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Separabilities of a Gaussian Radon measure

by

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SUMMARY. — Let (X, Y) be a dual system of real linear spaces, C(X, Y) the cylindrical σ-algebra of X, and W(X, Y) the weak Borel field of X.

The main purposes of this paper are to prove the separability of $L^2(\mu)$ for a Gaussian measure on (X, C(X, Y)) under some assumptions and to prove for a Gaussian Radon measure on (X, W(X, Y)) the separability of $L^2(\mu)$ and the τ(X, Y)-separability of the support, where τ(X, Y) is the Mackey topology.

1. INTRODUCTION AND NOTATIONS

Let (X, Y) be a *pair* of real linear spaces X and Y with a bilinear form $\langle x, \xi \rangle$ on X × Y, and let C(X, Y) be the minimal σ-algebra of subsets of X that makes all functions $\{ \langle \cdot, \xi \rangle ; \xi \in Y \}$ measurable. Furthermore, if the bilinear form satisfies the separation axioms;

$$\begin{aligned} \langle x_0, \xi \rangle = 0 \quad &\text{for all} \quad \xi \in Y \quad \text{implies} \quad x_0 = 0, \\ \langle x, \xi_0 \rangle = 0 \quad &\text{for all} \quad x \in X \quad \text{implies} \quad \xi_0 = 0, \end{aligned}$$

we call (X, Y) a *dual system.* In this case, we denote the weak topology on X by σ(X, Y) and the Mackey topology by τ(X, Y).

We say that a dual system (X, Y) is *topological* if X is a topological linear space such that all functions in Y are continuous, in other words, the

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topology of X is finer than $\sigma(X, Y)$, and we denote the Borel field of X by $B(X, Y)$. Any dual system is a topological dual system if X is equipped with $\sigma(X, Y)$ and we denote by $W(X, Y)$ the Borel field of X for $\sigma(X, Y)$. Evidently, $W(X, Y)$, *a fortiori*, $C(X, Y)$ is included in $B(X, Y)$ for any topological dual system.

Let U be a topological space and $B(U)$ be the Borel field of U. Then we say that a measure μ on $(U, B(U))$ is *Radon* if μ is a finite measure such that

$$\mu(A) = \sup \{ \mu(K) ; K \subset A, \text{ compact} \}$$

for every A in $B(U)$.

For a topological linear space X we denote the algebraic dual space of X by X^a and the topological dual space by X'.

Let (X, Y) be a pair of real linear spaces with a bilinear form $\langle x, \xi \rangle$. Then a *Gaussian measure* on $(X, C(X, Y))$ is a probability measure such that for every $\xi \in Y$, $\langle \cdot, \xi \rangle$ obeys a Gaussian law with mean $m(\xi)$ and variance $v(\xi)$. We call $m(\xi)$ *the mean functional* and $v(\xi)$ *the variance functional* of μ. In particular, if $m(\xi) \equiv 0$, we say the Gaussian measure is *centered.*

Let (X, Y) be a topological dual system. Then a *Gaussian Radon measure* on $(X, B(X, Y))$ is a Radon measure such that the restriction to $C(X, Y)$ is Gaussian.

In Section 2 of this paper we prove the separability of $L^2(\mu)$ for a Gaussian measure μ on $(X, C(X, Y))$ under the assumption of the existence of an *admissible metric* on Y where (X, Y) is a pair of linear spaces. In particular, we show that if X is a metrizable locally convex space and Y is a linear subspace of X', then $L^2(\mu)$ is separable.

Let (X, Y) be a topological dual system. In Section 3, we remark the equivalent-singular dichotomy of two Gaussian Radon measures on $(X, B(X, Y))$; in Section 4, we prove the separability of $L^2(\mu)$ for a Gaussian Radon measure on $(X, B(X, Y))$.

