ANNALES DE L'I. H. P., SECTION B

ROBERT J. ELLIOTT
ALLANUS H. TSOI

Time reversal of non-Markov point processes

Annales de l'I. H. P., section B, tome 26, nº 2 (1990), p. 357-373
<http://www.numdam.org/item?id=AIHPB_1990__26_2_357_0>

Ann. Inst. Henri Poincaré,

Vol. 26, n° 2, 1990, p. 357-373. *Probabilités et Statistiques*

Time reversal of non-Markov point processes

by

Robert J. ELLIOTT (1) **and Allanus H. TSOI** (2)

Department of Statistics and Applied Probability,
University of Alberta,
Edmonton, Alberta, Canada T6G 2G1

ABSTRACT. — Time reversal is considered for a standard Poisson process, a point process with Markov intensity and a point process with a predictable intensity. In the latter case an analog of the Fréchet derivative for functionals of a Poisson process is introduced and used in techniques of integration-by-parts to obtain formulate similar to those of Föllmer in the Wiener space situation.

Key words : Point processes, Poisson process, predictable intensity, non-Markov, integration-by-parts, Fréchet derivative.

RÉSUMÉ. — Le retournement du temps est considéré pour un processus de Poisson, un processus ponctuel avec intensité markovienne et un processus ponctuel avec intensité prévisible. Pour le dernier cas, nous introduisons une sorte de dérivée Fréchet pour les fonctionnels d'un processus de Poisson et l'utilisons dans les méthodes d'intégration par parties pour obtenir des formules qui sont similaires à celles de Föllmer pour la situation brownienne.

Classification A.M.S. : 60 G 55, 60 J 75.

(1) Research partially supported by N.S.E.R.C. Grant A-7964, the Air Force Office of Scientific Research, United States Air Force, under contract AFOSR-86-0332, and the U.S. Army Research Office under contract DAAL 03-87-0102.

(2) Research partially supported by N.S.E.R.C. Grant A-7964 and a University of Alberta Dissertation Fellowship.

1. INTRODUCTION

The time reversal of stochastic processes has been investigated for some years. One motivation comes from quantum theory, and this is discussed in the book of Nelson [11]. The time reversal of Markov diffusions is treated in, for example, the papers of Elliott and Anderson [4], and Haussman and Pardoux [8]. However, the first discussion of time reversal for a non-Markov process on Wiener space appears in the paper by Föllmer [7], in which he uses an integration-by-parts formula related to the Malliavin calculus.

In the present paper an analog of the Fréchet derivative is introduced for functionals of a Poisson process. The integration-by-parts formula on Poisson space, *see* [6], is formulated in terms of this derivative and counterparts of Föllmer's formulae are obtained.

In Section 2 the time reversed form of the standard Poisson process is derived. Section 3 considers a point (counting) process N with Markov intensity $h(N_t)$, so that $Q_t = N_t - \int_0^t h(N_s)\,ds$ is a martingale, and obtains the reverse time decomposition of Q for $t \in (0, 1]$. Finally, in Section 4, the situation when h is predictable is considered using the "Fréchet" derivative and integration-by-parts techniques mentioned above.

2. TIME REVERSAL UNDER THE ORIGINAL MEASURE

Consider a standard Poisson process $N = \{N_t : 0 \leqq t \leqq 1\}$ on $(\Omega, \mathscr{F}, P)$. We take $N_0 = 0$. Let $\{\mathscr{F}_t\}$ be the right-continuous, complete filtration generated by N. Let $G_t^0 = \sigma\{N_s : t \leqq s \leqq 1\}$ and $\{G_t\}$ be the left-continuous completion of $\{G_t^0\}$.

The following result is well known; *see*, for example, Theorem 2.6 in [9]. For completeness we sketch the proof.

THEOREM 2.1. — *Under* P, N *is a reverse time* G_t*-quasimartingale, and it has the decomposition:*

$$N_t = N_1 + M_t - \int_t^1 \frac{N_s}{s}\, ds,$$

where M *is a reverse time* G_t*-martingale.*

Proof. — Since N is Markov, we have, for $\varepsilon > 0$,

$$\begin{aligned} E[N_{t-\varepsilon} - N_t \mid G_t] &= E[N_{t-\varepsilon} - N_t \mid N_t] \\ &= -\frac{\varepsilon}{t} N_t \end{aligned} \tag{2.1}$$

(*see* [5] and [10]). Thus

$$\int_0^t E \mid E[N_{s-\varepsilon} - N_s \mid G_s] \mid ds = O(\varepsilon).$$

By Stricker's theorem [12], N_t is a reverse time G_t-quasimartingale. Considering approximate Laplacians we see it has the decomposition

$$N_t = N_1 + M_t + \int_t^1 \alpha_s\, ds \tag{2.2}$$

where from (2.1) and (2.2), for almost all t

$$\begin{aligned} \alpha_t &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{t-\varepsilon}^t E[\alpha_s \mid G_t]\, ds \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E[N_{t-\varepsilon} - N_t \mid G_t] \\ &= -\frac{N_t}{t}. \quad \square \end{aligned}$$

3. TIME REVERSAL AFTER A CHANGE OF MEASURE: THE MARKOV CASE

Consider a process $h_t = h(N_t)$ which satisfies: There exist positive constants A, $K > 0$ such that $0 < A < h(N_t) \leqq K$ for all t, a.s.

