Annales de l'I. H. P., section B

Lajos Takács

On the distribution of the supremum for stochastic processes

Annales de l'I. H. P., section B, tome 6, n° 3 (1970), p. 237-247
<http://www.numdam.org/item?id=AIHPB_1970__6_3_237_0>

Ann. Inst. Henri Poincaré, Section B :
Vol. VI, n° 3, 1970, p. 237-247. *Calcul des Probabilités et Statistique.*

On the distribution of the supremum for stochastic processes

by

Lajos TAKÁCS (*)
Case Western Reserve University, Cleveland, Ohio.

SUMMARY. — In this paper mathematical methods are given for finding the distribution of the supremum for compound recurrent processes and for stochastic processes with stationary independent increments. New and simple proofs are given for a result of H. Cramér and for a result of G. Baxter and M. D. Donsker. The paper also contains some extensions of these results.

1. INTRODUCTION

Our aim is to give mathematical methods for finding the distribution of $\eta(t) = \sup_{0 \leq u \leq t} \xi(u)$ for separable stochastic processes $\{ \xi(u), 0 \leq u < \infty \}$. Such methods were given by H. Cramér [*2*] in the case where $\{ \xi(u), 0 \leq u < \infty \}$ is a compound Poisson process and by G. Baxter and M. D. Donsker [*1*] in the case where $\{ \xi(u), 0 \leq u < \infty \}$ is a stochastic process with stationary independent increments. In this paper we shall give new and simple proofs for the results of H. Cramér [*2*] and G. Baxter and M. D. Donsker [*1*]. Furthermore we shall give various extensions of these results. In particular, we shall find the joint distribution of $\eta(t)$ and $\xi(t)$ for stochastic processes with stationary independent increments and for compound recurrent processes.

(*) This research was supported by the National Science Foundation under Contract No. GP-7847.

2. COMPOUND RECURRENT PROCESSES

First we shall consider the case where $\{ \xi(u), 0 \leqq u < \infty \}$ is a separable compound recurrent process which is defined in what follows. Let us suppose that $\tau_n - \tau_{n-1}$ $(n = 1, 2, \ldots, \tau_0 = 0)$ is a sequence of mutually independent and identically distributed positive random variables with distribution function $\mathbf{P}\{ \tau_n - \tau_{n-1} \leqq x \} = \mathrm{F}(x)$ and χ_n $(n = 1, 2, \ldots)$ is a sequence of mutually independent random variables with distribution function $\mathbf{P}\{ \chi_n \leqq x \} = \mathrm{H}(x)$. Furthermore let us suppose also that the two sequences $\{ \tau_n \}$ and $\{ \chi_n \}$ are independent.

Let us define

$$\xi(u) = \sum_{0 < \tau_n \leqq u} \chi_n - cu \tag{1}$$

for $u \geqq 0$ where c is a constant. We shall say that the process $\{ \xi(u), 0 \leqq u < \infty \}$ is a compound recurrent process.

We shall write

$$\eta(t) = \sup_{0 \leqq u \leqq t} \xi(u) \tag{2}$$

for $t \geqq 0$.

Let us introduce the following notation

$$\varphi(s) = \int_0^\infty e^{-sx} d\mathrm{F}(x) \tag{3}$$

for $\mathrm{Re}(s) \geqq 0$ and

$$\psi(s) = \int_{-\infty}^\infty e^{-sx} d\mathrm{H}(x) \tag{4}$$

for $\mathrm{Re}(s) = 0$.

Denote by $\mathrm{F}_n(x)$ $(n = 1, 2, \ldots)$ the n-th iterated convolution of $\mathrm{F}(x)$ with itself and by $\mathrm{H}_n(x)$ $(n = 1, 2, \ldots)$ the n-th iterated convolution of $\mathrm{H}(x)$ with itself. Let $\mathrm{F}_0(x) = \mathrm{H}_0(x) = 1$ for $x \geqq 0$ and $\mathrm{F}_0(x) = \mathrm{H}_0(x) = 0$ for $x < 0$.

