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Semi-Groups of Markov Operators


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by

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SUMMARY. — The paper states and proves some theorems for semi-groups of Markov operators (contractions on $L_1$) analogous to theorems known for a single operator:

(i) Let $\{P_t\}$ be a semi-group. $Q$ is said to be a convex combination of $\{P_t\}$ if

$$Qf(x) = \left(\int_0^\infty \phi(t)P_t f dt\right)(x) \quad (f \in L_\infty)$$

where $\phi(t) > 0$, $\phi(t) \searrow$,

$$\int_0^\infty \phi(t)dt = 1 \quad \text{and} \quad \int_0^\infty t\phi(t)dt < \infty.$$

(ii) $\{P_t\}$ is defined to be conservative, ergodic, a Harris process or quasi-compact if $Q$ has this property. Some theorems for such semi-groups analogous to theorems for single operator are proved.

(iii) A necessary and sufficient condition for the existence of a $\sigma$-finite invariant measure is given.
1. PRELIMINARIES

Let \((X, \Sigma, m)\) be a finite measure space. A Markov operator is a positive linear contraction \(P\) on \(L_1(X, \Sigma, m)\). \(P\) will be written to the right of its variable while its adjoint, acting on \(L_\infty(X, \Sigma, m)\) will be denoted by \(P^*\) and written to the left of its variable. Thus \(\langle uP, f \rangle = \langle u, Pf \rangle\) for \(u \in L_1\) and \(f \in L_\infty\).

The operator \(P\) acts on the space of the measures absolutely continuous with respect to \(m\), which is isometric to \(L_1(m)\) as follows

\[
\mu P(A) = \int P1_A d\mu.
\]

The same formula is defined for \(\sigma\)-finite measures. Our reference for ergodic theory of a single Markov operator is [6].

DÉFINITION 1.1. — A Markov Process is a strongly measurable semigroup \(\{P_t| t \geq 0\}\) of Markov operators.

By slight modifications of theorem 1.1 of [11] we have:

THEOREM 1.1. — Let \(\{P_t\}\) be a Markov process, then for every \(f \in L_\infty(m)\) there exists a function \(g(t, x)\) measurable on \([0, \infty)_x X\) (and uniquely defined with respect to the product of Lebesgue measure and \(m\)), such that for every function \(\phi(t) \geq 0\) on \([0, \infty)\) with

\[
\int_0^\infty \phi(t)dt < \infty, \quad \int_0^\infty \phi(t)g(t, x)dt = \left(\int_0^\infty \phi(t)P_t f dt\right)(x) \quad a. e. \ m \quad \text{on} \ X.
\]

DÉFINITION 1.2. — A Markov process is said to be conservative if for every \(0 \leq f\) we have

\[
\lim_{T \to \infty} \int_0^T P_t f dt = \begin{cases} 0 & \text{a. e.} \\ \infty & \text{a. e.} \end{cases}
\]

DÉFINITION 1.3. — A measure \(\mu\) is said to be invariant under \(\{P_t\}\) if \(\mu P_t = \mu, \forall t\).

2. CONVEX COMBINATION OF MARKOV PROCESSES

DÉFINITION 2.1. — Let \(\phi(t) > 0\) be a decreasing function on \([0, \infty)\) with

\[
\int_0^\infty \phi(t)dt = 1 \quad \text{and} \quad \int_0^\infty t\phi(t)dt < \infty,
\]

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Q is called a convex combination of the Markov processes \( \{ P_t \} \) if

\[
Qf(x) = \left( \int_0^\infty \phi(t)P_t f dt \right)(x).
\]

**Lemma 2.1.** — Let Q be a convex combination with the function \( \phi(t) \) as in the definition 2.1 then for every \( f \in L_\infty(m) \) and for every real number T we have

\[
\left| \int_0^T P_t (I - Q)f dt \right|_\infty \leq 4 \| f \|_\infty \cdot \int_0^\infty t \phi(t) dt
\]