Let (X, Y) be a dual system. In Section 5, we prove the $\tau(X, Y)$-separability of the support of a centered Gaussian Radon measure on $(X, W(X, Y))$. Furthermore, we prove the $\tau(X, Y)$-separability of the support of a non-centered Gaussian Radon measure on $(X, W(X, Y))$ under the following assumption:

(C.I) There exists an increasing sequence of $\sigma(X, Y)$-compact absolutely convex subsets $\{ F_n \}$ of X such that $\lim_n \mu(F_n) = 1$.

This is the case where X is $\tau(X, Y)$-quasi-complete, in particular, X is a Fréchet space and $Y = X'$, or Y is a Fréchet space and $X = Y'$.

For a Gaussian measure on (X, C(X, X')), where X is a separable or reflexive Banach space, the separability of the Hilbert space H_μ generated by the random variables $\langle \cdot, \xi \rangle$, $\xi \in X'$, is stated in H. Sato [7]. J. Kuelbs [4] has also stated the separability of H_μ for a centered Gaussian Radon measure on (X, B(X, X')) where X is a complete locally convex Hausdorff space. But they have the same error since every pre-Hilbert space has not a complete orthonormal system (A. Badrikian and S. Chevet [1]). In this paper, we have corrected it and obtained more general results.

2. SEPARABILITY OF $L^2(\mu)$

Let (X, Y) be a pair of real linear spaces and let μ be a Gaussian measure on (X, C(X, Y)) with the mean functional $m(\xi)$ and the variance functional $v(\xi)$. We say a metric ρ on Y is *admissible* if it defines a locally convex topology on Y such that

$$\mu^*((Y, \rho)' \cap X) = 1$$

where $(Y, \rho)'$ is the topological dual space of Y with ρ-topology $(Y, \rho)' \cap X = \{ x \in X ; \langle x, \cdot \rangle \text{ is continuous in } (Y, \rho) \}$ and μ^* is the outer measure, that is, for every $A \subset X$,

$$\mu^*(A) = \inf \{ \mu(E) : E \in C(X, Y) \text{ and } E \supset A \}.$$

In this case, for every sequence $\{ \xi_n \}$ convergent to ξ in (Y, ρ), the random sequence $\{ \langle x, \xi_n \rangle \}$ converges to $\langle x, \xi \rangle$ almost surely on the probability space $(X, C(X, Y), \mu)$. In fact, the set

$$A = \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \left\{ x \in X ; | \xi_k(x) - \xi(x) | \leqslant \frac{1}{n} \right\}$$

belongs to C(X, Y) and contains the set $(Y, \rho)' \cap X$, hence we have $\mu(A) = 1$, that is, $\langle x, \xi_n \rangle \to \langle x, \xi \rangle$ almost surely as $n \to \infty$. If a metric ρ on Y is admissible, there is a subset $E \subset X$ of outer measure one such that the topology ρ is finer than the pointwise convergence topology on E, that is, ρ is finer than the topology $\sigma(Y, E)$. Conversely, if a metrizable locally convex topology ρ on Y is finer than the one of pointwise convergence on a suitable subset E of outer measure one, ρ is admissible. In particular every metric on Y which defines a locally convex topology finer than the weak topology $\sigma(Y, X)$ is admissible.

LEMMA 2-1. — If ρ is an admissible metric on Y, then $m(\xi)$ and $v(\xi)$ are ρ-continuous.

Proof. — Since ρ is a metric, it is sufficient to show sequential continuity. Let $\{\xi_n\}$ be a sequence ρ-convergent to 0. Then the Gaussian random sequence $\{\langle \cdot, \xi_n \rangle\}$ converges to 0 almost surely and it is well-known that $m(\xi_n)$ and $v(\xi_n)$ also converge to 0.

This proves the lemma.

