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A simplified proof of the Sevastyanov theorem on branching processes


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by

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SUMMARY. — The paper presents a new proof of a well known theorem (Sevastyanov) on necessary and sufficient conditions for the degeneration of a branching processes with \( n \) types of particles.

SUMMAIRE. — Cet article présente une démonstration nouvelle d'un théorème bien connu (Sevastyanov) sur les conditions nécessaires et suffisantes pour la dégénérescence des processus en cascade avec \( n \) types de particules.

One of the most important theorems on branching processes with \( n \)-types of particles (\( n \)-dimensional Galton-Watson processes) is the Sevastyanov's theorem on degeneration. The original proof of this theorem, as presented in [I], is complicated and is based on both probabilistic and non-probabilistic arguments. It is the author's belief that the proof presented in this paper is simpler. Moreover, it used analytic tools only.

We shall consider discrete-time-parameter processes only, i. e. we shall suppose that the time-parameter \( t \) assumes the values \( t = 0, 1, 2, \ldots \). We shall denote by \( \mathcal{P} \) a Markovian homogeneous branching process with \( n \) types of particles. We shall call \( \mathcal{P} \) shortly a \( n \)-dimensional branching process. We shall distinguish the particles by indices \( i = 1, 2, \ldots, n \). The basic set of indices will be denoted by \( I = \{1, 2, \ldots, n\} \). If \( A = [0, 1, 2, \ldots] \), then \( A^n \) is the state space of \( \mathcal{P} \). The states of \( \mathcal{P} \) will be
denoted by $a = [a_1, a_2, \ldots, a_n]$. To denote special vectors, we shall write $0 = [0, \ldots, 0]$, $\mathbf{1} = [1, \ldots, 1]$ and $e_i$ will be the $i$-th unit vector. We shall denote by $P_i(t, a)$ the probability of transition from the state $e_i$ to the state $a$ after $t$ time units. For $B \subseteq \mathbb{A}^n$, $P_i(t, B)$ will be the corresponding probability of $B$, i.e.

$$P(t, B) = \sum_{a \in B} P_i(t, a).$$

We shall denote by $F_i(t, x)$ the generating function of $P_i(t, a)$, i.e.

$$F_i(t, x) = \sum_{a \in \mathbb{A}^n} x_1^{a_1} \ldots x_n^{a_n} P_i(t, a),$$

where $x = [x_1 \ldots x_n] \in [0, 1]^n$. We shall write $F(t, x)$ instead of $[F_1(t, x), \ldots, F_n(t, x)]$. It is well known that

$$(1) \quad F(s + t), x) = F(s, F(t, x)).$$

For $i, j \in I$, we shall write

$$M_{ij}(t) = \sum_{a \in \mathbb{A}^n} a_j P_i(t, a).$$

We shall suppose that all $M_{ij}(t)$ are finite and we shall denote by $M(t)$ the moment matrix $(M_{ij}(t))_{i,j \in I}$. In all symbols we have introduced the time-parameter $t$ will be omitted, if $t = 1$. It is well known that $M(t) = M'$. The maximal characteristic number of $M$ will be denoted by $R$. Since the branching process is uniquely determined by the basic vector $F(x) = [F_1(x) \ldots F_n(x)]$, we may speak of a branching process defined by the generating functions $F_1(x), \ldots, F_n(x)$.

The subsets of the basic index set $I = [1, 2, \ldots, n]$ will be denoted by $J, K$ or $I_p$. If $J \subseteq I$, $c(J)$ will denote the number of elements of $J$. If $x = [x_1, \ldots, x_n]$, then $x^{(J)}$ will denote the $c(J)$-dimensional vector the coordinate of which are $x_i$, $i \in J$. Generally, we shall express the fact that $x_i$ belongs to the $i$-th particle by the index $i$ only, not by the position of the coordinate $x_i$ in vector; f. i. $(x_1, x_2)$ and $(x_2, x_1)$ will be the same vector for our purposes. This will simplify the forming of new vectors by sub-vectors; f. i. if $I = [1, 2, 3, 4]$, $J = [1, 3]$, $K = [2, 4]$, $y^{(J)} = [y_1, y_3]$, $z^{(K)} = [z_2, z_4]$, then $x = [y^{(J)}, z^{(K)}] = (y_1, y_3, z_2, z_4) = (y_1, z_2, y_3, y_4)$.

