HIROSHI SATO
YOSHIAKI OKAZAKI
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Separabilities of a Gaussian Radon measure

by

Hiroshi SATO (*) and Yoshiaki OKAZAKI (**)

SUMMARY. — Let (X, Y) be a dual system of real linear spaces, C(X, Y) the cylindrical σ-algebra of X, and W(X, Y) the weak Borel field of X.

The main purposes of this paper are to prove the separability of $L^2(\mu)$ for a Gaussian measure on (X, C(X, Y)) under some assumptions and to prove for a Gaussian Radon measure on (X, W(X, Y)) the separability of $L^2(\mu)$ and the τ(X, Y)-separability of the support, where τ(X, Y) is the Mackey topology.

1. INTRODUCTION AND NOTATIONS

Let (X, Y) be a pair of real linear spaces X and Y with a bilinear form $\langle x, \xi \rangle$ on $X \times Y$, and let C(X, Y) be the minimal σ-algebra of subsets of X that makes all functions $\{ \langle \cdot, \xi \rangle ; \xi \in Y \}$ measurable. Furthermore, if the bilinear form satisfies the separation axioms;

$$\langle x_o, \xi \rangle = 0 \text{ for all } \xi \in Y \text{ implies } x_o = 0,$$

$$\langle x, \xi_0 \rangle = 0 \text{ for all } x \in X \text{ implies } \xi_0 = 0,$$

we call (X, Y) a dual system. In this case, we denote the weak topology on X by σ(X, Y) and the Mackey topology by τ(X, Y).

We say that a dual system (X, Y) is topological if X is a topological linear space such that all functions in Y are continuous, in other words, the
topology of $X$ is finer than $\sigma(X, Y)$, and we denote the Borel field of $X$ by $B(X, Y)$. Any dual system is a topological dual system if $X$ is equipped with $\sigma(X, Y)$ and we denote by $W(X, Y)$ the Borel field of $X$ for $\sigma(X, Y)$. Evidently, $W(X, Y)$, a fortiori, $C(X, Y)$ is included in $B(X, Y)$ for any topological dual system.

Let $U$ be a topological space and $B(U)$ be the Borel field of $U$. Then we say that a measure $\mu$ on $(U, B(U))$ is Radon if $\mu$ is a finite measure such that

$$\mu(A) = \sup \{ \mu(K) ; K \subset A, \text{ compact} \}$$

for every $A$ in $B(U)$.

For a topological linear space $X$ we denote the algebraic dual space of $X$ by $X'$ and the topological dual space by $X'$.

Let $(X, Y)$ be a pair of real linear spaces with a bilinear form $\langle \cdot , \cdot \rangle$. Then a Gaussian measure on $(X, C(X, Y))$ is a probability measure such that for every $\xi \in Y$, $\langle \cdot , \xi \rangle$ obeys a Gaussian law with mean $m(\xi)$ and variance $\nu(\xi)$. We call $m(\xi)$ the mean functional and $\nu(\xi)$ the variance functional of $\mu$. In particular, if $m(\xi) = 0$, we say the Gaussian measure is centered.

Let $(X, Y)$ be a topological dual system. Then a Gaussian Radon measure on $(X, B(X, Y))$ is a Radon measure such that the restriction to $C(X, Y)$ is Gaussian.

In Section 2 of this paper we prove the separability of $L^2(\mu)$ for a Gaussian measure $\mu$ on $(X, C(X, Y))$ under the assumption of the existence of an admissible metric on $Y$ where $(X, Y)$ is a pair of linear spaces. In particular, we show that if $X$ is a metrizable locally convex space and $Y$ is a linear subspace of $X'$, then $L^2(\mu)$ is separable.

Let $(X, Y)$ be a topological dual system. In Section 3, we remark the equivalent-singular dichotomy of two Gaussian Radon measures on $(X, B(X, Y))$; in Section 4, we prove the separability of $L^2(\mu)$ for a Gaussian Radon measure on $(X, B(X, Y))$.

Let $(X, Y)$ be a dual system. In Section 5, we prove the $\tau(X, Y)$-separability of the support of a centered Gaussian Radon measure on $(X, W(X, Y))$. Furthermore, we prove the $\tau(X, Y)$-separability of the support of a non-centered Gaussian Radon measure on $(X, W(X, Y))$ under the following assumption:

(C.1) There exists an increasing sequence of $\sigma(X, Y)$-compact absolutely convex subsets $\{ F_n \}$ of $X$ such that $\lim \mu(F_n) = 1$.