Define a linear transformation R_μ of Y into $L^2(\mu) = L^2(X, C(X, Y), \mu)$ by

$$R_\mu : \xi \in Y \to \langle \cdot, \xi \rangle \in L^2(\mu),$$

and H_μ the closure of $R_\mu Y$ in $L^2(\mu)$. Since we have

$$\| R_\mu \xi \|^2_{L^2(\mu)} = m(\xi)^2 + v(\xi), \qquad \xi \in Y,$$

it is easy to show the following lemma.

LEMMA 2-2. — If ρ is an admissible metric on Y, then R_μ is a ρ-continuous linear transformation of Y into H_μ.

Furthermore, by a slight modification of the proof of Proposition 3-4 of R. M. Dudley [*2*], we can prove the following key lemma.

LEMMA 2-3. — If ρ is an admissible metric on Y, then R_μ is a compact linear transformation of (Y, ρ) into H_μ and consequently the Hilbert space H_μ is separable.

Proof. — If $m(\xi)$ does not vanish, it is sufficient to show the compactness of the new transformation

$$R_0 \xi = R_\mu \xi - m(\xi) 1, \qquad \xi \in Y,$$

so that without loss of generality we may assume $m(\xi) \equiv 0$.

Since ρ defines a locally convex metric topology on Y, we can choose a countable increasing basis $\{p_n\}_{n=1}^\infty$ of continuous semi-norms in Y

$$p_1(\xi) \leqslant p_2(\xi) \leqslant \ldots \leqslant p_n(\xi) \leqslant \ldots .$$

For every n, put

$$\begin{aligned} S_n &= \{ \xi \in Y ; p_n(\xi) \leqslant 1 \} \\ \Gamma_n &= R_\mu S_n \\ O_n &= \{ x \in X ; \sup_{\xi \in S_n} | \langle x, \xi \rangle | \leqslant n \} . \end{aligned}$$

By lemma 2-2 we may assume that Γ_n is bounded in H_μ for every n. In order to prove the compactness of R_μ, it is sufficient to show the precompactness of Γ_N for some N. Assume Γ_n is not precompact for every n. Then by the boundness of Γ_n there exist a positive number ε and an infinite sequence $\{\tilde{\xi}_j^n\}_{j=1}^\infty$ in Γ_n such that the distance of $\tilde{\xi}_{j+1}^n$ from the linear span F_j^n of $\tilde{\xi}_1^n, \ldots, \tilde{\xi}_j^n$ is at least ε for all j. Let $\{ \xi_j^n \}_{j=1}^{\infty}$ be a sequence in S_n such that $\tilde{\xi}_j^n = R_\mu \xi_j^n$. Put

$$O'_n = \{ x \in X \, ; \sup_{j=1,2,\ldots} | \langle x, \xi_j^n \rangle | \leqslant n \} .$$

It is easy to show $\bigcup_{n=1}^{\infty} O_n = (Y, \rho)' \cap X$ and $O_n \subset O'_n$ for every n, hence $\bigcup_{n=1}^{\infty} O'_n \supset \bigcup_{n=1}^{\infty} O_n = (Y, \rho)' \cap X$. Since each O'_n belongs to $C(X, Y)$ and $\mu\left(\bigcup_{n=1}^{\infty} O'_n\right) \geqslant \mu^*((Y, \rho)' \cap X) = 1$, there exists a natural number N such that $\mu(O'_N) > 0$. Let

$$\tilde{\xi}_{j+1}^N = g_j^N + \sum_{k=1}^{j} a_{j,k} \tilde{\xi}_k^N$$

where $g_j^N \perp F_j^N$ and $\| g_j^N \|_H \geqslant \varepsilon$. Put

$$A_j^N = \{ x \in X \, ; \max_{1 \leqslant k \leqslant j} | \langle x, \xi_k^N \rangle | \leqslant N \} .$$

