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An operator-valued stochastic integral, III

by

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SOMMAIRE. — Plusieurs auteurs ont caractérisé la distribution gaussienne moyennant des formes linéaires indépendamment distribuées ou des statistiques linéaires distribuées de façon identique. Ceci a mené Laha et Lukacs à obtenir des caractérisations du processus de Wiener moyennant des intégrales stochastiques distribuées de façon identique. Dans ce travail nous définissons une intégrale stochastique à valeurs « opérateurs » par rapport à un processus stochastique à valeurs dans un espace hilbertien avec incréments indépendants, moyennant quoi nous caractérisons un processus de mouvement brownien à valeurs dans un espace hilbertien.

1. INTRODUCTION

Gaussian distribution has been characterized, among others, by Darmois, Linnik, Marcinkiewicz and Skitovich through independently distributed linear forms or identically distributed linear statistics. Motivated by the results of these authors Laha and Lukacs [8] obtained characterizations of the Wiener process through identically distributed stochastic integrals. Several of these characterization theorems are known in the scalar case. In this note we obtain a similar characterization of an Hilbert space valued Brownian motion (Wiener) process using an operator-valued stochastic integral.

An operator-valued stochastic integral with respect to an Hilbert space valued Wiener process has been defined by Kannan and Bharucha-Reid [*6*] and Kannan [7]. In [7] the author derives four more stochastic integrals from the operator-valued integral. In this article, following [*6*] and [7], we define an operator valued integral with respect to an Hilbert space valued stochastic process with independent increments. Using this operator valued integral we characterize an Hilbert valued Wiener process.

2. STOCHASTIC INTEGRALS

We use the following notations. $(\Omega, \mathscr{A}, \mu)$ is a complete probability space; $\{ \mathscr{A}_t, t \in \mathrm{I} = [0, 1] \}$ is an increasing family of sub-σ-algebras of $\mathscr{A}$ such that each $\mathscr{A}_t$ is complete with respect to the probability measure μ. $\mathscr{H}$ is a real separable Hilbert space with inner-product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. $\mathscr{B}(\mathscr{H})$ is the Banach algebra of endomorphisms of $\mathscr{H}$ with operator norm $\|\cdot\|_{\mathscr{B}}$. For $x, y \in \mathscr{H}$, $x \otimes y$ denotes the tensor product of x and y. $x \otimes y$ is an endomorphism of $\mathscr{H}$ defined by

$$(x \otimes y)h = \langle h, y \rangle x \tag{2.1}$$

for $h \in \mathscr{H}$. (For the properties of the tensor product of vectors of a Hilbert space we refer to Schatten [*11*]). $[\sigma c]$ denotes the Hilbert space of *Schmidt-class operators on* $\mathscr{H}$ with inner-product

$$\langle \mathrm{T}, \mathrm{U} \rangle_\sigma = \sum_{i=1}^{\infty} \langle \mathrm{T}e_i, \mathrm{U}e_i \rangle$$

where $\{ e_i, i \geq 1 \}$ is a complete orthonormal system in $\mathscr{H}$. $[\tau c]$ is the collection of *trace-class operators on* $\mathscr{H}$

$$\left([\tau c] = \{ \mathrm{T} \in \mathscr{B}(\mathscr{H}) \} : \sum_{i=1}^{\infty} | \langle \mathrm{T}e_i, e_i \rangle | < \infty \right).$$

For details of Schmidt-class and trace-class operators we refer to Dunford and Schwartz [*2*] and Schatten [*11*]. The following lemma is given in Kannan [7].

LEMMA 2.1. — *Let x and y be two integrable random elements in $\mathscr{H}$ and $\mathscr{J}$ be a sub-σ-algebra of $\mathscr{A}$. If x is $\mathscr{J}$-measurable, then,*

$$\mathscr{E} \{ (x \otimes y) | \mathscr{J} \} = x \otimes \mathscr{E} \{ y | \mathscr{J} \}. \tag{2.2}$$

By **H** we denote the Hilbert space of all (equivalence class of) second order stochastic processes $\{ \xi(t), t \in \mathrm{I} \}$ with norm

$$\| \xi \|_{\mathbf{H}} = \left[\int_0^1 \mathrm{E} \| \xi(t) \|^2 dt \right]^{1/2}.$$

