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Quasi-compactness and uniform ergodicity of Markov operators

by

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RÉSUMÉ. — Il est démontré que, pour des opérateurs Markoviens, la quasi-compacité est équivalente à la convergence ergodique uniforme vers une projection de rang fini. Pour des probabilités de transition ergodiques, la quasi-compacité est équivalente à la convergence ergodique forte, même pour un espace d'états de type non-dénombrable.

SUMMARY. — It is shown that, for Markov operators, uniform ergodicity with finite dimensional fixed points space is equivalent to quasi-compactness. For ergodic transition probabilities, strong convergence of the averages is shown equivalent to quasi-compactness, even when the σ -algebra is not countably generated.

The study of a quasi-compact linear operator T on a Banach space was given by Yosida and Kakutani [16], in order to obtain some of the limit theorems of Doeblin [3]. Horowitz [8] and Brunel [1] studied quasi-compact conservative and ergodic contractions in $L_1(m)$ (see also Lin [9]). Brunel and Revuz [2] studied the quasi-compactness of Harris recurrent transition probabilities on a countably generated σ -algebra.

The result of Yosida and Kakutani is that for T quasi-compact with

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$\|T\| \leq 1$, $\frac{1}{N} \sum_{j=1}^N T^j$ converges uniformly to a finite dimensional projection

(see [5] for the treatment of the case where only $T^n/n \rightarrow 0$ weakly is assumed). In this work we show that for Markov operators (on L_∞ or on $C(K)$), uniform ergodic convergence to a finite dimensional projection implies quasi-compactness. The result is used to remove the countability assumption from Brunel and Revuz' result [2].

An important tool is the following « uniform ergodic theorem » [10].

If $\|T\| \leq 1$, then $\frac{1}{N} \sum_{j=1}^N T^j$ converges uniformly if and only if $I - T$ has a closed range. The limit is a projection E on the fixed points of T .

We deal with $C(X)$ after treating the L_1 case (« abstract » Markov operators), for which we need the following lemma.

LEMMA 1. — Let T be a conservative and ergodic positive contraction of $L_1(X, \Sigma, m)$, with m σ -finite. If $|\lambda| = 1$ and λ is an eigenvalue of T^* , then $\{f \in L_\infty : T^*f = \lambda f\}$ is finite dimensional.

Proof. — If $T^*f = \lambda f$, then $T^*|f| \geq |T^*f| = |f|$, and $T^*|f| = |f|$ since T is conservative, hence $|f| = \text{constant}$ by ergodicity, and we may assume $|f| \equiv 1$.

Now identify L_∞ with a space $C(K)$, K compact Hausdorff. T^* corresponds to a positive contraction S on $C(K)$. If $T^*f_1 = \lambda f_1$ and $T^*f_2 = \lambda f_2$, with $|f_1| = |f_2| = 1$, then $S\hat{f}_1 = \lambda\hat{f}_1$, $S\hat{f}_2 = \lambda\hat{f}_2$ (with \hat{f}_i the images of f_i in $C(K)$). By a well-known result on unimodular eigenfunctions (see Schaefer [15]), $S(\hat{f}_1\hat{f}_2^{-1}) = \hat{f}_1\hat{f}_2^{-1}$, so $T(f_1f_2^{-1}) = f_1f_2^{-1}$. Hence $f_1f_2^{-1}$ is constant, and $f_1 = \alpha f_2$ with $|\alpha| = 1$, and the lemma holds.

Remark. — For λ a root of unity, one can avoid the representation of L_∞ and use the result of Foguel and Weiss [7] on the finiteness of the invariant sets of T^{*n} . The proof is not shorter.

THEOREM 2. — Let T be a positive contraction of $L_1(m)$. (i) T is quasi-compact if and only if $N^{-1} \sum_{j=1}^N T^j$ converges uniformly to a finite dimensional projection. (ii) In this case, $I - \lambda T$ has closed range for every $|\lambda| = 1$, and $\sigma(T) \cap \{|\lambda| = 1\}$ consists of finitely many points, which are all roots of unity, and are eigenvalues with finite dimensional eigenspaces.

Proof. — Assume $\left\| \frac{1}{N} \sum_{j=1}^N T^j - E \right\| \rightarrow 0$, with $E L_1(m)$ finite dimensional.