Let us define also

$$\xi_n = \chi_n - c(\tau_n - \tau_{n-1}) \tag{5}$$

for $n = 1, 2, \ldots$ and

$$\zeta_n = \xi_1 + \xi_2 + \ldots + \xi_n \tag{6}$$

for $n = 1, 2, \ldots$ and $\zeta_0 = 0$. Let

$$\eta_n = \max(\zeta_0, \zeta_1, \ldots, \zeta_n) \tag{7}$$

for $n = 0, 1, 2, \ldots$

THEOREM 1. — *If* $c \geqq 0$, $\mathrm{Re}(q) > 0$, $\mathrm{Re}(s + v) \geqq 0$ *and* $\mathrm{Re}(v) \leqq 0$, *then we have*

$$(q - cv)\int_0^\infty e^{-qt}\mathbf{E}\{e^{-s\eta(t)-v\xi(t)}\}\,dt \tag{8}$$
$$= [1 - \varphi(q - cv)]\cdot\exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\int_0^\infty e^{-(q-cv)u}\left[\int_{cu+0}^{\infty} e^{-s(x-cu)-vx}d\mathrm{H}_n(x)\right.\right.$$
$$\left.\left.+\int_{-\infty}^{cu+0} e^{-vx}d\mathrm{H}_n(x)\right]d\mathrm{F}_n(u)\right\}.$$

Proof. — First let us suppose that τ_n $(n=0, 1, 2, \ldots)$ and χ_n $(n=1, 2, \ldots)$ are numerical (non-random) quantities for which

$$0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_n < \ldots \quad \text{and} \quad \lim_{n\to\infty} \tau_n = \infty.$$

Let us define $\xi(t)$, $\eta(t)$ for $0 \leqq t < \infty$ and ζ_n, η_n for $0 \leqq n < \infty$ in exactly the same way as above. We shall show that if $c \geqq 0$, $\mathrm{Re}(q) > 0$, $\mathrm{Re}(s+v) \geqq 0$, $\mathrm{Re}(v) \leqq 0$, then the following basic relation holds between the functions $\xi(t)$, $\eta(t)$ $(0 \leqq t < \infty)$ and the sequences τ_n, ζ_n, η_n $(0 \leqq n < \infty)$

$$(q - cv)\int_0^\infty e^{-qt-s\eta(t)-v\xi(t)}dt = \sum_{n=0}^{\infty} e^{-q\tau_n - s\eta_n - v\zeta_n}[1 - e^{-(q-cv)(\tau_{n+1}-\tau_n)}]. \tag{9}$$

The proof of (9) is very simple. If $c \geqq 0$, then obviously $\eta(t) = \eta_n$ and $\xi(t) = \zeta_n - c(t - \tau_n)$ for $\tau_n < t < \tau_{n+1}$. Thus

$$(q - cv)\int_{\tau_n}^{\tau_{n+1}} e^{-qt-s\eta(t)-v\xi(t)}dt = e^{-q\tau_n - s\eta_n - v\zeta_n}[1 - e^{-(q-cv)(\tau_{n+1}-\tau_n)}] \tag{10}$$

for $n = 0, 1, 2, \ldots$ If we add (10) for $n = 0, 1, 2, \ldots$, then we get (9).

Next suppose that $\{\tau_n\}$ and $\{\chi_n\}$ are random variables as defined at the beginning of this section. Then (9) holds for almost all realizations of $\{\xi(t), \eta(t); 0 \leqq t < \infty\}$ and $\{\tau_n, \zeta_n, \eta_n; 0 \leqq n < \infty\}$. If we form the expectation of (9), then we obtain that

$$(q - cv)\int_0^\infty e^{-qt}\mathbf{E}\{e^{-s\eta(t)-v\xi(t)}\}\,dt = [1 - \varphi(q - cv)]\sum_{n=0}^{\infty}\mathrm{U}_n(s, v, q) \tag{11}$$

where

$$\text{(12)} \qquad U_n(s, v, q) = \mathbf{E}\{ e^{-s\eta_n - v\zeta_n - q\tau_n} \}.$$

Let us define a sequence of random variables η_n^* $(n = 0, 1, 2, \ldots)$ in the following way $\eta_0^* = 0$ and

$$\text{(13)} \qquad \eta_{n+1}^* = [\eta_n^* + \xi_{n+1}]^+$$

for $n = 0, 1, 2, \ldots$ where $[x]^+ = x$ for $x \geqq 0$ and $[x]^+ = 0$ for $x < 0$. Then we can write also that

$$\text{(14)} \qquad U_n(s, v, q) = \mathbf{E}\{ e^{-s\eta_n^* - v\zeta_n - q\tau_n} \}.$$

For (12) remains unchanged if we replace $\xi_1, \xi_2, \ldots, \xi_n$ by $\xi_n, \xi_{n-1}, \ldots, \xi_1$ respectively and by these substitutions η_n becomes η_n^*.