Then we have $O'_N \subset A_j^N$ for every j and hence

$$O < \mu(O'_N) \leqslant \mu(A_j^N) \qquad \text{for every } j .$$

On the other hand we have

$$\begin{aligned} &\mu(\{ x \in X \, ; | \langle x, \xi_{j+1}^N \rangle | \geqslant N \} \cap A_j^N) \\ &\geqslant \mu\left(\{ x \in X \, ; g_j^N(x) \geqslant N \} \cap \left\{ x \in X \, ; \left\langle x, \sum_{k=1}^{j} a_{j,k} \xi_j^N \right\rangle \geqslant 0 \right\} \cap A_j^N \right) \\ &= \frac{1}{2} \mu(\{ x \in X \, ; g_j^N(x) \geqslant N \}) \mu(A_j^N) . \end{aligned}$$

Now for some $\delta > 0$, we have for all j

$$\mu(\{ x \in X \, ; g_j^N(x) \geqslant N \}) \geqslant 2\delta$$

so

$$\mu(A_j^N) \leqslant (1 - \delta)^{j-1}$$

by induction. Hence we have

$$0 < \mu(O'_N) \leqslant \mu(A_j^N) \to 0 \qquad \text{as} \qquad j \to \infty .$$

This is a contradiction. Therefore Γ_n is precompact for some n, and this completes the proof.

This proves the lemma.

The above lemma can be extended.

LEMMA 2-4. — If Y is expressed as a union of at most countable numbers of linear subspaces $\{ Y_n \}$ and in each Y_n an admissible metric is defined, then H_μ is separable.

Proof. — For every n, by Lemma 2-3, $R_\mu Y_n$ is a separable subspace of H_μ so that

$$H_\mu = \overline{R_\mu Y} = \overline{\bigcup_{n=1}^{\infty} R_\mu Y_n}$$

is separable, where the closure is taken in $L^2(\mu)$.

This proves the lemma.

On the other hand, we can prove the following result.

LEMMA 2-5. — If H_μ is separable, $L^2(\mu)$ is also separable.

Proof. — Since H_μ is separable, we can choose a countable subset $\mathscr{Z} = \{ \zeta_n \}_{n=1}^{+\infty}$ in Y for which $\{ \langle \cdot, \zeta_n \rangle \}_{n=1}^{+\infty}$ is dense in H_μ. In order to prove the lemma, it is sufficient to show that the closed linear subspace of $L^2(\mu)$ generated from

$$\{ \cos \langle \cdot, \zeta_n \rangle , \sin \langle \cdot, \zeta_n \rangle ; n = 1, 2, 3, \ldots \}$$

is identical with $L^2(\mu)$.

Let $\phi(x)$ be a square summable function on $(X, C(X, Y), \mu)$ such that for every n,

$$\int_X \phi(x) \cos \langle x, \zeta_n \rangle d\mu(x) = 0 ,$$

$$\int_X \phi(x) \sin \langle x, \zeta_n \rangle d\mu(x) = 0 ,$$

so that

$$\int_X \phi(x) e^{i\langle x, \zeta_n \rangle} d\mu(x) = 0 .$$

Then we have only to show $\phi(x) = 0$, a. e. (μ).

Denote the collection of all finite subsets of Y by Γ. Then Γ is a directed set with respect to the inclusion. For every $\gamma = \{ \xi_1, \xi_2, \ldots, \xi_k \}$ in Γ and every real numbers $t_1, t_2, \ldots, t_k$, there exists a subsequence $\{ \zeta_{n_j} \}$ of $\mathscr{Z}$ such that

$$\sum_{l=1}^{k} t \langle \cdot, \xi_l \rangle = \lim_{j \to +\infty} \langle \cdot, \zeta_{n_j} \rangle$$

in H_μ and we have

$$\int_X e^{i\sum_{l=1}^{k} t_l \langle x, \xi_l \rangle} \phi(x) d\mu(x) = \lim_{j \to +\infty} \int_X e^{i\langle x, \zeta_{n_j} \rangle} \phi(x) d\mu(x) = 0 ,$$

in other words, the conditional expectation

$$\phi_\gamma(x) = \mathrm{E}[\phi(x) \mid \xi_1, \xi_2, \ldots, \xi_k] = 0, \qquad \text{a. e. } (\mu)$$

for every γ in Γ. By the convergence theorem of the filtered martingale of J. Neveu [5, Proposition V-1-2], ϕ_γ converges to ϕ in $L^2(\mu)$ so that we have $\phi(x) = 0$, a. e. (μ).