This is the case where $X$ is $\tau(X, Y)$-quasi-complete, in particular, $X$ is a Fréchet space and $Y = X'$, or $Y$ is a Fréchet space and $X = Y'$.
For a Gaussian measure on \((X, C(X, X'))\), where \(X\) is a separable or reflexive Banach space, the separability of the Hilbert space \(H_\mu\) generated by the random variables \(\langle \cdot, \xi \rangle\), \(\xi \in X'\), is stated in H. Sato [7]. J. Kuelbs [4] has also stated the separability of \(H_\mu\) for a centered Gaussian Radon measure on \((X, B(X, X'))\) where \(X\) is a complete locally convex Hausdorff space. But they have the same error since every pre-Hilbert space has not a complete orthonormal system (A. Badrikian and S. Chevet [1]). In this paper, we have corrected it and obtained more general results.

2. SEPARABILITY OF \(L^2(\mu)\)

Let \((X, Y)\) be a pair of real linear spaces and let \(\mu\) be a Gaussian measure on \((X, C(X, Y))\) with the mean functional \(m(\xi)\) and the variance functional \(\nu(\xi)\). We say a metric \(\rho\) on \(Y\) is \emph{admissible} if it defines a locally convex topology on \(Y\) such that

\[
\mu^*((Y, \rho)' \cap X) = 1
\]

where \((Y, \rho)' \cap X = \{ x \in X ; \langle x, \cdot \rangle \text{ is continuous in } (Y, \rho) \}\) and \(\mu^*\) is the outer measure, that is, for every \(A \subset X\),

\[
\mu^*(A) = \inf \{ \mu(E) : E \in C(X, Y) \text{ and } E \supset A \}.
\]

In this case, for every sequence \(\{\xi_n\}\) convergent to \(\xi\) in \((Y, \rho)\), the random sequence \(\{\langle x, \xi_n \rangle\}\) converges to \(\langle x, \xi \rangle\) almost surely on the probability space \((X, C(X, Y), \mu)\). In fact, the set

\[
A = \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \left\{ x \in X ; |\xi_k(x) - \xi(x)| \leq \frac{1}{n} \right\}
\]

belongs to \(C(X, Y)\) and contains the set \((Y, \rho)' \cap X\), hence we have \(\mu(A) = 1\), that is, \(\langle x, \xi, n \rangle \to \langle x, \xi \rangle\) almost surely as \(n \to \infty\). If a metric \(\rho\) on \(Y\) is admissible, there is a subset \(E \subset X\) of outer measure one such that the topology \(\rho\) is finer than the pointwise convergence topology on \(E\), that is, \(\rho\) is finer than the topology \(\sigma(Y, E)\). Conversely, if a metrizable locally convex topology \(\rho\) on \(Y\) is finer than the one of pointwise convergence on a suitable subset \(E\) of outer measure one, \(\rho\) is admissible. In particular every metric on \(Y\) which defines a locally convex topology finer than the weak topology \(\sigma(Y, X)\) is admissible.

**Lemma 2-1.** — If \(\rho\) is an admissible metric on \(Y\), then \(m(\xi)\) and \(\nu(\xi)\) are \(\rho\)-continuous.
Proof. — Since \( \rho \) is a metric, it is sufficient to show sequential continuity.

Let \( \{ \xi_n \} \) be a sequence \( \rho \)-convergent to 0. Then the Gaussian random sequence \( \{ \langle \cdot, \xi_n \rangle \} \) converges to 0 almost surely and it is well-known that \( m(\xi_n) \) and \( v(\xi_n) \) also converge to 0.

This proves the lemma.

Define a linear transformation \( R_\mu \) of \( Y \) into \( L^2(\mu) = L^2(\mu, C(X, Y), \mu) \) by

\[
R_\mu : \zeta \in Y \rightarrow \langle \cdot, \zeta \rangle \in L^2(\mu),
\]

and \( H_\mu \) the closure of \( R_\mu Y \) in \( L^2(\mu) \). Since we have

\[
\| R_\mu \zeta \|_{L^2(\mu)}^2 = m(\zeta)^2 + v(\zeta), \quad \zeta \in Y,
\]

it is easy to show the following lemma.