All the processes that we consider are centered. A process $\{ \beta(t), t \in \mathrm{I} \}$ is called *a process with independent increments* if, for all $t_0 < t_1 < \ldots < t_k$ in [0, 1], the random elements $\beta(t_0), \beta(t_1) - \beta(t_0), \ldots, \beta(t_k) - \beta(t_{k-1})$ are independent; that is, for any $h_0, h_1, \ldots, h_k \in \mathscr{K}$, the scalar random variables $\langle h_0, \beta(t_0) \rangle, \langle h_1, \beta(t_1) - \beta(t_0) \rangle, \ldots, \langle h_k, \beta(t_k) - \beta(t_{k-1}) \rangle$ are independent. Let $\{ \mathscr{A}_t, t \in \mathrm{I} \}$ be an increasing family of sub-σ-algebras of $\mathscr{A}$ such that (*a*) the processes that we consider are adapted to this family and (*b*) for any $t \in \mathrm{I}$, the random elements

$$\beta(t_1) - \beta(t), \ \beta(t_2) - \beta(t_1), \ldots, \beta(t_k) - \beta(t_{k-1})$$

are independent of $\mathscr{A}_t$, where $t_1, \ldots, t_k \in [t, 1]$, (such a family clearly exists).

LEMMA 2.2. — *Let $\beta(t) \in \mathbf{H}$ be a process with independent increments. Then, there is a self-adjoint positive trace-class operator* S *such that, for* $s < t$,

$$\mathscr{E}[(\beta(t) - \beta(s)) \otimes (\beta(t) - \beta(s))] = (t - s)\mathrm{S}. \tag{2.3}$$

The lemma obtains from the following

$$t\mathrm{S} = \begin{cases} \mathscr{E}[(\beta(t) - \beta(\tau_0)) \otimes (\beta(t) - \beta(\tau_0))], & \tau_0 \leq t \\ - \mathscr{E}[(\beta(t) - \beta(\tau_0)) \otimes (\beta(t) - \beta(\tau_0))], & \tau_0 > t. \end{cases} \tag{2.4}$$

Remark: For the purpose of defining the operator-valued stochastic integral we can even consider the above lemma in the following form: If $\beta(t) \in \mathbf{H}$ is a process with independent increments, then, there is an increasing family $\{ \mathrm{S}(t), t \in \mathrm{I} \}$ of trace-class operators such that, for $s < t$,

$$\mathscr{E}[(\beta(t) - \beta(s)) \otimes (\beta(t) - \beta(s))] = \mathrm{S}(t) - \mathrm{S}(s).$$

Here **H** is the Hilbert space of second order processes with norm

$$\| \xi \|_{\mathbf{H}} = \left[\int_0^1 \mathrm{E} \| \xi(t) \|^2 d\mathrm{S}(t) \right]^{1/2}$$

where the integral is the Bochner integral. We use this only in the form we stated in the Lemma 2.2.

We shall now define the operator-valued stochastic integral with respect to a process $\beta(t)$ with independent increments. Let S be the trace class operator associated with $\beta(t)$. We first define the integral for processes in $\mathbf{H}_0 \subset \mathbf{H}$ where $\mathbf{H}_0$ is the collection of simple processes in $\mathbf{H}$. Let

$$0 = t_0 < t_1 < \ldots < t_n = 1,$$

and

$$\xi(t) = \begin{cases} \xi(t_i) & \text{for} \quad t_i \leq t \leq t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Define $J_\xi(\omega) = \int_0^1 \xi(t, \omega) d\beta(t, \omega)$ by

$$\int_I \xi(t) d\beta(t) = \sum_{i=0}^{n-1} \xi(t_i) \otimes [\beta(t_{i+1}) - \beta(t_i)]. \tag{2.5}$$

LEMMA 2.3. — (1) J_ξ *is a linear operator from* $\mathbf{H}_0$ *into* $[\tau c] \subset [\sigma c]$;

(2) $$\mathscr{E}\{ J_\xi \} = 0;$$

(3) $$E\{ \| J_\xi \|_\sigma^2 \} = \| \xi \|_{\mathbf{H}}^2 \operatorname{tr} S \tag{2.6}$$

where tr S *denotes the trace of* S.

