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by

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SUMMARY. — Let $E$ be an infinite dimensional Banach space with norm $\| \cdot \|$. Then for each $\varepsilon > 0$, there exists a norm $N$ which is $(1 + \varepsilon)$-equivalent to $\| \cdot \|$, and a centered Gaussian measure $\mu$ on $E$ such that the distribution of $N(\cdot)$ for $\mu$ has an unbounded density with respect to Lebesgue measure.

RÉSUMÉ. — Soit $E$ un espace de Banach de dimension infinie avec la norme $\| \cdot \|$. Alors, pour chaque $\varepsilon > 0$, il y a une norme $N$ qui est $(1 + \varepsilon)$-équivalente à $\| \cdot \|$, et une mesure gaussienne centrée $\mu$ sur $E$ telle que la distribution de $N(\cdot)$ pour $\mu$ ait une densité non bornée par rapport à la mesure de Lebesgue.

1. INTRODUCTION

Consider an infinite dimensional Banach space, and $\mu$ a centered Gaussian measure on $E$, that is a Radon measure on $E$ such that for each $x^* \in E^*$ the law of $x^*$ is Normal centered. For $t \in \mathbb{R}^+$, let $B_t = \{ x \in E; \| x \| \leq t \}$, and $\phi(t) = \mu(B_t)$. The function $\phi(t)$ has remarkable properties. Let $\Phi(u)$ given by $\Phi(u) = (1/\sqrt{2\pi}) \int_{-\infty}^{u} \exp \left( -x^2/2 \right) dx$. A remarkable recent result
of A. Ehrhard [1] asserts that $\Psi = \Phi^{-1} \circ \phi$ is concave. It follows that for each $t_0 > 0$, there is a constant $C_0$ such that $|\Psi(t) - \Psi(u)| \leq C_0(t - u)$ for $t, u \geq t_0$. It follows that $|\phi(t) - \phi(u)| \leq C_0(t - u)$ for $t, u \geq t_0$ since $\Phi$ is lipschutz of constant $1/\sqrt{2\pi}$. This shows that the distribution of $|\cdot|$ has a bounded density with respect to Lebesgue measure on each interval $[t_0, \infty[$. (M. Talagrand recently showed that this density is continuous [8].) Let us consider the problem whether this density is bounded on $[0, \infty[$, that is whether there is $C$ such that for $0 \leq u \leq t$, we have

\begin{equation}
\mu(B_t \setminus B_u) \leq C(t - u)
\end{equation}

It has been shown by J. Kuelbs and T. Kurtz [3] that condition (*) holds for each gaussian $\mu$ when $E = l_2(N)$ provided with the usual norm. These results have been considerably generalized by the author and M. Talagrand, who showed that it is enough to assume the norm of $E$ is uniformly convex and that the modulus of uniform convexity is of power type (that is $\geq \varepsilon^p$ for some $p$ and $\varepsilon > 0$).

In the opposite direction, it has been shown independently by V. Pawlaskas and by the author and M. Talagrand that condition (*) fails in general [5]. A further example by the author and M. Talagrand exhibits a $\mathcal{C}^\infty$ renorming of $l_2(N)$, such that all the differentials of the norm remain bounded on the unit sphere, and still condition (*) fails for this renorming [6].

Closely connected to condition (*) is the problem of the rate of convergence in the central limit theorem (C. L. T.). If $X$ is an $E$-valued r. v. with zero expectation and moments of order 2, we say that $X$ is pregaussian if there exists a gaussian measure $\mu$ on $E$ with the same covariance as $S$, that is

\[ E(x(X) y(X)) = \int x(t) y(t) d\mu(t) \quad \text{for} \quad x^*, y^* \in E^* \]

If $(X_i)_{i \leq n}$ are i. i. d. copies of $X$, the rate of convergence in the C. L. T. is often estimated by

\[ \Delta_n = \sup_t \left| P \left\{ \left\| n^{-1/2} \sum_{i \leq n} X_i \right\| \leq t \right\} - \mu(B_t) \right| . \]

J. Kuelbs and T. Kutz showed that if condition (*) holds and the norm $\| \cdot \|$ is three times differentiable with these differentials bounded on the unit sphere, then $\Delta_n = O(n^{-1/6})$ if $X$ has a third moment. F. Gotze [7] reduced this bound to the best estimate $O(n^{-1/2})$ under slightly stronger conditions.