Define the family $\{\Lambda_t, 0 \leqq t \leqq 1\}$ of exponentials:

$$\Lambda_t = \prod_{0 \leqq u \leqq t} (1 + (h(N_{u-}) - 1)\,\Delta N_u) \exp\left(\int_0^t (1 - h(N_{u-}))\, du\right).$$

Then Λ is an $(\mathcal{F}_t)$-martingale under P, and is the unique solution of the equation

$$\Lambda_t = 1 + \int_0^t \Lambda_{u-} (h(\mathrm{N}_{u-}) - 1)(d\mathrm{N}_u - du).$$

Define a new probability measure P^h by

$$\frac{d\mathrm{P}^h}{d\mathrm{P}} = \Lambda_1.$$

Then under P^h, the process $\mathrm{H}_t = \mathrm{N}_t - \int_0^t h(\mathrm{N}_{u-})\,du$ is an $(\mathcal{F}_t)$-martingale (*see* [3]). Let $\beta(t) = \int_0^t h(\mathrm{N}_{u-})\,du$ so that β is positive and increasing in t because h is positive. Write

$$\psi(t) = \beta^{-1}(t),$$
$$\mathrm{N}'_t = \mathrm{N}_{\psi(t)},$$
$$\mathcal{F}'_t = \mathcal{F}_{\psi(t)}.$$

LEMMA 3.1. – *(N'_t) is a Poisson process under* $(\Omega, \mathcal{F}, (\mathcal{F}'_t), \mathrm{P}^h)$.

Proof. – Since $\mathrm{H}_t = \mathrm{N}_t - \beta(t)$ is an $(\mathcal{F}_t)$-martingale under P^h, $\mathrm{H}'_t = \mathrm{H}_{\psi(t)} = \mathrm{N}_{\psi(t)} - t$ is an $(\mathcal{F}'_t)$-martingale under P^h. By Itô's rule,

$$\mathrm{H}'^2 = 2\int_0^t \mathrm{H}'_{s-}\, d\mathrm{H}'_s + \sum_{s \leq t} (\Delta \mathrm{N}_{\psi(s)})^2$$
$$= 2\int_0^t \mathrm{H}'_{s-}\, d\mathrm{H}'_s + \mathrm{N}_{\psi(t)}.$$

Hence $\mathrm{H}^2_{\psi(t)} - t$ is also an $(\mathcal{F}'_t)$-martingale under P^h. Therefore, $\{\mathrm{N}'_t\}$ is Poisson by Lévy's characterization (Theorem 12.31 in [2]). □

LEMMA 3.2. – N *is Markov under* P^h.

Proof. – Consider any $\varphi \in \mathrm{C}_0^\infty(\mathbb{R})$. For $t \geq s$, by Bayes' formula,

$$\mathrm{E}^h[\varphi(\mathrm{N}_t) \mid \mathcal{F}_s] = \frac{\mathrm{E}[\Lambda_t \varphi(\mathrm{N}_t) \mid \mathcal{F}_s]}{\mathrm{E}[\Lambda_t \mid \mathcal{F}_s]}$$
$$= \mathrm{E}[\Lambda_s^t \varphi(\mathrm{N}_t) \mid \mathcal{F}_s]$$
$$= \mathrm{E}[\Lambda_s^t \varphi(\mathrm{N}_t) \mid \mathrm{N}_s],$$

because N is Markov under P, where

$$\Lambda_s^t = \prod_{s < u \leq t} (1 + (h(\mathrm{N}_u) - 1)\Delta \mathrm{N}_u) \exp\left(\int_s^t (1 - h(\mathrm{N}_u))\,du\right).$$

Hence

$$E^h[\varphi(N_t) \mid \mathscr{F}_s] = E^h[\varphi(N_t) \mid N_s]$$

and N is Markov under P^h. □

Note that

$$H_t = H_1 + N_t - N_1 + \int_t^1 h(N_s)\,ds. \qquad (3.1)$$

Thus H_t is a reverse time G_t-quasimartingale under P^h if and only if N_t is. To determine the reverse time decomposition we again investigate the approximate Laplacians, as in [4].

THEOREM 3.3.

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^h[N_{t-\varepsilon} - N_t \mid G_t] = -E^h\left[h(N_t - 1) \frac{N_t}{\int_0^t h(N_u)\,du} \,\middle|\, N_t \right]. \qquad (3.2)$$

Proof. — By Lemma 3.2,

$$E^h[N_t - N_{t-\varepsilon} \mid G_t] = E^h[N_t - N_{t-\varepsilon} \mid N_t].$$

Consider a bounded, differentiable function φ on $\mathbb{R}$ and its restriction to $\mathbf{Z}$ (the range of N). Now

$$\varphi(N_t) = \varphi(N_{t-\varepsilon}) + \int_{t-\varepsilon}^t (\varphi(N_{s-}+1) - \varphi(N_{s-}))\,dN_s.$$

So

$$\begin{aligned} \varphi(N_t)(N_t - N_{t-\varepsilon}) &= \int_{t-\varepsilon}^t (N_{s-} - N_{t-\varepsilon})(\varphi(N_{s-}+1) - \varphi(N_{s-}))\,dN_s \\ &\quad + \int_{t-\varepsilon}^t \varphi(N_{s-})\,dN_s + \sum_{t-\varepsilon < s \leq t} \Delta\varphi(N_s)\,\Delta N_s \\ &= \int_{t-\varepsilon}^t (N_{s-} - N_{t-\varepsilon})(\varphi(N_{s-}+1) - \varphi(N_{s-}))\,dN_s \\ &\qquad + \int_{t-\varepsilon}^t \varphi(N_{s-}+1)\,dN_s. \end{aligned}$$