Here $U_0(s, v, q) \equiv 1$ and $U_n(s, v, q)$ for $n = 1, 2, \ldots$ can be obtained recursively.

Let us define an operator **A** in the following way. If ζ is a complex random variable for which $\mathbf{E}\{ |\zeta| \} < \infty$ and η is a real random variable, then $\mathbf{E}\{ \zeta e^{-s\eta} \}$ exists for $\mathrm{Re}(s) = 0$ and it determines uniquely $\mathbf{E}\{ \zeta e^{-s\eta^+} \}$ for $\mathrm{Re}(s) \geqq 0$ where $\eta^+ = \max(0, \eta)$. Let us write

$$\text{(15)} \qquad \mathbf{E}\{ \zeta e^{-s\eta^+} \} = \mathbf{AE}\{ \zeta e^{-s\eta} \}$$

for $\mathrm{Re}(s) \geqq 0$. Here **A** is a linear operator. We note that for $\mathrm{Re}(s) > 0$ we can write that

$$\text{(16)} \qquad \mathbf{AE}\{ \zeta e^{-s\eta} \} = \frac{1}{2}\mathbf{E}\{ \zeta \} + \frac{s}{2\pi i} \lim_{\varepsilon \to 0} \int_{L_\varepsilon} \frac{\mathbf{E}\{ \zeta e^{-z\eta} \}}{z(s - z)} dz$$

where $L_\varepsilon = \{ z : z = iy, -\infty < y < -\varepsilon, \varepsilon < y < \infty \}$. Since (15) is continuous for $\mathrm{Re}(s) \geqq 0$, (16) determines (15) for $\mathrm{Re}(s) = 0$ by continuity.

Since $\eta_{n+1}^* = [\eta_n^* + \xi_{n+1}]^+$ and $\zeta_{n+1} = \zeta_n + \xi_{n+1}$ for $n = 0, 1, 2, \ldots$ we can write that for $\mathrm{Re}(q) > 0$, $\mathrm{Re}(s + v) \geqq 0$, $\mathrm{Re}(v) \leqq 0$ and $n = 0, 1, 2, \ldots$

$$\text{(17)} \qquad U_{n+1}(s, v, q) = \mathbf{A}\{ \psi(s + v)\varphi(q - cs - cv)U_n(s, v, q) \}$$

and here **A** operates on the variable s. The generating function of the sequence $\{ U_n(s, v, q) \}$ can be expressed as follows:

$$\text{(18)} \qquad \sum_{n=0}^{\infty} U_n(s, v, q)\rho^n = \exp\left\{ \sum_{n=1}^{\infty} \frac{\rho^n}{n} \mathbf{A}[\psi(s + v)\varphi(q - cs - cv)]^n \right\}$$

and (18) is convergent if $|\rho| \leqq 1$. It is easy to check that (18) is indeed the correct solution. Let us denote by $U(s, v, q, \rho)$ the right-hand side of (18). If we form the coefficient of ρ^n for $n = 0, 1, 2, \ldots$ in the power series expansion of $U(s, v, q, \rho)$ and apply the operator $\mathbf{A}$, then every coefficient remains unchanged. If we form the coefficient of ρ^n for $n = 0, 1, 2, \ldots$ in the power series expansion of $[1 - \rho\psi(s+v)\varphi(q - cs - cv)]U(s, v, q, \rho)$ and apply the operator $\mathbf{A}$, then we obtain 1 if $n = 0$ and 0 if $n \geqq 1$. Hence we can conclude that the coefficient of ρ^n in the expansion of $U(s, v, q, \rho)$ satisfies the same initial condition and the same recurrence relation as $U_n(s, v, q)$, that is the coefficient of ρ^n in the expansion of $U(s, v, q, \rho)$ is exactly $U_n(s, v, q)$. This proves (18). We note that (18) can also be proved by using the method of F. Pollaczek [5] or the method of F. Spitzer [6].

Since

$$(19)\quad \mathbf{A}\{[\psi(s+v)\varphi(q-cs-cv)]^n\}$$
$$= \mathbf{A}\left\{\int_{-\infty}^{\infty} e^{-(s+v)x} dH_n(x) \int_0^{\infty} e^{-(q-cs-cv)u} dF_n(u)\right\}$$
$$= \int_0^{\infty} e^{-(q-cv)u}\left[\int_{cu+0}^{\infty} e^{-s(x-cu)-vx} dH_n(x) + \int_{-\infty}^{cu+0} e^{-vx} dH_n(x)\right] dF_n(u),$$

we get (8) by (11) and (18). This completes the proof of theorem 1.

