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by

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ABSTRACT. — It $\Phi$ is the support of a continuous additive functional $(A_t)$ of a Markov process $(X_t)$, we use results on the structure of the processes $(\tau_i)$ and $(X_{\tau_i}, \tau_i)$ (where $\tau_i$ is the right continuous inverse of $A_t$) to describe the set $\mathcal{H} = \{ t : X_t \in \Phi \}$.

1. INTRODUCTION

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a standard process with state space $(E, \mathcal{E})$, $(A_t)_{t \geq 0}$ a continuous additive functional of $X$ and $\Phi$ its fine support (see [1] for definitions and notation).

Define $(\tau_i)_{i \geq 0}$ to be the right continuous inverse of $(A_t)$, that is,

$$\tau_i = \inf \{ s : A_s > t \},$$

and consider the random sets

$$I = \{ t : A_{t+\varepsilon} - A_t > 0 \text{ for all } \varepsilon > 0 \},$$

$$J = \{ t : A_{t+\varepsilon} - A_{t-\varepsilon} > 0 \text{ for all } \varepsilon > 0 \},$$

$$\mathcal{H} = \{ t : X_t \in \Phi \},$$

$$Q = \{ t < \infty : \tau_u = t \text{ for some } u \}.$$
It is well known (see [1], Ch. V) that a.s. \( Q = ICXCJ \). Moreover, 
\[ J - I = \{ \tau_i : \tau_i > 0 \} \] 
and hence
\[ H = \{ \tau_i ; t \geq 0 \} \cup \{ \tau_i : \tau_i \neq \tau_i ; X_{\tau_i} \in \Phi \} \]

These remarks show that \( H \) is essentially the range of \( \tau \) and so it 
is quite natural to try to describe the set \( H \) in terms of the process \( \tau \).

In the case \( \Phi = \{ x_0 \} \), with \( x_0 \) regular for itself, the sections of \( H \) can 
be described as follows (see [7], [8] and [9]).

i) a.s. \( P^{x_0} H \) is bounded or unbounded. We will say that \( H \), or \( x_0 \), is 
a.s. \( P^{x_0} \) transient or recurrent ;

ii) a.s. \( P^{x_0} H \) has Lebesgue measure zero or a.s. \( P^{x_0} H \) has positive 
Lebesgue measure. In the first case one calls \( H \) light and in the second 
case \( H \) is called heavy ;

iii) We will call \( H \) stable if its complement intersects every finite interval 
\((0, T)\) in a finite union of intervals, and unstable otherwise. Observe that 
\( H \) is stable if there are finitely many excursions from \( \{ x_0 \} \) in every finite 
interval. One has that a.s. \( P^{x_0} H \) is stable or a.s. \( P^{x_0} H \) is unstable.

The process \( (\tau_i)_{t \geq 0} \) is, in the present case, essentially a subordinator 
with respect to the law \( P^{x_0} \), and, using the structure of \( \tau \) one can give 
criteria for when is \( H \) going to be transient, recurrent, stable, etc. In fact, 
in [7] and [8] it is proved that if one considers the exponent 
\[ S(\theta) = \varepsilon \theta + \int_{(0, \infty)} (1 - e^{-\theta x}) \mu(dx) \]
in the Lévy representation of the distribution of \( \tau_i \) (i.e. \( e^{-rS(\theta)} = E^{x_0}[e^{-\theta \tau_i}] \)) 
then

i) \( H \) is recurrent \( \Leftrightarrow \mu \{ \infty \} = 0 \),

ii) \( H \) is heavy \( \Leftrightarrow \varepsilon > 0 \),

iii) \( H \) is stable \( \Leftrightarrow \mu \) is a finite measure.