**Proof**

\[
\left\| \int_0^T P_t (I - Q)f dt \right\|_\infty
\]

\[
= \left\| \int_0^\infty \left( P_t - P_t \int_0^\infty \phi(s)P_s ds \right) f dt \right\|_\infty
\]

\[
= \left\| \int_0^\infty \phi(s) \int_0^T (P_t - P_{t+s}) f dt ds \right\|_\infty
\]

\[
\leq \left\| \int_0^\infty \phi(s) \int_0^T (P_t - P_{t+s}) f dt ds \right\|_\infty + \left\| \int_0^\infty \phi(s) \int_0^T (P_t - P_{t+s}) f dt ds \right\|_\infty
\]

\[
\leq \left\| \int_0^\infty \phi(s) \int_0^s P_t f dt ds \right\|_\infty + \left\| \int_0^\infty \phi(s) \int_s^T P_t f dt ds \right\|_\infty
\]

\[
\leq 4 \| f \|_\infty \int_0^\infty s \phi(s) ds
\]

**Theorem 2.2.** — The Markov process \( \{ P_t \} \) is conservative if and only if its convex combination Q is conservative.

**Proof.** — If Q is not conservative then there exist a function \( f \geq 0 \) such that \( Qf \leq f \) and \( Qf \neq f \) (see [7]). Denote \( 0 \leq g = f - Qf \) by lemma 2.1

\[
\left\| \int_0^\infty P_t g dt \right\|_\infty < \infty,
\]

hence \( \{ P_t \} \) is not conservative.

On the other hand if \( \{ P_t \} \) is not conservative then there exists a function \( f \geq 0 \) such that \( \int_0^\infty P_t f dt < \infty \) (If \( \{ P_t \} \) is not conservative then by
theorem 2.1 of [I1] $P_{t_0}$ is not conservative, for each $t_0 > 0$, and hence there exists a function $h \geq 0$ with $P_{t_0}h \leq h$ and $P_{t_0}h \neq h$ take $f = h - P_{t_0}h$ and then $\int_0^\infty P_t f dt < \infty$. Denote $g = \int_0^\infty P_t f dt$

$$Qg = \int_0^\infty \phi(s)P_s \int_0^\infty P_t f dtds = \int_0^\infty \phi(s) \int_0^\infty P_t f dtds \leq g$$

and $Qg \neq g$. So $Q$ is not conservative.


Définition 2.2. — A conservative Markov process $\{P_t\}$ is said to be ergodic if $Qf = f$, $f \in L_\infty(m)$ $f = \text{const.}$ when $Q$ is any convex combination.

Lemme 2.3. — A conservative Markov process $\{P_t\}$ is ergodic if and only if $0 \neq f \geq 0 \int_0^\infty P_t f dt = \infty$ and hence the definition of ergodicity does not depend on the choice of the convex combination.

Proof. — If for each $0 \neq f \geq 0$ we have $\int_0^\infty \phi(t)P_t f dt > 0$. So, $Q$ is ergodic. On the other hand if there exist sets $A$ and $B$ such that $\int_0^\infty P_t 1_A dt = 0$ on $B$, then $Q^n 1_A = 0$ on $B$ for each $n$, because

$$Q^n 1_A = \int_0^\infty \phi \ast \phi \ast \ldots \ast \phi P_t 1_A dt = 0$$

(convolution $n$ times) on $B$ (see [5]) and $Q$ is not ergodic.

Remark. — In [5] is also proved that $\mu$ is an invariant measure under $\{P_t\}$ if and only if $\mu Q = \mu$.

