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by

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1. INTRODUCTION

Set $X_1, X_2, \ldots, X_N, \ldots$ be a sequence of independent identically distributed random vector with values in $\mathbb{R}^p$. We shall suppose that their common distribution has a density $f$ with respect to Lebesgue measure and that $f \in L^2(\mu)$, where $\mu$ is a Borel probability measure on $\mathbb{R}^p$ with density $r(x)$ with respect to Lebesgue measure.
If \( \{\phi_j\}_{j=1}^\infty \) is complete orthonormal system in \( L^2(\mu) \), the \( n \)-th partial sum of the respective Fourier series for \( f \) is

\[
f_n(x) = \sum_{j=1}^{n} a_j \phi_j(x) \quad x \in \mathbb{R}^p
\]

where

\[
a_j = \int_{\mathbb{R}^p} f(x) \phi_j(x) \mu(x) dx = \mathbb{E} \phi_j(X) \quad \phi_j(x) = \phi_j(x) \mu(x)
\]

Cencov [2] defines the following estimator of \( f_n \):

\[
\hat{f}_{n,N}(x) = \sum_{j=1}^{n} \hat{a}_j \phi_j(x)
\]

where the \( \hat{a}_j \)'s are estimators of \( a_j \) defined by

\[
\hat{a}_j = \int_{\mathbb{R}^p} \phi_j(x) d\hat{F}_N(x)
\]

\( \hat{F}_N \), being, as usual, the empirical distribution function of the sample \( X_1, X_2, \ldots, X_N \).

The aim of this paper is to give conditions under which

\[
\frac{N}{n^{1/2}} \| \hat{f}_{n,N} - f_n \|^2.
\]

When appropriately centered, has gaussian asymptotic distribution. The method we use is inspired by Naradaya [8], although instead of using the strong approximation of the empirical process by a Brownian bridge, we approximate the estimator by functions of Gaussian variables with values in \( L^2(\mu) \). For these ones, the result follows from the central limit theorem on the real line, and the approximation allows to study the behaviour of

\[
\frac{N}{\sqrt{n}} \| \hat{f}_{n,N} - f_n \|^2.
\]

The result is applied to various complete orthonormal sets.

Finally, we consider tests of goodness of fit upon \( \| \hat{f}_{n,N} - f_n \|^2 \) and the behaviour of \( g(n) = \| f_n - f \|^2 \), which permit together to study the asymptotic behaviour of the statistic \( \| \hat{f}_{n,N} - f \|^2 \).
2. ASSOCIATED GAUSSIAN VARIABLES

We have defined

\[ \hat{f}_{n,N}(x) = \sum_{j=1}^{n} \hat{a}_j \phi_j(x) \]

If we put

\[ Y_{n,k}(x) = \sum_{j=1}^{n} (\alpha_j(x_k) - a_j) \phi_j(x) \]

it is clear that

\[ \sqrt{N}(\hat{f}_{n,N} - f_n) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} Y_{n,k}(x) \]

If we consider the \( Y_{n,k} \) as independent identically distributed random variables with values in \( L^2(\mu) \) we have

\[ E(Y_{n,k}) = 0 \]

and

\[ \Gamma_n(g, h) = E\{ (g, Y_{n,k})(h, Y_{n,k}) \} \]

where \( \Gamma_n \) is the covariance of \( Y_{n,k} \) and \((\cdot, \cdot)\) is the scalar product in \( L^2(\mu) \).

It follows that

\[ \Gamma_n(\phi_i, \phi_j) = \int_{\mathbb{R}^p} f(x) \phi_i(x) \phi_j(x) r^2(x) dx - a_i a_j \quad i, j = 1, 2, \ldots, n. \]

Define the centered gaussian random variable \( Z_{1,n} \) with values in \( L^2(\mu) \) by

\[ Z_{1,n} = \sum_{i=1}^{n} \xi_i \phi_i \]

in such a way that \( Z_{1,n} \) and \( Y_{n,k} \) have the same covariance. Now, if \( \xi_0 \) is a normalized gaussian real random variable, independent from \( \{ \xi_i \}_{i=1}^{n} \), consider the random variable (with values in \( L^2(\mu) \)):

\[ Z_{2,n} = Z_{1,n} + \xi_0 \sum_{i=1}^{n} a_i \phi_i. \]
Them $Z_{2,n}$ is gaussian, $E(Z_{2,n}) = 0$ and its covariance $\Gamma^{(2)}_n$ satisfies

$$\Gamma^{(2)}_n(\phi_i, \phi_j) = \int_{\mathbb{R}^p} \phi_i(x)\phi_j(x)r^2(x)f(x)dx$$

