogous. Start from (1.3) and move the path of integration \( D_\omega \) to a circle with radius \( |\omega| > 1 \) and such that the first zero of
\[ \beta \left\{ \frac{1}{\alpha (1 - \omega)} \right\} - \omega \]
outside the circle \( |\omega| = 1 \) is an interior point of this circle.

Corollary to theorem 2. For \( a < 1 \) the variables \( \frac{1}{\beta \log n} V_n \), \( \frac{1}{\beta \log n} W_n \) and \( \frac{X_n}{\log n} \) converge for \( n \to \infty \) in probability to \( -\frac{1}{\varepsilon \beta}, -\frac{1}{\varepsilon \beta} \) and \( \frac{1}{\log (1 - ax)} \), respectively.
Proof. For every fixed \( x > 0 \) it follows from theorem 2 that for \( n \to \infty \)
\[ \Pr \left\{ \left| \frac{1}{\beta \log n} V_n + \frac{1}{\varepsilon \beta} + \frac{\log b_1}{\varepsilon \beta \log n} \right| > \frac{x}{-\varepsilon \beta \log n} \right\} \to \{ e^{-e^{-x}} + 1 - e^{-e^{-x}} \}, \]
so that for every \( z > 0 \)
\[ \Pr \left\{ \left| \frac{1}{\beta \log n} V_n + \frac{1}{\varepsilon \beta} + \frac{\log b_1}{\varepsilon \beta \log n} \right| > \frac{z}{-\varepsilon \beta} \right\} \to e^{-n^z} + 1 - e^{-n^z} \to 0 \]
for \( n \to \infty \),
and hence the statement for \( V_n \) follows; the other statements are proved similarly.

During a busy cycle a realisation of \( V_t \) may have a number of intersections with level \( K \). There are no intersections at all if during the busy cycle the virtual delay time is always less than \( K \). Denote by \( \Pi_k^{(j)} \) the number of intersections from above with level \( K \) of \( V_t \) in the \( j \)th busy cycle, \( j = 1, 2, \ldots \).
Obviously, the variables $\Pi_j^0, j = 1, 2, \ldots$, are independent and identically distributed variables. It has been shown in [8] that if $a \leq 1$ then

$$\Pr \{ \Pi_j^0 = m \} = f(0), \quad m = 0,$$

$$= \{ 1 - f(0) \} \{ 1 - h(0) \}^{m-1} h(0), \quad m = 1, 2, \ldots,$$

where

$$f(0) = \Pr \{ x_{\text{max}} < K \},$$

$$h(0) = \left[ \frac{1}{2\pi i} \int_{C_n} e^{\eta K} \frac{x\eta}{\beta(\eta) + x\eta - 1} \right]^{-1}, \quad \text{Re}\ \eta > 0.$$  

Denote by $E_K$ the state with $K$ customers left behind in the system at a departure. Let $\Delta_j^0$ represent the number of times that state $E_K$ occurs during the $j$th busy cycle. Obviously, $\Delta_j^0, j = 1, 2, \ldots$, are independent and identically distributed variables. It has been shown in [9] that if $a \leq 1$ then

$$\Pr \{ \Delta_j^0 = m \} = f(1), \quad m = 0,$$

$$= \{ 1 - f(1) \} \{ 1 - h(1) \}^{m-1} h(1), \quad m = 1, 2, \ldots,$$

where

$$f(1) = \Pr \{ x_{\text{max}} \leq K \},$$

$$h(1) = \left[ \frac{1}{2\pi i} \int_{D_\omega} d\omega \frac{\beta \left\{ \frac{1}{\alpha} (1 - \omega) \right\}}{\omega^{K+1} \beta \left\{ \frac{1}{\alpha} (1 - \omega) \right\} - \omega} \right]^{-1}, \quad |\omega| < 1.$$  

Define

$$P_{K,n} \coloneqq \max_{1 \leq j \leq n} \Pi_j^0, \quad L_{K,n} \coloneqq \max_{1 \leq j \leq n} \Delta_j^0,$$

then we have :

**Theorem 3.** — If $a \leq 1$ then

$$\lim_{n \to \infty} \Pr \left\{ \frac{x + \log \left\{ n \frac{1 - f(0)}{1 - h(0)} \right\}}{-\log \left\{ 1 - h(0) \right\}} < P_{K,n} \right\} = e^{-e^{-x}}, \quad -\infty < x < \infty,$$

$$\lim_{n \to \infty} \Pr \left\{ \frac{x + \log \left\{ n \frac{1 - f(1)}{1 - h(1)} \right\}}{-\log \left\{ 1 - h(1) \right\}} < L_{K,n} \right\} = e^{-e^{-x}}, \quad -\infty < x < \infty.$$  

**Proof.** It is easily verified that

$$\Pr \{ \Pi_j^0 \geq m \} = \frac{1 - f(0)}{1 - h(0)} \exp \{ m \log (1 - h(0)) \}.$$
from which the statement of the theorem follows as in the preceding theorem. Similarly for \(L_{K,n}\).

