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Hunt processes and analytic potential theory on rigged Hilbert spaces (*)

by

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ABSTRACT. — The analytic potential theory for energy forms $\int \nabla f \nabla f d\mu$ given by positive probability Radon measures μ on separable rigged Hilbert spaces is studied, adapting methods of Fukushima. The forms define self-adjoint operators in $L^2(d\mu)$ which generate, after elimination of polar sets, Hunt processes with continuous sample paths.

RÉSUMÉ. — On étudie la théorie analytique du potentiel pour des formes d'énergie $\int \nabla f \nabla f d\mu$ données par des mesures de probabilité de Radon sur des espaces d'Hilbert séparables, en adaptant des méthodes de Fukushima. Ces formes définissent des opérateurs auto-adjoints dans l'espace $L^2(d\mu)$ qui engendrent, après élimination d'ensembles polaires, des processus de Hunt aux trajectoires continues.

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1. INTRODUCTION

In this paper we study the analytic potential theory given by energy forms generated by positive probability Radon measures associated with rigged Hilbert spaces. We already considered such energy forms in previous papers [1]-[3], particularly from the point of view of their relations with quantum fields. In the present paper ⁽¹⁾ we understand analytic potential theory in very much the same spirit as in the recent work of M. Fukushima [5]-[9] and M. Silverstein [10]-[11], which continues and extends the classical work of H. Cartan [12] and A. Beurling-J. Deny [13]-[14] ⁽²⁾. The processes considered are symmetric Hunt processes i. e. a regular case of Dynkin's standard processes, constructed by eliminating first sets of capacity zero. For general background we refer, besides the already quoted references, to [20]-[24] ⁽³⁾.

Let us now shortly summarize the content of our paper.

In Section 2 we introduce the rigged Hilbert space $Q \subset K \subset Q'$, where K is a real separable Hilbert space and Q is a real complete Hausdorff locally convex vector space, with topological dual Q' . The rigging is, as usual, a natural structure for the localization of the support of the measures we are considering. We consider positive probability Radon measures μ on Q' , understood in the sense of L. Schwartz [36] and required to be quasi invariant with respect to translations by elements in Q . We give conditions on μ such that μ defines an energy form $\int \nabla f \nabla g d\mu$ which is closable in $L^2(d\mu)$, where ∇ is the gradient, naturally defined on C_b^1 cylinder functions as a map from $L^2(d\mu)$ into the Hilbert tensor product $K \otimes L^2(d\mu)$. For any closable energy form we consider the self-adjoint positive operator $H = \nabla^* \bar{\nabla}$ in $L^2(d\mu)$, the « energy operator » uniquely associated with the closure of the energy form. (Here ∇^* and $\bar{\nabla}$ denote respectively the adjoint and the closure of ∇). We find that a sufficient condition for H to be defined (i. e. for the energy form to be closable) is that the limit in $L^2(d\mu)$ of $t^{-1} \left[\left(\frac{d\mu(\cdot + tq)}{d\mu(\cdot)} \right)^{1/2} - 1 \right]$ as $t \rightarrow 0$ exists, for all $q \in Q$. Call $(2i)^{-1} \beta(q)(\cdot)$

⁽¹⁾ The main results of this paper have been summarized in the report [4].

⁽²⁾ Some other references in this line are e. g. [15]-[19].

⁽³⁾ For other approaches to diffusion processes, potential theory and related questions on Hilbert spaces see e. g. [25]-[36].

this limit. Let f be a C_b^2 cylinder function with base spanned by the orthonormal vectors e_1, \dots, e_n in $Q \subset K$, and write

$$(\beta \cdot \nabla)f(\cdot) \equiv \sum_{i=1}^n \beta(e_i)(\cdot)(e_i \cdot \nabla f)(\cdot),$$

where $e_i \cdot \nabla$ is the derivative in the direction e_i . Then we have

$Hf = -\Delta f - (\beta \cdot \nabla)f$, where Δf is naturally defined as $\sum_{i=1}^n (e_i \cdot \nabla)^2 f$. So

is H exhibited as a diffusion operator on C_b^2 cylinder functions. Conditions for μ in order that also all polynomials be in the domain of H are also given.

In Section 3 we show that any closable energy form has a closure which is a Dirichlet form in the sense of Fukushima [8], thus the associated energy operator H is the infinitesimal generator of a strongly continuous positivity preserving semigroup on $L^2(d\mu)$, conservative in the sense that $e^{-tH}1 = 1$, $t \geq 0$. We study the problem of when the Dirichlet form is regular in the sense of Fukushima and Silverstein, so that there exists a Hunt process properly associated to it [8], [10] (i. e. such that the state space is a locally compact separable Hausdorff space modulo a polar set and the transition semigroup associated with the process is a quasi continuous modification of the semigroup generated by the energy operator, when restricted to act on continuous $L^2(d\mu)$ -functions). We show that for the Dirichlet form to be regular it is sufficient that the following conditions be satisfied: 1) there exists a strictly positive compact operator A on K such that

$$\sum_i \lambda_i(a_i, \xi)^2 \equiv |\xi|_A^2$$

converges in $L^2(d\mu)$, where a_i are the normalized eigenvectors of A and λ_i the correspondent eigenvalues; 2) the closure of K in the $|\cdot|_A$ -norm is compactly contained in a separable Banach space continuously injected in Q' ; 3) the support X of μ is bounded in the $|\cdot|_A$ -norm. In such a case X is, in the topology induced from the one of the Banach space, a compact separable Hausdorff space and μ is a Radon everywhere dense measure. The regular Dirichlet form which then is given by μ has a locality property which yields, by the results of Fukushima and Silverstein, continuity in the above Banach norm of the sample paths of the strong Markov process properly associated with the form, so that this process is indeed a diffusion

process. A reflecting barrier process corresponding to it is obtained by restricting μ to any $|\cdot|_A$ -open subset of positive measure and considering the corresponding closable energy form. If Q is separable then X can be taken as the uniform closure of continuous cylinder functions. X differs then from the closure in X of $\bigcup_x \{ \xi \in Q' \mid |\xi|_A \leq \alpha \}$ only by a polar set.

2. THE ENERGY FORM

We say that $Q \subset K \subset Q'$ is a *rigging* of the real separable Hilbert space K iff Q is a real complete Hausdorff locally convex vector space with dual Q' , such that the dualization (q, ξ) , $q \in Q$, $\xi \in Q'$ between Q and Q' coincides, when restricted to $Q \times K$, with the inner product in K , and moreover the injections $Q \hookrightarrow K$ and $K \hookrightarrow Q'$ are both continuous with dense ranges.

We shall say that a measure μ belongs to $\mathcal{P}(Q')$ if μ is a positive probability Radon measure on Q' , in the sense of [36]. For the relation between Radon measures and cylindrical measures see [36]. If $\mu \in \mathcal{P}(Q')$ then there is an increasing sequence of compact subsets of Q' such that the μ -measure of their union X is equal to one. Hence X is a locally compact subset of Q' such that $\mu(X) = 1$. Thus we may identify $L^p(Q', d\mu)$ with $L^p(X, d\mu)$.

Let P be an orthogonal projection in K with finite dimensional range contained in Q . It is easily seen that P is continuous in the topology of Q' ,

since $Pk = \sum_{i=1}^n (e_i, k)e_i$, for any $k \in K$ and orthogonal base e_1, \dots, e_n in

the range of P . We shall denote its continuous extension to Q' again by the symbol P . In the following the symbol C_b^k (and $C_b \equiv C_b^0$) will stand for k -times continuously differentiable functions with bounded i -th derivatives, $i = 0, 1, \dots, k$.

We say that a function $f \in C(Q')$ is in $FC^n(Q')$ if for some orthogonal projection P with finite dimensional range in Q we have $f(\xi) = f(P\xi)$ for all ξ in Q' and $f(Pk)$, $k \in K$, as a function on PK , is in $C_b^n(PK)$. We also say that such an f is based on PK . Since (q, ξ) for $q \in Q$ separates points in Q' we have that it also separates points in K , so that FC^n and also $FC^\infty = \bigcap_n FC^n$ are, by the Stone-Weierstrass theorem, dense in the space of functions which are continuous and bounded on compact subsets of Q' . Therefore FC^n and FC^∞ are dense in $L^p(Q', d\mu)$ for all $1 \leq p < \infty$, by the properties of Radon measures.