This proves the lemma.

Summing up the above lemmas, we have the following theorem.

THEOREM 2-6. — Let (X, Y) be a pair of linear spaces X and Y with a bilinear form $\langle , \rangle$, μ be a Gaussian measure on (X, C(X, Y)) and assume that Y is expressed as a union of at most countable numbers of linear subspaces $\{ Y_n \}$ and in each Y_n an admissible metric is defined. Then $L^2(\mu)$ is a separable Hilbert space.

As corollaries of the above theorem, we have the following theorems.

THEOREM 2-7. — Let (X, Y) be a pair of linear spaces X and Y with a bilinear form $\langle , \rangle$, μ be a Gaussian measure on (X, C(X, Y)) and assume that there exists a locally convex metrizable topology on Y finer than the weak topology $\sigma(Y, X)$. Then $L^2(\mu)$ is a separable Hilbert space.

THEOREM 2-8. — Let X be a metrizable locally convex topological linear space, Y be a linear subspace of X' and μ be a Gaussian measure on (X, C(X, Y)). Then $L^2(\mu)$ is a separable Hilbert space.

Proof. — Since X is a metrizable locally convex space, we may regard X as a dense subspace of the reduced projective limit $\varprojlim_n X_n$ of Banach spaces $\{ X_n \}$ and we have $X' = \bigcup_n X'_n$ as a set. For every n, X'_n is equiped with a norm topology finer than the weak topology $\sigma(X', X)$. Therefore we have $Y = \bigcup_{n=1}^{\infty} (Y \cap X'_n)$ where $Y \cap X'_n$ has an admissible metric so that Theorem 2-6 is applicable.

This proves the theorem.

3. EQUIVALENCE OF GAUSSIAN RADON MEASURES

In this section we prove the equivalent-singular dichotomy for Gaussian Radon measures for later use.

To begin with, we prepare two lemmas. Let K be a compact Hausdorff space, C(K) be the space of all continuous functions on K and $B_0(K)$ be the Baire field, that is, the minimal σ-algebra of subsets of K that makes all functions in C(K) measurable. We have the following lemma.

LEMMA 3-1. — Let K be a compact Hausdorff space and let μ_1 and μ_2 be Radon measures on (K, B(K)). Then for every A in B(K) there exists A_0 in $B_0(K)$ such that

$$\mu_1(A \triangle A_0) = \mu_2(A \triangle A_0) = 0$$

where $A \triangle A_0 = A \cup A_0 - A \cap A_0$.

Proof. — Let A be in B(K). Then, since μ_i $(i = 1, 2)$ is Radon, there exists a decreasing sequence of open sets $\{ O_n^i \,;\, n = 1, 2, 3, \dots \}$ such that

$$A \subset O_n^i,$$
$$\mu_i(O_n^i - A) < \frac{1}{n}, \qquad n = 1, 2, 3, \dots, \qquad i = 1, 2.$$

Put $O_n = O_n^1 \cap O_n^2$, $n = 1, 2, 3, \dots$. Then we have simultaneously

$$A \subset O_n,$$
$$\mu_i(O_n - A) < \frac{1}{n}, \qquad n = 1, 2, 3, \dots, \qquad i = 1, 2.$$