If $v = 0$ in (8), then we get the Laplace transform of $\mathbf{E}\{e^{-s\eta(t)}\}$, and $\mathbf{E}\{e^{-s\eta(t)}\}$ and $\mathbf{P}\{\eta(t) \leqq x\}$ can be obtained by inversion.

By (8) we can write also that

$$(20)\quad (q - cv)\int_0^{\infty} e^{-qt}\mathbf{E}\{e^{-s\eta(t)-v\xi(t)}\}\,dt$$
$$= [1 - \varphi(q - cv)]\cdot\exp\{-\mathbf{A}\log[1 - \psi(s+v)\varphi(q - cs - cv)]\}$$

if $c \geqq 0$, $\mathrm{Re}(q) > 0$, $\mathrm{Re}(s + v) \geqq 0$ and $\mathrm{Re}(v) \leqq 0$.

Finally, for the sake of completeness we shall mention the following result which, however, will not be used in this paper. If $c \leqq 0$, $\mathrm{Re}(q) > 0$, $\mathrm{Re}(s + v) \geqq 0$ and $\mathrm{Re}(v) \leqq 0$, then we have

$$(21)\quad (q - cv)\int_0^{\infty} e^{-qt}\mathbf{E}\{e^{-s\eta(t)-v\xi(t)}\}\,dt$$
$$= Q(s, s+v, q) + \frac{cs}{q - c(s+v)}Q\left(s + v - \frac{q}{c}, s + v, q\right)$$

where

(22) $$Q(s, v, q) = 1-\varphi(q-cv)\cdot \mathbf{A}\{[1-\psi(v-s)] \exp\{-\mathbf{A}\log[1-\psi(v-s)\varphi(q+cs-cv)]\}\}$$

and **A** operates on the variable s.

3. COMPOUND POISSON PROCESSES

Let us suppose that in the previous section

(23) $$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geqq 0, \\ 0 & \text{for } x < 0, \end{cases}$$

that is, $\varphi(s) = \lambda/(\lambda + s)$ for $\mathrm{Re}(s) \geqq 0$. Here λ is a positive constant. In this case $\{\xi(u), 0 \leqq u < \infty\}$ is called a compound Poisson process. Then

(24) $$F_n(x) = \int_0^x e^{-\lambda u} \frac{(\lambda u)^{n-1}}{(n-1)!} \lambda du$$

for $x \geqq 0$,

(25) $$\mathbf{P}\{\xi(u) \leqq x\} = \sum_{n=0}^{\infty} e^{-\lambda u} \frac{(\lambda u)^n}{n!} H_n(cu + x)$$

for $u \geqq 0$ and

(26) $$\mathbf{E}\{e^{-s\xi(u)}\} = e^{cus-\lambda u[1-\psi(s)]}$$

for $\mathrm{Re}(s) = 0$.

In this case we have the following result.

THEOREM 2. — *If* $\mathrm{Re}(q) > 0$, $\mathrm{Re}(s + v) \geqq 0$, *and* $\mathrm{Re}(v) \leqq 0$, *then we have*

(27) $$q\int_0^{\infty} e^{-qt}\mathbf{E}\{e^{-s\eta(t)-v\xi(t)}\} dt = \exp\left\{\int_0^{\infty} \frac{e^{-qu}}{u}\left[\int_{+0}^{\infty} e^{-(s+v)x} d\mathbf{P}\{\xi(u) \leqq x\} + \int_{-\infty}^{+0} e^{-vx} d\mathbf{P}\{\xi(u) \leqq x\} - 1\right] du\right\}.$$

If $v = 0$, *then* (27) *reduces to*

$$(28)\quad q\int_0^\infty e^{-qt}\mathbf{E}\{e^{-s\eta(t)}\}dt=\exp\left\{\int_0^\infty \frac{e^{-qu}}{u}\left[\int_{-0}^\infty e^{-sx}d\mathbf{P}\{\xi(u)\leqq x\}-1\right]du\right\}.$$