We shall write for $J \subseteq I$,

$$M^{(J)} = (M_{ij})_{i,j \in J}$$
$M^{(J)}$ is a $c(J)$-dimensional matrix and we shall denote its maximal characteristic number by $R^{(J)}$.

If $J \subset I$, we shall denote by $\mathcal{P}^{(J)}$ the $c(J)$-dimensional branching process (with particle-indices $i \in J$), defined by the generating functions

$$F^{(J)}_i(x^{(J)}) = F_i(x^{(J)}, \bar{1}^{(I-J)}), \quad i \in J.$$  

Let $P_i^{(J)}(a)$ be the transition probabilities of $\mathcal{P}^{(J)}$. Then

$$P_i^{(J)}(B) = P_i(B \times A^{(I-J)})$$

for each $B \subset A^{(J)}$.

It follows from (3) that

(4) $M^{(J)}$ (defined by (2)) is the moment matrix of $\mathcal{P}^{(J)}$.

Let $J \subset K \subset I$; J will be called closed in $K$ (with respect to $\mathcal{P}$), if

$$P_i\left(\{a : a_j = 0 \text{ for all } j \in K - J\}\right) = 1$$

for each $i \in J$.

It follows from (3) that

(5) $J$ is closed in $K$ with respect to $\mathcal{P}$ if and only if it is closed in $K$ with respect to $\mathcal{P}^{(K)}$.

An index set $J \subset I$ will be called decomposable (with respect to $\mathcal{P}$), if there exist two non-empty and disjoint set $J_1 \subset J$, $J_2 \subset J$ such that $J_1 \cup J_2 = J$ and $J_1$ is closed in $J$. J will be called indecomposable if it is not decomposable. Clearly

(6) $J$ is indecomposable if and only if $M^{(J)}$ is indecomposable.

Also

(7) $J$ is indecomposable with respect to $\mathcal{P}$ if and only if it is indecomposable with respect to $\mathcal{P}^{(J)}$.

An index set $J$ will be called final (with respect to $\mathcal{P}$), if it is indecomposable and if

$$P_i\left(\left\{ a : \sum_{j \in J} a_j = 1 \right\}\right) = 1$$

for each $i \in J$.

It follows again from (3) that

(8) $J$ is final with respect to $\mathcal{P}$ if and only if it is final with respect to $\mathcal{P}^{(J)}$. 
We shall call
\[ P_i = \lim_{t \to \infty} P_i(t, 0) = \lim_{t \to \infty} F_i(t, 0) \quad (i = 1, \ldots, n) \]
the *degeneration probabilities of* \( \mathcal{P} \) and \( p = (p_1, \ldots, p_n) \) the *degeneration-probability vector* of \( \mathcal{P} \). We shall call \( \mathcal{P} \) *degenerate* if \( p = \bar{1} \). It is well known that
\[ F(p) = p \]
and that
\[ \text{(10)} \quad \text{P is degenerate if and only if } \bar{1} \text{ is the only solution} \]
in \([0, 1]^n\) of the system \( F(x) = x \).

In the proof of the main theorem we shall need two lemmas.

**Lemma A.** — If \( J \subset I, K \subset I, J \cap K = \emptyset, J \cup K = I, J \text{ closed in } I \) and if both \( \mathcal{P}^{(J)} \) and \( \mathcal{P}^{(K)} \) are degenerate, then \( \mathcal{P} \) is degenerate.

**Proof.** — Let \( p = (p_1, \ldots, p_n) \) be the degeneration-probability vector of \( \mathcal{P} \). Since \( J \) is closed in \( I \), \( F_i(x) \) with \( i \in J \) does not depend on \( x_j \) with \( j \in K \) and, consequently
\[ F_i(p, x) = F_i(p^{(J)}, \bar{1}^{(K)}) = F_i(p) = p_i \]
for all \( i \in J \) by (9). From (10), (11) and the assumption that \( \mathcal{P}^{(J)} \) is degenerate it follows that
\[ p_i = 1 \quad \text{for } i \in J \]
Then
\[ F_i(p^{(K)}) = F_i(\bar{1}^{(J)}, p^{(K)}) = F_i(p) = p_i \quad \text{for all } i \in K \]
and again
\[ p_i = 1 \quad \text{for } i \in K. \]
Hence \( p = \bar{1} \).