Since

$$\begin{aligned} H_t &= N_t - \int_0^t h(N_s)\,ds \\ &= N_t - \int_0^t h(N_{s-})\,ds \end{aligned}$$

is a martingale under P^h,

$$
\begin{aligned}
& E^h[\varphi(N_t)(N_t - N_{t-\varepsilon})] \\
& \quad = E^h\left[\int_{t-\varepsilon}^{t} (N_{s-} - N_{t-\varepsilon})(\varphi(N_{s-}+1) - \varphi(N_{s-}))\, h(N_{s-})\, ds\right] \\
& \qquad\qquad + E^h\left[\int_{t-\varepsilon}^{t} \varphi(N_{s-}+1)\, h(N_{s-})\, ds\right]. \quad (3.3)
\end{aligned}
$$

Now, if $|\varphi| \leqq C$,

$$
\begin{aligned}
& \left| E^h\left[\int_{t-\varepsilon}^{t} (N_{s-} - N_{t-\varepsilon})(\varphi(N_{s-}+1) - \varphi(N_{s-}))\, h(N_{s-})\, ds\right]\right| \\
& \quad \leqq 2\,KC \int_{t-\varepsilon}^{t} E^h[|N_{s-} - N_{t-\varepsilon}|]\, ds \\
& \quad \leqq 2\,KC \int_{t-\varepsilon}^{t} E^h\left[\left| N_{s-} - N_{t-\varepsilon} - \int_{t-\varepsilon}^{s-} h(N_{u-})\, du\right|\right] \\
& \qquad\qquad + E^h\left[\left|\int_{t-\varepsilon}^{s-} h(N_{u-})\, du\right|\right] ds \\
& \quad \leqq 2\,KC \int_{t-\varepsilon}^{t} \left\{\left[E^h\left| N_{s-} - N_{t-\varepsilon} - \int_{t-\varepsilon}^{s-} h(N_{u-})\, du\right|^2\right]^{1/2} + K\,\varepsilon\right\} ds \\
& \quad \leqq 2\,KC \int_{t-\varepsilon}^{t} \left\{E^h\left[\int_{t-\varepsilon}^{t} h(N_{u-})\, du\right]^{1/2} + K\,\varepsilon\right\} ds \\
& \quad \leqq 2\,KC \int_{t-\varepsilon}^{t} ((K\,\varepsilon)^{1/2} + K\,\varepsilon)\, ds \leqq K'\,\varepsilon^{3/2} + K''\,\varepsilon^2.
\end{aligned}
$$

Thus from (3.3),

$$
\begin{aligned}
\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^h[\varphi(N_t)(N_t - N_{t-\varepsilon})] &= E^h[\varphi(N_{t-}+1)\, h(N_{t-})] \\
&= E^h[\varphi(N_t+1)\, h(N_t)]. \quad (3.4)
\end{aligned}
$$

However,

$$
\begin{aligned}
E^h[\varphi(N_t+1)\, h(N_t)] &= E^h[\varphi(N_{\psi(\beta(t))}+1)\, h(N_{\psi(\beta(t))})] \\
&= E^h[\varphi(N'_{\beta(t)}+1)\, h(N'_{\beta(t)})] \\
&= E^h[E^h[\varphi(N'_{\beta(t)}+1)\, h(N'_{\beta(t)}) \mid \beta(t)]].
\end{aligned}
$$

And

$$
\begin{aligned}
&\mathrm{E}^h[\varphi(\mathrm{N}'_{\beta(t)}+1)\,h(\mathrm{N}'_{\beta(t)})\,|\,\beta(t)]\\
&\qquad=\sum_{k=0}^{\infty}\varphi(k+1)\,h(k)\,\frac{\beta(t)^k e^{-\beta(t)}}{k!}\\
&\qquad=\sum_{l=0}^{\infty}\varphi(l)\,h(l-1)\,\frac{\beta(t)^l e^{-\beta(t)}}{l!}\,\frac{l}{\beta(t)}\\
&\qquad=\mathrm{E}^h\left[\varphi(\mathrm{N}'_{\beta(t)})\,h(\mathrm{N}'_{\beta(t)}-1)\,\frac{\mathrm{N}'_{\beta(t)}}{\beta(t)}\,\middle|\,\beta(t)\right]\\
&\qquad=\mathrm{E}^h\left[\varphi(\mathrm{N}_t)\,h(\mathrm{N}_t-1)\,\frac{\mathrm{N}_t}{\beta(t)}\,\middle|\,\beta(t)\right].
\end{aligned}
$$

Hence,

$$
\mathrm{E}^h[\varphi(\mathrm{N}_t+1)\,h(\mathrm{N}_t)]=\mathrm{E}^h\left[\varphi(\mathrm{N}_t)\,h(\mathrm{N}_t-1)\,\frac{\mathrm{N}_t}{\int_0^t h(\mathrm{N}_u)\,du}\right]. \tag{3.5}
$$