(4) *If* $\{ \xi_n(t) \}$ *is a Cauchy sequence of simple processes in* $\mathbf{H}$, *then, the corresponding sequence* $\{ J_n = J_{\xi_n} \}$ *is a Cauchy sequence in* $L_2(\Omega, [\sigma c])$.

For $\xi(t) \in \mathbf{H}$, there is a sequence $\{ \xi_n(t) \}$ of simple processes converging strongly to $\xi(t)$. By the above lemma there is a $J \in L_2(\Omega, [\sigma c])$ such that $J_{\xi_n} \xrightarrow{L_2} J$. Now we define the integral of $\xi(t)$ by

$$J = \int_I \xi(t) d\beta(t).$$

Above definition can be summarized as follows.

THEOREM 2.1. — *There is a unique isometric operator (upto a constant factor* tr S) *from* $\mathbf{H}$ *into* $L_2(\Omega, [\sigma c])$, *denoted by*

$$\xi \to \int_I \xi(t) d\beta(t)$$

such that, for $t \in I$,

$$J_\xi(t) = \int_0^t \xi(\tau) d\beta(\tau) = \int_0^1 \chi_{[0,t]} \xi(\tau) d\beta(\tau).$$

Next we shall find out the covariance operator of J_ξ. Let X be a separable real Hilbert space with inner-product $(.\,|\,.)$ and $\mathscr{X}$ be the σ-algebra of

Borel subsets of X. Let $x(\omega)$ be a second order random element in X. $x(\omega)$ induces a probability measure v_x on $(X, \mathscr{X})$, $v_x = \mu \circ x^{-1}$. The *covariance operator* S_x *of* x (or v) is defined by

$$(S_x g \mid g) = \int_X (f \mid g)^2 dv_x(f).$$

using a result of Kannan and Bharucha-Reid [5] we give the following theorem.

THEOREM 2.2. — *The covariance operator* S_J *of the stochastic integral* J_ξ *is given by*

$$\begin{aligned}\langle S_J T, U \rangle_\sigma &= \int_\Omega \operatorname{tr}\,[(J_\xi \otimes J_\xi)(T \otimes U)]d\mu(\omega) \\ &= \int_\Omega \langle T, J_\xi \rangle_\sigma \operatorname{tr}\,[J_\xi \otimes U]d\mu(\omega)\end{aligned} \qquad (2.7)$$

where $T, U \in [\sigma c]$.

3. CHARACTERIZATION OF WIENER PROCESS

Let $\beta(t)$ be a $\mathscr{K}$-valued process with independent increments and $\mu_{t,\tau}$ be the probability measure on $\mathscr{K}$ induced by $\beta(t + \tau) - \beta(t)$. In general $\mu_{t,\tau}$ and the characteristic functional $\hat{\mu}_{t,\tau}$ of $\mu_{t,\tau}$ depend on the parameters t and τ. The process $\beta(t)$ is said to be homogeneous if $\mu_{t,\tau}$ (hence $\hat{\mu}_{t,\tau}$) depends only on τ (and is independent of t).

Throughout this section $\beta(t, \omega)$ will denote a centered second order homogeneous process with independent increments. Also in the operator-valued stochastic integral

$$J_\xi = \int_I \xi(t)d\beta(t)$$

we consider $\xi(t)$ as a function of t alone (i. e. free from ω).

Divide any interval $[t, t + \tau] \subset [0, 1]$ into n sub-intervals

$$\left[t + \frac{i-1}{n}\tau,\ t + \frac{i}{n}\tau\right]$$

of length τ/n. Then, for $h \in \mathscr{K}$,

$$\hat{\mu}_\tau(h) = [\hat{\mu}_{\tau/n}(h)]^n.$$

Also, $\hat{\mu}_{\tau_1+\tau_2} = \hat{\mu}_{\tau_1}\hat{\mu}_{\tau_2}$. Since $\beta(t)$ is a process with independent increments, so is $\langle h, \beta(t) \rangle$ for each $h \in \mathscr{H}$. Hence the process $\langle h, \beta(t) \rangle$ is infinitely-divisible. (For the details of canonical representations of infinitely-divisible characteristic functionals we refer to Gnedenko and Kolmogorov [*3*], Parthasarathy [*10*] and Varadhan [*12*]). We recall the following (cf. [*10*]).