**Lemma 2.2.** If \( \rho \) is an admissible metric on \( Y \), then \( R_\mu \) is a \( \rho \)-continuous linear transformation of \( Y \) into \( H_\mu \).

Furthermore, by a slight modification of the proof of Proposition 3-4 of R. M. Dudley [2], we can prove the following key lemma.

**Lemma 2.3.** If \( \rho \) is an admissible metric on \( Y \), then \( R_\mu \) is a compact linear transformation of \( (Y, \rho) \) into \( H_\mu \) and consequently the Hilbert space \( H_\mu \) is separable.

**Proof.** — If \( m(\zeta) \) does not vanish, it is sufficient to show the compactness of the new transformation

\[
R_0 \zeta = R_\mu \zeta - m(\zeta)1, \quad \zeta \in Y,
\]

so that without loss of generality we may assume \( m(\zeta) \equiv 0 \).

Since \( \rho \) defines a locally convex metric topology on \( Y \), we can choose a countable increasing basis \( \{ p_n \}_{n=1}^{\infty} \) of continuous semi-norms in \( Y \)

\[
p_1(\zeta) \leq p_2(\zeta) \leq \ldots \leq p_n(\zeta) \leq \ldots.
\]

For every \( n \), put

\[
S_n = \{ \zeta \in Y ; p_n(\zeta) \leq 1 \}, \quad \Gamma_n = R_\mu S_n, \quad O_n = \{ x \in X ; \sup_{\zeta \in S_n} | \langle x, \zeta \rangle | \leq n \}.
\]

By lemma 2.2 we may assume that \( \Gamma_n \) is bounded in \( H_\mu \) for every \( n \). In order to prove the compactness of \( R_\mu \), it is sufficient to show the pre-compactness of \( \Gamma_n \) for some \( N \). Assume \( \Gamma_n \) is not precompact for every \( n \).

Then by the boundness of \( \Gamma_n \) there exist a positive number \( \epsilon \) and an infinite sequence \( \{ \tilde{\zeta}_j \} \) in \( \Gamma_n \) such that the distance of \( \tilde{\zeta}_{j+1} \) from the linear

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span $F_j^n$ of $\xi^n_1, \ldots, \xi^n_j$ is at least $\varepsilon$ for all $j$. Let \( \{ \xi^n_j \}_{n=1}^{\infty} \) be a sequence in $S_n$ such that $\xi^n_j = R_{p_j^n x^n_j}$. Put

\[
O'_n = \{ x \in X ; \sup_{j=1,2,\ldots} | \langle x, \xi^n_j \rangle | \leq n \}.
\]

It is easy to show $\bigcup_{n=1}^{\infty} O'_n = (Y, \rho)' \cap X$ and $O_n \subset O'_n$ for every $n$, hence $\bigcup_{n=1}^{\infty} O'_n \supset \bigcup_{n=1}^{\infty} O_n = (Y, \rho)' \cap X$. Since each $O'_n$ belongs to $C(X, Y)$ and $\mu \left( \bigcup_{n=1}^{\infty} O'_n \right) \geq \mu^*(Y, \rho)' \cap X) = 1$, there exists a natural number $N$ such that $\mu(O'_n) > 0$. Let

\[
\xi^{N+1}_j = g^N_j + \sum_{k=1}^{j} a_{j,k} \xi^N_k
\]

where $g^N_j \subset F_j^N$ and $\| g^N_j \|_H \geq \varepsilon$. Put

\[
A^N_j = \{ x \in X ; \max_{1 \leq k \leq j} | \langle x, \xi^N_k \rangle | \leq N \}.
\]

Then we have $O'_n \subset A^N_j$ for every $j$ and hence

\[
O < \mu(O'_n) \leq \mu(A^N_j) \quad \text{for every } j.
\]

On the other hand we have

\[
\mu( \{ x \in X ; | \langle x, \xi^{N+1}_j \rangle | \geq N \} \cap A^N_j)
\]

\[
\geq \mu \left( \{ x \in X ; g^N_j(x) \geq N \} \right) \cap \left\{ x \in X ; \left\langle x, \sum_{k=1}^{j} a_{j,k} \xi^N_k \right\rangle \geq 0 \right\} \cap A^N_j
\]

\[
= \frac{1}{2} \mu \left( \{ x \in X ; g^N_j(x) \geq N \} \right) \mu(A^N_j).
\]