Thus from (3.4) and (3.5),

$$
\lim_{\varepsilon\downarrow 0}\mathrm{E}^h\left[\varphi(\mathrm{N}_t)\,\frac{(\mathrm{N}_t-\mathrm{N}_{t-\varepsilon})}{\varepsilon}\right]=\mathrm{E}^h\left[\varphi(\mathrm{N}_t)\,h(\mathrm{N}_t-1)\,\frac{\mathrm{N}_t}{\int_0^t h(\mathrm{N}_u)\,du}\right],
$$

or

$$
\lim_{\varepsilon\downarrow 0}\mathrm{E}^h\left[\frac{\mathrm{N}_{t-\varepsilon}-\mathrm{N}_t}{\varepsilon}\,\middle|\,\mathrm{G}_t\right]=-\mathrm{E}^h\left[h(\mathrm{N}_t-1)\,\frac{\mathrm{N}_t}{\int_0^t h(\mathrm{N}_u)\,du}\,\middle|\,\mathrm{N}_t\right].\quad\square
$$

By Theorem 3.3 and an argument similar to that in [4], we see that N, and hence H, is a reverse time G_t-quasimartingale under P^h, and it has the decomposition

$$
\mathrm{H}_t=\mathrm{H}_1+\mathrm{M}_t+\int_t^1 \alpha_t\,d_t. \tag{3.6}
$$

Moreover, we have the following expression for α_t:

THEOREM 3.4. – *The integrand α_t that appears in* (3.6) *is given by*

$$
\alpha_t=h(\mathrm{N}_t)-\mathrm{E}^h\left[h(\mathrm{N}_t-1)\,\frac{\mathrm{N}_t}{\int_0^t h(\mathrm{N}_u)\,du}\,\middle|\,\mathrm{N}_t\right].
$$

Proof. – From (3.1) and (3.6),

$$E^h[H_{t-\varepsilon}-H_t \mid G_t]=E^h\left[\int_{t-\varepsilon}^t \alpha_s\,ds \mid G_t\right]$$

$$=E^h[N_{t-\varepsilon}-N_t \mid G_t]+E^h\left[\int_{t-\varepsilon}^t h(N_s)\,ds \mid G_t\right].$$

Thus for almost all t

$$\alpha_t=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}E^h\left[\int_{t-\varepsilon}^t \alpha_s\,ds \mid G_t\right]=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}E^h[N_{t-\varepsilon}-N_t \mid G_t]+h(N_t).$$

From Theorem 3.3, α_t has the stated form. □

4. TIME REVERSAL AFTER A CHANGE OF MEASURE: THE NON-MARKOV CASE

This section involves an integration by parts for Poisson processes which is effected by using a Girsanov transformation to change the intensity and then compensating by a time change. In contrast, the integration by parts considered in [1] is obtained by introducing a perturbation of the size of the jumps. The topic is further investigated in [6].

Suppose $\{N_t : 0\leqq t\leqq 1\}$ is a Poisson process with jump times $T_1\wedge 1,\ldots,T_n\wedge 1,\ldots$ Let $\{u_t\}$ be a real predictable process satisfying $\{u_t\}$ is positive and bounded a. s.

For $\varepsilon>0$, consider the family of exponentials:

$$\Lambda_t^\varepsilon=\prod_{0\leqq s\leqq t}(1+\varepsilon u_s \Delta N_s)\exp\left(-\int_0^t \varepsilon u_s\,ds\right).$$

Then $\{\Lambda_t^\varepsilon\}$ is an $\{\mathscr{F}_t\}$-martingale with $E[\Lambda_t^\varepsilon]=1$ (*see* [6]). Define a probability measure P^ε on $\mathscr{F}_1$ by

$$\frac{dP^\varepsilon}{dP}=\Lambda_1^\varepsilon.$$

Set

$$\varphi_\varepsilon(t)=\int_0^t (1+\varepsilon u_s)\,ds$$

and write

$$\psi_\varepsilon(t)=\varphi_\varepsilon^{-1}(t)=\int_0^t \frac{1}{1+\varepsilon\, u_{\psi_\varepsilon(s)}}\,ds$$

$$\mathscr{F}_t^\varepsilon=\mathscr{F}_{\psi_\varepsilon(t)}.$$

Then the process $N_t^\varepsilon=N_{\psi_\varepsilon(t)}$ is Poisson on $(\Omega, \mathscr{F}, (\mathscr{F}_t^\varepsilon), P^\varepsilon)$ with jump times $\varphi_\varepsilon(T_1)\wedge 1, \ldots, \varphi_\varepsilon(T_n)\wedge 1, \ldots$ (*see* [6]).

For $\{u_t\}$ as above, set $U_t=\int_0^t u_s\,ds$. Suppose $g_s(w)$ is an $\{F_t\}$-predictable function on [0, 1]. Then for $0\leqq s\leqq T_1\wedge 1$,

$$g_s(w)=g(s),$$

and in general, for $T_{n-1}\wedge 1<s\leqq T_n\wedge 1$,

$$g_s(w)=g(s, T_1\wedge 1, \ldots, T_{n-1}\wedge 1).$$

Note that by setting $g_s(0,0,\ldots)=g(s)$ for $0\leqq s\leqq T_1\wedge 1$, $g_s((s-T_1)\vee 0, \ldots, (s-T_{n-1})\vee 0), 0, 0, \ldots)$ for $T_{n-1}\wedge 1<s\leqq T_n\wedge 1$, etc., such a g can be written in the form

$$g_s(w)=g_s((s-T_1)\vee 0, (s-T_2)\vee 0, \ldots), \qquad s\in[0,1]. \tag{4.1}$$

Therefore, we shall consider a predictable function g of this form, and further assume that if

$$g=g_s(t_1, t_2, \ldots),$$

then all the partial derivatives $\frac{\partial g_s}{\partial t_i}$ exist for all s, and there is a constant $K>0$ such that

$$\left|\frac{\partial g_s}{\partial t_i}\right|<K \quad \text{for all } i, \text{and for all } s. \tag{4.2}$$

We now define the analog of the Fréchet derivative for functionals of the Poisson process.