THEOREM 3.1. — *A function $\hat{\nu}(h)$ is the characteristic functional of an infinitely-divisible probability measure ν on $\mathscr{H}$ if and only if it can be uniquely represented as follows*

$$\hat{\nu}(h) = \exp\left[i\langle m, h \rangle - \frac{1}{2}\langle S_\nu h, h \rangle + \int_{\mathscr{H}} K(f, h)dM(f) \right]$$

where $m \in \mathscr{H}$, S_ν is an S-operator, M is a σ-finite measure with finite mass outside every neighborhood of the origin, with

$$\int_{\{f:\|f\|\leq 1\}} \|f\|^2 dM(f) < \infty,$$

and

$$K(f, h) = e^{i\langle f,h\rangle} - 1 - i\frac{\langle f, h\rangle}{1 + \|f\|^2}.$$

In our case $\beta(t)$ is a centered second-order process. For second order measures one needs Kolmogorov's representation theorem. An extension of Kolmogorov's theorem in the Hilbert space case is

THEOREM 3.2. — *A second-order probability measure λ*

$$\left(\int_{\mathscr{H}} \|f\|^2 d\lambda(f) < \infty \right)$$

is infinitely divisible if and only if its characteristic functional $\hat{\lambda}(g)$ can uniquely be written in the form

$$\hat{\lambda}(g) = \exp\left[i\langle m, g \rangle + \int_{\mathscr{H}} K(f, g)dM(f) \right] \tag{3.1}$$

where $m \in \mathscr{H}$, $0 \leq M(\mathscr{H}) < \infty$, $M(\{\theta\}) = 1$,

$$K(\theta, g) = -\frac{1}{2}\langle S_0 g, g \rangle, \tag{3.2}$$

S_0 is an S-operator, and

$$K(f, g) = [e^{i\langle f,g\rangle} - 1 - i\langle f, g \rangle]\,\|f\|^{-2} \tag{3.3}$$

for $f \neq \theta$ and $f, g \in \mathscr{H}$.

The proof using standard arguments is omitted. In our case we take $m = \theta$.

DEFINITION 3.1. — A centered second order homogeneous process $\beta(t)$ with independent increments is said to be *a Wiener process* or *a Brownian motion process* if the characteristic functional of $\xi(t)$ is given by

$$\hat{\lambda}_t(h) = \exp\left[-\frac{t}{2} \langle S_\lambda h, h \rangle \right] \tag{3.4}$$

where S_λ is an S-operator.

We need the following results to prove the characterization theorem. As a consequence of Theorem 2.2 we obtain the following

LEMMA 3.1. — *Let* S *be the operator associated with* $\beta(t)$ (*Lemma* 2.2). *The covariance operator* S_J *of the stochastic integral*

$$J = \int_0^1 \xi(t) d\beta(t)$$

is given by

$$\langle S_J T, T \rangle_\sigma = \int_0^1 \langle TST^* \xi(t), \xi(t) \rangle \, dt. \tag{3.5}$$

Proof: Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be a partition of [0, 1]. Using elementary properties of $[\sigma c]$ and the fact that $\beta(t)$ has independent increments we obtain

$$\begin{aligned}
\langle S_J T, T \rangle_\sigma &= \int_\Omega \left\langle T, \int_0^1 \xi(t) d\beta(t) \right\rangle_\sigma \operatorname{tr}\left[T \otimes \int_0^1 \xi(t) d\beta(t) \right] d\mu(\omega) \\
&= \int_\Omega \left\langle T, \int_0^1 \xi(t) d\beta(t) \right\rangle_\sigma^2 d\mu(\omega) \\
&= \sum_{i,j=0}^{n-1} \int_\Omega \langle T, \xi(t_i) \otimes \beta(\Delta_i) \rangle_\sigma \langle T, \xi(t_j) \otimes \beta(\Delta_j) \rangle_\sigma d\mu(\omega), \\
&\qquad\qquad (\beta(\Delta_i) = \beta(t_{i+1}) - \beta(t_i)), \\
&= \sum_{i=0}^{n-1} \int_\Omega \langle T, \xi(t_i) \otimes \beta(\Delta_i) \rangle_\sigma^2 d\mu(\omega) \\
&= \sum_{i=0}^{n-1} \int_\Omega \langle T^* \xi(t_i), \beta(\Delta_i) \rangle^2 d\mu(\omega) \\
&= \sum_{i=0}^{n-1} \langle T^* \xi(t_i), (t_{i+1} - t_i) ST^* \xi(t_i) \rangle \\
&= \int_0^1 \langle TST^* \xi(t), \xi(t) \rangle \, dt
\end{aligned}$$