**Lemma 2.1.** — If $\|fr\|_\infty < \infty$, then

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} E \|Z_{2,n}\|^2 - \|Z_{1,n}\|^2 = 0$$

*Proof.* — $E \|Z_{2,n}\|^2 - \|Z_{1,n}\|^2 \leq \sum_{j=1}^{n} a_j^2 + 2E \xi_0 |E| \sum_{j=1}^{n} \xi_j a_j$

and since $E(\xi_0) < 1$,

$$\left( \sum_{j=1}^{n} \xi_j a_j \right)^2 \leq \int_{\mathbb{R}^p} \left( \sum_{i=1}^{n} \phi_i(x)a_i \right)^2 r^2(x)f(x)dx \leq \|fr\|_\infty \|f\|^2.$$

The result now follows from

$$E \|Z_{2,n}\|^2 - \|Z_{1,n}\|^2 \leq \|f\| (\|f\| + 2 \|fr\|^{1/2}).$$

Before studying the asymptotic behaviour of $Z_{1,n}$ let us define:

1. $A_n = (\Gamma^{(2)}_n(\phi_i, \phi_j)) = (C_{ij})$

2. $A_n = \frac{1}{n} \int_{\mathbb{R}^p} \phi_i(x)f(x)dx = \frac{1}{n} \text{tr}(A_n)$

3. $S_m(n) = \sum_{i_1}^{n} \ldots \sum_{i_m}^{n} C_{i_1i_2} C_{i_3i_4} \ldots C_{i_mi_1}$, evidently $S_m(n) = \text{Tr}(A_n^m)$.

4. $\sigma_n^2 = \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \int_{\mathbb{R}^p} \phi_i(x)\phi_j(x)r^2(x)f(x)dx \right)^2$

Note that

$$\sigma_n^2 = \frac{2}{n} S_2(n) = \frac{2}{n} \sum_{i=1}^{n} \lambda_{i,n}^2$$

if $\{ \lambda_{i,n} \}_{i=1}^{n}$ are the eigenvalues of $A_n$.

**Lemma 2.2.** — Suppose that there exists $m \geq 3$ such that $\frac{1}{m} S_m = O(1)$. Then, if $\lim_{n \to \infty} \sigma_n^2 = \sigma^2 > 0$, we have

$$\lim_{n \to \infty} \frac{n \sigma_n^2}{(\max_{1 \leq i \leq n} \lambda_{i,n})^2} = \infty.$$

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So that the result follows letting \( n \to \infty \).

**Theorem 2.3.** — Under the same hypothesis of Lemma 2.2 we have

\[
W - \lim_{n \to \infty} \frac{1}{n^{1/2}} \left( \| Z_{2,n} \|^2 - E \| Z_{2,n} \|^2 \right) = N(0, \sigma^2).
\]

**Proof.** — Since \( Z_{2,n} \) is gaussian we can find a basis \( \{ e_1, \ldots, e_n \} \) such that

\[
Z_{2,n} = \lambda_{1,n}^{1/2} \gamma_1 e_1 + \ldots + \lambda_{n,n}^{1/2} \gamma_n e_n.
\]

where \( \gamma_1, \ldots, \gamma_n \) are normalized independent gaussian random variables. So

\[
\| Z_{2,n} \|^2 = \lambda_{1,n} \gamma_1^2 + \ldots + \lambda_{n,n} \gamma_n^2
\]

and

\[
n^{-1/2}(\| Z_{2,n} \|^2 - E \| Z_{2,n} \|^2) = n^{-1/2} \sum_{i=1}^{n} \lambda_{i,n} (\gamma_i^2 - 1).
\]

The result follows from

\[
\lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} \lambda_{i,n}^2 = \sigma^2.
\]

Together with Lemma 2.2 and Lindeberg's Theorem on the line.

**Remark.** — From Lemma 2.1 and \( n^{-1/2}E \| Z_{2,n} \|^2 = \sqrt{n} \Delta_n \) we obtain

\[
W - \lim_{n \to \infty} (n^{-1/2} \| Z_{1,n} \|^2 - n^{1/2} \Delta_n) = N(0, \sigma^2).
\]

### 3. MAIN THEOREM

The following Theorem is due to Kuelbs and Kurtz [7] see also Giné and León [4]. The statement is adapted to our present needs.