Let C be the conservative part of T . Since $L_1(C)$ is T -invariant [6], the uniform ergodic theorem implies that there is an $f \in L_1(C)$, $f > 0$ a. e. on C , satisfying $Tf = f$. By taking an equivalent finite measure m' such that $dm'/dm = f$ on C , we may and do assume $T1_C = 1_C$ ($T'f' = (dm/dm')$ ($T(f'dm'/dm)$ is the necessary transformation). We shall assume also $m(C) = 1$. Clearly now $E1_C = 1_C$. The restriction of T to $L_1(C)$ is a contraction of $L_\infty(C)$, and since $\{f : Tf = f\}$ is finite dimensional, there are finitely many disjoint sets A_1, \dots, A_K with union C such that $TL_1(A_j) \subset L_1(A_j)$ and T is ergodic on A_i [6].

Let $|\lambda| = 1$, $1 \neq \lambda \in \sigma(T)$. We show that $\lambda I - T$ has closed range. Let $(\lambda I - T)f_n \rightarrow f$ in L_1 norm. Define $g_{nk} = |f_n - f_k| - |T(f_n - f_k)|$. Then we have,

$$\begin{aligned} \|g_{nk}\| &= \int \left| |f_n - f_k| - |T(f_n - f_k)| \right| dm \\ &\leq \int |\lambda(f_n - f_k) - T(f_n - f_k)| dm \xrightarrow{n,k \rightarrow \infty} 0. \end{aligned}$$

Since by definition $g_{nk} + T|f_n - f_k| - |f_n - f_k| \geq 0$ a. e.,

$$\begin{aligned} \|(I - T)|f_n - f_k|\| &\leq \|(I - T)|f_n - f_k| - g_{nk}\| + \|g_{nk}\| \\ &= \int [g_{nk} + T|f_n - f_k| - |f_n - f_k|] dm + \|g_{nk}\| \\ &\leq \int g_{nk} dm + \|g_{nk}\| \leq 2\|g_{nk}\| \xrightarrow{n,k \rightarrow \infty} 0. \end{aligned}$$

By the uniform ergodic theorem, $I - T$ is invertible on $(I - T)L_1$. Since $|f_n - f_k| - E|f_n - f_k| \in (I - T)L_1$, and $(I - T)E = 0$, we have

$$\|(I - T)\{|f_n - f_k| - E|f_n - f_k|\}\| \xrightarrow{n,k \rightarrow \infty} 0$$

and therefore

$$\||f_n - f_k| - E|f_n - f_k|\| \xrightarrow{h,k \rightarrow \infty} 0.$$

Denote $a_{nki} = \int_{A_i} |f_n - f_k| dm$. But $E|f_n - f_k| = \sum_{i=1}^K a'_{nki} 1_{A_i}$. Hence

$$\int_D |f_n - f_k| dm + \sum_{i=1}^K \int_{A_i} \left| |1_{A_i} f_n - 1_{A_i} f_k| - a_{nki} \right| dm \rightarrow 0.$$

Applying lemma VI.2 of [2] K times, we have that $\{f_n\}$ contains a Cauchy

subsequence. If $f_{n_i} \rightarrow g$, $(\lambda I - T)g = \lim (\lambda I - T)f_{n_i} = f$ and $(\lambda I - T)L_1$ is closed (We extend the argument of [2]). Applying the uniform ergodic theorem to λT , we have that λ is an eigenvalue of T .

We now have that if $\lambda \in \sigma(T)$ with $|\lambda| = 1$, λ is an eigenvalue of T , hence of T^* . λ is isolated in $\sigma(T)$, since $\lambda I - T$ is invertible on its range. Hence $\sigma(T) \cap \{\lambda : |\lambda| = 1\}$ consists of finitely many eigenvalues $\lambda_1, \dots, \lambda_r$.

If $\lambda \in \sigma(T)$ with $|\lambda| = 1$, there is an $f \in L_1$ with $Tf = \lambda f$, and $T|f| \geq |Tf| = |f|$, hence $T|f| = |f|$ a. e., and $\{|f| \neq 0\} \subset C$. To show $\{f \in L_1 : Tf = \lambda f\}$ is finite dimensional we may and do assume T to be conservative. Since $T(1_{A_j}f) = 1_{A_j}Tf = \lambda 1_{A_j}f$, we may restrict ourselves to the invariant set A_j , and have T also ergodic. Let T_j be the restriction of T to $L_1(A_j)$. If λ is an eigenvalue of T_j it is of T_j^* . Apply now the lemma to obtain that $\{f \in L_1(A_j) : Tf = \lambda f\}$ is finite dimensional, hence $X_\lambda = \{f \in L_1 : Tf = \lambda f\}$ is finite dimensional.