Write

$$g_s^\varepsilon=g_s((s-\varphi_\varepsilon(T_1))\vee 0, \ldots, (s-\varphi_\varepsilon(T_n))\vee 0, \ldots).$$

Then

$$\left.\frac{\partial g_s^\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0}=-\sum_{i=1}^{\infty}\frac{\partial}{\partial t_i}g_s((s-T_1)\vee 0, \ldots, (s-T_n)\vee 0, \ldots)$$
$$\times\int_0^{T_i} u_r\,dr\, I_{T_i<s}. \tag{4.3}$$

Define

$$\mu(dt) = -\sum_{i=1}^{\infty} \frac{\partial g_s}{\partial t_i} \mathrm{I}_{\mathrm{T}_i < s}\, \delta_{\mathrm{T}_i}(dt)$$

where δ_{T_i} is the point mass at T_i. Then

$$\begin{aligned}
\left.\frac{\partial g_s^{\varepsilon}}{\partial \varepsilon}\right|_{\varepsilon=0} &= \int_0^s \int_0^t u_r\, dr\, \mu(dt) \\
&= \int_0^s \int_0^s \mathrm{I}_{0 \leqq r \leqq t \leqq s}\, u_r\, dr\, \mu(dt) \\
&= \int_0^s \mu([r, s])\, u_r\, dr \\
&= -\int_0^s \sum_{i=1}^{\infty} \mathrm{I}_{r \leqq \mathrm{T}_i < s} \frac{\partial g_s}{\partial t_i} u_r\, dr \\
&= \int_0^s \mathrm{D}\, g_s(\,.\,, [r, s])\, u_r\, dr,
\end{aligned}$$

where

$$\mathrm{D}g_s(\,.\,, [r, s]) = -\sum_{i=1}^{\infty} \mathrm{I}_{r \leqq \mathrm{T}_i < s} \frac{\partial g_s}{\partial t_i}.$$

Write

$$\mathrm{D}g_s(\,.\,, \mathrm{U}) = \int_0^s \mathrm{D}g_s(\,.\,, [r, s])\, u_r\, dr.$$

Note that

$$\mathrm{D}g_{\mathrm{T}_i}(\,.\,, \mathrm{U}) = -\sum_{j=1}^{i-1} \frac{\partial g_{\mathrm{T}_i}}{\partial t_j} \int_0^{\mathrm{T}_j} u_r\, dr. \tag{4.4}$$

DEFINITION 4.1. – A process $\{g_s\}$ of the form (4.1) is said to be differentiable if it satisfies (4.2) and (4.3) for all u satisfying (i) and (ii) above, and for all s. We call $\mathrm{D}g_s(\,.\,, \mathrm{U})$ the derivative of g_s in the direction U. It is of interest to note that this concept of differentiability of a function of a Poisson process is an analog of the Fréchet derivative of a function of a continuous process. *See* Föllmer [7], where similar formulae arise using the Fréchet derivative.

Now suppose $\{h_s\}$ is a bounded, $\{F_t\}$-predictable process of the form given by (4.1), which satisfies:

(*a*) h is differentiable in the sense of Definition 4.1.

(*b*) $\frac{\partial h_s}{\partial s}$ exists, and there exists a constant $A>0$ such that $\left|\frac{\partial h_s}{\partial s}\right|<A$ for all s, a. s.

(*c*) There are constants $B>0$, $C>0$ such that $0<B<h_s<C$ for all s, a. s.

It is easy to check that $h_s = h_s((s-T_1) \vee 0, (s-T_2) \vee 0, \ldots)$ is predictable. Consider the family of exponentials:

$$\begin{aligned} G_t &= \prod_{0 \leq s \leq t} (1+(h_s-1)\Delta N_s) \exp\left(\int_0^t (1-h_s)\,ds\right) \\ &= \left(\prod_{0 \leq T_i \leq t} h_{T_i}\right) \exp\left(\int_0^t (1-h_s)\,ds\right). \end{aligned} \tag{4.5}$$

Then $\{G_t\}$ is a martingale with $E[G_t]=1$. Since for each fixed ω, if $T_{n-1}(\omega) < t \leq T_n(\omega)$, G_t is a function of $(t, T_1(\omega), \ldots, T_{n-1}(\omega))$, we see as above that G_t can be considered to be of the form

$$G_t = G_t((t-T_1) \vee 0, \ldots, (t-T_n) \vee 0, \ldots).$$

THEOREM 4.2. – (G_t) *defined in* (4.5) *is differentiable in the sense of Definition* 4.1.