3. ON QUASI-COMPACT SEMI-GROUPS

Définition 3.1. — Let $\{P_t\}$ be an ergodic and conservative Markov process, let $Q = \int_0^\infty \phi(t)P_t dt$ be a convex combination, $\{P_t\}$ is said to be quasi-compact if $Q$ is a quasi-compact operator.
THEOREM 3.1. — Let \( \{P_t\} \) be an ergodic and conservative Markov process, then the following are equivalent:

(a) \( \{P_t\} \) is quasi-compact.
(b) For every set \( B \) there exists \( \alpha = \alpha(B) > 0 \) and \( T = T(B) \) such that

\[
\int_0^T P_t 1_B dt \geq \alpha.
\]

(c) There exists a finite invariant measure \( \mu \) and for every function \( f \) with \( \int fd\mu = 0 \) we have

\[
\left\| \frac{1}{T} \int_0^T P_t f dt \right\|_\infty \to 0 \quad \text{as} \quad T \to \infty.
\]

(d) There exists a finite invariant measure and let \( E \) be the projection \( Ef = \int fd\mu \) then

\[
\left\| \frac{1}{T} \int_0^T P_t dt - E \right\|_\infty \to 0 \quad \text{in the operator norm.}
\]

Proof

(d) \( \Rightarrow \) (c) trivial.

(c) \( \Rightarrow \) (b) also obvious.

(b) \( \Rightarrow \) (a) For every set \( B \) there exists \( \alpha = \alpha(B) \) and \( T = T(B) \) such that

\[
\int_0^T P_t 1_B dt \geq \alpha
\]

and hence

\[
Q 1_B = \int_0^\infty \phi(t)P_t 1_B dt \geq \phi(T) \int_0^T P_t 1_B dt \geq \alpha \phi(T)
\]

and by theorem 4.1 of [10] \( Q \) is quasi-compact. (a) \( \Rightarrow \) (d) Let \( Q \) be quasi-compact, denote \( L^0_\infty = \{ f \mid \int fd\mu = 0 \} \) (by theorem 4.1 of [10] there exists a finite invariant measure \( \mu = \mu Q \)) and \( (I - Q)L^0_\infty = L^0_\infty \) and hence for every function \( f \) there exists a function \( g \in L^0_\infty \) such that

\[
g - Qg = f - \int fd\mu.
\]

Hence by lemma 2.1

\[
\left\| \frac{1}{T} \int_0^T P_t \left( f - \int fd\mu \right) dt \right\|_\infty = \left\| \frac{1}{T} \int_0^T P_t(I - Q)g dt \right\|_\infty
\]

\[
\leq \frac{4}{T} \|g\|_\infty \int_0^\infty t\phi(t)dt \leq \frac{4C}{T} \int_0^\infty t\phi(t)dt
\]
where \( C \) is the norm of the operator \((I - Q)^{-1}\) acting on \( L^0_\infty \). Thus

\[
\lim_{T \to \infty} \sup_{\|f\|_\infty \leq 1} \left\| \frac{1}{T} \int_0^T P_t f dt - \int f d\mu \right\|_\infty = 0
\]

**Corollary 1.** — The definition 3.1 does not depend on the choice of the convex combination.

**Remark.** — In [2] is proved that \( U^1 = \int_0^\infty e^{P_t} dt \) is quasi-compact if and only if \( \lambda U = \lambda \int_0^\infty e^{-\lambda t} P_t dt \) is for each \( \lambda \), this is a special case of this corollary.

**Corollary 2.** — Let \( \{ P_t \} \) be an ergodic and conservative Markov process and \( P_{t_0} \) is a quasi-compact operator for some \( t_0 \) then the process is quasi-compact.

**Proof.** — By theorem 4.1 of [10], for every function \( f \in L_\infty \) there exists a function \( g \in L_\infty \) with \( \int g d\mu = 0 \), where \( \mu \) is the invariant measure for \( P_{t_0} \), such that \( f - \int f d\mu = g - P_{t_0} g \). Hence

\[
\left\| \frac{1}{T} \int_0^T P_t \left( f - \int f d\mu \right) dt \right\|_\infty = \left\| \frac{1}{T} \int_0^T P_t(g - P_{t_0} g) dt \right\|_\infty \\
= \left\| \frac{1}{T} \int_0^{t_0} P_t g dt - \frac{1}{T} \int_T^{T+t_0} P_t g dt \right\|_\infty \leq \frac{2 t_0 \| g \|_\infty}{T} \xrightarrow{T \to \infty} 0
\]

and by theorem 3.1 the process is quasi-compact.