Since the indicator function χ_{O_n} is lower semi-continuous and μ_i $i = 1, 2)$ is Radon, there exists a sequence $\{ f_n^i \,;\, n = 1, 2, 3, \dots \}$, $i = 1, 2$, of continuous functions on K such that

$$O \leqslant f_n^i(x) \leqslant \chi_{O_n}(x), \qquad x \in K, \qquad i = 1, 2,$$
$$O \leqslant \mu_i(O_n) - \int_K f_n^i(x) d\mu_i(x) < \frac{1}{n}, \qquad n = 1, 2, 3, \dots, \qquad i = 1, 2.$$

For every n, let $A_n = \{ x \in K \,;\, f_n^1(x) > O \} \cup \{ x \in K \,;\, f_n^2(x) > O \} \in B_0(K)$ and let $A_0 = \bigcap_n \bigcup_{k=n}^{+\infty} A_k$. A_0 has the desired property.

LEMMA 3-2. — Let (X, Y) be a topological dual system and let μ_1 and μ_2 are Radon measures on (X, B(X, Y)). Then, for every A in B(X, Y), there exists A_0 in C(X, Y) such that

$$\mu_1(A \triangle A_0) = \mu_2(A \triangle A_0) = 0.$$

Proof. — Since μ_1 and μ_2 are Radon measures, there exists an increasing sequence of compact subsets $\{ K_n \}$ of X such that

$$\mu_i(X - K_n) < \frac{1}{n}, \qquad n = 1, 2, 3, \dots, \qquad i = 1, 2.$$

Let A be a set in B(X, Y). Then, by Lemma 3-1, for every n there exists A_n^0 in $B_0(K_n)$ such that

$$\mu_i((A \cap K_n) \triangle A_n^0) = 0, \qquad i = 1, 2.$$

On the other hand, using the Stone-Weierstrass theorem, we can easily prove that

$$B_0(K_n) = C(X, Y) \cap K_n, \qquad n = 1, 2, 3, \ldots.$$

Therefore, for every n, there exists A_n in C(X, Y) such that $A_n \cap K_n = A_n^0$ and put $A_0 = \bigcup_n \bigcap_{k=n}^{+\infty} A_k$. It is easy to see that A_0 has the desired property.

Utilising the above lemma and the well-known results concerning the equivalent-singular dichotomy of Gaussian measures on (X, C(X, Y)), we have the following theorem without difficulty (Ju. A. Rozanov [*6*]).

THEOREM 3-3. — Let (X, Y) be a topological dual system of real linear spaces X and Y, B(X, Y) be the Borel field of X, and let μ and μ' be Gaussian Radon measures on (X, B(X, Y)). Then μ and μ' are either equivalent (mutually absolutely continuous) or singular and they are equivalent if and only if their restrictions to C(X, Y) are equivalent.

The above theorem derives the same results as shown in [*6*] in our terminology. Let (X, Y) be a topological dual system and μ be a Gaussian Radon measure on (X, B(X, Y)). Furthermore let R_μ be a linear transformation of Y into $L^2(\mu) = L^2(X, B(X, Y), \mu)$ defined by

$$R_\mu : \xi \in Y \to \langle \cdot, \xi \rangle \in L^2(\mu),$$

let H_μ be the closure of the range of R_μ in $L^2(\mu)$, and let R_μ^* be the algebraic adjoint transformation of H_μ^a into Y^a. As usual, we identify the topological dual space of H_μ with H_μ.

THEOREM 3-4. — Let (X, Y) be a topological dual system, μ a centered Gaussian Radon measure on (X, B(X, Y)), and $\bar{B}(X, Y)$ the μ-completion of B(X, Y). Moreover, assume that $R_\mu^* H_\mu \subset X$ and that H_μ is separable. Then we have:

(1) For $x \in X$ let μ_x be a Gaussian Radon measure on (X, B(X, Y)) defined by

$$\mu_x(A) = \mu(A + x), \qquad A \in B(X, Y).$$

Then μ and μ_x are equivalent if and only if $x \in R_\mu^* H_\mu$.

(2) Let X_0 be a B(X, Y)-measurable linear subspace of X such that $\mu(X_0) = 1$. Then we have $R_\mu^* H_\mu \subset X_0$.