Now for some $\delta > 0$, we have for all $j$

\[
\mu( \{ x \in X ; g^N_j(x) \geq N \}) \geq 2\delta
\]

so

\[
\mu(A^N_j) \leq (1 - \delta)^{j-1}
\]

by induction. Hence we have

\[
0 < \mu(O'_N) \leq \mu(A^N_j) \rightarrow 0 \quad \text{as } \quad j \rightarrow \infty.
\]

This is a contradiction. Therefore $\Gamma_n$ is precompact for some $n$, and this completes the proof.

This proves the lemma.

The above lemma can be extended.
LEMMA 2-4. — If $Y$ is expressed as a union of at most countable numbers of linear subspaces $\{Y_n\}$ and in each $Y_n$, an admissible metric is defined, then $H_\mu$ is separable.

Proof. — For every $n$, by Lemma 2-3, $R_\mu Y_n$ is a separable subspace of $H_\mu$ so that

$$H_\mu = \overline{\bigcup_{n=1}^\infty R_\mu Y_n}$$

is separable, where the closure is taken in $L^2(\mu)$.

This proves the lemma.

On the other hand, we can prove the following result.

LEMMA 2-5. — If $H_\mu$ is separable, $L^2(\mu)$ is also separable.

Proof. — Since $H_\mu$ is separable, we can choose a countable subset $\mathcal{Z} = \{\zeta_n\}_{n=1}^\infty$ in $Y$ for which $\{\langle \cdot, \zeta_n \rangle\}_{n=1}^\infty$ is dense in $H_\mu$. In order to prove the lemma, it is sufficient to show that the closed linear subspace of $L^2(\mu)$ generated from

$$\{\cos \langle \cdot, \zeta_n \rangle, \sin \langle \cdot, \zeta_n \rangle; n = 1, 2, 3, \ldots\}$$

is identical with $L^2(\mu)$.

Let $\phi(x)$ be a square summable function on $(X, C(X, Y), \mu)$ such that for every $n$,

$$\int_x \phi(x) \cos \langle x, \zeta_n \rangle \, d\mu(x) = 0,$$

$$\int_x \phi(x) \sin \langle x, \zeta_n \rangle \, d\mu(x) = 0,$$

so that

$$\int_x \phi(x) e^{i\langle x, \zeta_n \rangle} \, d\mu(x) = 0.$$

Then we have only to show $\phi(x) = 0$, a.e. ($\mu$).

Denote the collection of all finite subsets of $Y$ by $\Gamma$. Then $\Gamma$ is a directed set with respect to the inclusion. For every $\gamma = \{\xi_1, \xi_2, \ldots, \xi_k\}$ in $\Gamma$ and every real numbers $t_1, t_2, \ldots, t_k$, there exists a subsequence $\{\zeta_{n_j}\}$ of $\mathcal{Z}$ such that

$$\sum_{j=1}^k t_j \langle \cdot, \xi_{n_j} \rangle = \lim_{j \to +\infty} \langle \cdot, \zeta_{n_j} \rangle$$

in $H_\mu$ and we have

$$\int_x e^{i \sum_{j=1}^k t_j \langle x, \xi_{n_j} \rangle} \phi(x) \, d\mu(x) = \lim_{j \to +\infty} \int_x e^{i \langle x, \zeta_{n_j} \rangle} \phi(x) \, d\mu(x) = 0.$$
in other words, the conditional expectation
\[ \phi_\gamma(x) = \mathbb{E}[\phi(x) \mid \xi_1, \xi_2, \ldots, \xi_\gamma] = 0, \quad \text{a.e.} (\mu) \]
for every \( \gamma \) in \( \Gamma \). By the convergence theorem of the filtered martingale of J. Neveu [5, Proposition V-1-2], \( \phi_\gamma \) converges to \( \phi \) in \( L^2(\mu) \) so that we have \( \phi(x) = 0 \), a.e. \( (\mu) \).

This proves the lemma.

Summing up the above lemmas, we have the following theorem.