**Theorem 3.1.** — Let \( \{ Y_i \}_{i=1}^{n} \) be independent identically distributed...
random variables with values in $L^2(\mu)$, $E(Y_1) = 0$, $E(|Y_1|^3) < \infty$ and $Z_1$ a centered gaussian variable with the same covariance as $Y_1$. Then, for each $t$ and $\delta > 0$,

$$
P \left\{ \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i \right\| \leq t \right\} - P \left\{ \| Z_1 \| < t \right\} = 0 \left( \delta^{-3} \frac{E(Y_1)^3}{\sqrt{N}} \right) + P \left\{ \| Z_1 \| - t \leq \delta \right\}$$

holds true.

We now prove our main result:

**THEOREM 3.2.** — Suppose that for some $\alpha > 0$

$$E(\| Y_{n,1} \|^3) = o(N^{1/2}) \quad \text{if} \quad n = o(N^2)$$

If, additionally, $\| f_r \|_r < \infty$, $\sigma^2_n \to \sigma^2 > 0$ and $\frac{1}{n} S_m(n) = O(1)$ for some $m \geq 3$, then

$$W - \lim_{n \to \infty} \left[ \frac{N}{n^{1/2}} \| f_{n,N} - f_n \|^2 - n^{1/2} \Delta_n \right] = N(0, \sigma^2)$$

**Proof.** — Define $G_{n,N}(t)$ and $G_n(t)$ in the following way:

$$G_{n,N}(t) = P \left\{ \frac{N}{n^{1/2}} \left\| \frac{1}{\sqrt{N}} \sum_{k=1}^{N} Y_{n,k} \right\|^2 \leq (t + n^{1/2} \Delta_n)^{1/2} \right\}$$

$$G_n(t) = P \left\{ \frac{1}{n^{1/2}} \| Z_{1,n} \|^2 - n^{1/2} \Delta_n \leq t \right\}$$

$$= P \left\{ \left\| \frac{1}{n^{1/4}} Z_{1,n} \right\| \leq (t + n^{1/2} \Delta_n)^{1/2} \right\}.$$  

By theorem 3.1,

$$|G_{n,N}(t) - G_n(t)| = 0 \left( \delta^{-3} \frac{E(Y_{1,n} \|^3)}{n^{3/4} N^{1/2}} \right) + P \left\{ \left\| \frac{1}{n^{1/4}} Z_{1,n} \right\| - (t + n^{1/2} \Delta_n)^{1/2} \right\} \leq \delta.$$  

But

$$P \left\{ \left\| \frac{1}{n^{1/4}} Z_{1,n} \right\| - (t + n^{1/2} \Delta_n)^{1/2} \right\} \leq \delta \right\} = P \left\{ \delta^2 - 2\delta \sqrt{t + n^{1/2} \Delta_n + t} \leq \frac{1}{n^{1/2}} \| Z_{1,n} \|^2 - n^{1/2} \Delta_n \leq \delta^2 + 2\delta \sqrt{t + n^{1/2} \Delta_n + t} \right\}.$$  

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If we choose \( \delta = \delta(n) \) such that \( \delta^2 n^{1/2} \to 0 \), the remark following theorem 2.3 and the fact that \( \Delta_n \) is bounded, since

\[
\Delta_n \leq \left( \frac{1}{n} \sum_{i=1}^{n} \lambda_{i,n}^2 \right)^{1/2} \to \frac{1}{\sqrt{2}} \sigma,
\]

imply:

\[
P\left( \left| \frac{1}{n^{1/4}} \| \mathbf{Z}_{1,n} \| - \left( t + n^{1/2} \Delta_n \right)^{1/2} \right| \leq \delta \right) \to \phi_\sigma(t) - \phi_\sigma(t) = 0.
\]

Where \( \phi_\sigma \) denotes the normal \((0, \sigma^2)\) distribution.

Finally, if we choose \( \delta(n) = n^{-1/4} \theta_n \) with \( \theta_n \to 0 \), \( \delta^2 n^{1/2} = \theta_n \to 0 \) holds, and the theorem will follow if the first term in (3.1) tends to zero. This is achieved if \( \theta_n \) tends to zero slowly enough.