We now define the mapping ∇ between the real Hilbert space $L^2(d\mu)$

and $\mathbf{K} \otimes L^2(d\mu)$, with domain FC^1 , given by $f \rightarrow \nabla f$, where ∇f is the gradient of f considered as a C^1 -function on the finite dimensional space \mathbf{PK} on which $f \in FC^1$ is based (thus \mathbf{P} is a finite dimensional orthogonal projection with range in \mathbf{Q} such that $f(\xi) = f(\mathbf{P}\xi)$).

Clearly a given function f in FC^1 can be looked upon as being based on different spaces \mathbf{PK} . However it is easily seen that the definition of ∇f is independent of the choice of a particular base for f . We also see that if we look at f as based on \mathbf{PK} and e is an element of \mathbf{PK} , then, denoting by $e \cdot \nabla$ the directional derivative in the direction e , we have

$$\nabla f = \sum_{i=1}^n e_i \otimes (e_i \cdot \nabla) f, \quad (2.1)$$

if n is the dimension of \mathbf{PK} and e_1, \dots, e_n is any orthonormal base in \mathbf{PK} . Note that ∇f is also independent of the choice of the basis e_1, \dots, e_n .

We shall say that a measure $\mu \in \mathcal{P}(\mathbf{Q}')$ is *admissible* if ∇ is closable as an operator from $L^2(d\mu) \equiv L^2(\mathbf{Q}', d\mu)$ to $\mathbf{K} \otimes L^2(d\mu)$. We define the *energy form* given by μ first on FC^1 by

$$E(f, g) \equiv (\nabla f, \nabla g) \equiv \int \nabla f \cdot \nabla g d\mu, \quad (2.2)$$

where $\nabla f \cdot \nabla g$ is the scalar product in a finite dimensional space on which f and g are based. Let (f, f) be the scalar product of f with itself in $L^2(d\mu)$, then we see that $E(f, f) + (f, f)$ defines the graph norm of ∇ , so that ∇ is a closable quadratic form.

We say that μ is quasi invariant if it is quasi invariant under the measurable transformations of \mathbf{Q}' given by $\xi \rightarrow \xi + q$ for arbitrary $q \in \mathbf{Q}$. If μ is quasi invariant we define the unitary translations by

$$(\mathbf{V}(q)f)(\xi) = \left[\frac{d\mu(\xi + q)}{d\mu(\xi)} \right]^{1/2} f(\xi + q), \quad (2.3)$$

for any $f \in L^2(d\mu)$, where $\frac{d\mu(\xi + q)}{d\mu(\xi)}$ is the Radon-Nikodym derivative of

the translated measure with respect to μ . $\mathbf{V}(q)$ is then obviously a unitary representation of the abelian group \mathbf{Q} in the Hilbert space $L^2(d\mu)$, which is strongly continuous when restricted to finite dimensional subspaces of \mathbf{Q} . A mapping of \mathbf{Q}' into \mathbf{Q}' which is continuous when restricted to finite dimensional subspaces of \mathbf{Q} is said to be vaguely continuous. So that $\mathbf{V}(q)$ is a vaguely continuous unitary representation of \mathbf{Q} in the strong

$L^2(d\mu)$ topology. We denote the self-adjoint infinitesimal generator for the one parameter unitary group $V(tq)$, $t \in \mathbb{R}$, by $\pi(q)$. It follows that π is a linear mapping from Q into the space of self-adjoint operators in the sense that $\pi(q_1) + \pi(q_2) \subset \pi(q_1 + q_2)$. We say that $\mu \in \mathcal{P}_1(Q')$ if μ is a quasi invariant probability measure and $1 \in D(\pi(q))$ for all $q \in Q$, where we denote the domain of an operator A by $D(A)$. We have the following

THEOREM 2.1. — If $\mu \in \mathcal{P}_1(Q')$ i. e. if μ is a quasi invariant probability measure such that $1 \in D(\pi(q))$ for all $q \in Q$, then μ is admissible.

Proof. — Define $\beta(q)(\xi) \equiv (2i\pi(q).1)(\xi)$. Then we have $\beta(q)(.) \in L^2(d\mu)$. Let $f, g \in FC^1$. A simple calculation then shows that for any $q \in Q$ we have

$$((q.\nabla)f, g) = (f, (-q.\nabla - \beta(q))g), \tag{2.4}$$

where $q.\nabla$ is the directional derivative in the direction q i. e.

$$((q.\nabla)f)(\xi) = \frac{d}{dt} f(\xi + tq) |_{t=0}.$$

From (2.4) we have that the operator $q.\nabla$ has a densely defined adjoint, hence it is closable. Consider now elements in $K \otimes L_2(d\mu)$ of the form

$u = \sum_{i=1}^n q_i \otimes g_i$ with $q_i \in Q$ and $g_i \in FC^1$. From (2.4) we get

$$(\nabla f, u) = \sum_{i=1}^n (f, (-q_i.\nabla - \beta(q_i))g_i) = (f, (-\text{div} - \beta).u), \tag{2.5}$$

so that the mapping ∇ from $L^2(d\mu)$ into $K \otimes L^2(d\mu)$ has a densely defined adjoint mapping, from $K \otimes L^2(d\mu)$ into $L^2(d\mu)$, which on elements of the form u is given by

$$-(\text{div} + \beta).u = - \sum_{i=1}^n (q_i.\nabla + \beta(q_i))g_i. \tag{2.6}$$

∇ is thus closable and the theorem is proven. ■

Let μ be an admissible Radon measure on Q' , and let $\bar{\nabla}$ be the closure of ∇ on FC^1 . Since $E(f, f) = (\nabla f, \nabla f)$ we have that $E(f, f)$ is closable and the closure \bar{E} of E is such that $\bar{E}(f, f) = \|\bar{\nabla}f\|^2$. Hence \bar{E} has the domain $D(\bar{E}) = D(\bar{\nabla})$. The adjoint ∇^* of ∇ is densely defined, ∇ being closable. We have $\nabla^{**} = \bar{\nabla}$. Thus $H \equiv \nabla^* \bar{\nabla}$ is a self-adjoint operator in $L^2(d\mu)$ such that $D(H^{1/2}) = D(\bar{E}) = D(\bar{\nabla})$. We shall call H the *energy operator* associated with the admissible Radon measure μ .

If we assume in addition that μ is in $\mathcal{P}_1(Q')$ then $\beta(q)(\cdot) \in L^2(d\mu)$ and we have easily as in (2.5) that ∇ maps FC^2 into the domain of ∇^* , in such a way that for $u = \nabla f$ we still have $\nabla^*u = -(\operatorname{div} + \beta)u$. This gives us the following proposition.

PROPOSITION 2.2. — If $\mu \in \mathcal{P}_1(Q')$ then $FC^2 \subset D(H)$, where H is the energy operator associated with μ , and for $f \in FC^2$ we have

$$Hf = -\Delta f - \beta \cdot \nabla f,$$

where Δ and $\beta \cdot \nabla$ are so defined that if P is an orthogonal projection with finite dimensional range in Q such that $f(\xi) = f(P\xi)$ and $\{e_1, \dots, e_n\}$ is an orthonormal base in the range of P , then $\Delta f = \sum_{i=1}^n (e_i \cdot \nabla)^2 f$ and $(\beta \cdot \nabla)f = \sum_{i=1}^n \beta(e_i)(e_i \cdot \nabla)f$. ■

Remark that both Δ and $\beta \cdot \nabla$ are obviously independent of the projections P and the orthonormal base $\{e_1, \dots, e_n\}$, the functions in FC^2 having continuous bounded second derivatives.

We also remark that the following sharper version of Theorem 2.1 holds.

THEOREM 2.3. — Let $\mu \in \mathcal{P}(Q')$ and assume that $\int (q \cdot \nabla) f d\mu$ is, for any $q \in Q$, a continuous linear functional on $L^2(Q', d\mu)$, defined for all $f \in FC^1$. Then μ is admissible. Moreover ∇ maps FC^2 into $D(\nabla^*)$ so that in particular $FC^2 \subset D(H)$ and for any $f \in FC^2$ we have $Hf = -\Delta f - \beta \cdot \nabla f$, where $\beta(q)(\cdot) \in L^2(Q', d\mu)$ is given by $\int (q \cdot \nabla) f d\mu = - \int \beta(q)(\xi) f(\xi) d\mu(\xi)$, for any $f \in FC^1$.

