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A new version of Doeblin's theorem

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ABSTRACT. — The aim of this note is to prove the existence of an operator-universal probability measure for infinitely divisible (i. d.) probability measures (p. m.) on a Banach space.

A famous theorem of W. Doeblin [*2*] asserts that there exists a p. m. belonging to the domain of partial attraction of every one-dimensional i. d. p. m. . The multi-dimensional version of this theorem is made by J. Barańska [*1*] in a Hilbert space and recently, by Ho Dang Phuc [*5*] in a Banach space.

Some techniques from Ho Dang Phuc [*5*] can be applied to prove a more general theorem. Namely, we replace the norming sequence in Doeblin's theorem by linear operators.

Throughout the paper we shall denote by $(X, \| . \|)$ a real separable Banach space and consider only σ-additive non-negative Borel measures on X. For a bounded linear operator A and a measure μ we shall denote by $A\mu$ a measure defined by $A\mu(\varepsilon) = \mu(A^{-1}\varepsilon)$ $(\varepsilon \subset X)$. In particular, if $Ax = cx$ for some $c \geqslant 0$ and for all $x \in X$ then $A\mu$ will be denoted by the usual symbol $T_c\mu$.

A p. m. μ on X is said to be infinitely divisible if for every $n = 1, 2, \ldots$ there exists a p. m. μ_n such that $\mu = \mu_n^{*n}$, where the asterisk $*$ denotes the convolution operation of measures. It is known ([*3*], [*9*]) that every i. d. p. m. μ on X has a unique representation.

$$\mu = \rho * \tilde{e}(M) \tag{1}$$

where ρ is a symmetric Gaussian p. m., $\tilde{e}(M)$ is a generalized Poisson p. m. corresponding to a Levy's measure M on $X \setminus \{ 0 \}$. In particular, if M is a finite measure then $e(M) = \tilde{e}(M)$ is a Poisson p. m..

Recall that a p. m. P belongs to the domain of partial attraction of a p. m. q on X if there exist a subsequence $\{ n_k \}$ of natural numbers and a sequence $\{ a_k \}$ of real numbers such that $\{ T_{a_k} P^{*n_k} \}$ is shift-convergent to q. Here and in the sequel the convergence of p. m.'s will be understood in the weak sense. Further, P is said to be universal for i. d. p. m.'s if it belongs to the domain of partial attraction of every i. d. p. m. q.

Let A be a bounded linear operator on X. A p. m. P is said to be A-universal for i. d. p. m.'s on X if for every i. d. p. m. q on X there exist subsequences $\{ n_k \}$ and m_k of natural numbers such that $\{ A^{n_k} P^{*m_k} \}$ is shift-convergent to q.

Now we shall prove the following generalized Doeblin theorem.

THEOREM. — For every invertible bounded linear operator A on X such that

$$\| A^n \| \to 0 \quad \text{as} \quad n \to \infty \tag{2}$$

there exists an A-universal p. m. P for i. d. p. m.'s on X.

Remark. — It is easy to show that if X is finite-dimensional and A is a linear operator on X then from the existence of an A-universal p. m. P for i. d. p. m.'s on X it follows that A is invertible and the relation (2) holds.

We precede the proof of the theorem by two lemmas. The following lemma is due to Ho Dang Phuc [5].

LEMMA 1. — Let ρ be a symmetric Gaussian p. m. on X. Then there exists a sequence $\{ \rho_n \}$ of Poisson p. m.'s convergent to ρ.

Proof. — Let Z be an X-valued random variable with distribution ρ. By Theorem 3 [7] it follows that there exist a sequence $\{ x_n \}$ of elements of X and a sequence $\{ \varphi_n \}$ of independent Gaussian random variables with distribution N(0, 1) such that

$$Z = \sum_n x_n \varphi_n$$

where the series is convergent almost surely.

Let μ_n be the distribution of the random variable

$$Z_n = \sum_{k=1}^{n} x_k \varphi_k$$

Then, by Theorem 4.1 [*6*], it follows that $\{ \mu_n \}$ is convergent to ρ. Moreover, μ_n is a Gaussian p. m. on the finite-dimensional space

$$E_n := \operatorname{lin} \{ a_1, \ldots, a_n \} \subset E$$

for all n. Hence for each n there exists a sequence $\mu_{n,m}$, $m = 1, 2, \ldots$, of Poisson p. m.'s convergent to μ_n. Finally, define $\rho_n = \mu_{n,n}$ we get a sequence $\{ \rho_n \}$ of Poisson p. m.'s convergent to ρ. The Lemma is thus proved.

LEMMA 2. — Let $\{ M_n \}$ be a sequence of finite measures on X such that

$$\int_X \| x \| M_n(dx) \to 0 \quad \text{as} \quad n \to \infty . \tag{3}$$

Then the sequence $e(M_n)$, $n = 1, 2, \ldots$, of Poisson measures on X is convergent to δ_0.

Proof. — For $r > 0$ let us denote $B_r = \{ x \in X : \| x \| \leqslant r \}$. It can be easily seen that the condition (3) implies the following two conditions:

$$\{ M_n |_{B_r^c} \} \quad \text{is weakly convergent to 0 and} \tag{4}$$

$$\lim_{r \downarrow 0} \limsup_{n} \int_{B_r} \| x \| M_n(dx) = 0 \tag{5}$$

which, by Corollary 1.8 [*4*], implies that $e(M_n)$ is convergent to δ_0. Thus the Lemma is proved.