Let $E_k = \lim \frac{1}{N} \sum_{j=1}^N \lambda_k^{-j} T^j$ (which exists uniformly). Then

$$L_1 = \left(\sum_{k=1}^r E_k \right) L_1 \oplus \bigcap_{k=1}^r (\lambda_k I - T) L_1$$

and putting $Q = \sum_{k=1}^r \lambda_k E_k$, we have

$$\|T^n - Q^n\| = \left\| T^n \left(I - \sum_{k=1}^r E_k \right) \right\| \rightarrow 0,$$

since on $\bigcap_{k=1}^r (\lambda_k I - T) L_1$ the restriction of T has no spectral points of unit modulus. Q is compact ($E_k L_1$ is finite dimensional) and T is quasi-compact.

Quasi-compactness implies the uniform ergodic theorem [16] [10].

The fact that the eigenvalues λ_k are roots of unity follows (looking at T_j^* as before), from the above mentioned lemma of [15], which shows that $\lambda_k^n \in \sigma(T)$ for every n . (One can use directly the result in [5], since T^* is also quasi-compact).

Remarks. — 1. When T is conservative, Horowitz' result [8] is that *strong*

convergence of $\frac{1}{N} \sum_{j=1}^N T^{*j}$ to a finite dimensional projection implies quasi-compactness.

2. For non-positive contractions the result is false. The example in [11] is valid in l_1 .

THEOREM 3. — *Let P be a positive contraction of $C(X)$, (X compact Hausdorff). The following conditions are equivalent.*

(i) P is quasi-compact.

(ii) $N^{-1} \sum_{i=1}^N P^i$ converges uniformly to a finite dimensional projection.

(iii) $(I - P)C(X)$ is closed and $\{ f : Pf = f \}$ is finite dimensional.

(iv) $(I - P^*)C(X)^*$ is closed and the space of invariant measures is finite dimensional.

(v) $(I - P)C(X)$ is closed and the space of invariant measures is finite dimensional.

Proof. — (i) \Rightarrow (ii) is due to Yosida and Kakutani [16] (see [10] for another proof). (ii) \Rightarrow (iii) is immediate, by the uniform ergodic theorem.

(iii) \Rightarrow (iv). By the uniform ergodic theorem $\frac{1}{N} \sum_{i=1}^N P^i$ converges uniformly to a projection E on $\{ f = Pf \}$. Hence $\dim E^*X^* = \dim EX < \infty$. Since $\frac{1}{N} \sum_{i=1}^N P^{*i} \rightarrow E^*$ uniformly, (iv) follows.

(iv) \Rightarrow (v). By the uniform ergodic theorem $\frac{1}{N} \sum_{i=1}^N P^{*i}$ converges uniformly, and so does $\frac{1}{N} \sum_{i=n}^N P^i$, implying $(I - P)C(X)$ is closed.

(v) \Rightarrow (ii). Follows again from the uniform ergodic theorem.

Assume now conditions (ii) – (v). We can extend P to the space of complex continuous functions, so we may and do assume that this has been done.

Let $|\lambda| = 1, \lambda \in \sigma(P)$. We show that $I - \lambda P$ has closed range. Let $(I - \lambda P^*)\mu_n \rightarrow \mu$. Define $m_1 = \sum_{n=0}^{\infty} \alpha_n |\mu_n| + \alpha |\mu|$, with $\alpha_n, \alpha > 0$ such that m_1 is a probability. Let $m = \sum_{n=0}^{\infty} 2^{-(n+1)} P^{*n} m_1$. Then $P^*m \ll m$ and

we obtain an operator T on $L_1(m)$ by $Tu = d(P^*v)/dm$ when $u = dv/dm$.

By (iv) $\frac{1}{N} \sum_{i=1}^N T^i$ converges to a finite dimensional projection. Since

$(I - \lambda T)u_n \rightarrow u$ in $L_1(m)$, when $u_n = d\mu_n/dm$, then by theorem 2, $u = (I - \lambda T)v$ with $v \in L_1(m)$, and $\mu = (I - \lambda P^*)v$ with $dv/dm = v$. Hence $I - \lambda P^*$ has

closed range, and $\frac{1}{N} \sum_{i=1}^N \lambda^i P^i$ converges, by the uniform ergodic theorem,

with limit $E_{\bar{\lambda}}$, which projects on $\{f : Pf = \bar{\lambda}f\}$.