Proof. — Since $\int (q \cdot \nabla) f d\mu$ is continuous in $L^2(Q', d\mu)$ we have that $q \otimes 1$ is in $D(\nabla^*)$. Set $\beta(q) \equiv -\nabla^*(q \otimes 1)$. Let $g \in FC^1$, then, using $\nabla(fg) = (\nabla f)g + f\nabla g$ on FC^1 , we have

$$\int g(q \cdot \nabla) f d\mu = - \int (q \cdot \nabla g) f d\mu - \int \beta(q) g f d\mu, \quad (2.7)$$

which then gives us that $q \otimes g \in D(\nabla^*)$ for arbitrary $q \in Q$. By taking the linear span we get that $D(\nabla^*)$ is dense in $K \otimes L^2(Q', d\mu)$, hence ∇ is closable thus μ is admissible. By the definition of ∇ we have that ∇f for

$f \in FC^2$ is of the form $\sum_{i=1}^n e_i \otimes (e_i \cdot \nabla)f$, hence ∇ maps FC^2 into $D(\nabla^*)$.

We have, for any $q \in Q$ and $g \in FC^1$:

$$\nabla^*(q \otimes g) = -q \cdot \nabla g - \beta(q)g.$$

By the definition $H = \nabla^* \bar{\nabla}$ we then have that $FC^2 \subset D(H)$ and for any $f \in FC^2$ we have $Hf = -\Delta f - \beta \cdot \nabla f$, which proves the theorem. ■

We shall say that $\mu \in \mathcal{P}(Q')$ has moments of p -th order if the linear functions (q, ξ) are in $L^p(Q', d\mu)$ for all $q \in Q$. The following theorem is a converse of the preceding one.

THEOREM 2.4. — Assume μ is admissible and has second order moments. If the linear functions (q, ξ) belong to $D(H)$ for all $q \in Q$ then ∇ maps FC^2 into the domain of ∇^* , so that $FC^2 \subset D(H)$. Moreover $q \otimes 1 \in D(\nabla^*)$ for any $q \in Q$ and, for any $f \in FC^2$,

$$Hf = -\Delta f - (\beta \cdot \nabla)f,$$

where $\beta(q) = -\nabla^*(q \otimes 1)$.

Proof. — Using the assumption $(q, \xi) \in L^2(Q', d\mu)$ we have that, for any real α , $\frac{1}{\alpha}(e^{i\alpha(q, \xi)} - 1)$ is in $D(\nabla)$ and converges as $\alpha \rightarrow 0$, strongly in $L^2(Q', d\mu)$, towards $i(q, \xi)$, while $\nabla \frac{1}{\alpha}(e^{i\alpha(q, \xi)} - 1) = iq \otimes e^{i\alpha(q, \xi)} \rightarrow iq \otimes 1$ as $\alpha \rightarrow 0$. Hence $(q, \xi) \in D(\bar{\nabla})$ and $\bar{\nabla}(q, \xi) = q \otimes 1$. By the definition $H = \nabla^* \bar{\nabla}$ and the assumption $(q, \xi) \in D(H)$ we have $q \otimes 1 \in D(\nabla^*)$. In particular, for any $f \in FC^1$, we have that $(\nabla f, q \otimes 1) = \int (q \cdot \nabla)f d\mu$ is continuous as a linear functional on $L^2(Q', d\mu)$. The rest of the theorem follows then from the preceding ones. ■

Let $\mathcal{P}_n(Q')$ be the space of probability measures μ quasi invariant under translations by elements in Q and such that $D(\pi(q)^n) \ni 1$ for any $q \in Q$.

We have the following

LEMMA 2.5. — If $\mu \in \mathcal{P}_2(Q')$ and $\pi(q) \cdot 1$ is in $D(\bar{\nabla})$ then

$$\beta(q) = 2i\pi(q) \cdot 1 \in L^4(Q', d\mu).$$

Proof. — By the assumption we have $\beta(q) \in D(\bar{\nabla})$, hence

$$(q \cdot \bar{\nabla})\beta(q) \in L^2(Q', d\mu).$$

Since $\mu \in \mathcal{P}_1(Q')$ we have $FC^1 \subset D(\pi(q))$ and, for $f \in FC^1$,

$$\pi(q)f = \frac{1}{i} \left(q \cdot \nabla + \frac{1}{2} \beta(q) \right) f,$$

recalling the definition of $\pi(q)$. From this it follows that if $f \in D(\pi(q)) \cap D(q.\bar{\nabla})$ then $\beta(q)f \in L^2(Q', d\mu)$. From the assumptions we have

$$2i\pi(q)1 = \beta(q) \in D(\pi(q))$$

and having already proven that $\beta(q) \in D(q.\bar{\nabla})$ we then have the conclusion of the Lemma. ■

By a *polynomial* on Q' we shall understand any element in the algebra generated by the constants and the linear functions (q, ξ) with $q \in Q$. By *linear functions* we mean elements of the form (q, ξ) , $q \in Q$. We remark that if $\mu \in \mathcal{P}(Q')$ has moments of all orders then the polynomials on Q' are in $L^2(Q', d\mu)$ and are dense in $L^2(Q', d\mu)$. The polynomials on Q' are in $L^p(Q', d\mu)$ for arbitrary p , $1 \leq p < \infty$, if μ has moments of all orders.

THEOREM 2.6. — If $\mu \in \mathcal{P}(Q')$ has moments of all orders and is admissible such that $(q, \xi) \in D(H)$ and $\beta(q) \in L^r(Q', d\mu)$ for some $r > 2$ and any $q \in Q$, then all polynomials on Q' are in the domain of H and one has

$$Hp = -\Delta p - \beta.\nabla p$$

for any polynomial p on Q' .

In particular if $\mu \in \mathcal{P}_2(Q')$ has moments of all orders and $\pi(q).1 \in D(\bar{\nabla})$ for all $q \in Q$, then all polynomials p on Q' are in the domain of H and one has

$$Hp = -\Delta p - \beta.\nabla p.$$

Proof. — By Theorem 2.4 we have $Hf = -\Delta f - \beta.\nabla f$ for any $f \in FC^2$. The assumption that μ has moments of all orders we may approximate any polynomial p by functions f_α in FC^2 in such a way that $\Delta f_\alpha \rightarrow \Delta p$ strongly in $L^2(Q', d\mu)$ and $q.\nabla f_\alpha \rightarrow q.\nabla p$ strongly in $L^r(Q', d\mu)$, with $\frac{1}{r'} + \frac{1}{r} = \frac{1}{2}$, by taking $f_\alpha(\xi) = \frac{-i}{\alpha}(e^{i\alpha p(\xi)} - 1)$. Let now P be an orthogonal projection with finite dimensional range in Q such that $p(\xi) = p(P\xi)$. Let $\{e_1, \dots, e_n\}$ be an orthogonal base in PK . Then for any $f \in FC^1$

such that $f(\xi) = f(P\xi)$ we have $\beta.\nabla f = \sum_{i=1}^n \beta(e_i)(e_i.\nabla)f$, so that the strong

convergence $q.\nabla f_\alpha \rightarrow q.\nabla p$ in $L^r(Q', d\mu)$ implies strong $L^2(Q', d\mu)$ -convergence of $\beta.\nabla f_\alpha$. This gives us that $Hf_\alpha = -\Delta f_\alpha - \beta.\nabla f_\alpha$ converges strongly to $-\Delta p - \beta.\nabla p$. Hence, H being closed, we have $p \in D(H)$ and $Hp = -\Delta p - \beta.\nabla p$, which proves the first part of the Theorem.

Part two of the Theorem follows from the first part and Lemma 2.5, provided we prove $(q, \xi) \in D(H)$ for all $q \in Q$. But we have, for $f \in FC^1$:

$$\pi(q)f = \frac{1}{i} \left(q.\nabla + \frac{1}{2}\beta(q) \right) f = \frac{1}{i} \left(- (q.\nabla)^* - \frac{1}{2}\beta(q) \right) f, \quad (2.8)$$

from which it follows that if $f \in D(\pi(q))$ and $f \in D(q \cdot \bar{\nabla})$ then $f \in D((q \cdot \nabla)^*)$. By assumption we have $1 \in D(\pi(q))$ and, since $1 \in D(\bar{\nabla})$, we then have $1 \in D((q \cdot \nabla)^*)$, i. e. $q \otimes 1 \in D(\bar{\nabla}^*)$. Since $(q, \xi) \in L^2(Q', d\mu)$ it follows as in the beginning of the proof of Theorem 2.4 that $(q, \xi) \in D(\bar{\nabla})$ and $\bar{\nabla}(q, \xi) = q \otimes 1$, hence, by $H = \nabla^* \bar{\nabla}$, we have $(q, \xi) \in D(H)$. This then completes the proof. ■

Remark. — We see that if $\mu \in \mathcal{P}(Q')$ is admissible then $q \cdot \nabla$ and $(q \cdot \nabla)^*$ have FC^1 in their domains, for any $q \in Q$. We may therefore consider the densely defined symmetric operator $\frac{1}{2i}(q \cdot \nabla - (q \cdot \nabla)^*)$. In the case where μ is quasi-invariant and $\mu \in \mathcal{P}_1(Q')$ we have $FC^1 \subset D(\pi(q))$, since $1 \in D(\pi(q))$ and on FC^1 we have $\pi(q) = \frac{1}{2i}(q \cdot \nabla - (q \cdot \nabla)^*)$. In the general case however $\frac{1}{2i}(q \cdot \nabla - (q \cdot \nabla)^*)$ may not have any self-adjoint extension at all.