**Lemma B.** — If, for \( J \subset I, \mathcal{P}^{(J)} \) is not degenerate, then \( \mathcal{P} \) is not degenerate.

**Proof.** — Let us write \( K = I - J \). For each \( x = (x_1, \ldots, x_n) \) and each \( i \in J \) we have
\[ F_i^{(J)}(x^{(J)}) = F_i(x^{(J)}, \bar{1}^{(K)}) \geq F_i(x). \]
By (1)
\[ F_i^{(J)}(2, x^{(J)}) = F_i^{(J)}(F^{(J)}(x^{(J)})) \geq F_i^{(J)}((F(x))^{(J)}) \]
\[ \geq F_i(F(x)) = F_i(2, x) \quad \text{for each } i \in J. \]
Generally
\[ \text{(12)} \quad F_i^{(J)}(t, x^{(J)}) \geq F_i(t, x) \quad \text{for each } i \in J \text{ and } t, \]
and denoting by $p_i$ the degeneration probabilities of $\mathcal{P}$ and by $q_i (i \in J)$ the degeneration probabilities of $\mathcal{P}^{(i)}$, we have by (12)

(13) \[ q_i \geq p_i \quad \text{for all} \quad i \in J. \]

According to the assumption, $q_i < 1$ for at least one $i \in J$, and then $p_i < 1$ by (13). Hence, $\mathcal{P}$ is not degenerate.

**Theorem (Sevastyanov).** $\mathcal{P}$ is degenerate if and only if (a) $R \leq 1$ and (b) there are no final index sets.

**Proof.** — Let us suppose that the conditions (a) and (b) are satisfied.

(i) We shall first assume that the moment matrix $M$ is indecomposable. Let $p = (p_1, \ldots, p_n)$ be the degeneration-probability vector and let $J$ be the set of all indices $i$ for which $p_i < 1$. Let us suppose that $J$ is non-empty.

Then for each $i \in I$

(14) \[ F_i(p) = 1 + \sum_{j \in J} M_{ij}(p_j - 1) + \frac{1}{2} \sum_{j,k \in J} \frac{\partial^2}{\partial x_j \partial x_k} F_i(q)(p_j - 1)(p_k - 1) \]

where $q = (q_1, \ldots, q_n)$ is a vector such that

(15) \[ p_i < q_i < 1 \quad \text{for} \quad i \in J \]

\[ q_i = 1 \quad \text{for} \quad i \notin J. \]

By (9) and (14) we have for each $i \in I$

(16) \[ \sum_{j=1}^{n} M_{ij}(1 - p_j) = \sum_{j \in J} M_{ij}(1 - p_j) = 1 - p_j + \frac{1}{2} \sum_{j,k \in J} \frac{\partial^2}{\partial x_j \partial x_k} F_i(q)(1 - p_j)(1 - p_k) \geq 1 - p_j \geq R(1 - p_j). \]

Hence, $M(\bar{I} - p) \geq R(\bar{I} - p)$ and since $J$ supposed to be non-empty, $\bar{I} - p$ is an eigen-vector belonging to $R$, according to a well-known theorem on non-negative matrices. But then

(17) \[ M(\bar{I} - p) = R(\bar{I} - p) \]

and since $M$ is indecomposable, $1 - p_i > 0$ for all $i \in I$, i. e. $J = I$. It follows now from (16) and (17) that $R = 1$ and

(18) \[ \frac{\partial^2}{\partial x_j \partial x_k} F_i(q) = 0 \quad \text{for all} \quad i, j, k \in I. \]
By (15), \( q_i > 0 \) for all \( i \in J = I \), and since \( F_i \) is a power series with non-negative coefficients \( P_i(a) \), it follows from (18), that

\[
P_i \left( \left\{ a : \sum_{j=1}^{n} a_j \geq 2 \right\} \right) = 0 \quad \text{for all} \quad i \in I,
\]

