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Ergodic theory for inner functions of the upper half plane


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ABSTRACT. — The real restriction of an inner function of the upper half plane leaves Lebesgue measure quasi-invariant. It may have a finite or infinite invariant measure. We give conditions for the rational ergodicity and exactness of such restrictions.

§ 0. INTRODUCTION

In this paper, we consider the ergodic properties of the real restrictions of inner functions on the open upper half plane:

\[ \mathbb{R}^2^+ = \{ x + iy : x, y \in \mathbb{R}, y > 0 \}. \]

Let \( f : \mathbb{R}^2^+ \to \mathbb{R}^2^+ \) be an analytic function. We say that \( f \) is an inner function on \( \mathbb{R}^2^+ \) if for \( \lambda \)-a. e. \( x \in \mathbb{R} \) the limit \( \lim_{y \to 0} f(x + iy) \) exists, and is real. (Here, and throughout the paper, \( \lambda \) denotes Lebesgue measure on \( \mathbb{R} \).) Consider the limit \( \lim_{y \to 0} f(x + iy) = T(x) \). This is defined \( \lambda \)-a. e. on \( \mathbb{R} \). We call this limit the (real) restriction of \( f \), and will sometimes write this as \( T = T(f) \).
We will denote the class of inner functions on $\mathbb{R}^2^+$ by $I(\mathbb{R}^2^+) = I$, and their real restrictions by $M(\mathbb{R})$. We note that $f \in I(\mathbb{R}^2^+)$ iff $\emptyset^{-1} f \emptyset(z)$ is an inner function of the unit disc, according to the definition on p. 370 of [9] \\
\left(\text{where } \emptyset(z) = i \left( \frac{1 + z}{1 - z} \right) \right).

The following characterisation of $I(\mathbb{R}^2^+)$ appears in [6] and [17].

$f \in I(\mathbb{R}^2^+)$ iff 

\begin{equation}
(0.1) \quad f(\omega) = \alpha \omega + \beta + \int_{-\infty}^{\infty} \frac{1 + t \omega}{t - \omega} d\mu(t)
\end{equation}

where $\alpha \geq 0$, $\beta \in \mathbb{R}$ and $\mu$ is a bounded, positive Borel measure, singular w. r. t. $\lambda$. Since we shall be refering to (0.1) rather a lot, we shall denote the class of bounded, positive, singular measures on $\mathbb{R}$ by $S(\mathbb{R})$.

G. Letac ([6]) has shown that a measurable transformation $T$ of $\mathbb{R}$ preserves the class of Cauchy distributions iff either $T \in M(\mathbb{R})$ or $-T \in M(\mathbb{R})$. In particular, if $dP_{a+ib}(x) = \frac{b}{\pi (x-a)^2 + b^2} dx$ for $a + ib \in \mathbb{R}^2^+$ and $T = T(f) \in M(\mathbb{R})$, then :

\begin{equation}
(0.2) \quad P_{\omega} \circ T^{-1} = P_{f(\omega)} \quad \text{for} \quad \omega \in \mathbb{R}^2^+
\end{equation}

This equation shows that $M(\mathbb{R})$ is a class of non-singular transformations of the measure space $(\mathbb{R}, \mathcal{B}, \lambda)$, and is therefore an object of ergodic theory.

Let $f \in I(\mathbb{R}^2^+)$ have a fixed point $\omega_0 \in \mathbb{R}^2^+$. By (0.2), $T(f)$ preserves the Cauchy distribution $P_{\omega_0}$. It was shown in [16], that if $f$ is $1 - 1$, then $T(f)$ is conjugate to a rotation of the circle, and shown in [15] that otherwise, $T(f)$ is mixing. We show in § 1 that if $f$ is not $1 - 1$ then $T(f)$ is exact.

In § 2 we recall some well known facts about inner functions of $\mathbb{R}^2^+$. The Denjoy-Wolff theorem (see [13], [14] and [18]) adapted to $\mathbb{R}^2^+$ shows that when studying the ergodic properties of $T(f)$, for $f \in I(\mathbb{R}^2^+)$ with no fixed points in $\mathbb{R}^2^+$, we may assume that $\alpha(f) \geq 1$. In case $\alpha(f) > 1$, $T(f)$ is dissipative, and when $\alpha(f) = 1$, $T(f)$ preserves Lebesgue measure.

In § 3, we consider the case $\alpha(f) = 1$. Here, the conservativity of a restriction $T(f)$ is sufficient for its rational ergodicity ([1]) (ergodicity was established in [15]). We also give sufficient conditions for exactness, and discuss the similarity classes ([1]) of restrictions.

The ergodic theory of certain restrictions has been considered in [2], [5], [7], [10], [11], [15] and [16].

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§ 1. MIXING RESTRICTIONS
PRESERVING FINITE MEASURES

The purpose of this section is to prove

THEOREM 1.1. — Let \( f \in I(\mathbb{R}^2^+) \) and assume that \( f \) is not \( 1 - 1 \). If \( f \) has a fixed point \( \omega_0 \in \mathbb{R}^2^+ \), then \((\mathbb{R}, \mathcal{B}, P_{\omega_0}, T(f))\) is an exact measure preserving transformation.

i.e. \( \bigcap_{n \geq 1} T^{-n} \mathcal{B} = \{ \phi, \mathbb{R} \} \mod \lambda \).

Before proving theorem 1.1, we shall need some auxiliary results. The first of these is Lin's criterion for exactness of Markov operators (theorem 4.4 in [8]) as applied to our case. To state this, we shall need some extra notation:

Let \( T \in M(\mathbb{R}) \), then \((\mathbb{R}, \mathcal{B}, \lambda, T)\) is a non-singular transformation, and so \( g \in L^\infty(\mathbb{R}, \mathcal{B}, \lambda) \) iff \( g \circ T \in L^\infty(\mathbb{R}, \mathcal{B}, \lambda) \). We define the dual operator of \( T, \hat{T} : L^1(\mathbb{R}, \mathcal{B}, \lambda) \to L^1(\mathbb{R}, \mathcal{B}, \lambda) \) by

\[
\int_{\mathbb{R}} \hat{T}h. g d\lambda = \int_{\mathbb{R}} h. g \circ T d\lambda \quad \text{for} \quad h \in L^1 \quad \text{and} \quad g \in L^\infty
\]

If we write, for \( \omega = a + ib \in \mathbb{R}^2^+ \)

\[
\frac{dP_{\omega}}{d\lambda}(x) = \phi_{\omega}(x) = \frac{b}{\pi} \cdot \frac{1}{(x - a)^2 + b^2}
\]

then equation (0.2) translates to:

(1.1) \( \hat{T}\phi_{\omega} = \phi_{f(\omega)} \) for \( T = T(f) \in M(\mathbb{R}) \)

Clearly, \( \hat{T} \) is a positive linear operator, \( \int_{\mathbb{R}} \hat{T}h d\lambda = \int_{\mathbb{R}} h d\lambda \) for \( h \in L^1 \).