The proof of the above theorem is the same as those shown in [6]. Out of completeness, we give the proof of (2).

Let X_0 be a $\bar{B}(X, Y)$-measurable linear subspace of X such that $\mu(X_0) = 1$, and assume that $R_\mu^* H_\mu$ is not included in X_0. Then there exists an element x_0 in $R_\mu^* H_\mu$ such that $x_0 \notin X_0$. Since X_0 is a linear subspace of X, X_0 and $X_0 + x_0$ are disjoint and, by (1) we have $\mu(X_0) = \mu(X_0 + x_0) = 1$. Consequently we have

$$1 \geqq \mu(X_0 \{ X_0 + x_0 \}) = \mu(X_0) + \mu(X_0 + x_0) = 1 + 1 = 2 .$$

This is a contradiction.

4. SEPARABILITY OF $L^2(\mu)$

In this section we prove the separability of the Hilbert space $L^2(\mu)$ for a Gaussian Radon measure μ.

Let (X, Y) be a topological dual system and μ be a Gaussian Radon measure on (X, B(X, Y)). Then, since the topology $\sigma(X, Y)$ is coarser than the topology of X, μ is also a Gaussian Radon measure on (X, W(X, Y)). Consequently, there exists the minimal $\sigma(X, Y)$-closed linear subspace X_μ of X such that $\mu(X_\mu) = 1$. We call X_μ *the topological linear support* of μ.

Let X_μ^0 be the polar set of X_μ in Y. Then we have the following lemma.

LEMMA 4-1. — For ξ in Y, ξ is in X_μ^0 if and only if $\langle \cdot, \xi \rangle = 0$, a. e. ($\mu$).

Proof. — Since $\mu(X_\mu) = 1$, the necessity is obvious.

Assume that $\langle \cdot, \xi \rangle = 0$, a. e. ($\mu$). Since $X^\xi = \{ x \in X ; \langle x, \xi \rangle = 0 \}$ is a $\sigma(X, Y)$-closed linear subspace of X and $\mu(X^\xi) = 1$, we have $X_\mu \subset X^\xi$ by the minimality of X_μ, in other words, $\xi \in X_\mu^0$.

This proves the lemma.

Define the linear transformation R_μ of Y into $L^2(\mu) = L(X, B(X, Y), \mu)$ and H_μ as in the previous section. Then, using the above lemma, we can easily prove that X_μ^0 is the kernel of R_μ.

Put $Y_\mu = Y/X_\mu^0$. Then we have $X_\mu \cap W(X, Y) = W(X_\mu, Y_\mu)$ and $X_\mu \cap C(X, Y) = C(X_\mu, Y_\mu)$. Furthermore, since the induced topology of $\sigma(X, Y)$ on X_μ is identical with $\sigma(X_\mu, Y_\mu)$, a subset of X_μ is $\sigma(X, Y)$-closed if and only if $\sigma(X_\mu, Y_\mu)$-closed and the induced topology of $\tau(X, Y)$ on X_μ is coarser than $\tau(X_\mu, Y_\mu)$ (H. H. Schaefer [8], Chap. IV, § 4).

On the other hand, by Lemma 3-2 we have

$$L^2(X, B(X, Y), \mu) = L^2(X, C(X, Y), \mu) = L^2(X_\mu, C(X_\mu, Y_\mu), \mu) .$$

Therefore, in order to prove either the separability of

$$L^2(\mu) = L^2(X, B(X, Y), \mu)$$

or the $\tau(X, Y)$-separability of X_μ, we have only to prove it in the case of

$$\text{(C.II)} \qquad X = X_\mu .$$

In this case we have $Y = Y_\mu$ and by Lemma 4-1 we can easily show that for ξ in Y

$$\text{(C.II}'\text{)} \qquad \langle \cdot, \xi \rangle = 0, \qquad \text{a. e. } (\mu) \qquad \text{if and only if } \xi = 0 ,$$

and that R_μ is an injection of Y into H_μ.