Moreover,

$$\begin{aligned} DG_1(.,U)\,G_1^{-1} &= \int_0^1 \gamma_s u_s G_1^{-1}\,ds \\ &= \int_0^1 \int_r^1 \left[\frac{\partial h_s}{\partial s} + \sum_{j=1}^{\infty} I_{\{T_j<s\}} \frac{\partial h_s}{\partial t_j} + D\,h_s(.,[r,s])\right] \frac{1}{h_s}\,dN_s\,u_r\,dr \\ &\quad - \int_0^1 \int_r^1 D\,h_s(.,[r,s])\,ds\,u_r\,dr, \quad \text{a. s.} \end{aligned} \tag{4.6}$$

where

$$\gamma_s = -\sum_{i=1}^{\infty} I_{s \leq T_i \leq 1} \frac{\partial}{\partial t_i} G_1((1-T_1) \vee 0, \ldots, (1-T_n) \vee 0, \ldots).$$

Proof. – The first identity follows from the definition and properties of the derivative. To determine $DG_t(.,U)$ we calculate the derivative of G_t^ε at $\varepsilon=0$. Write

$$h_s^\varepsilon = h_s((s-\varphi_\varepsilon(T_1)) \vee 0, \ldots, (s-\varphi_\varepsilon(T_n)) \vee 0, \ldots),$$

so

$$\begin{aligned} G^\varepsilon_t &= \prod_{0 \leqq s \leqq t} (1 + (h^\varepsilon_s - 1)\, \Delta N_{\psi_\varepsilon(s)}) \exp\left(\int_0^t (1 - h^\varepsilon_s)\, ds\right) \\ &= \Big(\prod_{0 \leqq \varphi_\varepsilon(T_i) \leqq t} h^\varepsilon_{\varphi_\varepsilon(T_i)}\Big) \exp\left(\int_0^t (1 - h^\varepsilon_s)\, ds\right) \\ &= \Big(\prod_{0 \leqq T_i \leqq \psi_\varepsilon(t)} h^\varepsilon_{\varphi_\varepsilon(T_i)}\Big) \exp\left(\int_0^t (1 - h^\varepsilon_s)\, ds\right). \end{aligned}$$

Then

$$\log G^\varepsilon_t = \sum_{i=1}^{\infty} I_{T_i \leqq \psi_\varepsilon(t)} \log h^\varepsilon_{\varphi_\varepsilon(T_i)} + \int_0^t (1 - h^\varepsilon_s)\, ds. \qquad (4.7)$$

Differentiate (4.7) with respect to ε, and then set $\varepsilon = 0$, to see

$$\begin{aligned} DG_t(\,.\,,U)\frac{1}{G_t} = \sum_{i=1}^{\infty} \Bigg\{ I_{T_i \leqq t} \Bigg[\frac{\partial h_{T_i}}{\partial t} \int_0^{T_i} u_r\, dr \\ + \sum_{j=1}^{i-1} \frac{\partial h_{T_i}}{\partial t_j} \left(\int_0^{T_i} u_r\, dr - \int_0^{T_j} u_r\, dr \right) \Bigg] \frac{1}{h_{T_i}} \Bigg\} \\ - \int_0^t D\, h_s(\,.\,,U)\, ds, \quad \text{a. s.} \end{aligned}$$

From (4.4) this is

$$\begin{aligned} = \sum_{i=1}^{\infty} \Bigg\{ I_{T_i \leqq t} \Bigg[\frac{\partial h_{T_i}}{\partial t} \int_0^{T_i} u_r\, dr \\ + \sum_{j=1}^{i-1} \frac{\partial h_{T_i}}{\partial t_j} \int_0^{T_i} u_r\, dr + D\, h_{T_i}(\,.\,,U) \Bigg] \frac{1}{h_{T_i}} \Bigg\} \\ - \int_0^t D\, h_s(\,.\,,U)\, ds = \int_0^t \Bigg[\frac{\partial h_s}{\partial s} \int_0^s u_r\, dr \\ + \sum_{j=1}^{\infty} I_{\{T_j < s\}} \frac{\partial h_s}{\partial t_j} \int_0^s u_r\, dr + Dh_s(\,.\,,U) \Bigg] \frac{1}{h_s} dN_s \\ - \int_0^t Dh_s(\,.\,,U)\, ds. \quad (4.8) \end{aligned}$$

(Formally, the differentiation of the indicator functions $I_{T_i \leqq \psi_\varepsilon(t)}$ introduces Dirac measures $\delta(t - T_i.)$ However, $P(T_i = t) = 0$ and we later will take expectations, so these can be ignored.) From (4.8),

$$\mathrm{D}\mathrm{G}_1(.,\mathrm{U})\mathrm{G}_1^{-1}=\int_0^1\left\{\frac{\partial h_s}{\partial s}\int_0^s u_r\,dr+\sum_{j=1}^{\infty}\mathrm{I}_{\{\mathrm{T}_j<s\}}\frac{\partial h_s}{\partial t_j}\int_0^s u_r\,dr\right.$$

$$\left.+\int_0^s \mathrm{D}h_s(.,[r,s])\,u_r\,dr\right\}\frac{1}{h_s}d\mathrm{N}_s-\int_0^1\int_0^s \mathrm{D}h_s(.,[r,s])\,u_r\,dr\,ds$$

$$=\int_0^1\int_0^1 \mathrm{I}_{0\leqq r\leqq s\leqq 1}\left\{\frac{\partial h_s}{\partial s}u_r\frac{1}{h_s}+\sum_{j=1}^{\infty}\mathrm{I}_{\{\mathrm{T}_j<s\}}\frac{\partial h_s}{\partial t_j}u_r\frac{1}{h_s}\right.$$