**Remark.** — The converse is not true, for example if \( \{ P_t \} \) is the semigroup of rotations on the circle then it is easy to see that the process is quasi-compact but each \( P_t \) is not.

**Theorem 3.2.** — Let \( \{ P_t \} \) be an ergodic and conservative Markov process and there exists no pure charge (a finite additive measure which does not dominate any measure) \( \nu \) such that \( \nu P_t = \nu \) for each \( t \) then the process is quasi-compact.

**Proof.** — By the Fixed Point Theorem there exists a positive functional \( \lambda \) on \( L_\infty \) such that \( \lambda P = \lambda \lambda \), as a functional on \( L_\infty \), can be written uniquely as a sum \( \lambda = \mu + \nu \) where \( \mu \) is a measure and \( \nu \) a pure charge. It is clear that \( \mu P_t \geq \mu \) and by the conservativity of \( P_t \), \( \mu P_t = \mu \) for each \( t \), and by
the ergodicity $\mu$ is a unique finite invariant measure. Define the space $L = \text{span} \{ (I - P_t)L_x | 0 < t < \infty \}$. The orthogonal compliment of $L$ is $L^\perp = \{ v \in L_x^* | vP = v, \forall t \}$ and by the conditions of the theorem we have that $L^\perp$ is the one dimensional space $\{ x\mu \}$. So, by the Hahn-Banach Theorem if $\int f d\mu = 0$ then $f \in L$ and hence for each $\varepsilon > 0$ there exist functions $f_1, f_2, \ldots, f_j \in L_\infty$ and real numbers $t_1, t_2, \ldots, t_j$ such that

$$\| (f_1 - P_{t_1}f_1) + (f_2 - P_{t_2}f_2) + \ldots + (f_j - P_{t_j}f_j) - f \|_\infty \leq \varepsilon$$

Thus,

$$\left| \frac{1}{T} \int_0^T P_t f dt \right|_\infty \leq \left| \frac{1}{T} \int_0^T P_t (f_1 - P_{t_1}f_1) dt \right|_\infty + \ldots + \left| \frac{1}{T} \int_0^T P_t (f_j - P_{t_j}f_j) dt \right|_\infty$$

the last element of the sum is less than $\varepsilon$ and for each $1 \leq i \leq j$ we have

$$\left| \frac{1}{T} \int_0^T P_t (f_i - P_{t_i}f_i) dt \right|_\infty \leq \left| \frac{1}{T} \int_0^{t_i} P_t f dt \right|_\infty + \left| \frac{1}{T} \int_{t_i}^{T+t_i} P_t f dt \right|_\infty \leq \frac{2t_i \| f_i \|}{T} \xrightarrow{T \to \infty} 0$$

and hence

$$\left| \frac{1}{T} \int_0^T P_t f dt \right|_\infty \xrightarrow{T \to \infty} 0$$

and by theorem 3.1 the process is quasi-compact.

4. HARRIS PROCESSES

A single Markov operator $P$ is said to be a Harris operator if there exist an integral operator $K$, $Kf(x) = \int k(x, y)f(y)m(dy)$ and an integer $n$ such that $0 < K \leq P^n$ (for details see [6], Chapter V). Let $P$ be a Markov operator and $A$ a set, define $P_A = I_A \sum_{n=0}^{\infty} (P_I)^n P_I A$ where $I_A$ is...
the operator $I_A f(x) = \begin{cases} f(x) & x \in A \\ 0 & x \notin A \end{cases}$ is shown that $P_A$ is a Markov operator on $(A, \Sigma_A, m_A)$.

**Définition 4.1.** — Let $\{P_t\}$ a Markov process and $Q$ a convex combination of it, $\{P_t\}$ is said to be a Harris process if $Q$ is a Harris operator. Since $Q$ is a Harris operator it has a unique $\sigma$-finite invariant measure $\mu$ (see [6], Chapter VI).