**Theorem 2-6.** — Let \((X, Y)\) be a pair of linear spaces \( X \) and \( Y \) with a bilinear form \( \langle \cdot, \cdot \rangle, \mu \) be a Gaussian measure on \((X, C(X, Y))\) and assume that \( Y \) is expressed as a union of at most countable numbers of linear subspaces \( \{ Y_n \} \) and in each \( Y_n \) an admissible metric is defined. Then \( L^2(\mu) \) is a separable Hilbert space.

As corollaries of the above theorem, we have the following theorems.

**Theorem 2-7.** — Let \((X, Y)\) be a pair of linear spaces \( X \) and \( Y \) with a bilinear form \( \langle \cdot, \cdot \rangle, \mu \) be a Gaussian measure on \((X, C(X, Y))\) and assume that there exists a locally convex metrizable topology on \( Y \) finer than the weak topology \( \sigma(Y, X) \). Then \( L^2(\mu) \) is a separable Hilbert space.

**Theorem 2-8.** — Let \( X \) be a metrizable locally convex topological linear space, \( Y \) be a linear subspace of \( X' \) and \( \mu \) be a Gaussian measure on \((X, C(X, Y))\). Then \( L^2(\mu) \) is a separable Hilbert space.

**Proof.** — Since \( X \) is a metrizable locally convex space, we may regard \( X \) as a dense subspace of the reduced projective limit \( \lim_n X_n \) of Banach spaces \( \{ X_n \} \) and we have \( X' = \bigcup_n X'_n \) as a set. For every \( n \), \( X'_n \) is equipped with a norm topology finer than the weak topology \( \sigma(X', X) \). Therefore we have \( Y = \bigcup_{n=1}^{\infty} (Y \cap X'_n) \) where \( Y \cap X'_n \) has an admissible metric so that Theorem 2-6 is applicable.

This proves the theorem.

**3. EQUIVALENCE OF GAUSSIAN RADON MEASURES**

In this section we prove the equivalent-singular dichotomy for Gaussian Radon measures for later use.

To begin with, we prepare two lemmas. Let \( K \) be a compact Hausdorff space, \( C(K) \) be the space of all continuous functions on \( K \) and \( B_0(K) \) be
the Baire field, that is, the minimal $\sigma$-algebra of subsets of $K$ that makes all functions in $C(K)$ measurable. We have the following lemma.

**Lemma 3.1.** Let $K$ be a compact Hausdorff space and let $\mu_1$ and $\mu_2$ be Radon measures on $(K, B(K))$. Then for every $A$ in $B(K)$ there exists $A_0$ in $B_0(K)$ such that

$$\mu_1(A \Delta A_0) = \mu_2(A \Delta A_0) = 0$$

where $A \Delta A_0 = A \cup A_0 - A \cap A_0$.

**Proof.** Let $A$ be in $B(K)$. Then, since $\mu_i$ ($i = 1, 2$) is Radon, there exists a decreasing sequence of open sets $\{O_n^i; n = 1, 2, 3, \ldots\}$ such that

$$A \subset O_n^i, \quad \mu_i(O_n^i - A) < \frac{1}{n}, \quad n = 1, 2, 3, \ldots, \quad i = 1, 2.$$ 

Put $O_n = O_n^1 \cap O_n^2$, $n = 1, 2, 3, \ldots$. Then we have simultaneously

$$A \subset O_n, \quad \mu_i(O_n - A) < \frac{1}{n}, \quad n = 1, 2, 3, \ldots, \quad i = 1, 2.$$ 

Since the indicator function $\chi_{O_n}$ is lower semi-continuous and $\mu_i$ ($i = 1, 2$) is Radon, there exists a sequence $\{f_n^i; n = 1, 2, 3, \ldots\}$, $i = 1, 2$, of continuous functions on $K$ such that

$$0 \leq f_n^i(x) \leq \chi_{O_n}(x), \quad x \in K, \quad i = 1, 2,$$

$$0 \leq \mu_i(O_n) - \int_K f_n^i(x) d\mu_i(x) < \frac{1}{n}, \quad n = 1, 2, 3, \ldots, \quad i = 1, 2.$$ 

For every $n$, let $A_n = \{ x \in K ; f_n^1(x) > 0 \} \cup \{ x \in K ; f_n^2(x) > 0 \} \in B_0(K)$ and let $A_0 = \bigcap_n \bigcup_{k=n}^{\infty} A_k$. $A_0$ has the desired property.