For $\alpha > 1$, a linear isomorphism $T$ from $E$ to $F$ is called an $\alpha$-isomorphism if for $x \in E$ we have $\| x \|/\alpha \leq \| T(x) \| \leq \alpha \| x \|$. We say that $E$ and $F$...
are $\alpha$-isomorphic if there exists an $\alpha$-isomorphism between $E$ and $F$. We say that two norms $\| \cdot \|$ on $E$ are $\alpha$-equivalent if the identity is an $\alpha$-isomorphism from $(E, \| \cdot \|)$ to $(E, N(\cdot))$.

**Theorem.** — Let $(E, \| \cdot \|)$ be an infinite dimensional Banach space. Let $\varepsilon > 0$ and $(\xi_n)$ be a sequence converging to zero. Then there exists a norm $N(\cdot)$ on $E$ and an $E$ valued r. v. $X$ such that

a) $N(\cdot)$ is $(1 + \varepsilon)$-equivalent to $\| \cdot \|$, 
b) $X$ is bounded and pregaussian,

c) if $\mu$ is the gaussian measure on $E$ with the same covariance as $X$, $\mu$ fails condition (*) for the norm $N(\cdot)$,

d) the inequality

$$\Delta_n = \sup_t \left| \Pr \left\{ N \left( n^{-1/2} \sum_{i \leq n} X_i \right) \leq t \right\} - \mu(\{x : N(x) \leq t\}) \right| \geq \xi_n$$

holds for infinitely many $n$.

2. SOME TOOLS

Let $l^n_2$ be the $n$ dimensional Hilbert space, and $(e_i)_{i \leq n}$ be the canonical basis. Let $\gamma_n$ be the gaussian measure on $l^n_2$ such that the dual functionals $e_i^*$ are independent and standard normally distributed. The following observations are crucial.

**Observation 1.** — Since the variable $(e_i^*)^2$ are equidistributed independent of expectation $1$ and variance $3$, the one-dimensional C. L. T. asserts that the distribution of $\| x \|^2 = \sum_{i \leq n} (x_i(x))^2$ is close to $N(n, \sqrt{3n})$. In particular

$$\gamma_n \{ x ; n^{1/2} - 10 < \| x \| < n^{1/2} \} > 1/3 \text{ for } n \text{ large and } \gamma_n \{ x ; \| x \| < 2n^{1/2} \} \to 1.$$ 

Notice also that

$$\int \| x \|^2 d\gamma_n(x) = n.$$

**Observation 2.** — Let $Y_n$ be a r. v. valued in $l^n_2$ such that for $i \in \{1, 2, 3, \ldots, n\}$ and $j \in \{-1, 1\}$, it takes the value $jn^{1/2}e_i$ with probability $1/2n$. Let $(Y_i')$ be i. i. d. like $Y_n$. If $q$ is much smaller than $n$, with probability close to $1$, the r. v. $S_{n,q} = q^{-1/2} \sum_{1 \leq i \leq q} Y_i'$ takes values of the type $\sum_{i \leq l} a_i e_i$, where

card $I = q$ and $|a_i| = n^{1/2}q^{-1/2}$, so $\|S_{n,q}\| = n^{1/2}$ in this case. So for $q$ fixed,
\[
\lim_{n \to \infty} P \{ \|S_{n,q}\| = n^{1/2} \} = 1.
\]
We shall also make essential use of the following Banach space result.

**Theorem 1.** — Let $E$ be an infinite dimensional Banach space, and $F$ be a finite dimensional subspace of $E$, $\tau > 1$ and $n \in \mathbb{N}$. Then there is an $n$-dimensional subspace $G$ of $E$ of dimension $n$, that is $\tau$-isomorphic to $\ell_2^n$ and such that for $x \in G$, $y \in F$ we have $\|x\| \leq \tau \|x + y\|$.