**Theorem 4.1.** — Let $\{P_t\}$ be an ergodic and conservative Markov process then the following are equivalent:

(a) $\{P_t\}$ is a Harris process.

(b) There exists a set $A$ such that for every set $B \subset A$ there exist $T = T(B)$ and $0 < \alpha = \alpha(B)$ such that $\int_0^T P_t 1_B dt \geq \alpha 1_A$.

(c) There exists a set $A$ and a constant $C$ such that if $\text{supp } f \subset A$ and $\int fd\mu = 0$ then $\left\| \int_0^T P_t f dt \right\|_{\infty} \leq C \| f \|_{\infty}$.

**Proof.** — (b) $\Rightarrow$ (a) Let $Q$ be the convex combination $Qf = \int_0^\infty \phi(t)P_t f dt$, let $B \subset A$ be a set, there exist $T = T(B)$ and $\alpha = \alpha(T)$ such that

$$\int_0^T P_t 1_B dt \geq \alpha 1_A$$

and hence

$$Q1_B = \int_0^\infty \phi(t)P_t 1_B dt \geq \phi(T) \int_0^T P_t 1_B dt \geq \alpha \phi(T) 1_A$$

and by theorem 3.4 of [10] $Q$ is a Harris operator.

(c) $\Rightarrow$ (b) Assume that there exist a set $A$ with $\mu(A) < \infty$ and a constant $C$ such that is $\text{supp } f \subset A$ and $\int fd\mu = 0$ then

$$\left\| \int_0^T P_t f dt \right\|_{\infty} \leq C \| f \|_{\infty} = K.$$ Let $E \subset A$, take $f = 1_A - \frac{\mu(A)}{\mu(E)} 1_E$, then $\int fd\mu = 0$ and $\text{supp } f \subset A$, and hence we have

$$\left\| \int_0^T P_t \left(1_A - \frac{\mu(A)}{\mu(E)} 1_E \right) dt \right\|_{\infty} \leq K.$$
where $K$ is a constant independent on $T$. By the conservativity and Egorov's Theorem there exists a set $B \subset A$ such that \( \int_0^T \mu(A) \, dt \xrightarrow{T \to \infty} \infty \) uniformly on $B$. Hence there exists an integer $N$ such that \( \int_0^N \mu(A) \, dt \geq 2K1_B \). Therefore
\[
2K1_B \leq \int_0^N \mu(A) \, dt \leq K + \frac{\mu(A)}{\mu(E)} \int_0^N \mu(E) \, dt
\]
or
\[
\int_0^N \mu(E) \, dt \geq \frac{\mu(E)}{\mu(A)} \cdot K1_B.
\]
So, for every set $E \subset B$ there exist an integer $N = N(E)$ and a positive number $\alpha = \alpha(E)$ such that \( \int_0^N \mu(E) \, dt \geq \alpha1_B \).

(a) \( \Rightarrow \) (c) $Q$ is a Harris operator. By theorem 5.2 of [10] there exists a set $A$ such that $Q_A$ is quasi-compact. By theorem 4.1 of [10] we have that for each $f \in L_\infty$ with supp $f \subset A$ and \( \int f \, d\mu = 0 \) there exist $g \in L_\infty$ with supp $g \subset A$ and \( \int g \, d\mu = 0 \) such that $(I_A - Q_A)g = f$ and $\|g\|_\infty \leq C \|f\|_\infty$,
where $C$ is a constant independent on $f$.

By the calculations of [3] we have
\[
(I_A - Q_A)g = (I - Q) \sum_{n=0}^\infty (I_A Q)^n I_A g \quad \text{where} \quad \left\| \sum_{n=0}^\infty (I_A Q)^n I_A g \right\|_\infty \leq \|g\|_\infty.
\]

By lemma 2.1 we have
\[
\left\| \int_0^T P_t f \, dt \right\|_\infty = \left\| \int_0^T P_t (I - Q) \sum_{n=0}^\infty (I_A Q)^n I_A g \, dt \right\|_\infty
\]
\[
\leq 4 \|g\|_\infty \int_0^\infty t\phi(t) \, dt \leq 4C \|f\|_\infty \int_0^\infty t\phi(t) \, dt.
\]