Lin's Criterion (for restrictions). — Let \( T = T(f) \in M(\mathbb{R}) \).
\( T \) is exact iff

(1.2) \( \| \hat{T}^n u \|_1 \to 0 \) for every \( u \in L^1 \), \( \int_{\mathbb{R}} u d\lambda = 0 \). Here, and throughout, \( \| u \|_1 = \int_{\mathbb{R}} |u| d\lambda \).

We shall also need the following (elementary) lemma.

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LEMMA 1.2. — If \( \omega_n \in \mathbb{R}^{2+} \) and \( \omega_n \to \omega \in \mathbb{R}^{2+} \) then:

\[
\| \phi_{\omega_n} - \phi_\omega \|_1 \to 0
\]

Proof of theorem 1.1. — We first show that \( f^n(\omega) \to \omega_0 \ \forall \omega \in \mathbb{R}^{2+} \), where \( f^1(\omega) = f(\omega) \) and \( f^{n+1}(\omega) = f(f^n(\omega)) \).

Let \( \phi : U = \{ |Z| < 1 \} \to \mathbb{R}^{2+} \) be a conformal map. Then \( g = \phi^{-1} f \phi : U \to U \) is analytic, and \( g(\phi(\omega_0)) = \phi(\omega_0) \). By the Schwartz lemma ([9]):

\[
|g'(\phi(\omega_0))| < 1 \quad \text{as} \quad g \text{ is not 1 - 1.}
\]

It is now not hard to see that

\[
g^n(Z) \to \phi(\omega_0) \quad \forall z \in U,
\]

and hence that \( f^n(\omega) \to \omega_0 \ \forall \omega \in \mathbb{R}^{2+} \).

Hence, by lemma 1.2

\[
\| \hat{T}^n \phi_{\omega_0} - \phi_{\omega_0} \|_1 = \| f^{n(\omega)} - \phi_{\omega_0} \|_1 \to 0 \quad \text{for} \quad \omega \in \mathbb{R}^{2+}.
\]

We will now establish that

\[
\| \hat{T}^n u \|_1 \to 0 \quad \text{for} \quad u \in L^1
\]

with \( \int u d\lambda = 0 \) which, by Lin's criterion, will ensure the exactness of \( T \).

Let \( u \in L^1 \) with \( \int u d\lambda = 0 \) and let \( \varepsilon > 0 \). By Wiener's Tauberian theorem (see [12], p. 357), there exist \( \alpha_1 \ldots \alpha_N, a_1 \ldots a_N \in \mathbb{Q} \) such that

\[
\| u - \sum_{j=1}^{N} \alpha_j \phi_{a_j+i} \|_1 < \varepsilon/2
\]

Clearly, this implies that \( \sum_{j=1}^{N} \alpha_j < \varepsilon/2 \) and so:

\[
\| \hat{T}^n u \|_1 \leq \| \hat{T}^n \left( u - \sum_{j=1}^{N} \alpha_j \phi_{a_j+i} \right) \|_1
\]

\[
+ \| \hat{T}^n \left( \sum_{j=1}^{N} \alpha_j (\phi_{a_j+i} - \phi_{\omega_0}) \right) \|_1 + \| \sum_{j=1}^{N} \alpha_j \phi_{\omega_0} \|_1 \leq \| u - \sum_{j=1}^{N} \alpha_j \phi_{a_j+i} \|_1
\]

\[
+ \sum_{j=1}^{N} |\alpha_j| \| \hat{T}^n \phi_{a_j+i} - \phi_{\omega_0} \|_1 + \sum_{j=1}^{N} |\alpha_j| < \varepsilon + o(1) \quad \text{as} \quad k \to \infty \quad \square
\]

Since \( \varepsilon > 0 \) was arbitrary:

\[
\| \hat{T}^n u \|_1 \to 0. \quad \square
\]
§ 2. BASIC CLASSIFICATION

PROPOSITION 2.1. [17]. — Let \( f \in I(\mathbb{R}^2^+) \). Then

\[
\frac{f(ib)}{ib} \rightarrow \begin{cases} 
\alpha(f) = x \in [0, \infty) & \text{as } b \rightarrow \infty \text{ (as in 0.1)} \\
\gamma(f) \in [x, \infty] & \text{as } b \downarrow 0.
\end{cases}
\]

Moreover

\[ \alpha = \gamma \quad \text{iff} \quad f(\omega) = \alpha \omega \]

Proof. — From the representation 0.1, we immediately calculate that:

\begin{equation}
(2.1) \quad \frac{f(ib)}{ib} = \alpha + \frac{\beta}{ib} + \frac{1 - b^2}{ib} \int_{-\infty}^{\infty} \frac{t d\mu(t)}{t^2 + b^2} + \int_{-\infty}^{\infty} \frac{1 + t^2}{t^2 + b^2} d\mu(t)
\end{equation}

It follows from elementary integration theory that

\[ \frac{f(ib)}{ib} \rightarrow \alpha = \alpha(f) \quad \text{as } b \rightarrow \infty. \]

To check the limit as \( b \rightarrow 0 \), we « flip » \( f \) to get:

\[ \tilde{f}(\omega) = -1/f(-1/\omega) \]

Since \( \tilde{f} \in I(\mathbb{R}^2^+) \), we have that

\[ \frac{\tilde{f}(ib)}{ib} \rightarrow \alpha(\tilde{f}) \in [0, \infty) \quad \text{as } b \rightarrow \infty \]

but this decodes to:

\[ \frac{f(ib)}{ib} \rightarrow \gamma(f) = \frac{1}{\alpha(f)} \in (0, \infty] \quad \text{as } b \downarrow 0. \]

Now, if \( \gamma(f) < \infty \) then, by 2.1:

\[ \gamma(f) = \alpha + \int_{-\infty}^{\infty} \frac{1 + t^2}{t^2} d\mu(t) \]

Hence \( \gamma(f) \geq \alpha(f) \) with equality iff \( \mu \equiv 0. \)

PROPOSITION 2.2. — Let \( f \in I(\mathbb{R}^2^+) \) and \( T = T(f) \). If \( \alpha(f) > 1 \) then \( T \) is dissipative.

Proof. — Write \( f^\ast(\omega) = u_\ast(\omega) + iv_\ast(\omega) \).