We shall need the following version of Dvoretzki’s theorem: Given $\alpha > 1$, and $p \in \mathbb{N}$, there is a number $q(p, \alpha)$ such that any finite dimensional Banach space $H$ of dimension $\geq q(p, \alpha)$ contains a subspace $\alpha$-isomorphic to $\ell_p^m$.

Let $H$ be a complement of $F$. Let $\alpha = \tau^{1/4}$. We can assume $n \geq 1 + \dim F$.

Let $G_1$ be a subspace of $H$ that is $\alpha$-isomorphic to $\ell_2^n$ with $q = q(2n, \alpha)$.

On $G_1$ consider the norm $\|\cdot\|_1$ given by $\|x\|_1 = \inf \{\|x + y\|; y \in F\}$.

Dvoretzki’s theorem gives a subspace $G_2$ of $G_1$ such that $(G_2, \|\cdot\|_1)$ is $\alpha$-isomorphic to $\ell_2^n$.

Let $T_1$ (resp. $T_2$) be an $\alpha$-isomorphism from $(G_2, \|\cdot\|_1)$ (resp. $(G_2, \|\cdot\|_1)$ to $\ell_2^n$ and let $T = T_2 \circ T_1^{-1}$). The quadratic form $Q(x) = \|T(x)\|^2$ on $\ell_2^n$ can be diagonalized in an orthonormal basis $f_1, f_2, \ldots, f_{2n}$. We can assume the eigenvalues are such that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n}$. For $i \leq n$, there exist $u_i, v_i$ with $u_i^2 + v_i^2 = 1$, $u_i \lambda_i + v_i \lambda_{2n-i-1} = \lambda_n$.

Let $G'$ be the space generated by the vectors $u_i e_i + v_i e_{2n-i-1}$. For $x \in G'$, we have $\|T(x)\|^2 = \lambda_n \|x\|^2$.

Let $G = T_1^{-1}(G')$. For $x \in G$, we have
\[
\|x\| \leq \alpha \|T_1(x)\| = \lambda_n^{1/2} \alpha \|T_2(x)\| \leq \lambda_n^{1/2} \alpha^2 \|x\|_1
\]
and similarly $\lambda_n^{1/2} \|x\|_1 \leq \alpha^2 \|x\|$. Since $\dim G > \dim F$, it follows from [4], lemma 2.8 C that there is $x_0 \in G$ with $\|x_0\| = 1$ and $\|x_0\|_1 = 1$. This shows that $\lambda_n^{1/2} \leq \alpha^2$. Hence $\|x\| \leq \tau \|x\|_1$ for $x \in G$.

3. CONSTRUCTION

Let $\beta_p$ be a sequence with $\beta_p > 1$, $\prod_{1 \leq i \leq \infty} \beta_i < 1 + \varepsilon$. By induction over $p$, we construct sets $B_p$ of $E$, integers $q(p)$, real numbers $a_p$, $\delta_p$ and $r$ for $Z_p$ such that the following conditions are satisfied.

(1) $B_p$ is convex balanced; $B_1$ is the unit ball of $E$; for $p \geq 2$,

$B_{p-1} \subset B_p \subset \beta_p B_{p-1}$.
(2) \( Z_p \) is valued in a finite dimensional space; for each \( \omega \), \( \| Z_p(\omega) \| \leq 2^{-p} \) and the sequence \( (Z_p) \) is independent.

(3) If \( \eta_p \) is the gaussian measure with the same covariance as \( Z_p \), then
\[
\int \| x \|^2 d\eta_p(x) \leq 2^{-p}.
\]

(4) If \( \nu_p \) is the gaussian measure with the same covariance as \( X_p = \sum_{i \leq p} Z_i \), and \( N_p \) is the gauge of \( B_p \), we have for \( r \leq p \),
\[
\nu_p \{ x; a_r - \delta_r < N_p(x) < a_r \} > 2 \zeta_{q(r)}.
\]

(5) If \( (X^i_p)_i \) are i. i. d. copies of \( X_p \), for \( r \leq p \), we have
\[
P \{ N_p \left( q(r)^{-1/2} \sum_{i \leq q(r)} X^i_p \right) \leq a_r \} < \zeta_{q(r)}.
\]

(6) \( \delta_p = \zeta_{q(p)}/p \).