Consider now a fixed element e in Q , with $(e, e) = 1$. Then one sees easily that $\frac{1}{2i}(e \cdot \nabla - (e \cdot \nabla)^*)$ and (e, ξ) satisfy the canonical commutation relations and (e, ξ) is essentially self-adjoint on FC^2 , if μ has second order moments. The question of when symmetric operators satisfying the canonical commutation relations are given by translations and multiplications is a classical problem and even in the case where $\frac{1}{2i}(e \cdot \nabla - (e \cdot \nabla)^*)$ is essentially self-adjoint μ need not necessarily be quasi invariant.

3. THE MARKOV PROPERTY, ANALYTIC POTENTIAL THEORY AND THE HUNT PROCESS

Let μ be a positive measure on a measure space X . A positive definite quadratic form $\varepsilon(f, f)$ defined on a dense subset of the real $L^2(d\mu)$ is said to be *Markov* if for any $\delta > 0$ there exists a non-decreasing function $\phi_\delta(t)$, $t \in \mathbb{R}$ with $\phi_\delta(t) = t$ for $0 \leq t \leq 1$, $|\phi_\delta(t)| \leq |t|$ and $-\delta \leq \phi_\delta(t) \leq 1 + \delta$ for all t such that for any function $f \in D(\varepsilon)$, the domain of ε , we have that $\phi_\delta(f) \in D(\varepsilon)$ and $\varepsilon(\phi_\delta(f), \phi_\delta(f)) \leq \varepsilon(f, f)$. We say that a bounded operator A on $L^2(d\mu)$ is Markov iff $0 \leq Af \leq 1$ for any $0 \leq f \leq 1$.

THEOREM 3.1 (Fukushima). — If ε is Markov and closable, then the self adjoint operator H_ε associated with the closed form is the infinitesimal

generator for a Markov semigroup e^{-tH_ε} , i. e. e^{-tH_ε} is Markov for each $t \geq 0$. Moreover if a strongly continuous semigroup is Markov then the closed form defined by its generator is Markov.

For the proof of this theorem see [8]. ■

If μ is a positive Radon measure on a locally compact separable space X , such that μ is everywhere dense i. e. that for any open $A \subset X$ we have that $\mu(A) > 0$, then we say that $\varepsilon(f, f)$ is a *regular Dirichlet form* if it is closed, Markov and $C_0(X) \cap D(\varepsilon)$ is dense in $C_0(X)$ as well as in $D(\varepsilon)$, respectively in the uniform norm and in the norm given by $\varepsilon(f, f) + (f, f)$, where $C_0(X)$ are the continuous functions on X which vanish at infinity.

For a closed Markov form ε on $L^2(X, d\mu)$ one defines the capacity of an open set of A as $\text{Cap}(A) = \inf \{ \varepsilon(u, u) + (u, u); u \in D(\varepsilon) \text{ such that } u \geq 1 \text{ on } A \}$. For an arbitrary subset $B \subset X$ the capacity is defined by

$$\text{Cap}(B) = \inf \{ \text{Cap}(A); B \subset A \text{ and } A \text{ is open} \}. \quad (3.1)$$

One says that $N \subset X$ is polar if $N \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ is the σ -algebra of Borel measurable sets in X , and $\text{Cap}(N) = 0$.

THEOREM 3.2 (Fukushima). — To any regular Dirichlet form ε on $L^2(X, d\mu)$ where μ is a Radon measure on a locally compact space X such that μ is everywhere dense, there exists a $Y \in \mathcal{B}(X)$ such that $\text{Cap}(X - Y) = 0$ and a Hunt process ξ_t with state space $(Y, \mathcal{B}(Y))$, properly associated with ε . In particular the Markov semigroup e^{-tH_ε} is given by the transition probabilities of ξ_t .

Remark. — For the definition of a Hunt process see [24], Chap. XIV and for the definition of properly associated with ε see [8] [9]. For the proof of the Theorem see [8], § 5. ■

A regular Dirichlet form $\varepsilon(f, g)$ on $L^2(X, d\mu)$ is said to be *local* if for any f and g in $D(\varepsilon) \cap C_0(X)$ we have that $\varepsilon(f, g) = 0$ whenever f and g has disjoint support. For the proof of the following theorem see [10], p. 124, [9].

THEOREM 3.3 (Fukushima and Silverstein). — Let the assumptions be as in theorem 3.2, then if ε is local the Hunt process ξ_t may be taken to have continuous paths. ■

Let us now again consider a rigged Hilbert space $Q \subset K \subset Q'$ as in section 2 with a measure $\mu \in \mathcal{P}(Q')$, such that μ is admissible. It is an easy exercise to prove that the corresponding energy form E is Markov. Hence we have by theorem 3.1 the following:

THEOREM 3.4. — If $\mu \in \mathcal{P}(Q')$ is admissible then the energy form E is

Markov so that e^{-tH} , where H is the energy operator is a strongly continuous Markov semigroup. ■

If the Hilbert space K is finite dimensional, then of course $Q = K = Q'$, and in this case one has the following theorem

THEOREM 3.5. — If K is finite dimensional and $\mu \in \mathcal{P}(K)$ such that μ is absolutely continuous with respect to the Lebesgue measure on K , everywhere dense and admissible, then the closure \bar{E} of E is a local regular Dirichlet form, in particular there is a continuous Hunt process properly associated with E .

For the proof of this result see [37]. ■

If however K is not finite dimensional we can not in general expect \bar{E} to be a regular Dirichlet form on $L^2(Q', d\mu)$, the reason for this being that in general one cannot hope to find a locally compact subset X on which the measure is concentrated and such that the elements in $C_0(X)$ are non trivial in $L^2(Q', d\mu)$.

We shall therefore continue as follows. Let us first assume that μ has compact separable support X in Q' . Since any open set A such that $\mu(A) = 0$ is in the complement of X , by the definition of the support of μ , we have that μ is everywhere dense in X . We may obviously now identify $L^2(Q', d\mu)$ with $L^2(X, d\mu)$ and we thus get the energy form E defined on $L^2(X, d\mu)$ where μ is a everywhere dense Radon measure on the compact separable Hausdorff space X . Moreover since the linear functions (q, ξ) separate points in Q' they separate points in X and therefore $FC^1(Q')$ is an algebra of continuous functions containing 1 which separates points in X . Thus by the Stone-Weierstrass theorem FC^1 is dense in $C(X)$, the space of all continuous functions on X . We therefore have the following lemma

LEMME 3.6. — Let $Q \subset K \subset Q'$ be a rigging and $\mu \in \mathcal{P}(Q')$ with compact separable support X such that μ is admissible, then the closure of the energy form $\bar{E}(f, f)$ defines a regular Dirichlet form on $L^2(X, d\mu)$. ■

If $\varepsilon(f, f)$ is a positive quadratic form on $L^2(X, d\mu)$ then we say, following Fukushima, that every unit contraction operates on ε if, for any $f \in D(\varepsilon)$, $g = (0 \vee f) \wedge 1 \in D(\varepsilon)$ and $\varepsilon(g, g) \leq \varepsilon(f, f)$. We say that every normal contraction operates on ε if $f \in D(\varepsilon)$, $g \in L_2$, $|g(x)| \leq |f(x)|$ and $|g(x) - g(y)| \leq |f(x) - f(y)|$ for almost all x and y in X implies that $g \in D(\varepsilon)$ and $\varepsilon(g, g) \leq \varepsilon(f, f)$. For the proof of the following theorem see [8], § 3.