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From the representation (0.1), we have:

\[ v_{n+1}(\omega) = \alpha v_n(\omega) + v_n(\omega) \int_{-\infty}^{\infty} \frac{(1 + t^2)d\mu(t)}{(t - u_n)^2 + v_n^2} \geq \alpha v_n \]

Hence \( v_n(\omega) \geq \alpha^n \) for \( n \geq 1 \), and

Clearly

\[ \hat{T}^n \phi_i(t) = \frac{v_n(i)}{\pi((t - \mu)^2 + \sigma^2)} \leq \frac{1}{\pi \alpha^n} \]

and so

\[ \sum_{n=1}^{\infty} \hat{T}^n \phi_i(t) \leq \frac{1}{(\alpha - 1)} \quad \forall t \in \mathbb{R} \]

PROPOSITION 2.3 (Letac [6]). — Let \( f \in I(\mathbb{R}^2^+), T = T(f) \).

If \( \alpha(f) = 1 \) then \( \lambda \circ T^{-1} = \lambda \).

Proof. — Let \( f(ib) = u(b) + iv(b) \) we have:

\[ \frac{u(b)}{b} \to 0 \quad \text{and} \quad \frac{v(b)}{b} \to 1 \quad \text{as} \quad b \to \infty . \]

Hence, for \( A \in \mathcal{B} \):

\[ \pi b P_{ib}(A) \to \lambda(A) \]

and

\[ \pi b P_{fib}(A) \to \lambda(A) \quad \text{as} \quad b \to \infty . \]

Since \( P_{ib}(T^{-1}A) = P_{f_ib}(A) \), we have that

\[ \lambda(T^{-1}A) = \lambda(A) \quad \text{for} \quad A \in \mathcal{B} \]

The next result is the Denjoy-Wolff theorem stated on \( \mathbb{R}^2^+ \), which shows that if \( f \in I(\mathbb{R}^2^+) \) has no fixed point in \( \mathbb{R}^2^+ \), then \( \exists \tilde{f} \in I(\mathbb{R}^2^+) \) with \( \alpha(\tilde{f}) \geq 1 \), and such that (\( \mathbb{R}, \mathcal{B}, \lambda, T(f) \)) and (\( \mathbb{R}, \mathcal{B}, \lambda, T(\tilde{f}) \)) are conjugate, (and therefore have the same ergodic properties).

THEOREM 2.4. — Let \( f \in I(\mathbb{R}^2^+) \) have no fixed points in \( \mathbb{R}^2^+ \), and assume that \( \alpha(f) < 1 \); then

\[ \exists ! \ t \in \mathbb{R} \quad \text{such that} \quad \alpha(f, T^{-1}) \geq 1 \]
where
\[ \phi_\omega = \frac{1 + t\omega}{t - \omega}. \]
(Note that \( \alpha(\phi_\omega^{-1}f_\phi) = 1/\gamma(f) \)).

**Proof.** — Let \( \phi(z) = i \frac{1 + z}{1 - z} \). Then \( g = \phi^{-1}f: U \to U \) is analytic, and has no fixed points in \( U \). The Denjoy-Wolff theorem on \( U \) (see [13] or [14]) shows that \( \exists ! \rho \in T \) such that
\[ (*) \quad \Re \left( \frac{\rho + g(Z)}{\rho - g(Z)} \right) \geq \Re \left( \frac{\rho + Z}{\rho - Z} \right) \quad \forall Z \in U \]

Now let \( t = \phi(\rho), \psi = i \frac{\rho + Z}{\rho - Z} \) and \( \tilde{f} = \psi g^{-1} \in I(\mathbb{R}^2) \). It follows that \( \phi\psi^{-1} = \phi_t^{-1} \) and hence that \( \tilde{f} = \phi_t f \phi_t^{-1} \). Also, \((*)\) means that \( \Im g(Z) \geq \Im \psi(Z) \) for \( Z \in U \), and hence \( \Im \tilde{f}(\omega) \geq \Im \omega \) for \( \omega \in \mathbb{R}^2 \), which implies \( \alpha(\tilde{f}) \geq 1 \). \( \square \)

If \( \alpha(\phi_t f \phi_t^{-1}) > 1 \) for some \( t \), then by proposition 2.2, \( T(f) \) is dissipative. If \( \alpha(\phi_t f \phi_t^{-1}) = 1 \), then, by proposition 2.3, \( T(\phi_t f \phi_t^{-1}) = \phi_t T(f) \phi_t^{-1} \) preserves Lebesgue measure. Hence \( T(f) \) preserves the measure \( \nu_t \), where \( d\nu_t(x) = dx/(x - t)^2 \). The rest of this section is devoted to odd restrictions.

(We say that a restriction \( T \) is **odd** if \( T(-x) = -T(x) \)).

**Lemma 2.5.** — Let \( f \in I(\mathbb{R}^2) \) and let \( T = T(f) \). The following are equivalent:

- i) \( T \) is odd
- ii) \( \Re f(ib) = 0 \) for \( b > 0 \)
- iii) \( f(-\bar{\omega}) = -\tilde{f}(\omega) \) for \( \omega \in \mathbb{R}^2 \)
- iv) \( f(\omega) = \alpha\omega + \int_{-\infty}^{\infty} \frac{1 + t\omega}{t - \omega} d\mu(t) \)

where \( \mu \in S(\mathbb{R}) \) is symmetric

**Proof.** — The implications \( iv \Rightarrow iii \Rightarrow i \) and \( iii \Rightarrow ii \) are elementary. That \( ii \Rightarrow iii \) is because of the Schwartz reflection principle (see [9]). The fact that for \( t \geq 0 \):
\[ e^{\mu f(\omega)} = \int_{-\infty}^{\infty} e^{\mu T(x)} \phi_\omega(x) dx \]
gives the implication \( i \Rightarrow iii \).

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We show that \( \text{iii)} \Rightarrow \text{iv)}. \) Assume \( \text{iii)}. \) It is evident that \( \beta = 0 \) in the representation 0.1, so we have

\[
f(\omega) = \alpha \omega + \int_{-\infty}^{\infty} \frac{1 + t\omega}{t - \omega} d\mu(t) \quad \text{where} \quad \alpha \geq 0 \quad \text{and} \quad \mu \in \mathcal{S}(\mathbb{R}).
\]

We must show that \( \mu \) is symmetric. To see this, we first rewrite the equation \( \nu(-a + ib) = \nu(a + ib) \) (implied by \( \text{iii)} \)) as:

\[(2.2) \quad \int_{-\infty}^{\infty} \phi_{b}(t - a)(1 + t^2)d\mu(t) = \int_{-\infty}^{\infty} \phi_{b}(t + a)(1 + t^2)d\mu(t) \]