We proceed to the first step of the construction. We choose \( q(1) \) such that \( \zeta_{q(1)} < 1/6 \), and \( \delta_1 = \zeta_{q(1)} \). It follows from observations 1 and 2 that there exists \( n \) such that \( 10n^{-1/2} < \delta_1 \) and that

(7) \( \gamma_n \{ n^{1/2} - 10 < \| x \| < n^{1/2} \} > 1/3 > 2 \zeta_{q(1)} \)

(8) \( P \left\{ \left\| q(1)^{-1/2} \sum_{i \leq q(1)} Y^i_n \right\| < n^{1/2} \right\} < \zeta_{q(1)} \).

There exists \( d \) with \( n^{1/2} > d > n^{1/2}/2 \) such that

(9) \( \gamma_n \{ n^{1/2} - 10 < \| x \| < d \} > 2 \zeta_{q(1)} \)

and automatically we have

(10) \( P \left\{ \left\| q(1)^{-1/2} \sum_{i \leq q(1)} Y^i_n \right\| \leq d \right\} < \zeta_{q(1)} \).

There is \( 1 < \alpha < 2 \) such that

(11) \( \gamma_n \{ (n^{1/2} - 10)\alpha < \| x \| < d/\alpha \} > 2 \zeta_{q(1)} \).

(12) \( P \left\{ \left\| q(1)^{-1/2} \sum_{i \leq q(1)} Y^i_n \right\| < \alpha d \right\} < \zeta_{q(1)} \).

From Dvoretzki’s theorem, there is a subspace \( G \) of \( E \) and an \( \alpha \)-isomorphism \( T \) from \( l_2^n \) to \( G \). Let \( b = 1/(8d) \) and \( Z_1 = bT(Y_n) \).
(2) follows from \( \| Z_1(\omega) \| \leq 2bn^{1/2} \leq 1/2 \).

We check (3). Since \( \eta_1 = b T(\gamma_n) \), we have
\[
\int \| x \|^2 d\eta_1(x) = b^2 \int \| T(x) \|^2 d\gamma_n(x) \leq 4b^2n \leq 2^{-1},
\]
so (3) holds. Let \( a_1 = 1/8 \). We check (4). Since \( \delta_1/b \geq 10 \), we have
\[
\eta_1 \{ x ; a_1 - \delta_1 < \| x \| < a_1 \} = \gamma_n \{ y ; a_1 - \delta_1 < b \| T(y) \| < a_1 \} \\
\geq \gamma_n \{ y ; n^{1/2} - 10 < \| T(y) \| < d \} \\
\geq \gamma_n \{ y ; \alpha(n^{1/2} - 10) \leq \| y \| \leq d/\alpha \} \\
> 2^{\xi_{q(1)}}
\]
and hence (4) holds. To check (5), we note that
\[
\left\| q(1)^{-1/2} \sum_{i \leq q(r)} X_i \right\| \leq a_1 \Rightarrow \left\| T \left( q(1)^{-1/2} \sum_{i \leq q(r)} Y_i \right) \right\| \leq d
\]
so (5) follows from (12). Finally (6) holds by construction. The first step is completed.