Hence, \( M_{ij} = P_i(e_j) \) for all \( i, j \in I \), and since

\[
\sum_{j=1}^{n} P_i(e_j) \leq 1,
\]

\( M \) is a sub-stochastic matrix. On the other hand, we have proved that \( R = 1 \), which implies that \( M \) is a stochastic matrix, i.e.

\[
\sum_{j=1}^{n} P_i(e_j) = \sum_{j=1}^{n} M_{ij} = 1.
\]

It follows that \( I \) is a final set of indices and this is a contradiction to the condition (b). We came to this contradiction on the basis of the assumption that \( J \) is non-empty. Hence, \( J \) must be empty, i.e. \( p = 1 \).

(ii) We shall suppose again that conditions (a) and (b) hold, but \( M \) will now be an arbitrary moment matrix. It is well-known that there exists index sets \( I_l \subset I \) (\( l = 1, 2, \ldots, k \)) such that \( I_l \) are disjoint,

\[
\bigcup_{l=1}^{k} I_l = I,
\]

\( M^{(l)} \) are indecomposable and

\[
M_{ij} = 0 \quad \text{for} \quad i \in I_l, \quad j \in I_{l+1} \cup \ldots \cup I_k \quad (l = 1, 2, \ldots, k - 1).
\]

We shall write \( J_l = I_1 \cup \ldots \cup I_l \). By (6) and (7), \( I_l \) is indecomposable with respect to \( \mathcal{P}^{(l)} \), and it is well known that \( R^{(l)} \supseteq R \). Hence, by (4), (8) and part (i) of this proof, each \( \mathcal{P}^{(l)} \) is degenerate. In particular, \( \mathcal{P}^{(1)} \) is degenerate. Let us suppose that we have already proved that \( \mathcal{P}^{(l)} \) is degenerate. It follows from (20) and (5) that \( J_l \) is closed in \( J_{l+1} \) with respect to \( \mathcal{P}^{(l+1)} \) and hence \( \mathcal{P}^{(l+1)} \) is degenerate according to Lemma A. By induction, \( \mathcal{P}^{(k)} = \mathcal{P} \) is degenerate.

We shall prove that conditions (a) and (b) are necessary

(iii) Let us suppose that there exists a final index set \( J \subset I \). According to the definition of a final set,

\[
F^{(l)}(\emptyset) = P^{(l)}(\emptyset) = 0 \quad \text{for all} \quad i \in J.
\]
Then \( F_i^{(0)}(2, 0) = F_i^{(0)}(F_j(0)) = F_i^{(0)}(0) = 0 \) and, generally, \( F_i^{(0)}(t, 0) = 0 \) for all \( i \in J \) and all \( t \). Hence, the process \( \mathcal{P}^{(0)} \) is not degenerate and according to Lemma B, \( \mathcal{P} \) is also not degenerate.

(iv) Let us suppose that \( R > 1 \). It is well-known from the spectral theory of non-negative matrices that there exists \( j \in I \) and \( s \) such that \( M_{jj}(s) > 1 \). Let \( \mathcal{P} \) be a new branching process with the index set \( I \), generated by basic generating functions \( F_j(x) = F_j(s, x) \). According to (1), the general generating functions of \( \mathcal{P} \) are \( F_i(t, x) = F_i(st, x) \) and, consequently,

\[
(i) \lim_{t \to \infty} F_i(t, 0) = \lim_{t \to \infty} F_i(st, 0) = \lim_{t \to \infty} F_i(t, 0), \quad i \in I.
\]

Let us write \( J = \{ j \} \). Then \( \mathcal{P}^{(1)} \) is a one-dimensional subprocess of \( \mathcal{P} \) with the first moment \( M_{jj}(s) > 1 \) and according to a well-known theorem on one-dimensional branching processes, \( \mathcal{P}^{(1)} \) is not degenerate. By Lemma B, \( \mathcal{P} \) is also not degenerate. But, according to (21), \( \mathcal{P} \) and \( \mathcal{P} \) have the same degeneration-probability vector and, consequently, \( \mathcal{P} \) is not degenerate.

REFERENCES


(Manuscrit reçu le 17 mars 1969).

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