Next, we take \( g(t) \) a continuous function of compact support and let \( g_{b}(t) = \phi_{ib} \ast g \) for \( b > 0 \). It follows from (2.2) that

\[
\int_{-\infty}^{\infty} g_{b}(-t)(1 + t^2)d\mu(t) = \int_{-\infty}^{\infty} g_{b}(t)(1 + t^2)d\mu(t).
\]

The symmetry of \( \mu \) is established by the (elementary) facts that

\[
g_{b}(t) \to g(t) \quad \text{as} \quad b \to 0
\]

\[
\sup_{t \in \mathbb{R}} (1 + t^2)|g_{b}(t)| < \infty \quad \square
\]

We denote the collection of those inner functions on \( \mathbb{R}^{2+} \) satisfying the conditions of the above lemma by \( \mathcal{I}_{0}(\mathbb{R}^{2+}) \), and remark that \( f \in \mathcal{I}_{0}(\mathbb{R}^{2+}) \) iff \( \Theta^{-1}f\Theta \) is an essentially real inner function of \( \mathcal{U} \). (Here \( \Theta(z) = i \left( \frac{1 + z}{1 - z} \right) \).

**Theorem 2.6.** — Let \( f \in \mathcal{I}_{0}(\mathbb{R}^{2+}) \) and \( T = T(f) \).

If \( \alpha(f) < 1 < \gamma(f) \) then \( T \) preserves a Cauchy distribution. Moreover, if \( \omega f(\omega) \) is not constant, then \( T \) is exact.

**Proof.** — If \( f \in \mathcal{I}_{0}(\mathbb{R}^{2+}) \) then it follows from the lemma

\[
\gamma(f) = \alpha(f) + \int_{-\infty}^{\infty} \frac{1 + t^2}{t^2} d\mu(t).
\]

Now since \( \alpha(f) < 1 < \gamma(f) \), we have that

\[
\int_{-\infty}^{\infty} \frac{1 + t^2}{t^2} d\mu(t) > 1 - \alpha > 0.
\]
But \( \int_{0}^{\infty} \frac{1 + t^2}{t^2 + b^2} \, d\mu(t) \downarrow 0 \) as \( b \to \infty \) so there is a \( b_0 > 0 \) such that
\[
\int_{0}^{\infty} \frac{1 + t^2}{t^2 + b^2} \, d\mu(t) = 1 - \alpha, \quad \text{i.e.} \quad f(b_0) = ib_0, \quad \text{hence} \quad P_{ib_0} \circ T^{-1} = P_{ib_0}.
\]

The result now follows from theorem 1.1. \( \square \)

To illustrate the results of this section, we consider \( T x = \alpha x + \beta \tan x \) where \( \alpha, \beta > 0 \). Here, \( \alpha(T) = \alpha \), and \( \gamma(T) = \alpha + \beta \).

If either \( \alpha > 1 \), or \( \alpha + \beta < 1 \), \( T \) is dissipative.

If \( \alpha < 1 < \alpha + \beta \), then \( T \) preserves a Cauchy distribution and is exact.
(This was established in [5] for \( \alpha = 0, \beta > 1 \)).

The remaining cases (\( \alpha = 1 \) and \( \alpha + \beta = 1 \)) are contained in the discussion of:

§ 3. RESTRICTIONS PRESERVING INFINITE MEASURES

In this section, we consider those restrictions preserving infinite measures with \( \alpha = 1 \), or \( \alpha(\phi, f \phi^{-1}) = 1 \) for some \( t \).

We will see that for these transformations, conservativity is sufficient for ergodicity and rational ergodicity ([I]), a stronger property (example 1.2 in [I]). We then give sufficient conditions for exactness.

Firstly, we recall the definition of rational ergodicity. Let \( (X, \mathcal{B}, m, \tau) \) be a conservative, ergodic, measure preserving transformation of a non-atomic, \( \sigma \)-finite measure space. We say that \( \tau \) is rationally ergodic if there is a set \( A \), of positive finite measure and \( K < \infty \) such that

\[
\int_A \left( \sum_{k=0}^{n-1} 1_A \circ \tau^k \right)^2 \, dm \leq K \left( \sum_{k=0}^{n-1} m(A \cap \tau^{-k}A) \right)^2 \quad \text{for} \quad n \geq 1
\]

For a rationally ergodic transformation \( \tau \), we let \( B(\tau) \) denote the collection of sets with the property (B). It was shown in [I] that there is a sequence \( \{ a_n(\tau) \} \) such that

\[
\frac{1}{a_n(\tau)} \sum_{k=0}^{n-1} m(A \cap \tau^{-k}A) \to m(A)^2 \quad \text{for every} \quad A \in B(\tau)
\]

The sequence \( \{ a_n(\tau) \} \) is known as a return sequence for \( \tau \) and the collection of all sequences asymptotically proportional to \( a_n(\tau) \)

\[
\left( \text{i.e. } \frac{a_n}{a_n(\tau)} \to c \in (0, \infty) \right)
\]
is known as the asymptotic type of \( \tau \) and denoted by \( \mathcal{A}(\tau) \). It was shown in [7] (theorem 2.4) that if \( \tau_1 \) and \( \tau_2 \) are rationally ergodic transformations which are both factors of the same measure preserving transformation, then

\[
\mathcal{A}(\tau_1) = \mathcal{A}(\tau_2) \quad \text{(i.e. } \exists \lim_{n \to \infty} \frac{a_n(\tau_1)}{a_n(\tau_2)} \in (0, \infty) \).
\]

We commence with the case \( \alpha(f) = 1 \).

**Lemma 3.1.** — Let \( f \in \text{I}(\mathbb{R}^2^+) \) be non-linear and let \( T = T(f) \),

\[
f^n(\omega) = u_n(\omega) + iv_n(\omega) \quad \text{for } n \geq 1 \ \omega \in \mathbb{R}^2^+.
\]

If \( \alpha = 1 \) then \( T \) is conservative

\[
\text{iff } \sum_{n=1}^{\infty} \frac{v_n(\omega)}{|f^n(\omega)|^2} = \infty \quad \forall \omega \in \mathbb{R}^2^+.
\]

**Proof.** — It will be more comfortable to work on the unit disc \( U \). Accordingly, we let \( M(z) = \theta^{-1}f(\theta(z)) \) where \( \theta(z) = i\left(\frac{1+z}{1-z}\right) \). Then \( M \) is an inner function on \( U \). Let \( M(re^{i\theta}) \to r e^{i\theta} \) as \( r \to 1 \) a.e. Denoting \( \text{Im} \left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) \) by \( q_z(\theta) \) and \( q_z(\theta) d\theta \) by \( d\pi_z(\theta) \), we see that \( \pi_z \circ \theta^{-1} = \pi_0 P_{\theta(z)} \) and this combined with the fact that \( \theta^{-1}T\theta = \tau \) gives us that:

\[
\pi_z \circ \tau^{-1} = \pi_{M(z)}.
\]

So \( \tau \) is a non-singular transformation of \( (T, \lambda) \), and is conservative iff \( T \) is conservative.