$$\left.+\mathrm{D}h_s(.,[r,s])\,u_r\frac{1}{h_s}\right\}dr\,dN_s-\int_0^1\int_0^1 \mathrm{I}_{0\leqq r\leqq s\leqq 1}\,\mathrm{D}\,h_s(.,[r,s])\,u_r\,dr\,ds$$

$$=\int_0^1\int_r^1\left[\frac{\partial h_s}{\partial s}+\sum_{j=1}^{\infty}\mathrm{I}_{\{\mathrm{T}_j<s\}}\frac{\partial h_s}{\partial t_j}+\mathrm{D}h_s(.,[r,s])\right]\frac{1}{h_s}d\mathrm{N}_s\,u_r\,dr$$

$$-\int_0^1\int_r^1 \mathrm{D}\,h_s(.,[r,s])\,ds\,u_r\,dr,$$

which is (4.6). □

Consider the family of exponentials defined by (4.5) and define a new probability measure P^h on $\mathscr{F}_1$ by:

$$\frac{d\mathrm{P}^h}{d\mathrm{P}}=\mathrm{G}_1.$$

Then (*see* [3]) the process

$$\begin{aligned}\mathrm{Z}_t&=\mathrm{N}_t-\int_0^t h_s\,ds\\&=\mathrm{Q}_t-\int_0^t (h_s-1)\,ds,\end{aligned}\tag{4.9}$$

where $\mathrm{Q}_t=\mathrm{N}_t-t$, is an $(\mathscr{F}_t)$-martingale under P^h. We want to show that Z_t is a reverse time G_t-quasimartingale under P^h, having the decomposition

$$\mathrm{Z}_t=\mathrm{Z}_1+\mathrm{M}_t+\int_t^1 \alpha_s\,ds.\tag{4.10}$$

From (4.9), we can write

$$\mathrm{Z}_t=\mathrm{Z}_1+\mathrm{Q}_t-\mathrm{Q}_1+\int_t^1 (h_s-1)\,ds.$$

Now for almost all t

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathrm{E}^h\left[\int_{t-\varepsilon}^t (h_s-1)\,ds\,|\,\mathrm{G}_t\right]=\mathrm{E}^h[h_t-1\,|\,\mathrm{G}_t].$$

Hence, to show that Z_t has the decomposition given by (4.10), it again suffices to consider approximate Laplacien as in [4] and show that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^h [Q_{t-\varepsilon} - Q_t \mid G_t]$$

exists.

THEOREM 4.3. – *For almost all* $t \in [0, 1]$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^h [Q_t - Q_{t-\varepsilon} \mid G_t] = \frac{1}{t} E^h [Q_t + a_t \mid G_t] - E^h [b_t \mid G_t] \qquad (4.11)$$

where

$$a_t = \int_0^t \int_s^1 \left[\frac{\partial h_r}{\partial r} + \sum_{j=1}^{\infty} I_{\{T_j < r\}} \frac{\partial h_r}{\partial t_j} + D\, h_r(., [s, r]) \right] \frac{1}{h_r} dN_r\, ds - \int_0^t \int_s^1 Dh_r(., [s, r])\, dr ds$$

and

$$b_t = \int_t^1 \left[\frac{\partial h_r}{\partial r} + \sum_{j=1}^{\infty} I_{\{T_j < r\}} \frac{\partial h_r}{\partial t_j} + Dh_r(., [t, r]) \right] \frac{1}{h_r} dN_r - \int_t^1 Dh_r(., [t, r])\, dr.$$

Proof. – First we note that if $H((1-T_1) \vee 0, \ldots, (1-T_n) \vee 0, \ldots)$ is a square integrable functional and its first partial derivatives are all bounded by a constant, then, using a similar argument as in [6], we have the integration by parts formula

$$E\left[\left(\int_0^1 u_s\, dQ_s\right) H\right] = -E[DH(., U)] \qquad (4.12)$$

where $DH(., U)$ is the derivative in direction U of Definition 4.1.

A direct consequence is the product rule

$$E\left[FH\left(\int_0^1 u_s\, dQ_s\right)\right] = -E[FDH(., U)] - E[HDF(., U)]. \quad (4.13)$$

Let $H = G_1$ be the Girsanov density, then (4.13) becomes

$$E^h\left[F \int_0^1 u_s\, dQ_s\right] = -E^h[DF(., U)] - E^h[FG_1^{-1} DG_1(., U)]. \quad (4.14)$$

Now fix $t_0 \in (0, 1)$. Write $T_k(t_0)$ for the k-th jump time of N_t greater than t_0. Suppose F is a bounded and G_{t_0} measurable function. Furthermore, we suppose that F is a differentiable function (in the sense of Definition

4.1) of the form

$$F((1-T_1(t_0))\vee 0,\ldots,(1-T_k(t_0))\vee 0,\ldots),$$

and that the derivatives of F are bounded. Then the measure $DF(.,dt)$ is concentrated on $[t_0, 1]$ and (4.14) holds for such an F. Take $u_s = I_{[t_0-\varepsilon, t_0]}(s)$ in (4.14). For such an F

$$\begin{aligned} DF(.,U) &= \int_{t_0-\varepsilon}^{t_0} DF(.,[r,1])\,dr \\ &= \int_{t_0-\varepsilon}^{t_0} DF(.,[t_0,1])\,dr \\ &= \varepsilon\, DF(.,[t_0,1]). \end{aligned}$$