**Lemma 3.2.** Let $(X, Y)$ be a topological dual system and let $\mu_1$ and $\mu_2$ are Radon measures on $(X, B(X, Y))$. Then, for every $A$ in $B(X, Y)$, there exists $A_0$ in $C(X, Y)$ such that

$$\mu_1(A \Delta A_0) = \mu_2(A \Delta A_0) = 0.$$ 

**Proof.** Since $\mu_1$ and $\mu_2$ are Radon measures, there exists an increasing sequence of compact subsets $\{K_n\}$ of $X$ such that

$$\mu_i(X - K_n) < \frac{1}{n}, \quad n = 1, 2, 3, \ldots, \quad i = 1, 2.$$
Let $A$ be a set in $B(X, Y)$. Then, by Lemma 3-1, for every $n$ there exists $A_n^0$ in $B_b(K_n)$ such that
\[ \mu_i((A \cap K_n) \triangle A_n^0) = 0, \quad i = 1, 2. \]

On the other hand, using the Stone-Weierstrass theorem, we can easily prove that
\[ B_b(K_n) = C(X, Y) \cap K_n, \quad n = 1, 2, 3, \ldots. \]
Therefore, for every $n$, there exists $A_n$ in $C(X, Y)$ such that $A_n \cap K_n = A_n^0$
and put $A_0 = \bigcup_{n} \bigcap_{k=n}^{+\infty} A_k$. It is easy to see that $A_0$ has the desired property.

Utilising the above lemma and the well-known results concerning the equivalent-singular dichotomy of Gaussian measures on $(X, C(X, Y))$, we have the following theorem without difficulty (Ju. A. Rozanov [6]).

**Theorem 3-3.** — Let $(X, Y)$ be a topological dual system of real linear spaces $X$ and $Y$, $B(X, Y)$ be the Borel field of $X$, and let $\mu$ and $\mu'$ be Gaussian Radon measures on $(X, B(X, Y))$. Then $\mu$ and $\mu'$ are either equivalent (mutually absolutely continuous) or singular and they are equivalent if and only if their restrictions to $C(X, Y)$ are equivalent.

The above theorem derives the same results as shown in [6] in our terminology. Let $(X, Y)$ be a topological dual system and $\mu$ be a Gaussian Radon measure on $(X, B(X, Y))$. Furthermore let $R_\mu$ be a linear transformation of $Y$ into $L^2(\mu) = L^2(X, B(X, Y), \mu)$ defined by
\[ R_\mu : \xi \in Y \rightarrow \langle \cdot, \xi \rangle \in L^2(\mu), \]
let $H_\mu$ be the closure of the range of $R_\mu$ in $L^2(\mu)$, and let $R_\mu^*$ be the algebraic adjoint transformation of $H_\mu^*$ into $Y^*$. As usual, we identify the topological dual space of $H_\mu$ with $H_\mu$.

**Theorem 3-4.** — Let $(X, Y)$ be a topological dual system, $\mu$ a centered Gaussian Radon measure on $(X, B(X, Y))$, and $B(X, Y)$ the $\mu$-completion of $B(X, Y)$. Moreover, assume that $R_\mu^*H_\mu \subset X$ and that $H_\mu$ is separable. Then we have:

1. For $x \in X$ let $\mu_x$ be a Gaussian Radon measure on $(X, B(X, Y))$ defined by
\[ \mu_x(A) = \mu(A + x), \quad A \in B(X, Y). \]
Then $\mu$ and $\mu_x$ are equivalent if and only if $x \in R_\mu^*H_\mu$.

2. Let $X_0$ be a $B(X, Y)$-measurable linear subspace of $X$ such that $\mu(X_0) = 1$. Then we have $R_\mu^*H_\mu \subset X_0$.
The proof of the above theorem is the same as those shown in [6]. Out of completeness, we give the proof of (2).