Let \( \hat{\tau} \) be the operator dual to \( \tau \), acting on \( L^1 \). Then \( \hat{\tau} q_z(t) = q_{M(z)}(t) \) and \( \tau \) is conservative iff

\[
\sum_{n=1}^{\infty} q_{M^n(z)}(t) = \infty \quad \text{a.e. } \forall z \in U.
\]

We next show that \( M^n(z) \to 1 \) as \( n \to \infty \) \( \forall z \in U \). This will follow from the fact that \( f^n(\omega) \to \infty \) as \( n \to \infty \) \( \forall \omega \in \mathbb{R}^2^+ \) which we now demonstrate. From 0.1:

\[
v_{n+1}(\omega) = v_n(\omega) + v_n(\omega) \int_{U_n} (1 + t^2) d\mu(t) \geq v_n(\omega).
\]
Hence \( v_n \uparrow v_\infty \). It is not hard to see that if \( v_\infty < \infty \), we must have \( |U_n| \to \infty \). Hence \( M^n(z) \to 1 \).

Now choose \( z \in U \) and let \( M^n(z) = r_n e^{i\theta_n} \). We have \( r_n \to 1 \) and \( \theta_n \to 0 \).

Also:

\[
q_{M^n(z)}(t) = \frac{1 - r_n^2}{1 - 2r_n \cos(\theta_n - t) + r_n^2} \sim \frac{1 - r_n}{1 - \cos t} \quad \text{as} \quad n \to \infty.
\]

For \( t \neq 0 \). Thus:

\[
(3.2) \quad T \text{ is conservative iff} \quad \sum_{n=1}^{\infty} 1 - |M^n(z)| = \infty \quad \forall z \in U.
\]

The second condition is the same as

\[
\sum_{n=1}^{\infty} 1 - |M^n(z)|^2 = \infty \quad \forall z \in U.
\]

Now if \( \omega = a + ib \in \mathbb{R}^2^+ \), then

\[
1 - \left| \frac{\omega - i}{\omega + i} \right|^2 = \frac{4b}{a^2 + (b + 1)^2}
\]

From the definition of \( M \), we have

\[
1 - \left| M^n \left( \frac{\omega - i}{\omega + i} \right) \right|^2 = \frac{4v_n(\omega)}{U_n(\omega) + (v_n + 1)^2} \sim \frac{4v_n(\omega)}{|f^n(\omega)|^2} \quad \text{as} \quad n \to \infty \quad \square
\]

**Theorem 3.2.** — Let \( f \in \mathcal{I}(\mathbb{R}^2^+) \) be non-linear, \( T = T(f) \) and \( \alpha(f) = 1 \). If \( T \) is conservative then \( T \) is rationally ergodic, and

\[
\mathcal{A}(T) = \left\{ \sum_{k=1}^{n} \frac{v_k(\omega)}{|f^k(\omega)|^2} \right\} \quad \text{for every} \quad \omega \in \mathbb{R}^2^+.
\]

**Proof.** — We first prove ergodicity, and here again, it is more comfortable to work on \( U \). We prove the ergodicity of \( \tau \) (as defined in Lemma 3.1). If \( T \) is conservative then by (3.2):

\[
\sum_{n=1}^{\infty} 1 - |M^n(z)| = \infty \quad \forall z \in U.
\]

Since \( M^n(z) \to 1 \), we must have that the points \( \{ M^n(z) \}_{n \geq 1} \) are distinct. Now, let \( h \in N(U) \) (defined on p. 303 of \([9]\)). If \( h(M(z)) = h(z) \) for all \( z \in U \) then by theorem 15-23 of \([9]\), \( h \) must be constant. The ergodicity of \( \tau \) is deduced from this as follows:

Let \( A \subseteq T \) be a \( \tau \)-invariant measurable set and let

\[
v(z) = \int_{\pi}^{\pi} q_{A}(\theta) 1_{A}(\theta) \frac{d\theta}{2\pi}.
\]

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Then \( v(M(z)) = v(z) \), \( \| v \circ M^n \|_\infty \leq 1 \ \forall n \geq 1 \), and \( v(re^{i0}) \to 1_A(\theta) \) a.e. as \( r \to 1 \). Now \( v \) can be regarded as the imaginary part of an analytic function \( F \in H(U) \). By theorem 17-26 of [9] \( F \in H^1(U) \subseteq N(U) \) and \( \| F \circ M^n \|_1 \leq A \ \forall n \geq 1 \).

Moreover: \( F(M(z)) = F(z) + c \) where \( c \in \mathbb{R} \).

Let \( F^*(e^{i\theta}) = \lim_{r \uparrow 1} F(re^{i\theta}) \), then \( F^*(\tau e^{i\theta}) = F^*(e^{i\theta}) + c \). The conservativity of \( \tau \) yields that \( c = 0 \) (since the set \([ F^* | \leq M ] \) has positive measure for some \( M \), and so every point of this set returns infinitely often to it under iterations of \( \tau \) — an impossibility if \( c \neq 0 \)). Thus, \( F \) is constant and hence also \( 1_A(0) \).

We now turn to rational ergodicity. Let

\[
b_n(\omega) = \frac{|f^n(\omega)|^2}{v_n(\omega)}
\]

Since \( f^n(\omega) \to \infty \), it is clear that:

\[
(3.3) \quad \pi b_n(\omega) \hat{T}^n \phi_\omega(t) \to 1
\]

uniformly on compact subsets of \( \mathbb{R} \). Let

\[
a_n(\omega) = \sum_{k=1}^{\infty} \frac{1}{\pi b_k(\omega)}.
\]

From (3.3) we have that

\[
(3.4) \quad \frac{1}{a_n(\omega)} \sum_{k=0}^{n-1} \hat{T}^k \phi_\omega \to 1
\]

uniformly on compact subset of \( \mathbb{R} \).

Now, since \( T \) is a conservative ergodic transformation, it follows that \( \hat{T} \) is a conservative ergodic Markov operator, and we have from (3.4), by the Chacon-Ornstein theorem (see [3]) that:

\[
(3.5) \quad \frac{1}{a_n(\omega)} \sum_{k=0}^{n-1} \hat{T}^k f \to \int \mathbb{R} f \, d\lambda \quad \text{a.e.} \quad \forall f \in L^1.
\]

Hence

\[
\exists a_n \to \infty \text{ s.t.} \quad \frac{a_n(\omega)}{a_n} \to 1 \quad \text{for every} \quad \omega \in \mathbb{R}^2_+.
\]

We will prove rational ergodicity of \( T \) by showing that bounded intervals are in \( B(T) \).