Proof. — First we shall prove (27) for $c \geqq 0$. In this case (27) is a particular case of (8). Now $1 - \varphi(q - cv) = (q - cv)/(\lambda + q - cv)$ in (8) and by using (24) and (25) the exponential factor in (8) can be expressed as follows:

$$(29)\quad \exp\left\{\sum_{n=1}^\infty \int_0^\infty \frac{e^{-\lambda u-(q-cv)u}}{u}\frac{(\lambda u)^n}{n!}\left[\int_{cu+0}^\infty e^{-s(x-cu)-vx}d\mathrm{H}_n(x)\right.\right.$$

$$\left.\left.+\int_{-\infty}^{cu+0} e^{-vx}d\mathrm{H}_n(x)\right]du\right\} = \exp\left\{\int_0^\infty \frac{e^{-qu}}{u}\left[\int_{+0}^\infty e^{-(s+v)x}d\mathbf{P}\{\xi(u)\leqq x\}\right.\right.$$

$$\left.\left.+\int_{-\infty}^{+0} e^{-vx}d\mathbf{P}\{\xi(u)\leqq x\} - e^{-\lambda u+cvu}\right]du\right\}.$$

Since

$$(30)\qquad \exp\left\{\int_0^\infty \frac{e^{-qu}}{u}(1 - e^{-\lambda u+cvu})du\right\} = 1 + \frac{\lambda - cv}{q},$$

we get (27) by (29).

For $c \leqq 0$ we can easily deduce (27) from the previous case. Since the finite dimensional distribution functions of the two processes

$$\{\xi(t) - \xi(u) \text{ for } 0 \leqq u \leqq t\} \quad\text{and}\quad \{\xi(t \dot{-} u) \text{ for } 0 \leqq u \leqq t\}$$

are identical, we can conclude that $\eta(t) - \xi(t)$ and $-\xi(t)$ have exactly the same joint distribution as $\sup_{0\leqq u\leqq t}[-\xi(u)]$ and $-\xi(t)$. Furthermore if $c \leqq 0$, then for the process $\{-\xi(u), 0 \leqq u < \infty\}$ we can apply (27) if we replace c by $-c$ and $\xi(u)$ by $-\xi(u)$. Thus we obtain that for $c \leqq 0$, $\mathrm{Re}(q) > 0$, $\mathrm{Re}(s + v) \geqq 0$ and $\mathrm{Re}(v) \leqq 0$

$$(31)\quad q\int_0^\infty e^{-qt}\mathbf{E}\{e^{-s[\eta(t)-\xi(t)]+v\xi(t)}\}dt$$

$$= \exp\left\{\int_0^\infty \frac{e^{-qu}}{u}\left[\int_{+0}^\infty e^{-(s+v)x}d\mathbf{P}\{-\xi(u)\leqq x\}\right.\right.$$

$$\left.\left.+\int_{-\infty}^{+0} e^{-vx}d\mathbf{P}\{-\xi(u)\leqq x\} - 1\right]du\right\}.$$

If we replace v by $-(v + s)$ in (31), then we obtain (27) for $c \leqq 0$. This completes the proof of theorem 2.

In the case where $\{ \xi(u), 0 \leqq u < \infty \}$ is a compound Poisson process the Laplace transform of $\mathbf{E}\{ e^{-s\eta(t)} \}$ has previously been found by H. Cramér [2]. Here we provided a new proof for Cramér's result and we also found the Laplace transform of $\mathbf{E}\{ e^{-s\eta(t)-v\xi(t)} \}$ which makes it possible to find the joint distribution of $\eta(t)$ and $\xi(t)$.

4. PROCESSES WITH STATIONARY INDEPENDENT INCREMENTS

Let us suppose now that $\{ \xi(u), 0 \leqq u < \infty \}$ is a separable stochastic process with stationary independent increments for which $\mathbf{P}\{ \xi(0)=0 \}=1$. Then for $\mathrm{Re}(s) = 0$ and $0 \leqq u < \infty$ we can write that

$$\mathbf{E}\{ e^{-s\xi(u)} \} = e^{u\Phi(s)} \tag{32}$$

where $\Phi(s)$ is an appropriate function.