In the next sections, we intend to use some of the results proved by 
Cinlar in [2], [3] and [4], and Rolin in [10], about the structure of the processes \( \tau_i \) and \( (X_{\tau_i}, \tau_i) \) in the case \( \Phi \) is a more general set (\( \Phi \) the support 
of \( (A_i) \)) to study the set \( H \). Our results will extend those in [7] and [8] 
for the case \( \Phi = \{ x_0 \} \). To be more specific: in section 3 we study the set \( H \) 
with respect to the measures \( P^x \), \( x \in \Phi \), by using the results of Cinlar on 
Lévy systems for \( (X_{\tau_i}, \tau_i) \), and, in section 4, we describe \( H \) conditional on 
the paths of the time changed process \( X_{\tau_i} \). We begin by stating some preli-
minary results that will be needed in these sections.
2. PRELIMINARIES

Consider, as in the introduction, a standard process $X$, a continuous additive functional $(A_t)$ of $X$, with fine support $\Phi$, and let $(\tau_t)$ be the right continuous inverse of $(A_t)$. Let $\Phi_\Delta = \Phi \cup \{ \Delta \}$.

Denote by $\Phi$ the Borel subsets of $\Phi$, $b\Phi$ the bounded Borel measurable functions on $\Phi$, $R_+$ the Borel subsets of $R_+$, and $bR_+$ the bounded Borel functions on $R_+$. $\Phi_\Delta$, etc. have similar meanings.

The joint process $(X_t, \tau_t)$ is a Markov additive process (see [2]). We assume $\Phi$ to be projective, in which case $X_t$ will be a Hunt process (see [1], Ch. V) and so it follows from the results in [3] (*) that there is a Lévy system $(H, L)$ for $(X_t, \tau_t)$ with $H$ a continuous additive functional of $(X_t)$, $L$ a kernel from $\Phi_\Delta \times R_+$ into $\Phi$ such that

\[
E^x \sum_{s \leq t} f(X_{t-s}, X_t, \tau_s - \tau_s^-).1_{\{X_{t-s} \neq X_{t-s}^-\}} = E^x \int_0^t dH_s \int_{\Phi_\Delta \times R_+} L(X_t, dy, du)f(X_t, y, u)
\]

for each $f$ in $b\Phi \times \Phi_\Delta \times R_+$.

The process $(\tau_t)$ can be decomposed as $\tau_t = \tau_t^c + \tau_t^d$ where $\tau_t^c$ (the continuous part of $\tau_t$) is a continuous additive functional of $(X_t)$ and $\tau_t^d$ is a pure jump increasing additive process (see [2] or [4]).

Let us put $C_t = H_t + \tau_t^c + t$; $(C_t)$ is a strictly increasing continuous additive functional of $(X_t)$.

It is proved in [4] that if we let $\sigma_t = \inf \{ s : C_s > t \}$, then the process $(\hat{X}_t, \hat{\tau}_t) = (X_{t_{\sigma_t}}, \tau_{\sigma_t})$ is again a Markov additive process and its Lévy system is such that the corresponding additive functional $\hat{H}_t$ is equal to $t \wedge \zeta$.

Now, we observe that if one defines $B_t = C_{A_t}$ then we obtain.

\[
(2.2) \text{ LEMMA.} - i) (B_t) \text{ is a continuous additive functional of } X.
\]

ii) The right continuous inverse of $B_t$ coincides with $\tau_{\sigma_t}$.

iii) $(A_t)$ and $(B_t)$ have the same support $\Phi$.

(*) See note at the end of the paper.

Proof. — i) To prove that \((B_t)\) is adapted see [6] section 2, Lemma 14. The additivity of \((B_t)\) follows from the fact that \((C_t)\) is a continuous additive functional of \((X_{t+})\) and \(A_t\) is a stopping time relative to \((\mathcal{F}_{t+})\).

ii) \(\inf \{ u : C_{A_u} > s \} = \inf \{ u : A_u > \sigma_s \} = \tau_{\sigma_s}\).

iii) This last assertion follows from the fact that \(\sigma_0 \equiv 0\), and so \(\Phi = \{ x : \mathbb{P}^x(\tau_0 = 0) = 1 \} = \{ x : \mathbb{P}^x(\tau_{\sigma_0} = 0) = 1 \} = \text{support } (B_t)\).