Let \( A = [a, b] \) where \(-\infty < a < b < \infty \).
Then $1_A \leq c \phi_i$
Hence, by (3.4), there is a $C_1 < \infty$ s.t.

\begin{equation}
\frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k 1_A(x) \leq C_1 \quad \text{for} \quad n \geq 1, \, x \in A.
\end{equation}

This, combined with (3.5), gives (by dominated convergence)

\begin{equation}
\frac{1}{a_n} \sum_{k=0}^{n-1} \lambda(A \cap T^{-k}A) \to \lambda(A)^2
\end{equation}

To complete the proof that $T$ is rationally ergodic, we show that:

\begin{equation}
\int_A \left( \sum_{k=0}^{n-1} 1_A \circ T^k \right)^2 d\mu \leq 2C_1 a_n^2 \quad \text{for} \quad n \geq 1.
\end{equation}

\begin{align*}
\int_A \left( \sum_{k=0}^{n-1} 1_A \circ T^k \right)^2 d\mu & \leq 2 \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \lambda(A \cap T^{-k}(A \cap T^{-l}A)) \\
& = 2 \sum_{l=0}^{n-1} \int_{A \cap T^{-l}A} \sum_{k=0}^{n-1} \hat{T}^k 1_A d\lambda \leq 2C_1 a_n^2 \quad \square
\end{align*}

We now turn to exactness. The following elementary lemma plays a similar role to that of lemma 1.2.

**Lemma 3.3.** — If $b_n \to \infty$, $B_n \sim b_n$ and

$$\frac{a_n}{b_n} \to 0 \quad \text{as} \quad n \to \infty,$$

then

$$\| \phi_{a_n+ib_n} - \phi_{ib_n} \|_1 \to 0 \quad \text{as} \quad n \to \infty.$$

**Theorem 3.4.** — Let $f \in \mathcal{I}(\mathbb{R}^2^+)$, $T = T(f)$ and assume

$$f(\omega) = \omega + \int_{-\infty}^{\infty} \frac{d\nu(t)}{t - \omega}$$

then: $T$ is exact, rationally ergodic and $\mathcal{A}(T) = \{ \sqrt{n} \}$.

**Proof.** — Let $L = \max \{ \nu(\mathbb{R}), \nu(\mathbb{R})^2 \}$ and assume that $K \geq \frac{1}{4}$. We
write \( f^n(\omega) = u_n(\omega) + iv_n(\omega) \). The assumption of the theorem means that

\[
\begin{align*}
\text{(3.9)} & \quad u_{n+1} = u_n + \int_{-K}^{K} \frac{t - u_n}{(t - u_n)^2 + v_n^2} \, dv(t) \\
& \quad v_{n+1} = v_n + v_n \int_{-K}^{K} \frac{dv(t)}{(t - u_n)^2 + v_n^2} 
\end{align*}
\]

The first part of the proof of this result consists of deducing the asymptotic behaviour of \( u_n \) and \( v_n \). For this, we assume that \( \omega = a + iL \) where \( a \in \mathbb{R} \). The recurrence relations (3.9) show us that

\[ v_n(\omega) \geq L \quad \text{for every} \quad n \geq 1. \]

And this enables us to deduce the boundless of \( |u_n(\omega)| \) as follows:

Noting that:

\[
\left| \int_{-K}^{K} \frac{t - u_n}{(t - u_n)^2 + v_n^2} \, dv(t) \right| \leq \frac{v(\mathbb{R})}{2v_n} \leq \frac{1}{2}
\]

we see that:

If \( u_n \geq K \) then \( -K \leq K - \frac{1}{2} \leq u_{n+1} \leq u_n \).

If \( u_n \leq -K \) then \( u_n \leq u_{n+1} \leq -K + \frac{1}{2} \leq K \).

If \( u_n \leq K \) then

\[
\begin{align*}
& \quad u_{n+1} \leq u_n + (K - u_n) \int_{-K}^{K} \frac{dv}{(t - u_n)^2 + v_n^2} \leq u_n + \frac{(K - u_n)}{v_n^2} v(\mathbb{R}) \leq K \\
& \quad u_{n+1} \leq K \\
& \quad u_{n+1} \leq -K.
\end{align*}
\]

Hence \( |u_n(a + iL)| \leq |a| V^K \) for \( n \geq 1 \).

The recurrence relations (3.9) now imply that \( v_n \to \infty \) as \( n \to \infty \) and hence

\[
\begin{align*}
v_{n+1}^2 - v_n^2 &= 2v_n^2 \int_{-K}^{K} \frac{dv(t)}{(t - u_n)^2 + v_n^2} \\
& \quad + v_n^2 \left( \int_{-K}^{K} \frac{dv(t)}{(t - u_n)^2 + v_n^2} \right)^2 \to 2v(\mathbb{R}) \quad \text{as} \quad n \to \infty
\end{align*}
\]

Hence \( v_n(a + iL) \sim \sqrt{2v_n} \) as \( n \to \infty \).

Lemma 3.3 now shows us that for every \( a \in \mathbb{R} \):

\[
\text{(3.19)} \quad \| \mathcal{T}^n \phi_{a+il} - \phi_{i,\sqrt{2v_n}} \| \to 0 \quad \text{as} \quad n \to \infty.
\]

We now obtain exactness by Lin's criterion by an argument similar
to that of theorem 1.1 (the rational ergodicity of $T$ has already been established, and its asymptotic type characterised, by theorem 3.2).

Let $u \in L^1$, $\int_{\mathbb{R}} u d\lambda = 0$, and $\varepsilon > 0$.

By Wiener's Tauberian theorem, there are $a_1 \ldots a_N$, $a_1 \ldots a_N \in \mathbb{R}$ such that

$$\left\| u - \sum_{k=1}^{N} a_k \phi_{a_k + iL} \right\|_1 < \varepsilon/2$$

Whence:

$$\left\| \hat{T}^n u \right\|_1 \leq \left\| \hat{T}^n \left( u - \sum_{k=1}^{N} a_k \phi_{a_k + iL} \right) \right\|_1$$

$$+ \left\| \hat{T}^n \sum_{k=1}^{N} a_k \phi_{a_k + iL} - \sum_{k=1}^{N} a_k \phi_{i \sqrt{2} n \pi \alpha_k} \right\|_1 + \left\| \sum_{k=1}^{N} a_k \phi_{i \sqrt{2} n \pi \alpha_k} \right\|_1$$

$$\left\| \hat{T}^n u \right\|_1 \leq \left\| u - \sum_{k=1}^{N} a_k \phi_{a_k + iL} \right\|_1$$

$$+ \sum_{k=1}^{N} a_k \left\| \hat{T}^n \phi_{a_k + iL} - \phi_{i \sqrt{2} n \pi \alpha_k} \right\|_1 + \left\| \sum_{k=1}^{N} a_k \phi_{i \sqrt{2} n \pi \alpha_k} \right\|_1 < \varepsilon + o(1) \quad \square$$

We note that the « generalized Boole transformation » (proven ergodic in [7]) falls within the scope of this last theorem.