As before we obtain.

**Corollary to theorem 3.** For \(a \leq 1\) the variables \(\frac{P_{K,n}}{\log n}\) and \(\frac{L_{K,n}}{\log n}\) converge for \(n \to \infty\) in probability to

\[
\frac{1}{-\log \{1 - h(0)\}} \quad \text{and} \quad \frac{1}{-\log \{1 - h(1)\}},
\]

respectively.

It is noted that if \(B(t) = 1 - e^{-t/\beta}\) for \(t > 0\) then \(\varepsilon \beta = -(1 - a), b_1 = (1 - a), b_2 = a(1 - a), b_3 = a^{-1}(1 - a), 1 - \alpha \varepsilon = a^{-1},\)

\[
f(0) = \frac{1 - e^{-(1-a)K/\beta}}{1 - ae^{-(1-a)K/\beta}}, \quad h(0) = \frac{1 - a}{1 - ae^{-(1-a)K/\beta}}, \quad a < 1,
\]

\[
= \frac{K/\beta}{1 + K/\beta}, \quad a = 1,
\]

\[
f(1) = \frac{1 - a^K}{1 - a^{K+1}}, \quad h(1) = \frac{1 - a}{1 - a^{K+1}}, \quad a < 1,
\]

\[
= \frac{K}{1 + K}, \quad a = 1.
\]

### 3. Extreme Value Distributions for G/M/1

Denote by \(A(t)\) the distribution function of the interarrival times for the queueing system G/M/1;

\[
\alpha(\rho) = \int_0^\infty e^{-\rho t}dA(t), \quad \Re \rho \geq 0, \quad A(0 +) = 0, \quad \alpha = \int_0^\infty tdA(t) < \infty.
\]

For the system G/M/1 the variables \(v_{\max}, w_{\max}\) and \(x_{\max}\) will have the same meaning as those for the system M/G/1, and similarly for \(V_n, W_n\) and \(X_n\).

For \(a \leq 1\) we have (cf. Cohen [3], [5], [6]),

\[
(3.1) \quad 1 - \Pr \{ v_{\max} < v \} = 0, \quad v < 0,
\]

\[
= \left\{ \frac{1}{2\pi i} \int_{C_{\varepsilon}} e^{\xi v} \frac{\beta d\xi}{\alpha(\xi) + \beta \xi - 1} \right\}^{-1}, \Re \varepsilon > \psi, \quad v > 0,
\]

\[
(3.2) \quad 1 - \Pr \{ w_{\max} < w \} = 0, \quad w < 0,
\]

\[
= \left\{ \frac{1}{2\pi i} \int_{C_{\varepsilon}} e^{\xi v} \frac{\beta d\xi}{1 - \beta \xi \alpha(\xi) + \beta \xi - 1} \right\}^{-1} \frac{1}{\beta} \quad \Re \varepsilon > \psi, \quad w > 0,
\]

\[
(3.3) \quad 1 - \Pr \{ x_{\max} \leq x \}
\]

\[
= \left\{ \frac{1}{2\pi i} \int_{D_{\omega}} \frac{d\omega}{\omega^{\varphi}} \frac{1}{\alpha \left\{ \frac{1 - \omega}{\beta (1 - \omega)} \right\} - \omega} \right\}^{-1}, \quad |\omega| < \varphi, \quad x = 1, 2, \ldots,
\]

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here $\Psi$ is the larger zero of $\alpha(\xi) + \beta \xi - 1$ with $\Re \xi \geq 0$, and $\phi$ is the smaller zero of $\alpha \left\{ \frac{1}{\beta} (1 - \omega) \right\} - \omega$ with $|\omega| \leq 1$. If $a = 1$ then $\psi = 0$, $\phi = 1$, whereas for $a < 1$ both $\phi$ and $\psi$ are positive with multiplicity one. Put

$$N(t) \overset{\text{def}}{=} 0, \quad t < 0,$$

$$= \int_0^t \left\{ 1 - A(u) \right\} \frac{du}{\alpha}, \quad t > 0,$$

and

$$M(t) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \left\{ N(t) \right\}^{n*}, \quad t > 0,$$