**Corollary 3.3.** — If in addition to the hypothesis of theorem 3.2, one has \( g(n) = \| f_n - f \|^2 = o(n^{1/2} N^{-1}) \), then

\[
W - \lim_{n \to \infty} \left[ \frac{N}{n^{1/2}} \| f_n - f \|^2 - n^{1/2} \Delta_n \right] = N(0, \sigma^2).
\]

We shall use the following theorem to verify that

\[
E(\| Y_{n,1} \|^3) = o(N^{1/2}).
\]

**Theorem 3.4.** — Define \( v_i = \sup_x | \alpha_i(x) | \). Suppose that

\[
\sum_{i=1}^{n} v_i^2 = o(n^{-2} N) \quad \text{and} \quad \| fr \|_{\infty} < \infty
\]

then

\[
E(\| Y_{n,1} \|^3) = o(N^{1/2}).
\]

**Proof.** —

\[
E(\| Y_{n,1} \|^3) = E \left( \sum_{j=1}^{n} (\alpha_j(X_1) - a_j)^2 \right)^{3/2} \leq 2^{5/2} E \left( \sum_{j=1}^{n} \alpha_j^2(X_1) \right)^{3/2}
\]

now

\[
\sum_{j=1}^{n} \alpha_j^2(X_1) \leq \sum_{j=1}^{n} v_j^{2/3} \alpha_j^{4/3}(X_1)
\]

on applying Hölder's inequality it follows that

$$\mathbb{E}\left(\sum_{j=1}^{n} \alpha_j^2(X_1)\right)^{3/2} \leq \left(\sum_{j=1}^{n} \psi_j^2\right)^{1/2} \mathbb{E}\left(\sum_{j=1}^{n} \alpha_j^2(X_1)\right)$$

but

$$\mathbb{E}\left(\sum_{j=1}^{n} \alpha_j^2(X_1)\right) = o(n)$$

then

$$\mathbb{E}(\|Y_{n,1}\|^3) \leq K\left(n^2 \sum_{j=1}^{n} \psi_j^2\right)^{1/2} = o(N^{1/2})$$

according to the hypothesis.

4. EXAMPLES AND APPLICATIONS

a) Let us consider the space $L^2((-\pi, \pi)^2)$ with respect to Lebesgue measure, and the complete orthonormal set $\left\{ \frac{1}{2\pi} e^{i(mx+ny)} \right\}_{(m,n)\in\mathbb{Z}^2}$. With the obvious modifications in the previous sections to be able to include complex valued functions $f$, we may study the estimators

$$\tilde{f}_{n,N}(x, y) = \sum_{j,k = -n}^{n} \tilde{C}_{k,j} e^{i(kx+jy)}$$

(Here, $C_{k,j}$ are the Fourier coefficients of $f$, and $\tilde{C}_{k,j}$ their estimators).

We easily verify

$$\Delta_n = \frac{2}{4\pi^2}$$

and

$$\sigma_n^2 = \frac{1}{2\pi^2(2n+1)^2} \sum_{-2n \leq j,k \leq 2n} (2n+1-|k|)(2n+1-|j|)|C_{k,j}|^2$$

and by Beppo-Levi's Theorem:

$$\sigma_n^2 \rightarrow \frac{1}{2\pi^2} \|f\|^2$$
The bound \( \sup \left| D_n(x) \right| = 0(\log n) \), for the Dirichlet-Kernel ([10], p. 151) gives:

\[
\frac{S_m(n)}{n} \leq (\text{const}) \left( \frac{\log n}{n} \right)^{2m} = O(1) \quad (m \geq 3)
\]

Moreover, \( v_{k,j} = 1 \) and if we put \( v = (2n + 1)^2 \)

\[
\sum_{-n \leq k, j \leq n} v_{k,j}^2 = o\left(v^{-2}N\right)
\]

is verified if \( n = N^\alpha \) with \( \alpha < 1/6 \).

Finally if \( f \) is periodic continuously differentiable in \( C((-\pi, \pi)^2 \) until the second order and it has three derivatives in \( L^2 \) with respect to each variable i.e. \( \|f_{xxx}\|_2 < \infty \) and \( \|f_{yyy}\|_2 < \infty \), then is easy to verify that

\[g(n) = O(n^{-6}) = (n^{1/2}N^{-1})\]

if \( n = N^\alpha \) with \( \alpha > \frac{1}{7} \).

Then, under the above conditions for \( f \), we get

\[
W = \lim_{N \to \infty} \left[ \frac{N}{2n + 1} \| \hat{f}_{n,N} - f \|^2 - \frac{2n + 1}{4\pi^2} \right] = N(0, \sigma^2)
\]

if \( n = N^\alpha \), \( \frac{1}{7} < \alpha < 1/6 \), and \( \sigma^2 = \frac{1}{2\pi^2} \| f \|^2 \).