If we added $\beta \neq 0$ to $f$ in theorem 3.4, we would obtain that for $\text{Im } \omega$ large enough $|u_\omega(\omega)| \geq c_1 n$ and $v_\omega(\omega) \leq c_2 \log n$ (where $f^n(\omega) = u_\omega(\omega) + iv_\omega(\omega)$). The methods of lemma 3.1 would yield that $T(f)$ is dissipative.

The following corollary follows immediately from lemma 3.1 and theorem 3.2.

**Corollary 3.5.** — Let $f \in L^2(\mathbb{R}^2)$ and let $T = T(f)$, $f^n(i) = iv_n(i)$. If $\alpha(f) = 1$ then:

$$T \text{ is conservative iff } \sum_{n=1}^{\infty} \frac{1}{v_n(i)} = \infty$$

and in this case, $T$ is rationally ergodic with

$$\mathcal{A}(T) = \left\{ \sum_{k=1}^{n} \frac{1}{\pi v_k(i)} \right\}.$$
Moreover, in case $f \in \mathcal{I}_0$ and $\alpha(f) = 1$ we have that $v_n \to \infty$ and so:

\[
v_{n+1}^2 - v_n^2 = 2v_n^2 \int_{-\infty}^{\infty} \frac{1 + t^2}{t^2 + v_n^2} \, d\mu(t) + \frac{v_n^2}{\left( \int_{-\infty}^{\infty} \frac{1 + t^2}{t^2 + v_n^2} \, d\mu(t) \right)^2} \to 2 \int_{-\infty}^{\infty} (1 + t^2) \, d\mu(t) \leq \infty\]

Hence:

\[
\frac{v_n(i)}{\sqrt{n}} \to \sqrt{2 \int_{-\infty}^{\infty} (1 + t^2) \, d\mu(t) \leq \infty}
\]

which means:

a) $T \times T \times T$ is dissipative

b) $\frac{a_n(T)}{\sqrt{n}} \to c \in [0, \infty)$ as $n \to \infty$ (in case $T$ is r.e.).

These last two properties are held in common with the restrictions of theorem 3.4, and with the Markov shifts of random walks on $\mathbb{Z}$.

The following example does not fall within the scope of theorem 3.4, (though theorem 3.2 does apply).

**Example 3.6.** — $Tx = x + \alpha \tan x$ is exact, rationally ergodic with $a_n(T) \sim \frac{\log n}{\alpha}$ for $\alpha > 0$.

**Proof.** — Let $f(\omega) = \omega + \alpha \tan \omega$ and $f^n(\omega) = u_n(\omega) + iv_n(\omega)$. Then:

\[
u_{n+1} = u_n + \frac{2\alpha \sin 2u_ne^{2v_n}}{e^{4v_n} - 2 \cos 2u_ne^{2v_n} + 1}
\]

and

\[
v_{n+1} = v_n + \frac{\alpha e^{4v_n}}{e^{4v_n} - 2 \cos 2u_ne^{2v_n} + 1}
\]

Whence:

\[
v_{n+1} - v_n \geq \alpha \tanh v_n \geq \alpha \tanh v_0 > 0
\]

so

\[
v_n \sim \alpha n \quad \text{as } n \to \infty.
\]

On the other hand:

\[
|u_{n+1} - u_n| \leq \frac{2\alpha e^{2v_n}}{(e^{2v_n} - 1)^2} \leq 4\alpha e^{-2v_n} \leq 4\alpha e^{-an} \quad \text{for } n \text{ large}.
\]

Hence $u_n \to u_\infty$, and the argument that $T$ is exact now proceeds identically to the last argument of theorem 3.4. □
The following lemma will give examples of \( f \in I_0(\mathbb{R}^2^+) \) with \( \alpha(f) = 1 \) and \( T = T(f) \) dissipative, and also uncountably many dissimilar \( \Gamma.e. \) (see [1]) restrictions \( T(f) \) with \( f \in I_0(\mathbb{R}^2^+) \), \( \alpha(f) = 1 \).

**Lemma 3.7.** — Let \( \mu \in S(\mathbb{R}) \) be symmetric with
\[
c(x) = \mu(|t| \geq x) \sim \frac{1}{x^\alpha} \quad \text{where} \quad 0 < \alpha < 2.
\]

Let
\[
f_a(\omega) = \omega + \int_{-\infty}^{\infty} \frac{1 + t\omega}{t - \omega} d\mu(t) \quad \text{and} \quad f^n(i) = iv_n.
\]

Then: \( v_n \sim cn^{1/\alpha} \) where \( c \) depends only on \( \alpha \).

**Proof.** — We have
\[
v_{n+1} = v(1 + F(v_n))
\]
where
\[
F(b) = \int_{-\infty}^{\infty} \frac{1 + t^2}{t^2 + b^2} d\mu(t).
\]

It is not difficult to see that
\[
F(b) = \frac{\mu(\mathbb{R})}{b^2} + 2(b^2 - 1) \int_{0}^{\infty} \frac{xc(x)}{(x^2 + b^2)^2} dx
\]

We first show that \( F(b) \sim \frac{c_1}{b^\alpha} \) as \( \alpha \to \infty \)

Let \( \varepsilon > 0 \), and \( M \) be such that
\[
\frac{1 - \varepsilon}{x^\alpha} \leq c(x) \leq \frac{1 + \varepsilon}{x^\alpha} \quad \forall x \geq M
\]
Writing
\[
L_M(b) = \int_{M}^{\infty} \frac{x^{1-\alpha}}{(x^2 + b^2)^2} dx
\]
we have that:
\[
(1 - \varepsilon)L_M(b) = \int_{M}^{\infty} \frac{xc(x)dx}{(x^2 + b^2)^2} \leq (1 + \varepsilon)L_M(b).
\]

Now
\[
L_M(b) = \int_{M}^{\infty} \frac{x^{1-\alpha}}{(x^2 + b^2)^2} dx = \frac{1}{b^{2+\alpha}} \int_{M/b}^{\infty} \frac{x^{1-\alpha}dx}{(x^2 + 1)^2} \sim \frac{c}{b^{2+\alpha}} \quad \text{as} \quad b \to \infty
\]

where
\[
c = \int_{0}^{\infty} \frac{x^{1-\alpha}dx}{(x^2 + 1)^2}
\]
Since \( \varepsilon > 0 \) was arbitrary and \( \alpha < 2 \), we have that

\[
F(b) \sim \frac{c}{b^2} \quad \text{as} \quad b \to \infty.
\]

Clearly, \( v_\alpha \to \infty \), hence:

\[
v_{n+1}^\alpha - v_n^\alpha = v_n^\alpha (1 + F(v_n))^\alpha - 1
\sim \alpha v_n^\alpha F(v_n) \quad \text{as} \quad n \to \infty
\to \alpha c \quad \text{as} \quad n \to \infty
\]

Thus \( v_\alpha \sim (\alpha c n)^{1/\alpha} \) as \( n \to \infty \) \( \square \)

We now let \( T_\alpha = T(f_\alpha) \).