so that $M(t)$ is the renewal function of the renewal process with $N(t)$ as renewal distribution. As in section 1 (cf. the derivation of (1.10), . . . , (1.12)) we have from (3.1), . . . , (3.3) for $a = 1$

$$1 - \Pr \left\{ V_{\max} < v \right\} = \left\{ M(v) \right\}^{-1}, \quad v > 0,$$

$$1 - \Pr \left\{ W_{\max} < w \right\} = \left\{ \int_0^\infty M(w + t)e^{-t/\beta} \frac{dt}{\beta} \right\}^{-1}, \quad w > 0,$$

$$1 - \Pr \left\{ X_{\max} \leq x \right\} = \left\{ \int_0^\infty \frac{(t/\beta)^{x-1}}{(x-1)!} e^{-t/\beta} M(t) \frac{dt}{\beta} \right\}^{-1}, \quad x = 1, 2, \ldots$$

If the second moment $\alpha_2$ of $A(t)$ is finite then from renewal theory

$$\lim_{t \to \infty} \frac{M(t)}{t} = \frac{2\alpha}{\alpha_2}.$$

The same argumentation as used in the proof of theorem 1 leads immediately to

THEOREM 4. — If $a = 1$ and $\alpha_2 < \infty$ then the distribution functions of

$$\frac{2\alpha}{n\alpha_2} V_n$$

and of

$$\frac{2\alpha^2}{n\alpha_2} X_n$$

all converge to $G(x)$ for $n \to \infty$.

Further

THEOREM 5. — If $a < 1$ then for $-\infty < x < \infty$,

$$\lim_{n \to \infty} \Pr \left\{ \frac{1}{\beta} V_n < \frac{x + \log (nc_1)}{\psi \beta} \right\} = e^{-e^{-x}},$$

$$\lim_{n \to \infty} \Pr \left\{ \frac{1}{\beta} W_n < \frac{x + \log (nc_2)}{\psi \beta} \right\} = e^{-e^{-x}},$$

$$\lim_{n \to \infty} \Pr \left\{ X_n < \frac{x + \log (nc_1)}{-\log (1 - \alpha \psi)} \right\} = e^{-e^{-x}},$$
with \[
c_1 = \frac{\alpha'(\psi) + \beta}{\beta}, \quad c_2 = \frac{\alpha'(\psi) + \beta}{\beta} (1 - \beta \psi),
\]
\[
\alpha'(\rho) = - \int_0^\infty te^{-\rho t} dA(t), \quad \Re \rho \geq 0.
\]

Proof. From (3.1) we have for \( \Re \zeta > \psi > \Re \eta > 0, \nu > 0 \)
\[
\frac{1}{2\pi i} \oint_{C_\zeta} e^{v \zeta} \frac{\beta d\zeta}{\alpha(\zeta) + \beta \zeta - 1} = \frac{\beta e^{\psi \nu}}{\alpha'(\psi) + \beta} + \int_{C_n} e^{v \eta} \frac{\beta d\eta}{\alpha(\eta) + \beta \eta - 1},
\]
so that, since \( \alpha(\eta) + \eta \beta - 1 \) has no zeros for \( 0 < \Re \eta < \psi \), it immediately follows from (3.1) that
\[
\lim_{v \to \infty} e^{v \psi} \left\{ \Pr \left\{ \Upsilon_{\max} < \psi \right\} \right\} = \frac{\alpha'(\psi) + \beta}{\beta} = \frac{1}{\beta} \left\{ \alpha'(\psi) + \frac{1 - \alpha(\psi)}{\psi} \right\} > 0.
\]

From this relation the statement for \( V_n \) follows as in the proof of theorem 2. The proof of the statement for \( W_n \) is similar. To prove the statement for \( X_n \), move the path of integration \( D_\infty \) to a circle with radius \( |\zeta| \) and such that \( \varphi < |\zeta| < 1 \), and observe that \( \varphi = 1 - \alpha \psi \). The statement for \( X_n \) is now easily derived.

**Corollary to theorem 5.** For \( a < 1 \) the variables \( \frac{1}{\beta \log n} \frac{V_n}{\psi}, \frac{1}{\beta \log n} \frac{W_n}{\psi} \) and \( \frac{X_n}{\log n} \) converge for \( n \to \infty \) in probability to \( \frac{1}{\psi \beta^2}, \frac{1}{\psi \beta} - \log (1 - \alpha \psi) \), respectively.

The proof is analogous to that of the corollary of theorem 2 in the preceding section.

**REFERENCES**


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