THEOREM 3.3. — *The stochastic integral* $J = \int_I \xi(t)d\beta(t, \omega)$ *is infinitely divisible. If* $\hat{\varphi}(g)$ *denotes the characteristic functional of* $\beta(1, \omega)$ *and* $v(g) = \log \hat{\varphi}(g)$, *then the characteristic functional of* J *is given by*

$$\int_I v(T^*\xi(t))dt \tag{3.6}$$

for $T \in [\sigma c]$.

Proof: Corresponding to $\xi(t)$, there is a sequence $\{ \xi_n(t) \}$ of simple functions such that $\| \xi_n - \xi \|_H \to 0$ as $n \to \infty$ and

$$J_n = \int_I \xi_n(t)d\beta(t) \to \int_I \xi(t)d\beta(t) = J$$

in $L_2(\Omega, [\sigma c])$. If λ_n and λ denote the characteristic functionals of J_n and J, then $J_n \to J$ in $L_2(\Omega[\sigma c])$ implies that $\lambda_n \to \lambda$.

Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be a partition of [0, 1] and

$$\xi_n(t) = \begin{cases} \xi_n(t_i) & \text{if} \quad t_i \leq t \leq t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$J_n = \int_I \xi_n(t)d\beta(t) = \sum_{i=0}^{n-1} \xi_n(t_i) \otimes \beta(\Delta_i).$$

Now

$$\begin{aligned}
\lambda_n(T) &= E\left[\exp i \left\langle \sum_{j=0}^{n-1} \xi_n(t_j) \otimes \beta(\Delta_j), T \right\rangle_\sigma \right] \\
&= E\left[\prod_{j=0}^{n-1} \exp i \langle \xi_n(t_j) \otimes \beta(\Delta_j), T \rangle_\sigma \right] \\
&= E\left[\prod_{j=0}^{n-1} \exp i \langle T^*\xi_n(t_j), \beta(\Delta_i) \rangle_{\mathscr{H}} \right] \\
&= \prod_{j=0}^{n-1} \hat{\varphi}_{\Delta_j}(T^*\xi_n(t_j)) \\
&= \prod_{j=0}^{n-1} [\hat{\varphi}(T^*\xi_n(t_j))]^{\Delta_j}
\end{aligned}$$

where $\Delta_j = t_{j+1} - t_j$, $\beta(\Delta_j) = \beta(t_{j+1}) - \beta(t_j)$ and $\hat{\varphi}_t$ is the characteristic functional of $\beta(t)$. Hence

$$\begin{aligned}\Lambda_n(\mathrm{T}) &= \log \lambda_n(\mathrm{T}) \\ &= \sum_{j=0}^{n-1} \Delta_j \log \hat{\varphi}(\mathrm{T}^*\xi_n(t_j)) \\ &= \sum_{j=0}^{n-1} v(\mathrm{T}^*\xi_n(t_j))\Delta_j \\ &= \int_{\mathrm{I}} v(\mathrm{T}^*\xi_n(t))dt \\ &\to \int_{\mathrm{I}} v(\mathrm{T}^*\xi(t))dt = \Lambda(\mathrm{T})\end{aligned}$$

Hence the theorem.

Finally we give the characterization theorem.