THEOREM 4-2. — Let (X, Y) be a topological dual system and μ be a Gaussian Radon measure on (X, B(X, Y)). Then $L^2(\mu) = L^2(X, B(X, Y), \mu)$ is a separable Hilbert space.

Proof. — We may assume (C.II) without loss of generality.

Since μ is also a Gaussian Radon measure on (X, W(X, Y)), there exists an increasing sequence of $\sigma(X, Y)$-compact subsets $\{ K_n \}$ of X such that

$$\lim_n \mu(K_n) = 1 .$$

Let Z be the linear hull of $\bigcup_n K_n$ and denote by ρ the topology on Y of uniform convergence on all K_n. Then, using (C.II′), we can easily show that ρ is locally convex metrizable and finer than $\sigma(Y, Z)$. On the other hand, we have $Z \subset (Y, \rho)' \cap X$ and $\mu^*(Z) = 1$. Therefore ρ is an admissible metric on Y and consequently, by Theorem 2-6 and Lemma 3-2, $L^2(X, B(X, Y), \mu) = L^2(X, C(X, Y), \mu)$ is separable.

This proves the theorem.

5. SEPARABILITY OF THE SUPPORT OF A GAUSSIAN RADON MEASURE

Using the preceding results, we prove the following theorem.

THEOREM 5-1. — Let (X, Y) be a dual system and μ be a centered Gaussian Radon measure on (X, W(X, Y)). Then the topological linear support X_μ of μ is $\tau(X, Y)$-separable.

Proof. — As stated in § 4, we may assume the condition (C.II), that is, $X = X_\mu$. The measure μ can be extended to a centered Gaussian Radon measure on $(Y^a, W(Y^a, Y))$. Then X is $\overline{W}(Y^a, Y)$-measurable, hence by theorem 3-4 (2), we have $R_\mu^* H_\mu \subset X$. It is known $\mu(\overline{R_\mu^* H_\mu}^{\tau(X,Y)}) = 1$, where $-\tau(X, Y)$ means the closure for $\tau(X, Y)$ in X, remark that $\overline{R_\mu^* H_\mu}^{\tau(X,Y)} = R_\mu^* H_\mu^{00}$ (the bipolar) $= \{ x \in X ; \langle x, \xi \rangle = 0$ for all $\xi \in R_\mu^* H_\mu^0 \}$ and for every $\xi \in R_\mu^* H_\mu^0$ $\mu(\{ X \in X \mid \langle x, \xi \rangle = 0 \}) = 1$. The minimality of $X = X_\mu$ implies that $R_\mu^* H_\mu$ is $\tau(X, Y)$-dense in X. By theorem 4-2, H_μ is separable and $X = X_\mu$ is $\tau(X, Y)$-separable.

This proves the theorem.

For non-centered Gaussian Radon measures we have the following result.

THEOREM 5-2. — Let (X, Y) be a dual system and μ be a Gaussian Radon measure on (X, W(X, Y)) satisfying the condition (C. I). Then the topological linear support X_μ of X is $\tau(X, Y)$-separable.

Proof. — As stated in Section 4, we have only to prove in the case $X = X_\mu$. Let $\{ F_n \}$ be the increasing sequence of $\sigma(X, Y)$-compact absolutely convex subsets of X given in (C. I). Then the topology ρ on Y of the uniform convergence on all F_n is finer than $\sigma(Y, X)$ and coarser than $\tau(Y, X)$ and therefore an admissible metric on Y. By Lemma 2-3, R_μ is a compact linear transformation of (Y, ρ), *a fortiori*, of $(Y, \tau(Y, X))$ into H_μ and H_μ is separable. Consequently R_μ^* is also a compact linear transformation of H_μ into $(X, \tau(X, Y))$ with dense range and this proves the theorem.

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