By corollary 3.5:

If \( 0 < \alpha < 1 \) then \( T_\alpha \) is dissipative.

If \( 1 \leq \alpha < 2 \) then \( T_\alpha \) is rationally ergodic and

\[
\mathcal{A}(T_\alpha) = \begin{cases} 
\{ \log n \} & \text{if} \quad \alpha = 1 \\
\{ n^{1-1/\alpha} \} & \text{if} \quad 1 < \alpha < 2.
\end{cases}
\]

If follows from theorem 2.4 of \([1]\) that if \( 1 \leq \alpha_1 < \alpha_2 < 2 \) then \( T_{\alpha_1} \) and \( T_{\alpha_2} \) are not factors of the same measure preserving transformation.

**Theorem 3.8.** Let \( f \in \mathcal{L}(\mathbb{R}^2) \) and \( T = T(f) \).

Suppose \( x_0 \in \mathbb{R} \) and \( f \) is analytic in a neighbourhood around \( x_0 \).

If \( T(x_0) = x_0, T'(x_0) = 1 \) and \( T''(x_0) = 0 \) then \( T \) preserves the measure \( \nu_{x_0} \)

where \( d\nu_{x_0}(x) = \frac{dx}{(x - x_0)^2} \), and is exact, rationally ergodic with asymptotic type \( \{ \sqrt{n} \} \).

**Remarks.** The conditions \( T(x_0) = x_0 \) and \( T'(x_0) = 1 \) correspond to:

\( \lambda(\phi_{x_0}, f \phi_{x_0}^{-1}) = 1 \). If, in this situation, \( T''(x_0) \neq 0 \); then \( T \) is dissipative.

By possibly considering \( g(\omega) = f(\omega + x_0) - x_0 \) we may (and do) assume \( x_0 = 0 \).

**Proof.** Let

\[
f(\omega) = \omega + \sum_{n=3}^{\infty} a_n \omega^n \quad \text{for} \quad |\omega| \text{ small}.
\]

Then

\[
\frac{1}{f(\omega)} - \frac{1}{\omega} = \frac{\omega - f(\omega)}{f(\omega)} = \frac{\omega}{f(\omega)} \sum_{n=3}^{\infty} a_n \omega^n
\to 0 \quad \text{as} \quad \omega \to 0.
\]
Hence
\[ \frac{1}{f(\omega)} = \frac{1}{\omega} + \sum_{n=1}^{\infty} b_n \omega^n \quad \text{for } |\omega| \text{ small}. \]

Let \( \tilde{f}(\omega) = -1/\left(-\frac{1}{\omega}\right) \).

Then:
\[ \tilde{f}(\omega) = \omega + \sum_{n=1}^{\infty} b_n \omega^{-n} \quad \text{for } |\omega| \text{ large}, \]
say \( |\omega| \geq K \) and, since \( \tilde{f} \in I(\mathbb{R}^2^+) \), \( \alpha(\tilde{f}) = 1 \):

\[ \tilde{f}(\omega) = \omega + \beta + \int_{-\infty}^{\infty} \frac{1 + t\omega}{t - \omega} d\mu(t) \quad \text{where } \mu \in S(\mathbb{R}), \beta \in \mathbb{R} \]

In order to prove the theorem by applying theorem 3.4, we will show that
\[ \tilde{f}(\omega) = \omega + \int_{-K}^{K} \frac{dv(t)}{t - \omega} \quad \text{where } v \in S(\mathbb{R}). \]

Firstly, let \( g(\omega) = \tilde{f}(\omega) - \omega \). By (3.11):

\[ -ibg(ib) \rightarrow b_1 \quad \text{as } b \rightarrow \infty \]

But by (3.12):

\[ -ibg(ib) = -ib\left(\beta - b^2 \int_{-\infty}^{\infty} \frac{t \mu(t)}{t^2 + b^2}\right) + ib \int_{-\infty}^{\infty} \frac{t \mu(t)}{t^2 + b^2} + b^2 \int_{-\infty}^{\infty} \frac{1 + t^2}{t^2 + b^2} d\mu(t). \]

Hence, we obtain, from the convergence of the real part, that
\[ \int_{-\infty}^{\infty} (1 + t^2) \mu(t) < \infty \]
and from the convergence of the imaginary part that:

\[ b^2 \int_{-\infty}^{\infty} \frac{t \mu(t)}{t^2 + b^2} \rightarrow \beta \quad \text{as } b \rightarrow \infty. \]

which convergence, when combined with the previous one, gives
\[ \int_{-\infty}^{\infty} t \mu(t) = \beta. \]
Now, let $dv(t) = (1 + t^2)d\mu(t)$, then $v \in S(\mathbb{R})$ and it follows easily that

\[\int(\omega) = \omega + \int_{-\infty}^{\infty} \frac{dv(t)}{t - \omega} \]

(3.14)

Now, let $h_b(a) = \text{Im} g(a + ib) = b$.

By (3.11) $g$ is uniformly continuous on compact subsets of $[|\omega| \geq K]$, and so $h_b(a) \to 0$ as $b \to 0$ uniformly on compact subsets of $[|a| > K]$.

Let $dQ_b(x) = h_b(x)dx$, then $Q_b = P_{ib} \ast v$, and so $Q_b(A) \to v(A)$ for $A$ a compact set. If $A$ is a compact subset of $[|x| > K]$, then

\[v(A) = \lim_{b \to 0} Q_b(A) = \lim_{b \to 0} \int_A h_b(x)dx = 0.\]

Thus $v$ is concentrated on $[-K, K]$ and (3.13) is established.

The transformations $T_\alpha x = \alpha x + (1 - \alpha) \tan x$ for $0 \leq \alpha < 1$ fall within the scope of theorem 3.9 (it was shown in [11] that $T_0$ is ergodic). It follows from asymptotic type considerations that the above transformations are dissimilar to $T_\alpha x = x + \alpha \tan x$.

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