Let \( X_0 \) be a \( B(X, Y) \)-measurable linear subspace of \( X \) such that \( \mu(X_0) = 1 \), and assume that \( R_\mu^* H_\mu \) is not included in \( X_0 \). Then there exists an element \( x_0 \) in \( R_\mu^* H_\mu \) such that \( x_0 \in X_0 \). Since \( X_0 \) is a linear subspace of \( X \), \( X_0 \) and \( X_0 + x_0 \) are disjoint and, by (1) we have \( \mu(X_0) = \mu(X_0 + x_0) = 1 \). Consequently we have

\[
1 \geq \mu(X_0 \{ X_0 + x_0 \}) = \mu(X_0) + \mu(X_0 + x_0) = 1 + 1 = 2.
\]

This is a contradiction.

4. SEPARABILITY OF \( L^2(\mu) \)

In this section we prove the separability of the Hilbert space \( L^2(\mu) \) for a Gaussian Radon measure \( \mu \).

Let \((X, Y)\) be a topological dual system and \( \mu \) be a Gaussian Radon measure on \((X, B(X, Y))\). Then, since the topology \( \sigma(X, Y) \) is coarser than the topology of \( X \), \( \mu \) is also a Gaussian Radon measure on \((X, W(X, Y))\). Consequently, there exists the minimal \( \sigma(X, Y) \)-closed linear subspace \( X_\mu \) of \( X \) such that \( \mu(X_\mu) = 1 \). We call \( X_\mu \) the topological linear support of \( \mu \).

Let \( X_\mu^0 \) be the polar set of \( X_\mu \) in \( Y \). Then we have the following lemma.

**Lemma 4-1.** — For \( \zeta \) in \( Y \), \( \zeta \) is in \( X_\mu^0 \) if and only if \( \langle \cdot, \zeta \rangle = 0 \), a.e. \((\mu)\).

**Proof.** — Since \( \mu(X_\mu) = 1 \), the necessity is obvious.

Assume that \( \langle \cdot, \zeta \rangle = 0 \), a.e. \((\mu)\). Since \( X_\xi = \{ x \in X; \langle x, \zeta \rangle = 0 \} \) is a \( \sigma(X, Y) \)-closed linear subspace of \( X \) and \( \mu(X_\xi) = 1 \), we have \( X_\mu \subset X_\xi \) by the minimality of \( X_\mu \), in other words, \( \zeta \in X_\mu^0 \).

This proves the lemma.

Define the linear transformation \( R_\mu \) of \( Y \) into \( L^2(\mu) = L(X, B(X, Y), \mu) \) and \( H_\mu \) as in the previous section. Then, using the above lemma, we can easily prove that \( X_\mu^0 \) is the kernel of \( R_\mu \).

Put \( Y_\mu = Y/X_\mu^0 \). Then we have \( X_\mu \cap W(X, Y) = W(X_\mu, Y_\mu) \) and \( X_\mu \cap C(X, Y) = C(X_\mu, Y_\mu) \). Furthermore, since the induced topology of \( \sigma(X, Y) \) on \( X_\mu \) is identical with \( \sigma(X_\mu, Y_\mu) \), a subset of \( X_\mu \) is \( \sigma(X, Y) \)-closed if and only if \( \sigma(X_\mu, Y_\mu) \)-closed and the induced topology of \( \tau(X, Y) \) on \( X_\mu \) is coarser than \( \tau(X_\mu, Y_\mu) \) (H. H. Schaefer [8], Chap. IV, § 4).

On the other hand, by Lemma 3-2 we have

\[
L^2(X, B(X, Y), \mu) = L^2(X, C(X, Y), \mu) = L^2(X_\mu, C(X_\mu, Y_\mu), \mu).
\]
Therefore, in order to prove either the separability of

\[ L^2(\mu) = L^2(X, B(X, Y), \mu) \]

or the \( \tau(X, Y) \)-separability of \( X_\mu \), we have only to prove it in the case of

\[ (C.\ II) \quad X = X_\mu. \]

In this case we have \( Y = Y_\mu \) and by Lemma 4-1 we can easily show that for \( \xi \) in \( Y \)

\[ \langle \cdot, \xi \rangle = 0, \quad \text{a. e. (}\mu\text{)} \quad \text{if and only if} \quad \xi = 0, \]

and that \( R_\mu \) is an injection of \( Y \) into \( H_\mu \).

\textbf{THEOREM 4-2.} — Let \((X, Y)\) be a topological dual system and \( \mu \) be a Gaussian Radon measure on \((X, B(X, Y))\). Then \( L^2(\mu) = L^2(X, B(X, Y), \mu) \) is a separable Hilbert space.