In view of Lemma (2.2) we will assume that the Lévy system for \((X_{t+}, \tau_i)\) is such that \(H_t = t \wedge \zeta\), so that (2.1) can be rewritten as follows:

\[
E^x \sum_{s \leq t} f(X_{t+s-}, X_{t+s}, \tau_s - \tau_{s-}) 1_{\{X_{t+s-} \neq X_{t+s}\} \cup \{\tau_s \neq \tau_{s-}\}} = E^x \int_0^\infty d(s \wedge \zeta) \int_{\Phi_\Delta \times \mathbb{R}_+} L(X_{t+s}, dy, du) f(X_{t+s}, y, u).
\]

By means of an approximation argument one can get a more general relation than (2.3), namely, one can show that if \(Z_s\) is adapted to \((\mathcal{F}_{t+})\) positive and left continuous, then one has for \(f \in b \mathbb{X} \times \mathbb{R}_+\)

\[
E^x \sum_{0 \leq s \leq t} Z_s f(X_{t+s-}, X_{t+s}, \tau_s - \tau_{s-}) 1_{\{X_{t+s-} \neq X_{t+s}\} \cup \{\tau_s \neq \tau_{s-}\}} = E^x \int_0^t Z_s ds \wedge \zeta \int_{\Phi_\Delta \times \mathbb{R}_+} L(X_{t+s}, dy, du) f(X_{t+s}, y, u)
\]

Finally, observe that since \(t = C_{\sigma_1} = H_{\sigma_1} + \tau_{\sigma_1}^c + \sigma_{\tau_1}^c\), \(\tau_{\sigma_1}^c\) is absolutely continuous with respect to \(t\), so we may also assume that

\[
\tau_{\sigma_1}^c = \int_0^t d(X_{t+s}) ds \text{ where } a \text{ is positive and } \Phi \text{ measurable.}
\]

3. THE SET \(\mathcal{H}\)

We will now study the set \(\mathcal{H}\) with respect to the laws \(\mathbb{P}^x\) for \(x \in \Phi\). The notations and definitions will be the ones introduced in the preceding sections.

**Weight**

It follows from the fact that \(\{ s : X_s \in \Phi \} \) differs from \(\{ s : \Delta \tau_{A_s} = 0 \} \) by countably many points, that the « occupation time » of \(\Phi\) is related to \(\tau_i\) as follows:

\[
\tau_i = \int_0^t 1_{\Phi}(X_s) ds \text{ a. s. } \mathbb{P}^x, x \in \Phi \text{ (see [10], chap. IV).}
\]
From (3.1), one gets that $\mathcal{H}$ is heavy a. s. $P^x \lhd \tau_\infty$ is positive a. s. $P^x$.
It is clear that if a. s. $P^x$, the process spends a positive time in a given subset of $\Phi$, then $\mathcal{H}$ will be heavy a. s. $P^x$.

By writing $\tau^c$ in terms of the time changed process $X_{\tau^c}$ as in (2.5) namely

$$\tau^c = \int_0^\tau a(X_{\tau^c})ds$$

one gets that $\mathcal{H}$ is light a. s. $P^x \forall x \in \Phi$ if $a \equiv 0$. On the other hand, if we let $D = \{ a > 0 \}$ then it is easy to see that

$$\int_0^\infty a(X_{\tau^c})ds = \int_0^\infty 1_D(X_s)ds$$

($a$ is defined to be zero outside of $\Phi$), and hence $\mathcal{H}$ will be heavy a. s. $P^x$ for $x \in D$ if $D$ is finely open.

Observe that in the case $\Phi = \{ x_0 \}$ ($x_0$ regular), for all $t$, one has $X_{\tau^c} = x_0$, $\tau^c = et$, $a(X_{\tau^c}) = e$; so it is clear from (3.1) and (2.5) that a. s. $P^x \mathcal{H}$ is heavy or light, and, $\mathcal{H}$ is heavy $\Leftrightarrow e > 0$, which coincides with the results given in [7] and [8].

If $\Phi$ is a finite set, $\Phi = \{ x_1, \ldots, x_n \}$, with all the $x_i$ being regular, then, if we let $a(x_i) = \varepsilon_i$, we see that $x_i$ is heavy a. s. $P^{x_i} \lhd \varepsilon_i > 0$.

Recurrence

We observe that a. s. $P^x \tau_{\varepsilon_{\infty}} = \sup \{ s \leq t : X_s \in \Phi \}$ from which it follows that the last exit from $\Phi$ coincides with $\tau_{\varepsilon_{\infty}}$, i.e. $\tau_{\varepsilon_{\infty}} = \sup \{ s \geq 0 : X_s \in \Phi \}$.