THEOREM 3.4. — *Let $\{ \beta(t), t \in \mathrm{I} \}$ be a $\mathscr{K}$-valued centered second-order homogeneous process with independent increments and* S *be the positive definite trace-class operator on $\mathscr{K}$ associated with $\{ \beta(t) \}$. Let $\xi(t)$ and $\eta(t)$ be two bounded functions in* H *and*

$$\mathrm{J}_\xi = \int_{\mathrm{I}} \xi(t)d\beta(t) \qquad \textit{and} \qquad \mathrm{J}_\eta = \int_{\mathrm{I}} \eta(t)d\beta(t).$$

Then, the Hilbert-Schmidt-class random operators J_ξ and J_η are identically distributed if, and only if

(1) *$\beta(t)$ is a Wiener process, and*

(2) $$\int_{\mathrm{I}} \langle \mathrm{TST}^*\xi(t), \xi(t) \rangle \, dt = \int_{\mathrm{I}} \langle \mathrm{TST}^*\eta(t), \eta(t) \rangle \, dt$$

Proof: The *if* part is clear. To see the *only if* part let J_ξ and J_η be identically distributed.

If

$$\Lambda(\mathrm{T}) = \log \lambda(\mathrm{T}) \qquad \text{and} \qquad v(g) = \log \hat{\varphi}(g)$$

then, by the hypothesis

$$\int_0^1 v(\mathrm{T}^*\xi(t))dt = \int_0^1 v(\mathrm{T}^*\eta(t))dt \tag{3.7}$$

for $\mathrm{T} \in [\sigma c]$. From Theorem 3.3, both sides of (3.7) are the logarithms of the characteristic functionals of the infinitely divisible J_ξ and J_η. From the extension of Kolmogorov's representation theorem (Theorem 3.2), let

$$v(g) = \int_{\mathscr{X}} \mathrm{K}(f, g) d\mathrm{M}(f).$$

Then

$$\begin{aligned}
\Lambda_\xi(\mathrm{T}) &= \int_{\mathrm{I}} v(\mathrm{T}^*\xi(t))dt \\
&= \int_0^1 \left[\int_{\mathscr{X}} \mathrm{K}(f, \mathrm{T}^*\xi(t)) d\mathrm{M}(f) \right] dt \\
&= \int_0^1 \left[\int_{\mathscr{X}\backslash\theta} \{ e^{i\langle f, \mathrm{T}^*\xi(t)\rangle} - 1 - i\langle f, \mathrm{T}^*\xi(t)\rangle \} \| f \|^{-2} d\mathrm{M}(f) \right] dt \\
&\quad - \frac{1}{2}\int_0^1 \langle \mathrm{ST}^*\xi(t), \mathrm{T}^*\xi(t)\rangle \, dt,
\end{aligned}$$

where S is the positive definite trace-class operator associated with $\beta(t)$,

$$\begin{aligned}
&= \int_0^1 \left[\int_{\mathscr{X}\backslash\theta} \{ e^{i\langle \mathrm{T}f, \xi(t)\rangle} - 1 - i\langle \mathrm{T}f, \xi(t)\rangle \} \| f \|^{-2} d\mathrm{M}(f) \right] dt \\
&\quad - \frac{1}{2}\int_0^1 \langle \mathrm{TST}^*\xi(t), \xi(t)\rangle \, dt \\
&= \int_0^1 \left[\int_{\mathscr{X}\backslash\theta} \{ e^{i\langle g, \xi(t)\rangle} - 1 - i\langle g, \xi(t)\rangle \} \| \mathrm{T}^{-1}g \|^{-2} d\mathrm{M}(\mathrm{T}^{-1}g) \right] dt \\
&\quad - \frac{1}{2}\int_0^1 \langle \mathrm{TST}^*\xi(t), \xi(t)\rangle \, dt \\
&= \int_0^1 \left[\int_{\mathscr{X}\backslash\theta} \{ e^{i\langle g, \xi(t)\rangle} - 1 - i\langle g, \xi(t)\rangle \| g \|^{-2} (\| g \|^2 \| \mathrm{T}^{-}1g \|^{-2}) d\mathrm{M}(\mathrm{T}^{-1}g) \right] dt \\
&\quad - \frac{1}{2}\int_0^1 \langle \mathrm{TST}^*\xi(t), \xi(t)\rangle \, dt \qquad (3.8)
\end{aligned}$$