\textit{Proof.} — We may assume \((C.\ II)\) without loss of generality.

Since \( \mu \) is also a Gaussian Radon measure on \((X, W(X, Y))\), there exists an increasing sequence of \( \sigma(X, Y) \)-compact subsets \( \{ K_n \} \) of \( X \) such that

\[ \lim_n \mu(K_n) = 1. \]

Let \( Z \) be the linear hull of \( \bigcup_n K_n \) and denote by \( \rho \) the topology on \( Y \) of uniform convergence on all \( K_n \). Then, using \((C.\ II')\), we can easily show that \( \rho \) is locally convex metrizable and finer than \( \sigma(Y, Z) \). On the other hand, we have \( Z \subset (Y, \rho)' \cap X \) and \( \mu^*(Z) = 1 \). Therefore \( \rho \) is an admissible metric on \( Y \) and consequently, by Theorem 2-6 and Lemma 3-2, \( L^2(X, B(X, Y), \mu) = L^2(X, C(X, Y), \mu) \) is separable.

This proves the theorem.

\section*{5. SEPARABILITY OF THE SUPPORT OF A GAUSSIAN RADON MEASURE}

Using the preceding results, we prove the following theorem.

\textbf{THEOREM 5-1.} — Let \((X, Y)\) be a dual system and \( \mu \) be a centered Gaussian Radon measure on \((X, W(X, Y))\). Then the topological linear support \( X_\mu \) of \( \mu \) is \( \tau(X, Y) \)-separable.

\textit{Proof.} — As stated in § 4, we may assume the condition \((C.\ II)\), that is, \( X = X_\mu \). The measure \( \mu \) can be extended to a centered Gaussian Radon

measure on \((Y^a, W(Y^a, Y))\). Then \(X\) is \(W(Y^a, Y)\)-measurable, hence by theorem 3-4 (2), we have \(R^*_\mu H_{\mu} \subset X\). It is known \(\mu(R^*_\mu H_{\mu}^{(X,Y)}) = 1\), where \(\tau(X, Y)\) means the closure for \(\tau(X, Y)\) in \(X\), remark that \(\mu(R^*_\mu H_{\mu}^{(X,Y)}) = R^*_\mu H_{\mu}^{00}\) (the bipolar) = \{ \(x \in X ; \langle x, \xi \rangle = 0\) for all \(\xi \in R^*_\mu H_{\mu}^0\) \} and for every \(\xi \in R^*_\mu H_{\mu}^0\) \(\mu(\{ X \in X \mid \langle x, \xi \rangle = 0 \}) = 1\). The minimality of \(X = X_{\mu}\) implies that \(R^*_\mu H_{\mu}\) is \(\tau(X, Y)\)-dense in \(X\). By theorem 4-2, \(H_{\mu}\) is separable and \(X = X_{\mu}\) is \(\tau(X, Y)\)-separable.

This proves the theorem.

For non-centered Gaussian Radon measures we have the following result.

**Theorem 5-2.** — Let \((X, Y)\) be a dual system and \(\mu\) be a Gaussian Radon measure on \((X, W(X, Y))\) satisfying the condition \((C. I)\). Then the topological linear support \(X_{\mu}\) of \(X\) is \(\tau(X, Y)\)-separable.

**Proof.** — As stated in Section 4, we have only to prove in the case \(X = X_{\mu}\).

Let \(\{ F_n \}\) be the increasing sequence of \(\sigma(X, Y)\)-compact absolutely convex subsets of \(X\) given in \((C. I)\). Then the topology \(\rho\) on \(Y\) of the uniform convergence on all \(F_n\) is finer than \(\sigma(Y, X)\) and coarser than \(\tau(Y, X)\) and therefore an admissible metric on \(Y\). By Lemma 2-3, \(R_{\mu}\) is a compact linear transformation of \((Y, \rho)\), and \((Y, \tau(Y, X))\) into \(H_{\mu}\) and \(H_{\mu}\) is separable. Consequently \(R^*_\mu\) is also a compact linear transformation of \(H_{\mu}\) into \((X, \tau(X, Y))\) with dense range and this proves the theorem.

**REFERENCES**