We show that $E_{\bar{\lambda}}^*$ has finite dimensional range. If not, let $\{v_n\}_{n=1}^\infty$ be a sequence of linearly independent (complex) finite measures with $P/v_n = \bar{\lambda}v_n$.

Assume $\|v_n\| = 1$. Let $m_1 = \sum_{n=1}^\infty 2^{-n} |v_n|$ and $m = \sum_{n=0}^\infty 2^{-(n+1)} P^{*n} m_1$.

Again, we have a positive contraction T on $L_1(m)$, with $Tu_n = \bar{\lambda}u_n$, where $u_n = dv_n/dm$. But by theorem 2, all eigenvalues of T have finite dimensional

eigenspaces, which yields a contradiction. Since $\frac{1}{N} \sum_{n=1}^N \lambda^{-n} P^n$ converges

uniformly for $\lambda \in \sigma(P)$ with $|\lambda| = 1$, λ is isolated in $\sigma(P)$, hence $\sigma(P) \cap \{\lambda : |\lambda| = 1\}$ is finite.

The end of the proof of theorem 2 can be used to show that P is quasi-compact.

Remark. — Horowitz' result [8] (see previous remarks) fails for general positive contractions of $C(X)$, as observed in [10].

COROLLARY 4. — Let P be a positive contraction of $C(X)$. If $\frac{1}{N} \sum_{j=1}^N P^j$

converges uniformly to a finite dimensional projection, then, for every $k \geq 1$,

$\frac{1}{N} \sum_{j=1}^N P^{kj}$ converges uniformly.

Remarks. — Sawashima and Niuro [14] give an example where the corollary fails if the limit is infinite dimensional. Their example also shows that a positive contraction T on L_1 may satisfy the uniform ergodic theorem, while T is not quasi-compact on $(I - T)L_1$.

For comparison, we note the following result concerning the Markov

operator $Pf(x) = \int f(y)P(x, dy)$ induced by a transition probability. We omit the additional requirement of a countably generated σ -algebra which appears in [2]. The Harris condition and the existence of an invariant measure need not be assumed *a priori*. For simplification we assume P to be ergodic, i. e., $Pf = f$ implies f is constant.

THEOREM 5. — *Let Σ be a σ -algebra of subsets of X and let $P(x, A)$ be a transition probability inducing an ergodic (Markov) operator on the space $B(X, \Sigma)$ of bounded measurable functions. Then the following conditions are equivalent.*

- (i) P is quasi-compact.
- (ii) $\frac{1}{N} \Sigma P^i$ converges uniformly (necessarily to a one dimensional projection).
- (iii) $(I - P)B(X, \Sigma)$ is closed.
- (iv) $(I - P^*)M(X, \Sigma)$ is closed. ($M(X, \Sigma) = \{ \text{finite signed measures} \}$).
- (v) Every P^* -invariant functional is a measure.
- (vi) The space of P^* invariant functionals is one-dimensional.
- (vii) $\frac{1}{N} \sum_{i=1}^N P^i$ converges strongly.
- (viii) P satisfies Doeblin's condition.

Proof. — (i) \Rightarrow (ii) is well-known [16] [10].

(iii) \Leftrightarrow (ii). By the uniform ergodic theorem. and then (ii) \Rightarrow (iv).

(iv) \Rightarrow (ii). Let T be the restriction of P^* to $M(X, \Sigma)$. By the uniform ergodic theorem $\frac{1}{N} \sum_{i=1}^N T^i$ converges uniformly, so does $\frac{1}{N} \Sigma T^{*i}$, which is $\frac{1}{N} \sum_{i=1}^N P^i$ when restricting T^* to $B(X, \Sigma)$. Hence (ii), (iii) and (iv) are equivalent.

(ii) \Rightarrow (v). $\left\| \frac{1}{N} \sum_{i=1}^N P^i - E \right\| \rightarrow 0$, and Ef is invariant for any f , so it is constant by ergodicity.

Let m be any probability on Σ . Let

$$\mu(A) = E1_A = \int \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N P^i 1_A dm = \lim_{n \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N P^{*i} m(A).$$

Then μ is a probability, and $\langle \mu, f \rangle = Ef$, showing $P^* \mu = \mu$.

If $P^* \nu = \nu$ then

$$\nu(f) = \left\langle \frac{1}{N} \sum_{i=1}^N P^{*i} \nu, f \right\rangle = \left\langle \nu, \frac{1}{N} \sum_{i=1}^N P^i f \right\rangle \rightarrow \langle \nu, Ef \rangle = Ef \cdot \nu(1).$$

Hence $\nu = \nu(1) \cdot \mu$ and is a measure.