THEOREM 3. — *If $\{ \xi(u), 0 \leqq u < \infty \}$ is a separable stochastic process with stationary independent increments, then formulas* (27) *and* (28) *hold unchangeably.*

Proof. — We can find a sequence of separable compound Poisson processes $\{ \xi_k(u), 0 \leqq u < \infty \}$ such that the finite dimensional distribution functions of $\{ \xi_k(u), 0 \leqq u < \infty \}$ converge to the corresponding finite dimensional distribution functions of $\{ \xi(u), 0 \leqq u < \infty \}$. If

$$\mathbf{E}\{ e^{-s\xi_k(u)} \} = e^{c_k su - \lambda_k[1-\psi_k(s)]u} \tag{33}$$

for $\mathrm{Re}(s) = 0$, then we can achieve the desired convergence by choosing c_k, λ_k and $\psi_k(s)$ in such a way that

$$\lim_{k\to\infty} \{ c_k s - \lambda_k[1 - \psi_k(s)] \} = \Phi(s) \tag{34}$$

for $\mathrm{Re}(s) = 0$.

If $\eta_k(t) = \sup_{0 \leqq u \leqq t} \xi_k(u)$, then it follows that the joint distribution of $\eta_k(t)$ and $\xi_k(t)$ $(k = 1, 2, \ldots)$ converges to the joint distribution of $\eta(t)$ and $\xi(t)$ as $k \to \infty$. Since (27) holds for all $\xi_k(t)$ and $\eta_k(t)$ $(k = 1, 2, \ldots)$, if follows by the continuity theorem for Laplace-Stieltjes transforms that (27) holds also for $\xi(t)$ and $\eta(t)$. This completes the proof of the theorem.

In a somewhat different form formula (28) was found by G. Baxter and M. D. Donsker [1]. Formula (27) for stochastic processes is similar to a result of F. Spitzer [6] for partial sums of mutually independent and identically distributed random variables.

Note. — If we take into consideration that for $\operatorname{Re}(q) > 0$ and $\operatorname{Re}(s)=0$

$$\int_0^\infty \frac{e^{-qu}}{u}(1 - e^{\Phi(s)u})du = \log\left(1 - \frac{\Phi(s)}{q}\right), \tag{35}$$

where the principal value of the logarithm is used, then by (27) we obtain that

$$q\int_0^\infty e^{-qt}\mathbf{E}\{e^{-s\eta(t)-v\xi(t)}\}\,dt = \exp\left\{-\mathbf{A}\log\left(1 - \frac{\Phi(s+v)}{q}\right)\right\}. \tag{36}$$

for $\operatorname{Re}(q) > 0$, $\operatorname{Re}(s+v) \geqq 0$, $\operatorname{Re}(v) \leqq 0$ and the operator $\mathbf{A}$ is defined by (16).

5. EXAMPLES

Suppose that $\{\xi(u), 0 \leqq u < \infty\}$ is a separable stochastic process with stationary independent increments and $\xi(1)$ has a stable distribution with parameters α and β where $0 < \alpha \leqq 2$ and $-1 \leqq \beta \leqq 1$. Then

$$\mathbf{E}\{e^{-s\xi(u)}\} = e^{u\Phi(s)} \tag{37}$$

for $\operatorname{Re}(s) = 0$ and $0 \leqq u < \infty$ where for $\alpha \neq 1$, $0 < \alpha \leqq 2$ and $-1 \leqq \beta \leqq 1$

$$\Phi(s) = -|s|^\alpha\left(1 - \beta\frac{s}{|s|}\tan\frac{\alpha\pi}{2}\right) \tag{38}$$

and for $\alpha = 1$ and $-1 \leqq \beta \leqq 1$

$$\Phi(s) = -|s|\left(1 - \frac{2\beta s}{\pi|s|}\log|s|\right). \tag{39}$$

In this case the distribution of $\eta(t) = \sup\limits_{0 \leqq u \leqq t} \xi(u)$ has been determined by several authors in various particular cases. We refer to the works of D. A. Darling [3], G. Baxter and M. D. Donsker [1] and C. C. Heyde [4]. We can always use formulas (28) or (36) to find the Laplace-Stieltjes transform of $\mathbf{P}\{\eta(t) \leqq x\}$ and by inversion we can obtain $\mathbf{P}\{\eta(t) \leqq x\}$. However, if $\Phi(s)$ is given by (38) or by (39) with $\beta = 0$, then $\eta(t)$ has the same distribution as $t^{1/\alpha}\eta(1)$ and this observation makes possible some simplifications. Following D. A. Darling [3] we can also express the left-hand side of (28) in another way. Let