THEOREM 3.7 (Fukushima). — Let ε be a closed symmetric form on an $L^2(X, d\mu)$. Then the following three conditions are equivalent

- a) ε is Markov,
- b) every unit contraction operates on ε ,
- c) every normal contraction operates on ε . ■

Let now $Q \subset K \subset Q'$ be a rigging and let A be a strictly positive compact operator on K with normalized eigenvectors a_i and eigenvalues $\lambda_i > 0$ such that $(a_i, \xi) \in L^2(d\mu)$ for each $i = 1, 2, \dots$. We say that A defines an $L^p(d\mu)$ -form $(\xi, A\xi)$ if $(\xi, A\xi) = \sum_i \lambda_i (a_i, \xi)^2$ converges in $L^p(d\mu)$. Since A is

compact it follows by dominated convergence that if A defines an $L^p(d\mu)$ -form so does A^r for any $r \geq 1$. Let $h \in C_b^1(\mathbb{R})$ then

$$|h((\xi, A\xi) - h((\xi, A\xi)_n)| \leq |(\xi, A\xi) - (\xi, A\xi)_n| \|h'\|_\infty \tag{3.1}$$

where $(\xi, A\xi)_n = \sum_{i=1}^n \lambda_i (a_i, \xi)^2$. So that if A defines an L^p -form then

$h((\xi, A\xi)_n) \rightarrow h((\xi, A\xi))$ strongly in L^p . Moreover $h((\xi, A\xi)_n) \in FC^1(Q')$ and

$$\nabla h((\xi, A\xi)_n) = 2 \sum_{i=1}^n \lambda_i a_i \otimes (a_i, \xi) h'((\xi, A\xi)_n). \tag{3.2}$$

Now $\left\| \sum_{i=n+1}^\infty \lambda_i a_i \otimes (a_i, \xi) \right\|^2 = \sum_{i=n+1}^\infty \lambda_i^2 (a_i, \xi)^2$, which converges to zero

in $L^p(d\mu)$. So that $\sum_{i=1}^n \lambda_i^2 a_i (a_i, \xi)$, considered as a function from Q' into K ,

converges to $\sum_{i=1}^\infty \lambda_i^2 a_i (a_i, \xi)$ in $L^p(Q', d\mu; K)$. Let now $h \in C_b^2(\mathbb{R})$, then we

know that $h'((\xi, A\xi)_n)$ converges to $h'((\xi, A\xi))$ in L^p . Hence we get from (3.2), if $p \geq 4$, that $\nabla h((\xi, A\xi)_n)$ converges in $L^2(Q', d\mu; K) \cong K \otimes L^2(d\mu)$. Hence $h((\xi, A\xi)) \in D(\bar{\nabla})$ and

$$\nabla h((\xi, A\xi)) = 2 \sum_{i=1}^\infty \lambda_i a_i \otimes (a_i, \xi) h'((\xi, A\xi)). \tag{3.3}$$

Let now $f \in FC^1(Q')$ and $h \in C_b^2(\mathbb{R})$, then $f(\xi)h((\xi, A\xi)_n) \rightarrow f(\xi)h((\xi, A\xi))$ in $L^2(d\mu)$, moreover $\nabla(fh((\xi, A\xi)_n)) = (\nabla f)h((\xi, A\xi)_n) + f\nabla h((\xi, A\xi)_n)$, which by the results above converge in $L^2(d\mu)$, so that $fh((\xi, A\xi)) \in D(\bar{\nabla})$ and

$$\bar{\nabla}(fh((\xi, A\xi))) = (\nabla f)h + 2A\xi h'((\xi, A\xi))f. \tag{3.4}$$

Since $|A\xi|_{\mathbb{K}}^2 = (\xi, A^2\xi) \leq \|A\|(\xi, A\xi)$ we see that

$$A\xi h'((\xi, A\xi)) \in L^\infty(Q', d\mu; \mathbb{K})$$

if $h \in C_0^1(\mathbb{R})$, i. e. if h is in the space of differentiable functions of compact support in \mathbb{R} , so that $\|2A\xi h'((\xi, A\xi))f\|_2 \leq 2\|A\|^{1/2}\|xh'(x)\|_\infty\|f\|_2$. We therefore have by (3.4) that the mapping $f \rightarrow \bar{\nabla}(h((\xi, A\xi))f)$ is continuous in the $D(\bar{V})$ norm hence, since \bar{V} is closed, it follows that $h((\xi, A\xi))f \in D(\bar{V})$ for any $f \in D(\bar{V})$ and that (3.4) extends by continuity to a continuous mapping of $D(\bar{V})$ into itself.

Let now $\eta \in Q'$ be such that $(\eta, A\eta) < \infty$, then from $|(\eta, A\xi)|^2 \leq (\eta, A\eta)(\xi, A\xi)$ we have $(\eta, A\xi) \in L^2(d\mu)$, if A defines an $L^p(d\mu)$ -form. We may thus, for $h \in C_0^2(\mathbb{R})$, consider $h(|\xi - \eta|_{\mathbb{A}}^2)f$ for $f \in FC^1$, where $|\xi - \eta|_{\mathbb{A}}^2 = ((\xi - \eta), A(\xi - \eta))$. We then find as above that $h(|\xi - \eta|_{\mathbb{A}}^2)$ maps FC^1 into $D(\bar{V})$, and

$$\bar{\nabla}(h(|\xi - \eta|_{\mathbb{A}}^2)f) = h(|\xi - \eta|_{\mathbb{A}}^2)\bar{\nabla}f + 2A(\xi - \eta)h'(|\xi - \eta|_{\mathbb{A}}^2)f(\xi). \tag{3.5}$$

Using now $(\eta, A^2\eta) \leq \|A\|(\eta, A\eta)$ we get in the same way as above that the mapping $f(\xi) \rightarrow h(|\xi - \eta|_{\mathbb{A}}^2)f(\xi)$ extends to a mapping from $D(\bar{V})$ into $D(\bar{V})$, continuous in the strong topology. Hence we have the following theorem.

THEOREM 3.8. — Let $Q \subset K \subset Q'$ be a rigging and let $\mu \in \mathcal{P}(Q')$ be an admissible measure on Q' . Let A be a strictly positive compact operator on K such that A defines an L^2 -form $(\xi, A\xi)$. Then for any $h \in C_0^2(\mathbb{R})$ and any $\eta \in Q'$ with $|\eta|_{\mathbb{A}} < \infty$ we have that the mapping $f(\xi) \rightarrow h(|\xi - \eta|_{\mathbb{A}}^2)f(\xi)$ defines a continuous mapping from $D(\bar{V})$ into $D(\bar{V})$ with its strong topology and one has

$$\begin{aligned} \bar{\nabla}(h(|\xi - \eta|_{\mathbb{A}}^2)f(\xi)) \\ = h(|\xi - \eta|_{\mathbb{A}}^2)\bar{\nabla}f(\xi) + 2A(\xi - \eta)h'(|\xi - \eta|_{\mathbb{A}}^2)f(\xi). \quad \blacksquare \end{aligned}$$

Let A be as in theorem 3.8 and let us also assume that the injection $K \hookrightarrow Q'$ is continuous in $|\eta|_{\mathbb{A}}$; in which case we shall say that it is *A-continuous*. Then we have that the subset of Q' where $|\xi|_{\mathbb{A}} < \infty$ is a Hilbert space $K_{\mathbb{A}}$ with the $|\cdot|_{\mathbb{A}}$ -norm and the injections $K \hookrightarrow K_{\mathbb{A}} \hookrightarrow Q'$ are both continuous. Since $|\xi|_{\mathbb{A}}^2$ is in $L^2(Q', d\mu)$ we have that all open sets in $K_{\mathbb{A}}$ are μ -measurable and $\mu(K_{\mathbb{A}}) = 1$. Hence we may restrict the measure to $K_{\mathbb{A}}$ and since $K_{\mathbb{A}} \hookrightarrow Q'$ is continuous we have that the open sets in $K_{\mathbb{A}}$ with the $|\cdot|_{\mathbb{A}}$ topology generate the σ -algebra of μ -measurable sets, and we may thus identify $L^p(Q', d\mu)$ with $L^p(K_{\mathbb{A}}, d\mu)$. For $f \in L^p(d\mu)$ we define $\text{supp}_{\mathbb{A}} f \subset K_{\mathbb{A}}$ as the smallest $|\cdot|_{\mathbb{A}}$ -closed set in $K_{\mathbb{A}}$ such that $f(\eta) = 0$ for a. a. $\eta \notin \text{supp}_{\mathbb{A}} f$.