Proof of the Theorem. — By the condition (2) it follows that there exist constants $c > 0$ and $\alpha > 1$ such that for each $n = 1, 2, \ldots$

$$\| A^n \| \leqslant C\alpha^{-n} . \tag{6}$$

Let $\{ q_n \}$ be a countable dense subset of the set of all i. d. p. m.'s on X. We may assume that the corresponding Levy's measure M_n to q_n in the representation (1) is concentrated at $B_n = \{ x \in X : \| x \| \leqslant n \}$ and that $M_n(X) \leqslant n$ for $n = 1, 2, \ldots$. By virtue of Lemma 1 we may suppose that

$$q_n = e(M_n) * \delta_{x_n} \tag{7}$$

for some sequence $\{ x_n \} \subset X$.

Define

$$(8) \qquad M = \sum_{n=1}^{\infty} [\alpha^{n^2}]^{-1} A^{-n^3} M_n$$

where $[\alpha^{n^2}]$ denotes the integer part of α^{n^2} and α is determined by (6). It is clear that M is a finite measure on X. Then we put

$$(9) \qquad P = e(M).$$

We claim that P is A-universal for i. d. p. m.'s on X. Let q be an arbitrary i. d. p. m. on X. Then there is a subsequence $\{ n_k \}$ of natural numbers such that the sequence $\{ q_{n_k} \}$ is convergent to q. Let $t_k = [\alpha^{n_k^2}]$. Our further aim is to prove that the sequence

$$(10) \qquad v_k := A^{n_k^3} p^{*t_k}, \qquad k = 1, 2, \ldots,$$

is shift-convergent to q.

Accordingly, putting

$$(11) \qquad N_k^1 = \sum_{n > n_k} t_k [\alpha^{n_k^2}]^{-1} A^{n_k^3 - n^3} M_n$$

and

$$(12) \qquad N_k^2 = \sum_{n < n_k} t_k [\alpha^{n^2}]^{-1} A^{n_k^3 - n^3} M_n$$

and taking into account the definition of v_k in (10) we have the equation

$$(13) \qquad v_k = e(M_n) * e(N_k^1) * e(N_k^2)$$

We shall prove that $\lim_k N_k^1(X) = 0$ and $\lim_k \int_X \| x \| N_k^2(dx) = 0$, which, by virtue of (13) and Lemma 2, implies that $\{ v_k \}$ is shift-convergent to q.

The first limit is clear, because

$$\begin{aligned} N_k^1(X) &\leqslant \sum_{n > n_k} n t_k [\alpha^{n^2}]^{-1} \\ &\leqslant \sum_{n=1}^{\infty} (n_k + n) [\alpha^{n_k^2}] [\alpha^{(n_k+n)^2}]^{-1} \\ &\leqslant \frac{\alpha}{\alpha - 1} \sum_{n=1}^{\infty} (n_k + n) \alpha^{-(2(n_k+n)n)} \to 0 \qquad \text{as} \qquad k \to \infty . \end{aligned}$$

On the other hand, we have, by (6),

$$\begin{aligned}
\int_X \| x \| N_k^2(dx) &= \sum_{n < n_k} \int_X \| A^{n_k^3 - n^3} x \| t_k [\alpha^{n^2}]^{-1} M_n(dx) \\
&\leqslant C \sum_{n < n_k} n^2 \alpha^{n^3 - n_k^3} [\alpha^{n_k^2}] [\alpha^{n^2}]^{-1} \\
&\leqslant \frac{C\alpha}{\alpha - 1} \sum_{n < n_k} n^2 \alpha^{n^3 - n_k^3} \alpha^{n_k^2 - n^2} \\
&\leqslant \frac{C\alpha}{\alpha - 1} \sum_{n < n_k} n^2 \alpha^{-n^2} \alpha^{(n_k - 1)^3 - n_k^3 + n_k^2} \\
&\leqslant \frac{C\alpha}{\alpha - 1} \alpha^{-2n_k^3 + 3n_k - 1} \sum_{n=1}^{\infty} n^2 \alpha^{-n^2} \to 0 \quad \text{as} \quad k \to \infty .
\end{aligned}$$

Thus the theorem is fully proved.

REFERENCES

[1] J. BARAŃSKA, Domain of partial attraction for infinitely divisible distributions in a Hilbert space. *Colloq. Math.*, t. **XXVIII**, 1973, p. 317-322.

[2] W. DŒBLIN, Sur l'ensemble de puissances d'une loi de probabilité. *Bull. Sci. Math.*, t. **6**, 1939, p. 71-96.

[3] E. DETTWEILER, Grenzwentsätze für Wahrscheinlichkeitsmasse auf Badrikianschen Räumen. *Z. Wahrscheinlichkeitstheory verw. Gebiete*, t. **34**, 1976, p. 285-311.

[4] E. GINÉ, *A survey on the general central limit problem in Banach spaces*. Séminaire sur la géométrie des espaces de Banach, 1977-1978. École Polytechnique.

[5] HO DANG PHUC, *Universal distribution for infinitely divisible distributions in a Banach space* (Prepint). Graduate work at the Wroclaw University, 1978.

[6] K. ITO and M. NISIO, On the convergence of sums of independent Banach space valued random variables. *Osaka J. of Math.*, t. **5**, 1968, p. 35-48.

[7] N. S. JAIN and G. KALLIANPUR, Norm convergent expansion for Gaussian processes in Banach spaces. *Proceed. Amer. Math. Soc.*, t. **25**, 1970, p. 890-895.

[8] R. R. PARTHASARATHY, *Probability measures on metric spaces*, New York, London, 1967.

[9] A. TORTRAT, Structure des lois indéfiniment divisible dans un espace vectoriel topologique (separe) X. Symposium on Probability methods in analysis. *Lecture Notes in Math.*, p. 299-328.

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