Thus, if we say that $\Phi$ is transient for $x$ if $\mathcal{H}$ is bounded a. s. $P^x$, and recurrent for $x$ if $\mathcal{H}$ is unbounded a. s. $P^x$, we get that $\Phi$ is transient for $x$ if $\tau_{\varepsilon_{\infty}} < \infty$ a. s. $P^x$.

In terms of the Lévy system for $(X_{\tau^c}, \tau_t)$ one has the following results.

(3.2) PROPOSITION. — For all $x \in \Phi$ the following equality holds

$$E\left[e^{-\tau_{\varepsilon_{\infty}}} \right] = E^x \int_0^\infty e^{-y} L(X_s, \Phi_{\Delta}, \{ \infty \})dA_s$$

Proof.

$$E\left[e^{-\tau_{\varepsilon_{\infty}}} \right] = E^x \sum_{s > 0} e^{-s} \int_0^\infty 1_{(\infty)}(\Delta_s)$$
by (2.4) this last term equals
\[
\mathbb{E}^x \int_0^\infty ds \int_{\Phi_\Delta \times \mathbb{R}^+} L(X_{t_s}, dy, du) e^{-rs} 1_{\{\infty\}}(u)
\]
\[
= \mathbb{E}^x \int_0^\infty L(X_{t_s}, \Phi_\Delta, \{\infty\}) e^{-rs} ds
\]
\[
= \mathbb{E}^x \int_0^\infty e^{-s} L(X_s, \Phi_\Delta, \{\infty\}) d\Lambda_s
\]

The last equality follows from a well known time change formula (see [1], Ch. V).

It follows from proposition (3.2) that \(\tau_{A^-_\infty} = \infty\) a. s. \(P^x \leftrightarrow L(X_{t_s}, \Phi_\Delta, \{\infty\})\)
is \(P^x\) indistinguishable from 0. Or, equivalently

\[
\tau_{A^-_\infty} = \infty\ \text{a. s.}\ \ P^x \leftrightarrow L(\cdot, \Phi_\Delta, \{\infty\}) = 0\ \text{a. s.}\ P^x
\]

Observe that in the case \(\Phi = \{x_0\}\) these conditions reduce to the condition for recurrence given in [7] and [8] namely that \(\mu\{\infty\} = 0\).

Let us denote by \(\overline{X}_t\), the left limit \(X_{t^-}\), then, when \(\Phi\) is transient for \(x\), one has the following expression for the joint distribution of

\[
\tau_{A^-_\infty}, \overline{X}_{\tau_{A^-_\infty}}.
\]

(3.3) Proposition. — Let \(\Phi\) be transient for \(x\), then, if \(g \in b \Phi\) and \(b > 0\), one has

(3.4)
\[
\mathbb{E}^x [g(\overline{X}_{\tau_{A^-_\infty}}), b < \tau_{A^-_\infty}] = \mathbb{E}^x \int_0^\infty g(X_s) L(X_s, \Phi_\Delta, \{\infty\}) d\Lambda_s
\]

Proof.

\[
\mathbb{E}^x [g(X_{\tau_{A^-_\infty}}), b < \tau_{A^-_\infty}] = \mathbb{E}^x \sum_{s > 0} g(\overline{X}_{t_s^-}) 1_{(b, \infty)}(\tau_s^-) 1_{\{\infty\}}(\Delta \tau_s)
\]
\[
= \mathbb{E}^x \int_0^\infty g(X_s) 1_{(b, \infty)}(\tau_s) ds \int_{\Phi_\Delta \times \mathbb{R}^+} L(X_{t_s}, dy, du) 1_{\{\infty\}}(u)
\]
\[
= \mathbb{E}^x \int_0^\infty g(X_s) 1_{(b, \infty)}(\tau_s) L(X_{t_s}, \Phi_\Delta, \{\infty\}) ds = \mathbb{E}^x \int_b^\infty g(X_s) L(X_s, \Phi_\Delta, \{\infty\}) d\Lambda_s
\]

(3.5) Remark. — One may check that proposition (3.3) also holds if \(x \in E - \Phi\).