Consider a sequence of partitions of $\mathrm{I} = [0, 1]$:

$$0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{k_n}^{(n)} = 1, \qquad n \geq 1,$$

such that the mesh of the partition goes to zero as $n \to \infty$. Let

$$\begin{aligned}
\mathrm{I}_n(\mathrm{T}) = \sum_{j=0}^{k_n-1} (t_{j+1}^{(n)} - t_j^{(n)}) \int_{\mathscr{X}\backslash\theta} [e^{i\langle g, \xi(t_j^{(n)})\rangle} \\
-1 - i\langle g, \xi(t_j^{(n)})\rangle] \| g \|^{-2} (\| g \|^2 \| \mathrm{T}^{-1}g \|^{-2}) d\mathrm{M}(\mathrm{T}^{-1}g) \\
- \frac{1}{2}\sum_{j=0}^{k_n-1} \langle \mathrm{TST}^*\xi(t_j^{(n)}), \xi(t_j^{(n)})\rangle (t_{j+1}^{(n)} - t_j^{(n)}) \qquad (3.9)
\end{aligned}$$

We note that the right side of (3.9) converges to the right side of (3.8). Also the first sum in (3.9) is

$$= \int_{\mathscr{H}\setminus\theta} [e^{i\langle g, \xi(t_j^{(n)})\rangle} - 1 - i\langle g, \xi(t_j^{(n)})\rangle] \, \| g \|^{-2} dM_n(g) \tag{3.10}$$

where for any Borel set B in $\mathscr{H}\setminus\theta$,

$$\begin{aligned} M_n(B) &= \sum_{j=0}^{k_n-1} (t_{j+1}^{(n)} - t_j^{(n)}) \int_B \| g \|^2 \, \| T^{-1} g \|^{-2} dM(T^{-1}g) \\ &= \sum_{j=0}^{k_n-1} (t_{j+1}^{(n)} - t_j^{(n)}) \int_{T^{-1}B} \| Tg \|^2 \, \| g \|^{-2} dM(g) \\ &\to \int_0^1 \left[\int_{T^{-1}B} \| Tg \|^2 \, \| g \|^{-2} dM(g) \right] dt \\ &= \int_0^1 \left[\int_B \| g \|^2 \, \| T^{-1} g \|^{-2} dM(T^{-1}g) \right] dt \end{aligned} \tag{3.11}$$

$I_n(T)$ is the logarithm of an infinitely divisible characteristic functional and $I_n(T) \to \Lambda_\xi(T)$. Hence, for any Borel set B of $\mathscr{H}\setminus\theta$,

$$M_n(B) \to M(B) \quad \text{as} \quad n \to \infty.$$

Thus we have

$$M(B) = \int_0^1 \left[\int_B \| g \|^2 \, \| T^{-1} g \|^{-2} dM(T^{-1}g) \right] dt \tag{3.12}$$

Now

$$\Lambda_\xi(T) = \int_0^1 \int_{\mathscr{H}\setminus\theta} K(g, \xi(t)) dM_\xi(g) dt - \frac{1}{2} \int_0^1 \langle TST^*\xi(t), \xi(t) \rangle \, dt$$

$$\Lambda_\eta(T) = \int_0^1 \int_{\mathscr{H}\setminus\theta} K(g, \eta(t)) dM_\eta(g) dt - \frac{1}{2} \int_0^1 \langle TST^*\eta(t), \eta(t) \rangle \, dt.$$

Since these representations are unique and $\Lambda_\xi(T) = \Lambda_\eta(T)$ we have $M_\xi = M = M_\eta$. From (3.12) we note that

$$M(B) \leq \| T \|^2 M(T^{-1}B),$$

for any Borel set B in $\mathscr{H}\setminus\theta$ and $T \in [\sigma c]$. This implies that

$$M(\mathscr{H}\setminus\theta) \leq \| T \|^2 M(\mathscr{H}\setminus\theta) \leq \| T \|^4 M(\mathscr{H}\setminus\theta) \leq \ldots \leq \| T \|^{2n} M(\mathscr{H}\setminus\theta) \leq \ldots$$

Since T is an arbitrary operator from $[\sigma c]$, we see that $M(\mathscr{H}\setminus\theta) = 0$. This fact together with (3.8) and Lemma 3.1, completes the proof.

In conclusion we remark that one can give the caracterization theorem in terms of the trace-integral, evaluation-integrals and inner-product-integral associated with the operator-valued integral (cf. [7]). Suitable modifications of the proof are obvious.

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