Let us now assume that the first \( p \) steps have been completed. There exist two numbers \( 1 < \alpha < \beta_p \) and \( b > 0 \) such that for \( r \leq p \) we have
\[
(1 - b)v_p \{ x ; \alpha(a_r - \delta_r + 16b) < N_p(x) < a_r - 16b \} > 2^{\xi_{q(r)}}.
\]
\[
P \left\{ N_p \left( q(r)^{-1/2} \sum_{i \leq q(r)} X_p^i \right) \leq \alpha a_r + 8b \right\} < \xi_{q(r)}.
\]
We can assume \( b \leq 2^{-p-4} \). Let \( c = b/q(p) \). Let \( q(p + 1) \) be large enough that
\[
6^{\xi_{q(p+1)}} < v_p \{ x ; \| x \| < c(\alpha - 1)/2 \} \quad \text{and} \quad 12^{\xi_{q(p+1)}} < c.
\]
Let \( \delta_{p+1} = \xi_{q(p+1)}/(p + 1) \). From observations 1 and 2, there exists \( n \) with \( \delta_{p+1} \geq 20n^{-1/2} \) and
\[
\gamma_n \{ x ; n^{1/2} - 10 < \| x \| < n^{1/2} \} > 1/3,
\]
\[
\gamma_n \{ x ; \| x \| \geq 2n^{1/2} \} \leq b,
\]
\[
\int \| x \|^2 d\gamma_n(x) = n,
\]
\[
P \left\{ \left\| q(p + 1)^{-1/2} \sum_{i \leq q(p+1)} Y_i^p \right\| < n^{1/2} \right\} < \xi_{q(p+1)}.
\]
Let $d$ with $n^{1/2}/2 < d < n^{1/2}$ and

\begin{equation}
\gamma_n \{ x; n^{1/2} - 10 < \| x \| < d \} > 1/3 .
\end{equation}

Let $\tau$ with $\tau^3 = (\alpha + 1)/2$.

Let $F$ be a finite dimensional space of $E$ in which $X_p$ is valued. We use Theorem 1 for $(E, N_p)$. So there is a finite dimensional space $G$ of $E$ and $\tau$-isomorphism $T$ from $l_2^n$ to $G$ such that for $x \in G$, $y \in F$ we have

\[ N_p(x) \leq \tau N_p(x + y) . \]

We define $B_{p+1}$ as the closed convex hull of the set

\[ B_p \cup \{ x + y; x \in G, y \in F, \| T^{-1}(x) \| = \tau^2, \| y \| \leq (\alpha - 1)/2 \} . \]

For $x \in G$, $y \in F$, $\| T^{-1}(x) \| \leq \tau^2$, $\| y \| \leq (\alpha - 1)/2$, we have $N_p(x) \leq \tau^3$, so $N_p(x + y) \leq N_p(x) + N_p(y) \leq N_p(x) + \| y \| \leq \alpha$.

In particular $B_p \subset B_{p+1} \subset \alpha B_p$, so (1) holds since $\alpha < \beta_p$.

Moreover for $x \in E$ we have $N_p(x)/\alpha \leq N_{p+1}(x) \leq N_p(x)$. We now propose the following fact.

**Fact.** — For $x \in G$, $\| T^{-1}(x) \| = \tau^2$, $y \in F$, $\| y \| \leq (\alpha - 1)/2$, we have $N_{p+1}(x + y) = 1$.

We already know that $N_{p+1}(x + y) \leq 1$. There is a linear functional $\phi_1$ on $G$ such that $\phi_1(x) = 1$ while $\phi_1(x') \leq 1$ when $\| T^{-1}(x') \| \leq \tau^2$, so there is a linear functional $\phi_2$ on $F + G$ such that $\phi_2(x) = 1$, $\phi_2(x') \leq 1$ whenever $\| T^{-1}(x') \| \leq 1$, $x' \in G$ and $\phi_2 = 0$ on $F$. In particular $\phi_2(x + y) = 1$. If $x' + y' \in B_p$, then $\| x' \| \leq \tau$, so $\| T^{-1}(x') \| \leq \tau^2$, so $\phi_2(x' + y') \leq 1$. This shows that $N_p(\phi_2) \leq 1$. So $\phi_2$ can be extended on $E$ by a $\phi$ with $N_p(\phi) \leq 1$. Since $\phi(x' + y') \leq 1$ for $x' \in G$, $\| T^{-1}(x') \| \leq \tau^2$ and $y' \in F$ and since $\phi \leq 1$ on $B_p$, the definition of $B_{p+1}$ shows that $N_{p+1}(\phi) \leq 1$. As $\phi(x + y) = 1$, the fact is proved.