(3.6) Remark. — By taking \(b = 0\) and \(g = 1\) in (3.4) one gets that

\[
\mathbb{P}^x(\tau_{A^-_\infty} > 0) = u_C(x)
\]
where

\[ C_t = \int_0^t L(X_s, \Phi, \{ \infty \})dA_s \]

is a natural potential. Hence, if \( \tau_{\Delta} < \) a. s.,

\( \Phi \) is transient in the usual sense (see [5]).

We observe that the fact that the condition for transience is simpler in this case is due to the fact that \( \Phi \) is the support of a continuous additive functional.

Moreover, with the notation we just introduced, one can rewrite (3.4) as follows.

\[
E_x[g(X_{t\Delta}) \mid \tau_{\Delta} > b] = E_x \int_b^\infty g(X_s) L(X_s, \Phi, \{ \infty \})dA_s = E_x \int_b^\infty g(X_s)dC_s
\]

where \( \mathcal{U}_c(x) = E_x \int g(X_s)dC_s \), which provides another proof of proposition (3.3) in Getoor-Sharpe’s [5], plus an explicit representation of the additive functional \( (C_t) \) in terms of probabilistic objects.

**Stability**

It follows from the fact that \( (A_t) \) increases when \( X_t \in \Phi \) and the definition of \( (\tau_t) \), that, in order to study the excursions from the set \( \Phi \) in \([0, t]\), one can examine the jumps of \( \tau_s \) up to \( \tau_t \), that is

\[
\sum_{s > 0} 1_{(0, A_s]}(s)1_{(0, \infty)}(\Delta \tau_s) = \sum_{s > 0} 1_{(0, t]}(\tau_s)1_{(0, \infty)}(\Delta \tau_s).
\]

Taking expectations in (3.7) and using (2.4) we obtain

\[
E_x \sum_{s > 0} 1_{(0, t]}(\tau_s - + \Delta \tau_s)1_{(0, \infty)}(\Delta \tau_s) = E_x \int_0^\infty ds \int_0^\infty L(X_s, \Phi, du)1_{(0, t]}(\tau_s + u)
\]

where \( \mathcal{U}(x, f, g) = E_x \int_0^\infty f(X_s)g(\tau_s)dt \).

This last calculation shows that the excursions from \( \Phi \) can also be studied in terms of the Lévy system for \((X_t, \tau_t)\). There are some obvious remarks that we can make, namely (3.7) will be finite for all \( t \) if Vol. XV, n° 1 - 1979.
L(x, (0, ∞)) is bounded for all x ∈ Φ, and infinite for t = ∞ if L(x, (0, ∞)) = oo for all x ∈ Φ. However, since (X_t, τ_t) may behave differently for different points in Φ, and L(x, (0, ∞)) varies with x ∈ Φ, we will give a definition of stability that takes into account this local behaviour.

For F, G ∈ Φ, consider

\[ R_t = \sum_s 1_G(X_{\tau_s})1_F(X_{t_0})1_{(0,t]}(\tau_s)1_{(0,\infty)}(\Delta \tau_s) \]

then

\[
E^x(R_t) = E^x \int_0^\infty 1_{(0,t]}(\tau_s)1_G(X_{\tau_s})L(X_{\tau_s}, F, (0, t - \tau_s)]d\tau
\]

\[
= \int G \int_0^t U(x, dy, du)L(X_{\tau_s}, F, (0, t - u)]
\]

**DEFINITION.** — We will say that \( \mathcal{H} \) is stable for \( (x, F, G) \) if the right hand side of (3.8) is finite. Otherwise we will say that it is unstable.

**Remark.** — It is clear that when \( \Phi = \{ x_0 \} \) we get the criteria in [7].

### 4. DESCRIPTION OF \( \mathcal{H} \) IN TERMS OF CONDITIONAL PROBABILITIES

We will now briefly discuss the weight, recurrence and stability of \( \mathcal{H} \) given the paths of the time changed process \( (X_{\tau_t}) \).