We say that a closed mapping T from $L^2(d\mu)$ into $L^2(d\mu)$ with dense domain $D(T)$ is A -local if for any $f \in D(T)$ we have

$$\text{supp}_A T f \subset \text{supp}_A f. \quad (3.6)$$

THEOREM 3.9. — Let the assumptions be as in theorem 3.8. If in addition the injection $K \hookrightarrow Q'$ is A -continuous then \bar{V} is A -local.

Proof. — Let $C = \text{supp}_A f$ and $f \in D(\bar{V})$. Let $\eta \notin C$ and let $d = \text{dist}(\eta, C)$. Then $d > 0$ since C is closed in K_A . Let $h \in C_0^2(\mathbb{R})$ such that $\text{supp } h \subset (-\infty, d^2)$. Then $h(|\xi - \eta|_A^2)f(\xi) = h'(|\xi - \eta|_A^2)f(\xi) = 0$, μ -a. e. Hence by theorem 3.8 we have that $h(|\xi - \eta|_A^2)(\bar{V}f)(\xi) = 0$ μ -a. e. Since h was arbitrary in $C_0^2(-\infty, d^2)$ we find that the ball $b(\eta, d) = \{\xi \in K_A; |\xi - \eta|_A < d\}$ is in the complement of C . Hence we have proved that for any $\eta \notin C$ there is an open ball $b(\eta, d)$ such that $b(\eta, d)$ is in the complement of C . This proves the theorem. ■

Let us now assume we have two compact operators A_1 and A_2 such that $A_1^{-1/2}A_2A_1^{-1/2}$ is bounded on K . Then there is a constant C such that $(\xi, A_2\xi) \leq C^2(\xi, A_1\xi)$ hence $|\xi|_{A_2} \leq C|\xi|_{A_1}$, and if the injection $K \hookrightarrow Q'$ is A_i -continuous for $i = 1$ and $i = 2$ then we have the situation that $K \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow Q'$ where all the injections are continuous and $\mu(K_{A_1}) = \mu(K_{A_2})$. Moreover for any $f \in L^2(d\mu)$ we have that $\text{supp}_{A_2} f$ is the $|\cdot|_{A_2}$ -closure of $\text{supp}_{A_1} f$ so that we always have

$$\text{supp}_{A_1} f \subset \text{supp}_{A_2} f. \quad (3.7)$$

Let us now assume that $A_1^{-1/2}A_2A_1^{-1/2}$ is compact on K , in which case the injection $K_{A_1} \hookrightarrow K_{A_2}$ is compact. Hence the ball $b_\lambda = \{\eta \in K_{A_1}; |\eta|_{A_1} \leq \lambda\}$ is a compact subset of K_{A_2} . Moreover, since K_{A_2} is separable, b_λ is separable and compact. Let us now assume that there is a λ such that $\mu(b_\lambda) = 1$. Consider b_λ in the $|\cdot|_{A_2}$ topology, and let X be the smallest $|\cdot|_{A_2}$ closed subset of b_λ such that $\mu(X) = 1$. Then X is a separable and compact metric space with a Radon measure μ which is everywhere dense. Since the linear functions (q, ξ) for $q \in Q$ separate points in Q' they obviously separate points of $X \subset Q'$. Hence $FC^1(Q')$ restricted to X is an algebra of continuous functions on the compact X which separates points. Hence, by Stone-Weierstrass' theorem, FC^1 is uniformly dense in $C(X)$, the space of all continuous functions on X .

Let now in addition $\mu \in \mathcal{P}(Q')$ be admissible. Then \bar{E} is a closed Markov form on $L^2(X, d\mu) = L^2(Q', d\mu)$ such that FC^1 is dense in $D(\bar{E})$ as well as in $C(X)$. Hence we have that \bar{E} considered as a form on $L^2(X, d\mu)$ is a regular Dirichlet form.

Let now f and g be in $D(\bar{E}) \cap C(X)$ such that f and g have disjoint supports, which means that $\text{supp}_{A_2} f$ and $\text{supp}_{A_2} g$ are disjoint. By theorem 3.9 however we have that $\text{supp}_{A_2} \bar{\nabla} f \subset \text{supp}_{A_2} f$ and $\text{supp}_{A_2} \bar{\nabla} g \subset \text{supp}_{A_2} g$. Since by definition we have $\bar{E}(f, g) = (\bar{\nabla} f, \bar{\nabla} g)$ and since $\bar{\nabla} f$ and $\bar{\nabla} g$ have disjoint supports we have $(\bar{\nabla} f, \bar{\nabla} g) = 0$. Thus we see that the closure \bar{E} of the energy form is a local regular Dirichlet form. Hence by theorem 3.2 there is a regular Hunt process ξ_t with continuous paths in X properly associated with \bar{E} . Here continuity means continuity in the $|\cdot|_{A_2}$ metric. The process has state space $(Y, \mathcal{B}(Y))$, where Y is a Borel subset of X such that $\text{Cap}(X - Y) = 0$ and $\mathcal{B}(Y)$ are the Borel sets in Y generated by the norm $|\cdot|_{A_2}$.

THEOREM 3.10. — Let $Q \subset K \subset Q'$ be a rigging and $\mu \in \mathcal{P}(Q')$ be admissible. Let A_1 and A_2 be compact strictly positive definite operators on K such that the injection $K \hookrightarrow Q'$ is continuous in the norms $|\eta|_{A_i} = (\eta, A_i \eta)^{1/2}$, $i = 1$ and $i = 2$, and such that A_1 defines an $L^2(d\mu)$ form $(\xi, A_1 \xi)$ and $A_1^{-1/2} A_2 A_1^{-1/2}$ is compact. If μ has support X which is bounded in the A_1 -norm $|\cdot|_{A_1}$ then X in the topology given by the A_2 norm is a compact separable space with μ as an everywhere dense Radon measure, and the closure of the energy form \bar{E} on $L^2(X, d\mu)$ is a local regular Dirichlet form. In particular there is a Hunt process ξ_t properly associated with \bar{E} with state space $(Y, \mathcal{B}(Y))$, where Y is a A_2 -Borel subset of X such that $\text{Cap}(X - Y) = 0$ and $\mathcal{B}(Y)$ is the σ -algebra of the A_2 -Borel subsets of Y , i. e. the Borel sets generated by the norm $|\cdot|_{A_2}$ on Y . Moreover ξ_t is continuous in the metric given by the A_2 -norm. ■

We see that in the proof of theorem 3.10 we did not make any use of the fact that $|\cdot|_{A_2}$ was given by a quadratic form $(\xi, A_2 \xi)$. In fact the only thing we made use of was that the mapping $K_{A_1} \hookrightarrow K_{A_2}$ was compact, and that the balls in K_{A_1} were separable in the A_2 -norm. Hence we have the following corollary

COROLLARY. — Let $Q \subset K \subset Q'$ be a rigging and $\mu \in \mathcal{P}(Q')$ be admissible. Let A_1 be as in theorem 3.10 and let K_{A_1} be the closure of K in the A_1 -norm and assume that μ is supported by a set $X \subset Q'$, i. e. $\mu(Q' - X) = 0$ and X is a bounded subset of K_{A_1} . Let B be a separable Banach space with norm $\|\cdot\|_0$ such that $K_{A_1} \subset B \subset Q'$ with the injection $K_{A_1} \hookrightarrow B$ compact and the injection $B \hookrightarrow Q'$ continuous. Then the conclusion of theorem 3.10 holds with $\|\cdot\|_0$ replacing the A_2 -norm. ■

Let us now consider the case where μ does not have support in any of the balls b_λ but $\mu(K_{A_1}) = 1$ so that $\bigcup_{\lambda > 0} b_\lambda = K_{A_1}$ is a locally compact

subset of K_{A_2} which supports the measure μ . After throwing again away, if necessary, a maximal open subset of $\bigcup_{\lambda} b_{\lambda}$ with μ -measure zero, we are left with a closed locally compact subset X of $\bigcup_{\lambda} b_{\lambda}$ which supports the measure and on which μ is everywhere dense. Remember that the topology on $\bigcup_{\lambda} b_{\lambda}$ is the one given by $|\cdot|_{A_2}$ and the same goes for X .