**Remark.** — This fact motivated the choice of $B_{p+1}$.

We set $Z_{p+1}(\omega) = (c/d)T(Y(\omega))$. Now (2) follows from

\[ \| Z_{p+1}(\omega) \| \leq (1 + \epsilon)N_p(Z_{p+1}(\omega)) \leq 2(c/d)\| T \| \cdot \| Y(\omega) \| \leq 4cn^{1/2}/d \leq 8b/q(p) \leq 2^{-p-1} . \]

Also, (3) follows from

\[ \int \| x \|^2 d\eta_{p+1}(x) \leq 4(c/d)^2 \int \| y \|^2 d\gamma_n(y) \leq 2^{-p-1} \quad \text{with} \quad \eta_{p+1} = 2(c/d)T^{-1}(\gamma_n) . \]

We now check (4). We first show that for \( r \leq p \), we have

\[
v_{p+1} \{ z \mid a_r - \delta_r < N_{p+1}(z) < a_r \} > 2\xi_q(r).
\]

We notice that \( v_{p+1} \) is a measure on \( F + G \), that identifies to \( v_p \otimes \eta_{p+1} \). Let

\[
A = \{ x \in G ; ||x|| \leq 16c \}.
\]

For \( z \in F \), \( ||z|| \leq 2n^{1/2} \), we have \( ||(c/d)T(z)|| \leq 16c \).

It follows from (17) and the fact that \( \eta_{p+1} = (c/d)T(\gamma) \) that \( \eta_{p+1}(A) \geq 1 - b \). Let

\[
B = \{ y \in F ; \alpha(a_r - \delta_r + 16b) < N_p(y) < a_r - 16b \}.
\]

For \( x \in A \), since \( N_{p+1}(x) \leq ||x|| \), we have \( N_{p+1}(x) \leq 16c \leq 16b \). For \( y \in B \), since \( N_{p+1}(y) \leq N_p(y) \leq \alpha N_{p+1}(y) \), we have

\[
a_r - \delta_r + 16b < N_{p+1}(y) < a_r - 16b.
\]

So, for \( x \in A \), \( y \in B \), we have \( a_r - \delta_r < N_{p+1}(x + y) < a_r \).

It follows from (13) that

\[
v_{p+1} \{ z ; a_r - \delta_r < N_{p+1}(z) < a_r \} \geq v_p(B)\eta_{p+1}(A) > 2\xi_q(p+1).
\]

Let \( a_{p+1} = c/\tau^2 \). To finish the proof that (4) holds at rank \( p + 1 \), it remains to show if

\[
H = \{ z ; a_{p+1} - \delta_{p+1} < N_{p+1}(z) < a_{p+1} \}
\]

then \( v_{p+1}(H) > 2\xi_q(p+1) \). Let

\[
C = \{ x \in G ; a_{p+1} - \delta_{p+1} < N_{p+1}(z) < a_{p+1} \}.
\]

For \( x \in G \), \( N_{p+1}(x) = ||T^{-1}(x)||/\tau^2 \), so

\[
C \ni \{ x \in G ; c - \delta_{p+1} < ||T^{-1}(x)|| < c \}
\]

\[
\ni \{ x \in G ; d - 10 < ||(d/c)T^{-1}(x)|| < d \}
\]

since \( (d/c)\delta_{p+1} \geq d\delta_{p+1} \geq 10 \). In particular, \( \eta_{p+1}(C) \geq 1/3 \) from (20).

Let \( D = \{ y \in F ; ||y|| \geq (\alpha - 1)/3 \} \). We have \( \delta_{p+1} \leq c/6 \), and we can assume \( \tau^2 < 4/3 \). We then have \( (\alpha - 1)/3 \leq a_{p+1} - \delta_{p+1} \).

It follows that for \( x \in C \), \( y \in D \), we have \( N_{p+1}(x + y) = N_{p+1}(x) \), so \( x + y \in H \). Hence \( v_{p+1}(H) > v_p(D)\eta_{p+1}(C) > 2\xi_q(p+1) \) from (15), so (4) holds.