It is proved in [2] and [10] that if we let \( \mathcal{X} \) denote the \( \sigma \)-algebra generated by \( (X_{\tau_t})_{t \geq 0} \) completed with respect to the family of measures \( P^\mu \) (\( \mu \) a finite measure on \( \Phi \)), and \( \mathcal{L} \) the same \( \sigma \)-algebra but with respect to the process \( (X_{\tau_t}, \tau_t)_{t \geq 0} \), then there is a regular version of the conditional probability \( P^\omega[\cdot | \mathcal{X}] \) on \( \mathcal{L} \), which is independent of \( x \in \Phi \). Denote this version by \( P^\omega(\cdot) \) when evaluated at \( \omega \in \Omega \), and let \( E^\omega \) denote expectation with respect to \( P^\omega \).

The process \( (\tau_t) \) is a process with independent increments on \( (\Omega, \mathcal{L}, P^\omega) \) and, one has the following representation

\[
E^\omega[e^{-x\tau_t}] = \exp \left[ -\alpha e_t^\omega(\omega) - \int_0^\infty (1 - e^{-2u})v_t^\omega(du) \right]
\]

where

\[
v_t^\omega(A) = E^\omega \sum_{s \leq t} 1_A(\Delta \tau_s)
\]

for \( A \), a Borel set in \( \mathbb{R}^+ \) (see [9] and [2] for proof).
Just as in the case $\Phi = \{ x_0 \}$, $\nu_t$ enables us to study the recurrence and stability as follows:

Let

$$\hat{\zeta} = \inf \{ t : X_t = \Delta \},$$

then, $A_{\infty} = \hat{\zeta}$, from which it follows that

$$P^\omega(\tau_{\hat{\zeta}} \leq \infty) = P^\omega(\Delta \tau_{\hat{\zeta}} = \infty) = E^\omega[1_{\{-\infty}\}(\Delta \tau_{\hat{\zeta}})] = E^\omega \sum_{0 < s < \hat{\zeta}} 1_{(0, \infty)}(\Delta \tau_s) = \nu_{\hat{\zeta}}^\omega \{ \infty \}$$

Hence, $\mathcal{H}$ is transient or recurrent with respect to $P^\omega$ according as to $\nu_{\hat{\zeta}}^\omega \{ \infty \}$ is zero or one.

On the other hand, it follows from (4.2) that

$$\nu_t^\omega(0, \infty) 1_{[t < \hat{\zeta}]} = E^\omega \left[ \sum_{0 < s < t \leq \hat{\zeta}} 1_{(0, \infty)}(\Delta \tau_s) ; t < \hat{\zeta} \right]$$

hence, $\mathcal{H}$ is stable or unstable for $P^\omega$ according as to $\nu_t^\omega(0, \infty)$ is finite or infinite for all $t$.

With regards to the weight of $\mathcal{H}$ one has that a. s. $P^\omega$ (for each $\omega$) $\mathcal{H}$ is heavy or light, in fact:

$$P^\omega(\tau_{\infty}^c > 0) = E^\omega[1_{\{ \tau_{\infty}^c > 0 \}} | \mathcal{H}] = 1_{\{ \tau_{\infty}^c > 0 \}}$$

where the last equality follows from the fact that $\tau_{\infty}^c$ is a continuous additive functional of $(X_t)$.

Finally, we observe that in the case $\Phi = \{ x_0 \}$, $\nu_t = P^\omega$ for almost all $\omega \in \Omega$ (see [10]) and we obtain the criteria in [7] and [8].

Note. — We wish to thank Prof. B. Maisonneuve for the following remark: In order to apply Cinlar’s results on the existence of a Lévy system one has to prove that $(\tau_t)$ is quasileft continuous with respect to the family $(\mathcal{F}_t)$. Let $D_t = \inf \{ s > t : X_s \in \Phi \}$ and let $T_n$ be an increasing sequence of stopping times of $(\mathcal{F}_t)$ with limit $T$.

Then, $\tau_{T_n^-}$ and $\tau_{T^-}$ are stopping times of $(\mathcal{F}_{D_t})$ and $\tau_{T_n^-} \uparrow \tau_{T^-}$. Note now that $\tau_t = D_{\tau_t}$ and use the quasileft continuity of the process $(D_t)$ with respect to $(\mathcal{F}_{D_t})$, which is proved in B. Maisonneuve’s, Systèmes régénératifs, Astérisque, 1974, vol. 15, p. 27.

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