Choose now a fixed λ_0 , set $X_n = X \cap b_{n\lambda_0}$, and let μ_n be the restriction of μ to X_n , normalized so that $\mu_n(X_n) = 1$. We shall see that if μ is admissible so is μ_n . In fact let $h \in C_0^2(-\infty, (n\lambda_0)^2)$ and $f \in FC^1(Q')$, then $h((\xi, A_1\xi))f(\xi) \in D(\bar{V})$ in $L^2(d\mu)$, by theorem 3.8. Let us now assume that μ is admissible with second order moments, then by theorem 2.4 and theorem 2.3 we have that, for any $g \in FC^1$ and any $q \in Q$, $q \otimes g \in D(\nabla^*)$ in $L^2(Q', d\mu)$, moreover

$$\nabla^*(q \otimes g) = -(q \cdot \nabla)g(\xi) - \beta(q)(\xi)g(\xi). \quad (3.8)$$

Since $\beta(q) \in L^2(d\mu)$ we have that $g \rightarrow \beta(q)g$ is strongly continuous from $L^\infty(d\mu)$ into $L^2(d\mu)$. Since obviously $\|(q \cdot \nabla)g\| \leq |q|_K \|\nabla g\|$ we have that ∇^* is continuous in the natural strong topology on $L^\infty(d\mu) \cap D(\bar{V})$. Since ∇^* is closed we therefore get that $L^\infty(d\mu) \cap D(\bar{V}) \subset D(\nabla^*)$ and formula (3.8) extends by continuity to $L^\infty(d\mu) \cap D(\bar{V})$. Since $h((\xi, A_1\xi))f(\xi)$ is in $D(\bar{V})$ and obviously also in $L^\infty(d\mu)$ we have therefore that it is in $D(\nabla^*)$ as well. Now since $h \in C_0^2(-\infty, (n\lambda_0)^2)$ we have, for $g \in FC^1$ and $q \in Q$, that

$$\int_{X_n} h((\xi, A_1\xi))f(\xi)(q \cdot \nabla)g(\xi) d\mu_n = \mu(X_n)^{-1} \int_X h((\xi, A_1\xi))f(\xi)(q \cdot \nabla)g(\xi) d\mu. \quad (3.9)$$

By the formula (3.8) together with theorem 3.8 we may now compute the right hand side of (3.9), and we get

$$\nabla^*(q \otimes (h((\xi, A_1\xi))f(\xi))) = -h((\xi, A_1\xi))q \cdot \bar{\nabla}f(\xi) - 2(q, A_1\xi)h'((\xi, A_1\xi))f(\xi) - \beta(q)(\xi)h((\xi, A_1\xi))f(\xi). \quad (3.10)$$

We see that the right hand side has support in X_n , in fact it lies in $L^2(X_n, d\mu_n)$. From this it follows that the right hand side of (3.9) is continuous in g in the strong $L^2(X_n, d\mu_n)$ -topology.

We shall see below, in the proof of Theorem 3.11, that the functions of the form $h((\xi, A_1\xi))f(\xi)$ for $h \in C_0^2(-\infty, (\lambda_0 n)^2)$ and $f \in FC^1$ are dense in $L^2(X_n, d\mu_n)$, thus we get that the linear span of the elements of the form

$q \otimes h(|\xi - \eta|_{\mathbb{A}}^2) f(\xi)$ is dense in $\mathbb{K} \otimes L^2(X_n, d\mu_n)$. Hence we have proven that ∇_n^* , the adjoint of ∇_n in $L^2(X_n, d\mu_n)$, is densely defined and therefore ∇_n is closable, thus μ_n admissible. This then gives us that, X_n being compact, there exists a continuous Hunt process ξ_t properly associated with \bar{E}_n , where E_n is the energy form given by μ_n in $L^2(X_n, d\mu_n)$. We pause to give the following theorem.

THEOREM 3.11. — Let $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{Q}'$ be a rigging and $\mu \in \mathcal{P}(\mathbb{Q}')$ be admissible and have second order moments. Let A be any compact strictly positive definite operator on \mathbb{K} such that A defines an $L^2(d\mu)$ -form $(\xi, A\xi)$, such that the injection $\mathbb{K} \hookrightarrow \mathbb{Q}'$ is continuous in the norm $|\eta|_{\mathbb{A}} = (\eta, A\eta)^{1/2}$. If μ_{Ω} is the restriction of μ to an A -open subset $\Omega \subset \mathbb{Q}'$ of positive measure, i. e. Ω is open in the $|\cdot|_{\mathbb{A}}$ norm and $\mu(\Omega) > 0$, then μ_{Ω} is an admissible measure in $\mathcal{P}(\mathbb{Q}')$.

By theorem 3.10 and its corollary we have then that the energy form E_{Ω} in $L^2(d\mu_{\Omega})$ has a closure which is a regular local Dirichlet form in $L^2(X_{\Omega}, d\mu_{\Omega})$, where X_{Ω} is the support in $\mathbb{K}_{\mathbb{A}}$ of $d\mu_{\Omega}$ with the topology induced by any Banach norm $\|\cdot\|_0$ such that the A -bounded sets are separable and compact in $\|\cdot\|_0$ and $\mathbb{K}_{\mathbb{A}} \hookrightarrow \mathbb{Q}'$ is continuous in $\|\cdot\|_0$. Moreover the corresponding Hunt process ξ_t^{Ω} is continuous in such a Banach norm $\|\cdot\|_0$.

Proof. — In the argument preceding theorem 3.11 we proved that for any ball $b_{\lambda} = \{\eta; |\eta|_{\mathbb{A}} \leq \lambda\}$ the restriction of μ to b_{λ} is admissible. The proof for any A -open Ω proceeds as follows. Take $\eta \in \Omega$ and let $d = \inf \{|\eta - \xi|_{\mathbb{A}}; \xi \in \mathbb{Q}' - \Omega\}$, then for $h \in C_0^2(-\infty, d^2)$ we have for f and $g \in FC^1$ and $q \in \mathbb{Q}$ that

$$\int_{\Omega} h(|\xi - \eta|_{\mathbb{A}}^2) f(\xi) (q\nabla)g(\xi) d\mu_{\Omega} = \mu(\Omega)^{-1} \int_{\mathbb{Q}'} h(|\xi - \eta|_{\mathbb{A}}^2) f(\xi) (q\nabla)g(\xi) d\mu. \quad (3.11)$$

Again by theorem 3.8 and the formula (3.9) we compute the right hand side of (3.11) and find that in fact, as a function of g , it is strongly continuous in $L^2(\mathbb{Q}', d\mu_{\Omega})$. From this we have that

$$h(|\xi - \eta|_{\mathbb{A}}^2) f(\xi) \in D(\nabla_{\Omega}^*).$$

To see that $D(\nabla_{\Omega}^*)$ is dense in $\mathbb{K} \otimes L^2(d\mu_{\Omega})$ we proceed as follows. Consider the function of the form $h(|\xi - \eta|_{\mathbb{A}}^2)$, $h \in C_0^2(-\infty, d^2)$. The σ -algebra generated by these functions is obviously the σ -algebra generated by the A -open sets in Ω . The assumption that $\mathbb{K} \hookrightarrow \mathbb{Q}'$ is continuous when \mathbb{K} has the A -norm implies that $\Omega \hookrightarrow \mathbb{Q}'$ is continuous when Ω has the topology given by the A -open sets. Hence the A -open sets in Ω generate the whole σ -algebra of μ_{Ω} -measurable sets. From this it follows that the characteristic

functions for open A -balls in Ω span a dense subspace of $L^2(d\mu_\Omega)$. Any such characteristic function may of course be approximated in $L^2(d\mu_\Omega)$ by functions of the form $h(|\xi - \eta|_\Lambda^2)$, so the functions $h(|\xi - \eta|_\Lambda^2)$ span a dense linear subspace of $L^2(d\mu_\Omega)$, and therefore the linear span of elements of the form $q \otimes h(|\xi - \eta|_\Lambda^2)$ is dense in $K \otimes L^2(d\mu_\Omega)$, and is contained in $D(\nabla_\Omega^*)$. Hence ∇_Ω has a densely defined adjoint ∇_Ω^* and is therefore closable and thus μ is admissible. This proves the theorem. ■

Let $Q \subset K \subset Q'$ be a Hilbert rigging and $\mu \in \mathcal{P}(Q')$ be admissible and let A be as in theorem 3. 11. Let Ω be an A -open set. The process ξ_t^Ω constructed in theorem 3. 11 may be considered as the reflecting barrier process corresponding to the process ξ_t of the unrestricted measure μ . Of course we have yet to construct the process ξ_t corresponding to a measure μ for the case where μ does not have bounded support with respect to some square integrable form. and we shall return to this point later.