We now check (5). We first show that for \( r \leq p \), we have

\[
P \left\{ N_{p+1} \left( q(r)^{-1/2} \sum_{i \leq q(r)} X_{p+1}^i \right) \leq a_r \right\} < \xi_q(r).
\]

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We have seen that \( \| Z_{p+1}(\omega) \| \leq 8b/q(p) \), so since \( X_{p+1} = X_p^i + Z_{p+1}^i \), we get

\[
N_p \left( q(r)^{-1/2} \sum_{i \leq q(r)} X_p^i \right) \leq N_p \left( q(r)^{-1/2} \sum_{i \leq q(r)} X_p^i \right) + 8b \leq \alpha N_{p+1} \left( q(r)^{-1/2} \sum_{i \leq q(r)} X_p^i \right) + 8b
\]

so the result follows from (14). To check (5), it remains to show that

\[
P \left\{ N_{p+1} \left( q(p+1)^{-1/2} \sum_{i \leq q(p+1)} X_{p+1}^i \right) \leq a_{p+1} \right\} < \xi_{q(p+1)}.
\]

We first note that for \( x \in G \), \( y \in F \), we have \( N_{p+1}(x + y) \geq N_{p+1}(x) \) since \( N_{p+1}(x + \lambda y) \) is a convex function of \( \lambda \) that is equal to \( N_{p+1}(x) \) for small \( \lambda \). Hence

\[
N_{p+1} \left( q(p+1)^{-1/2} \sum_{i \leq q(p+1)} X_{p+1}^i \right) \leq N_{p+1} \left( q(p+1)^{-1/2} \sum_{i \leq q(p+1)} Z_{p+1}^i \right)
\]

and the result follows from (19) and the definition of \( Z \). The construction is complete since (6) holds by construction.

### 4. PROOF OF THE THEOREM

It follows from (2) that one can define a bounded r. v. \( X \) by \( X(\omega) = \sum \mathcal{Z}_i(\omega) \).

For each \( l \), let \( U_l \) be a Gaussian r. v. with the same covariance as \( Z_l \), and such that the sequence \( (U_l) \) is independent. It follows from (3) that the series \( (U_l) \) is summable in \( L_2(\mathbb{E}) \). Its sum \( V \) is Gaussian, and has the same covariance as \( X \), so \( X \) is pregaussian. Let \( \mu \) be the distribution of \( V \).

Let \( N(x) = \lim_p N_p(x) \) and \( \theta_p = \prod_{i \leq p} \beta_i \). From condition (1) we get

\( N(x) \leq N_p(x) \leq \theta_p N(x). \) It follows from (4) that for \( q \leq p, r \leq p \) we get

\[
v_q \left\{ x ; a_r - \delta_r < N(x) < \theta_q a_r \right\} > 2 \xi_{q(r)}.
\]

Since \( v_q \) is the distribution of \( V_p = \sum_{l \leq p} U_l \), we get by letting \( p \to \infty \)

\[
\mu \left\{ x ; a_r - \delta_r \leq N(x) \leq \theta_q a_r \right\} \geq 2 \xi_{q(r)}.
\]
and letting $q \to \infty$ gives
\[ \mu \left\{ x; a_r - \delta_r \leq N(x) \leq a_r \right\} \geq 2 \xi_{q(r)}. \]

It follows from (6) that condition (*) fails for $\mu$. It follows from (5) that for $r \leq p$,
\[ P \left\{ \theta_p N \left( q(r)^{-1/2} \sum_{i \leq q(r)} X_i \right) \leq a_r \right\} \leq \xi_{q(r)}. \]

Letting $p \to \infty$ gives
\[ P \left\{ N \left( q(r)^{-1/2} \sum_{i \leq q(r)} X_i \right) \leq a_r \right\} \leq \xi_{q(r)}. \]

In particular,
\[ \left| \mu \left\{ x; N(x) \leq a_r \right\} - P \left\{ N \left( q(r)^{-1/2} \sum_{i \leq q(r)} X_i \right) \leq a_r \right\} \right| \geq \xi_{q(r)} \]

which completes the proof of the theorem.

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