(v) \Rightarrow (vi). Let $\nu \neq \mu$ be two invariant probabilities. Let $m = \nu + \mu$. Then $P^* m = m$, and T defined in $L_1(m)$ by $T(d\mu/dm) = d(P^* \mu/dm)$ is a positive contraction of $L_1(m)$, with $T1 = 1$. Let $f \in B(X, \Sigma)$ with $\int f d\mu \neq \int f d\nu$. $T^* f = P f m$ a. e., and by the ergodic theorem $\lim \frac{1}{N} \sum T^{*i} f = \hat{f}$ exists a. e. and $T^* \hat{f} = \hat{f}$ a. e. (m). But $T^*(d\mu/dm) = d\mu/dm$, $T^*(d\nu/dm) = d\nu/dm$, so that

$$\int \hat{f} d\mu = \int \lim_{i \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N T^{*i} f d\mu = \int f d\mu \neq \int \hat{f} d\nu.$$

Let $g = \limsup \frac{1}{N} \sum_{i=1}^N P^i f$. Then $Pg \geq \limsup \frac{1}{N} \sum_{i=2}^{n+1} P^i f = g$, and $h = \lim P^k g$

is P invariant. Hence h is constant (P is ergodic). Since $h = \hat{f}$ a. e. (m), $\int \hat{f} d\mu = \int h d\mu = \int h d\nu = \int \hat{f} d\nu$, a contradiction. Since the positive and negative parts of an invariant functional are invariant, (vi) is proved.

(vi) \Rightarrow (vii). Let ν be a functional on $B(X, \Sigma)$ such that $\nu(f) = 0$ for $f \in (\overline{(I - P)B(X, \Sigma)}) \oplus \{ \text{constants} \}$. Then $P^* \nu = \nu$ and $\nu(1) = 0$. But condition (vi) implies $\nu = \nu(1)\mu$ where μ is an invariant positive functional. Hence $\nu = 0$. By the Hahn-Banach theorem

$$\overline{(I - P)B(X, \Sigma)} \oplus \{ \text{constants} \} = B(X, \Sigma),$$

and (vii) follows.

(vii) \Rightarrow (ii). Let $(I - P)f_n \rightarrow g$ in norm. By Doob [4, p. 209] there is a countably generated σ -algebra Σ_0 such that for $A \in \Sigma_0 P(x, A)$ is Σ_0 measurable, and $f_n g$ are Σ_0 -measurable (Σ_0 is the admissible σ -algebra generated by the sets $\{ r \leq f_n \leq s \}$, $\{ r \leq g \leq s \}$, r, s rationals).

Let $P_0(x, A) = P(x, A)$ for $x \in X, A \in \Sigma_0$. For $f \in B(X, \Sigma_0), P_0 f = P f$, so P_0 satisfies (vii). (vii) \Rightarrow (v), so by Brunel and Revuz [2] (who use the fact that Σ_0 is countably generated) $\frac{1}{N} \sum_{i=1}^N P_0^i$ converges uniformly and

$I - P_0$ has closed range. Hence $g = (I - P_0)f$ with $f \in B(X, \Sigma_0)$, so $g = (I - P)f$. Therefore $I - P$ has closed range and (ii) holds by the uniform ergodic theorem.

(ii) \Rightarrow (i). $B(X, \Sigma)$ is isometrically and order-isomorphic to $C(K)$ and theorem 3 applies, since the limit projection has one-dimensional range by ergodicity.

(i) \Leftrightarrow (viii) is shown in [13].

Remarks. — 1. Condition (v) is equivalent to. (v') There is no purely finitely additive P^* -invariant functional (since the countably additive part of an invariant functional is invariant).

2. Condition (vii) clearly implies the Harris condition:

$$m(A) = \lim \frac{1}{N} \sum_{n=1}^N P^n 1_A$$

is (the unique) invariant probability, and

$$m(A) > 0 \Rightarrow \sum_{n=1}^{\infty} P^n 1_A(x) = \infty .$$

Note that Moy [12] establishes the existence of a kernel under this condition without Σ being countably generated. It is not clear how to obtain a compact operator in $B(X, \Sigma)$, bounded by P^{n_0} (which is easy in $L_{\infty}(m)$) without the countable additivity, and this is why we use Σ_0 in proving (vii) \Rightarrow (ii).

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