We now prove the following theorem.

THEOREM 3.12. — Let $Q \subset K \subset Q'$ be a rigged Hilbert space and $\mu \in \mathcal{P}(Q')$ be admissible. Let A be a strictly positive definite operator on K such that A defines an $L^2(d\mu)$ -form, and such that the injection $K \hookrightarrow Q'$ is continuous and compact with the A -norm $|\eta|_A = (\eta, A\eta)^{1/2}$ on K . Let $K_A \subset Q'$ be the completion of K in the A -norm. Then $\mu(Q' - K_A) = 0$ and $\text{Cap}(Q' - K_A) = 0$.

Proof. — Since by assumption $(\xi, A\xi) \in L^2(d\mu)$, we have that the subset of Q' on which $(\xi, A\xi) = \infty$ has measure zero. This set is in $Q' - K_A$. Let now $\phi \in C_0^1(0, 1)$ be such that $\phi(x) \geq 0$ with $\int_0^1 \phi(x) dx = 1$. Set $h_n(x) = \int_0^x \phi(t-n) dt$. It follows by a slight modification of the argument preceding theorem 3. 8 that $h_n((\xi, A\xi)) \in L^2(d\mu)$ and is in $D(\bar{\nabla})$, moreover

$$\bar{\nabla} h_n((\xi, A\xi)) = 2A\xi h_n'((\xi, A\xi)). \quad (3.12)$$

Obviously $h_n((\xi, A\xi))$ differs from 1 only on the set $(\xi, A\xi) \leq n + 1$, hence it is ≤ 1 on the A -ball $b_{n+1} = \{ \xi \in K_A; (\xi, A\xi) \leq n + 1 \}$. By the assumption of the compactness of the injection $K \hookrightarrow Q'$ relative to the A -norm on K it follows that the injection $K_A \hookrightarrow Q'$ is compact, hence the sets b_{n+1} are compact in Q' and therefore closed, so their complements are open and obviously $h_n((\xi, A\xi))$ is identical 1 on the complement of b_{n+1} . We have therefore by the definition of the capacity of an open set that

$$\text{Cap}(Q' - b_{n+1}) \leq \| \bar{\nabla} h_n((\xi, A\xi)) \|^2 + \| h_n((\xi, A\xi)) \|^2. \quad (3.13)$$

By (3.12) we have that this is equal to

$$4 \int_{(\xi, A\xi) \geq n} (\xi, A^2 \xi) h_n'^2((\xi, A\xi)) d\mu + \int_{(\xi, A\xi) \geq n} h_n^2((\xi, A\xi)) d\mu \\ \leq (4 \|A\| \|\xi, A\xi\|_2 \|\phi\|_\infty^2 + 1) \mu\{(\xi, A\xi) \geq n\}. \quad (3.14)$$

By the definition of the capacity of an arbitrary set and the fact that $Q' - K_A$ is contained in the open set $Q' - b_{n+1}$ we have that

$$\text{Cap}(Q' - K_A) \leq \text{Cap}(Q' - b_{n+1}),$$

so that

$$\text{Cap}(Q' - K_A) \leq (4 \|A\| \|\xi, A\xi\|_2 \|\phi\|_\infty^2 + 1) \mu\{(\xi, A\xi) \geq n\}.$$

Now since $(\xi, A\xi) \in L^2(d\mu)$ the right hand side above tends to zero as $n \rightarrow \infty$. This proves the theorem. ■

Let now $Q \subset K \subset Q'$ be a rigged Hilbert space such that Q is separable then it follows immediately that \overline{FC} , the uniform closure of the continuous functions with finite dimensional base ⁽⁴⁾ in Q , is a countably generated algebra of continuous functions on Q' which separates points in Q' and contains the constants. By the Gelfand representation theorem we then know that $\overline{FC} \cong C(X)$ where X is a compact separable Hausdorff space and the points of X are the characters of \overline{FC} . $C(X)$ is the space of all continuous bounded functions on X . Since \overline{FC} separates points in Q' we may identify $\xi \in Q'$ with the character with kernel given by the maximal ideal of functions vanishing at ξ . This is then a continuous injection $Q' \hookrightarrow X$

so we may regard X as a compactification of Q' . We have that $\int f d\mu$ defines a continuous-linear positive functional on \overline{FC} which is 1 on 1, and by the Riesz representation theorem and the identification $\overline{FC} \equiv C(X)$ we get a measure on X which we also denote by μ , and it is easy to see that μ is Radon measure on X . Let $X_0 \subset X$ be the support of μ in X i. e. the minimal closed set such that $\mu(X - X_0) = 0$. If we consider μ as a measure on X_0 it is obviously everywhere dense.

Since FC is dense in $L^p(Q', d\mu)$ we may again identify $L^p(Q', d\mu)$ with $L^p(X_0, d\mu)$.

Let us now also assume that μ is admissible then since FC^1 is dense in $D(\bar{E}) = D(\bar{V})$ and in $L^2(d\mu)$ we get that \bar{E} defines a closed Markov form on $L^2(X_0, d\mu)$ such that $D(\bar{E}) \cap C(X_0)$ is dense in $C(X_0)$ as well as in $D(\bar{E})$ in their respective topologies. Hence \bar{E} defines a regular Dirichlet

⁽⁴⁾ (of the form $f(\langle e_1, \xi \rangle, \dots, \langle e_n, \xi \rangle)$, $\{e_i\}$, $e_i \in Q$ a fixed c. o. s. in K).

form on $L^2(X_0, d\mu)$. Since X_0 is actually a compact separable Hausdorff space we get the following theorem.

THEOREM 3.13. — Let $Q \subset K \subset Q'$ be a rigged Hilbert space, such that Q is separable. Let $\mu \in \mathcal{P}(Q')$ be admissible. Let X be the character space of \overline{FC} , the uniform closure of the continuous functions on Q' with finite base in Q , then X is a separable compact Hausdorff space and μ extends to a unique Radon measure on X . Let X_0 be the support of μ in X , there is then a natural identification of $L^2(Q', d\mu)$ with $L^2(X_0, d\mu)$ and this identification takes the energy form \bar{E} into a regular Dirichlet form on $L^2(X_0, d\mu)$. In particular we have that there is a Hunt process ξ_t properly associated with \bar{E} in $L^2(X_0, d\mu)$. ■

Let now A be as in Theorem 3.12 i. e. A is a compact operator on K with eigenvalues λ_i and eigenvectors $a_i \in Q$, such that $\sum_{i=0}^{\infty} \lambda_i(a_i, \xi)^2 \in L^2(d\mu)$.

Now obviously $\exp\left(-\sum_{i=1}^n \lambda_i(a_i, \xi)^2\right) \in FC$, hence it extends to a function $\exp\left(-\sum_{i=1}^n \lambda_i(a_i, x)^2\right) \in C(X)$, so that $\exp\left(-\sum_{i=1}^n \lambda_i(a_i, x)^2\right) \geq e^{-\alpha}$ are closed sets in X .

The intersection of these sets is therefore also closed in X and we denote it by $(x, Ax) \leq \alpha$. It is easily seen that this set is actually the closure in X of the subset of ξ in Q' such that $(\xi, A\xi) \leq \alpha$. The union over all α of the sets $(x, Ax) \leq \alpha$ is therefore a Borel subset of X , which we denote by $\tilde{K}_A \subset X$. We have the following Theorem

THEOREM 3.14. — $X - \tilde{K}_A$ is a polar set.

Proof. — We have already seen that $X - \tilde{K}_A$ is a Borel set, \tilde{K}_A being a Borel set. The computation of $\text{Cap}(X - \tilde{K}_A)$ is done as in Theorem 3.12 by first estimating the capacity of the open set $X - \tilde{C}_\alpha$, where $\tilde{C}_\alpha = \{x \in X \mid (x, Ax) \leq \alpha\}$. We find then that

$$\text{Cap}(X - \tilde{C}_\alpha) \leq \left\{ 4 \|A\| \|\xi, A\xi\|_2 \|\phi\|_\infty^2 + 1 \right\} \mu \{ \xi \mid (\xi, A\xi) > \alpha \}$$

and since $\text{Cap}(X - \tilde{K}_\alpha) \leq \text{Cap}(X - \tilde{C}_\alpha)$ the theorem is proven.

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