**Note.** — This result was obtained by Naradaya [8] in the univariate case, although our method improves the choice of the exponent \( \alpha \). With minor changes the same proof applies to densities on \( \mathbb{R}^p \).

b) As a second example we consider an asymptotic test of goodness of fit for uniform distribution on a sphere.

The basis for \( L^2(S^2) \) with the measure invariant by rotations is denoted by \( \{Y^n_m(\theta, \phi)\}_{m=-n}^n \) \( n = 0, 1, \ldots \) \( 0 \leq \theta < 2\pi, -\frac{\pi}{2} \leq \phi < \pi/2 \) and constructed from the spherical harmonics ([3], p. 511).

We put, with the obvious notations

\[
\hat{f}_{n,N}(\theta, \phi) = \sum_{k=0}^n \sum_{m=-k}^k \hat{C}_{m,k} Y^n_m(\theta, \phi)
\]

\[
\hat{C}_{m,k} = \frac{1}{N} \sum_{i=1}^N Y^n_m(\theta_i, \phi_i)
\]

where \( (\theta_1, \phi_1), \ldots, (\theta_N, \phi_N) \) is the observed sample.
The statistic $T_{n,N} = \| \hat{f}_{n,N} - I \|_{L^2(S^1)}^2$ can be used to test uniformity.

One easily verifies that $A_n = I_d$, so that $\Delta_n = 1$, $\sigma_n^2 = 2$, $\frac{S_m(n)}{n} = 1$. Moreover

$$\sum_{j=1}^{n} v_j^2 = 0(n^3) = o(n^{-2}N) \quad \text{(with } v = n^2 + 1) \text{ for } n = N^2 \text{ and } \alpha < 1/7.$$

Hence, Theorem 3.2 gives

$$W = \lim_{N \to \infty} \left[ \frac{N}{\sqrt{n^2 + 1}} T_{n,N} - (n^2 + 1)^{1/2} \right] = N(0, 2)$$

c) As a final example, let $f \in L^2(-1, 1)$, $r(x) = 1$ and $\phi_j = (2j + 1)^{1/2}p_j, p_j$ the sequence of Legendre polynomials

$$v_j \leq \sqrt{2j + 1}$$

So that

$$\sum_{j=1}^{n} v_j^2 = 0(n^{-2}) = o(n^{-2}N) \quad \text{for } n = N^2 \text{ and } \alpha < 1/4.$$

The remaining conditions can be verified using the following Theorem ([5], p. 116).

**THEOREM 4.1.** — With the above notations and $f \in C [-1, 1]$, consider Toeplitz matrices.

$$A_n(f) = \left( \int_{-1}^{1} \phi_i(x)\phi_j(x)f(x)dx, \quad i, j = 1, \ldots, n \right)$$

If $\lambda_i^{(n)} (i = 1, \ldots, n)$ are the eigenvalues of $A_n(f)$, then for each $m \geq 1$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (\lambda_i^{(n)})^m = \frac{1}{\pi} \int_{-1}^{1} f^m(x) \frac{1}{\sqrt{1 - x^2}} dx$$

holds true.

In our case, we get

$$\lim_{n \to \infty} \sigma_n^2 = \frac{2}{\pi} \int_{-1}^{1} f^2(x) \frac{1}{\sqrt{1 - x^2}} dx$$

and

$$S_m(n) = \sum_{i=1}^{m} (\lambda_i^{(n)})^m = 0(n).$$
If assume that \( f^4 \in C[-1, 1] \), then \( g(n) = 0(n^{-5}) \) (see [6], p. 209) and \( g(n) = o(n^{1/2}N^{-1}) \) if \( n = N^x, \alpha > 2/11 \).

Summing up, if \( f^4 \in C[-1, 1], n = N^x, \frac{2}{11} < \alpha < 1/4 \), then

\[
W - \lim_{N \to \infty} \left( \frac{N}{n^{1/2}} \left\| f_{n,N} - f \right\|^2 - n^{1/2}\Delta_n \right) = N(0, \sigma^2)
\]

with

\[
\sigma^2 = \frac{2}{\pi} \int_{-1}^{1} \frac{f^2(x)}{\sqrt{1 - x^2}} \, dx.
\]

5. ASYMPTOTIC BEHAVIOUR
UNDER CONTIGUOUS ALTERNATIVES

Suppose that one wants to test the null hypothesis

\[ H_0 : f = f^0 \]

against the sequence of alternatives

\[ H_N : f^N(x) = f^0(x) + \delta_N \Phi(x) \]

where \( \Phi \) is a fixed function in \( L^2(\mu) \) and \( \delta_N \to 0 \).