$$\mathrm{W}(x) = \mathbf{P}\{\eta(1) \leqq x\} \tag{40}$$

and suppose that $\mathbf{P}\{\eta(t) \leqq x\} = \mathrm{W}(xt^{-1/\alpha})$ where $0 < \alpha \leqq 2$. If $q = 1$ in (28) then the left-hand side can be expressed as

$$\int_0^\infty e^{-t}\mathbf{E}\{e^{-s\eta(t)}\}\,dt = \int_0^\infty \mathrm{I}\left(\frac{1}{sx}\right)d\mathrm{W}(x) \tag{41}$$

for $\mathrm{Re}(s) > 0$ where

$$\mathrm{I}(s) = \int_0^\infty e^{-u-u^{1/\alpha}/s}du \tag{42}$$

for $\mathrm{Re}(s) > 0$. We observe that for $0 \leqq x < \infty$ the function $\mathrm{I}(x)$ is a distribution function of a positive random variable. If we write $s = 1/y$ in (41) where $0 < y < \infty$, then

$$\mathrm{G}(y) = \int_0^\infty \mathrm{I}\left(\frac{y}{x}\right)d\mathrm{W}(x) \tag{43}$$

can be interpreted as the distribution function of the product of two independent positive random variables having distribution functions $\mathrm{I}(x)$ and $\mathrm{W}(x)$ respectively.

On the other hand $\mathrm{G}(x)$ can be obtained by using (28) or (36). In (36) if $q = 1$, then

$$\mathbf{A}\log[1 - \Phi(s)] = \frac{s}{2\pi}\int_0^\infty \frac{\mathrm{Im}\{(s+iy)\log[1-\Phi(iy)]\}}{y(y^2+s^2)}dy \tag{44}$$

for $\mathrm{Re}(s) > 0$. If we denote the right-hand side of (44) by $\mathrm{L}(s)$, then we can write that

$$\mathrm{G}(x) = \exp\left\{-\mathrm{L}\left(\frac{1}{x}\right)\right\} \tag{45}$$

for $0 < x < \infty$.

Finally the unknown $\mathrm{W}(x)$ can be obtained from (43) by using Mellin-Stieltjes transform. Since

$$\int_0^\infty x^s d\mathrm{I}(x) = \Gamma(1-s)\Gamma\left(1 - \frac{1}{\alpha} + \frac{s}{\alpha}\right) \tag{46}$$

for $1 - \alpha < s < 1$, we obtain from (43) that

$$\int_0^\infty x^s dW(x) = \frac{1}{\Gamma(1-s)\Gamma\left(1 - \frac{1}{\alpha} + \frac{s}{\alpha}\right)} \int_0^\infty x^s dG(x) \tag{47}$$

for $1 - \alpha < s < 1$ and $|s| < \alpha$. By inversion we can obtain $W(x)$.

If, in particular, $\alpha = 1$, then (47) reduces to

$$\int_0^\infty x^s dW(x) = \frac{\sin \pi s}{\pi} \int_0^\infty x^s dG(x) \tag{48}$$

for $0 < s < 1$ hence

$$W(x) = \frac{G(xe^{-\pi i}) - G(xe^{\pi i})}{2\pi i} \tag{49}$$

for $x > 0$, where the definition of $G(s)$ is extended by analytical continuation to the complex plane cut along the negative real axis from the origin to infinity.

REFERENCES

[*1*] G. Baxter and M. D. Donsker, On the distribution of the supremum functional for processes with stationay independent increments. *Trans. Amer. Math. Soc.*, **85**, 1957, 73-87.

[*2*] H. Cramer, *On some questions connected with mathematical risk. University of California Publications in Statistics*, **2**, 1954, 99-124.

[*3*] D. A. Darling, The maximum of sums of stable random variables. *Trans. Amer. Math. Soc.*, **83**, 1956, 164-169.

[*4*] C. C. Heyde, On the maximum of sums of random variables and the supremum functional for stable processes. *Journ. Appl. Prob.*, **6**, 1969, 419-429.

[*5*] F. Pollaczek, Fonctions caractéristiques de certaines répartitions définies au moyen de la notion d'ordre. Application à la théorie des attentes. *Comptes Rendus Acad. Sciences* (Paris), **234**, 1952, 2334-2336.

[*6*] F. Spitzer, A combinatorial lemma and its application to probability theory. *Trans. Amer. Math. Soc.*, **82**, 1956, 323-339.

(Manuscrit reçu le 8 décembre 1969).