**Corollary.** — The definition 3.1 does not depend on the choice of the convex combination.

**Remark.** — Theorem 4.1 is a generalization of some theorems of [1], [4] and [12].

5. ON \(\sigma\)-FINITE INVARIANT MEASURES

**Theorem 5.1.** — A necessary and sufficient condition for the existence of a \(\sigma\)-finite invariant measure \(\mu\) for the conservative and ergodic Markov process \(\{P_t\}\) which is finite on the set \(A\) is that for each \(0 \leq f \in L_\infty\) with \(\text{supp } f \subset A\) we have:

\[
\lim_{T \to \infty} \frac{\int_0^T P_t f dt}{\int_0^T P_t 1_A dt} \neq 0
\]

**Proof.** — If a \(\sigma\)-finite invariant measure exists then by the ratio limit theorem (see [I]) \(\lim_{T \to \infty} \frac{\int_0^T P_t f dt}{\int_0^T P_t 1_A dt}\) exists and is different from zero on a set of positive measure. Hence the condition is necessary. Let us prove that the condition is sufficient. Let \(Q\) be the convex combination

\[
Qf = \int_0^\infty e^{-\lambda} P_t f dt.
\]

By lemma 1.1 of [I] \(\mu\) is a \(\sigma\)-finite invariant measure for \(\{P_t\}\) if and only if it is an invariant measure for \(Q\), so, it is sufficient to show that there exists a \(\sigma\)-finite invariant measure for \(Q\), which is finite on the set \(A\). It is known (see for example [6], Chapter VI, theorem C) that there exists such a measure for \(Q\) if and only if there exists a finite invariant measure for \(Q_A\). It is also known (see for example [9] lemma 1) that if there exists no finite invariant measure for the Markov operator \(P\), then the space \((I - P)L_\infty\) contains positive functions. Hence if there exists no \(\sigma\)-finite invariant measure for \(Q\) which is finite on \(A\) then there exists \(f \geq 0\) with \(\text{supp } f \subset A\) such that for each \(\varepsilon > 0\) there exists \(g \in L_\infty\) with \(\text{supp } g \subset A\) such that \(|f - g + Q_A g| \leq \varepsilon 1_A\) and we have:

\[
\left| \frac{\int_0^T P_t f dt}{\int_0^T P_t 1_A dt} \right| \leq \left| \frac{\int_0^T P_t (I - Q_A) g dt}{\int_0^T P_t 1_A dt} \right| + \left| \frac{\int_0^T P_t (f - g + Q_A g) dt}{\int_0^T P_t 1_A dt} \right|
\]

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The second element of the sum in the left-hand side of the inequality is less than \( e \), while for the numerator of the first element we have by the calculations of [3]

\[
(I_A - Q_A)g = (I - Q) \sum_{n=0}^{\infty} (I_AQ)^n I_A g
\]

where

\[
\left\| \sum_{n=0}^{\infty} (I_AQ)^n I_A g \right\|_\infty \leq \|g\|_\infty
\]

and by lemma 2.1 we have

\[
\left\| \int_0^T P_t(I_A - Q_A)g dt \right\|_\infty = \left\| \int_0^T P_t(I - Q) \sum_{n=0}^{\infty} (I_AQ)^n I_A g dt \right\|_\infty \leq 4 \|g\|_\infty.
\]

So

\[
\lim_{T \to \infty} \left\| \int_0^T P_t(I_A - Q_A)g dt \right\|_\infty \leq \lim_{T \to \infty} \frac{4 \|g\|_\infty}{\int_0^T P_t1_A dt} \equiv 0
\]

and hence

\[
\lim_{T \to \infty} \frac{\int_0^T P_t f dt}{\int_0^T P_t1_A dt} \equiv 0
\]

and the theorem is proved.

**Remark.** — The theorem of [9] is the analogous theorem for a single Markov operator.

**REFERENCES**


scheinlichkeitstheorie*.


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