The following Theorem states that \( T_{n,N} = - \frac{1}{n} \left\| f_{n,M} - f_n \right\|^2 \) is asymptotically gaussian under \( H_N \). The proof follows the lines of [1], th. 4.2 and [8], th. 4.2, and the result can be applied to the previous examples.

We must define before

\[
\tilde{\sigma}^2_n = \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \int_{\mathbb{R}^p} \phi_i(x) \phi_j(x) r^2(x) f^N(x) \, dx \right)^2
\]

**Theorem 5.1.** Under \( H_N \), if \( \Delta_n = \Delta + o\left( \frac{1}{\sqrt{n}} \right) \), \( \tilde{\sigma}^2_n \to \sigma_0^2 > 0 \), suppose also that the hypothesis of the Theorem 3.2 are satisfied for \( n = N^x, 0 < \alpha < \alpha_0 \) and \( \delta_N = (N^{(2-\alpha)}) \) then

\[
W - \lim_{N \to \infty} \sqrt{n} \left( \frac{T_{n,N} - \Delta}{\sigma_0} \right) = N\left( \frac{1}{\sigma_0} \left\| \Phi \right\|^2, 1 \right)
\]

**Proof.** Denote

\[ E_{H_n}(f_{n,N}) = f_n^N. \]
Where $E_{H_n}$ denotes the expectation when the true underlying distribution has density $f^N$.

Let $\{\phi_i\}$ a complete orthonormal basis for $L^2(\mu)$ and $\gamma_i = (\Phi, \phi_i)$ the Fourier coefficients of the function $\Phi$. Define

$$\tilde{T}_{n,N} = \frac{N}{n} || \hat{f}_{n,N} - f_n^N ||^2.$$ 

We have

$$|| \hat{f}_{n,N} - f_n^0 ||^2 = || \hat{f}_{n,N} - f_n^N ||^2 + 2 \langle \hat{f}_{n,N} - f_n^N, f_n^N - f_n^0 \rangle + || f_n^N - f_n^0 ||^2$$

So that:

$$\sqrt{n} \left( \frac{T_{n,N} - \Delta_0}{\sigma_0} \right)$$

$$= \sqrt{n} \left( \frac{\tilde{T}_{n,N} - \Delta_0}{\sigma_0} \right) + \frac{2}{\sqrt{n} \sigma_0} \langle \hat{f}_{n,N} - f_n^N, f_n^N - f_n^0 \rangle + \frac{N}{\sigma_0 \sqrt{n}} || f_n^N - f_n^0 ||^2$$

but

$$\frac{N}{\sigma_0 \sqrt{n}} || f_n^N - f_n^0 ||^2 = \frac{\delta^2_{N,1}}{\sigma_0 \sqrt{n}} || \Phi_n ||^2 \xrightarrow{n \to \infty} \frac{|| \Phi ||^2}{\sigma_0}$$

moreover

$$E \left[ \frac{2}{\sqrt{n} \sigma_0} \langle \hat{f}_{n,N} - f_n^N, f_n^N - f_n^0 \rangle \right] = 0,$$

and

$$E \left[ \frac{2}{\sqrt{n} \sigma_0} \langle \hat{f}_{n,N} - f_n^N, f_n^N - f_n^0 \rangle \right]^2 = \frac{4N \delta^2_{N,1}}{n \sigma_0^2} E(\sqrt{N} (\hat{f}_{n,N} - f_n^N), \Phi_n)^2$$

$$= \frac{4N \delta^2_{N,1}}{n \sigma_0^2} \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \int \phi_i \phi_j f^N dx - a_i a_j \right) \phi_i \phi_j \right)$$

$$\leq k \frac{1}{\sqrt{n}} || r f_n ||_\infty || \Phi_n ||^2 \leq k \frac{1}{\sqrt{n}} || r f ||_\infty || \Phi ||^2$$

then we can conclude

$$\frac{2}{\sqrt{n} \sigma_0} \langle \hat{f}_{n,N} - f_n^N, f_n^N - f_n^0 \rangle \xrightarrow{N \to \infty} 0$$

and finally applying Theorem 3.2

$$W \xrightarrow{N \to \infty} \left( \frac{\tilde{T}_{n,N} - \Delta}{\sigma_0} \right) = N(0, 1).$$
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REFERENCES


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