Therefore, we have from (4.14)

$$E^h[(Q_{t_0}-Q_{t_0-\varepsilon})F] = -\varepsilon E^h[DF(.,[t_0,1])] + E^h\left[FG_1^{-1}\int_{t_0-\varepsilon}^{t_0}\sum_{i=1}^{\infty} I_{s\leqq T_i<1}\frac{\partial G_1}{\partial t_i}\,ds\right]. \quad (4.15)$$

From (4.15), for almost all t

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}[(Q_{t_0}-Q_{t_0-\varepsilon})F] = -E^h[DF(.,[t_0,1])] + E^h\left[FG_1^{-1}\sum_{i=1}^{\infty} I_{t_0\leqq T_i<1}\frac{\partial G_1}{\partial t_i}\right]. \quad (4.16)$$

Using (4.15) again with $\varepsilon = t_0 = t$, we have

$$-E^h[DF(.,[t,1])] = \frac{1}{t}E^h[Q_t F] - \frac{1}{t}E^h\left[FG_1^{-1}\int_0^t\sum_{i=1}^{\infty} I_{s\leqq T_i<1}\frac{\partial G_1}{\partial t_i}\,ds\right]. \quad (4.17)$$

Now let $u_s = I_{[0,t]}(s)$ in Theorem 4.2 to obtain

$$\begin{aligned} -\int_0^t\left(\sum_{i=1}^{\infty} I_{s\leqq T_i<1}\frac{\partial G_1}{\partial t_i}\right)G_1^{-1}\,ds &= \int_0^t\int_s^1\left[\frac{\partial h_r}{\partial r}+\sum_{j=1}^{\infty} I_{\{T_j<r\}}\frac{\partial h_r}{\partial t_j}+Dh_r(.,[s,r])\right]\frac{1}{h_r}\,dN_r\,ds \\ &\quad -\int_0^t\int_s^1 Dh_r(.,[s,r])\,dr\,ds. \end{aligned}$$

Hence (4.17) becomes

$$-E^h[DF(.,[t,1])] = \frac{1}{t}E^h[Q_t F] + \frac{1}{t}E^h[a_t F]. \qquad (4.18)$$

Now take $u_s = I_{[t-\varepsilon, t]}(s)$ in Theorem 4.2 to obtain

$$-\int_{t-\varepsilon}^{t}\left(\sum_{i=1}^{\infty} I_{s \leqq T_i < 1}\frac{\partial G_1}{\partial t_i}\right)G_1^{-1}\,ds$$
$$= \int_{t-\varepsilon}^{t}\int_{s}^{1}\left[\frac{\partial h_r}{\partial r} + \sum_{j=1}^{\infty} I_{\{T_j < r\}}\frac{\partial h_r}{\partial t_j} + Dh_r(.,[s,r])\right]\frac{1}{h_r}dN_r\,ds$$
$$-\int_{t-\varepsilon}^{t}\int_{s}^{1} Dh_r(.,[s,r])\,dr\,ds. \qquad (4.19)$$

Multiply both sides of (4.19) by F, and then take expectations

$$-E^h\left[F\int_{t-\varepsilon}^{t}\left(\sum_{i=1}^{\infty} I_{s \leqq T_i < 1}\frac{\partial G_1}{\partial t_i}\right)G_1^{-1}\,ds\right] = E^h\left[F\int_{t-\varepsilon}^{t} b_s\,ds\right]. \qquad (4.20)$$

Divide both sides of (4.20) by ε, and then let $\varepsilon \downarrow 0$, to obtain for almost all t

$$-E^h\left[F\left(\sum_{i=1}^{\infty} I_{t \leqq T_i < 1}\frac{\partial G_1}{\partial t_i}\right)G_1^{-1}\right] = E^h[b_t F]. \qquad (4.21)$$

Combining (4.16), (4.18) and (4.21), we have

$$\lim_{\varepsilon \downarrow 0}\frac{1}{\varepsilon}E^h[(Q_t - Q_{t-\varepsilon})F] = \frac{1}{t}E^h[(a_t + Q_t)F] - E^h[b_t F].$$

Thus we have proved (4.11). □

As a consequence of Theorem 4.3, Z_t is a reverse time G_t-quasimartingale having the decomposition given by (4.10). It follows immediately that the integrand α_t in (4.10) is given by

$$\alpha_t = E^h[b_t + h_t - 1 \mid G_t] - \frac{1}{t}E^h[a_t + Q_t \mid G_t].$$

REFERENCES

[1] K. BICHTELER, J.-B. GRAVEREAUX and J. JACOD, Malliavin Calculus for Processes with Jumps, *Stochastics Monographs*, Vol. **2**, Gordon and Breach, New York, London, 1987.

[2] R. J. Elliott, Stochastic Calculus and Applications, *Applications of Mathematics*, Vol. **18**, Springer-Verlag, Berlin, Heidelberg, New York, 1982.
[3] R. J. Elliott, *Filtering and control for point process observations*, Research Report No. 09.86.1, Department of Statistics and Applied Probability, University of Alberta, 1986.
[4] R. J. Elliott and B. D. O. Anderson, Reverse time diffusions, *Stoch. Proc. Appl.*, Vol. **19**, 1985, pp. 327-339.
[5] R. J. Elliott and P. E. Kopp, *Eauivalent Martingale measures for bridge processes*, Technical Report No. 89.17, Department of Statistics and Applied Probability, University of Alberta, 1989.
[6] R. J. Elliott and A. H. Tsoi, *Integration by parts for Poisson processes*, Technical Report No. 89.09, Department of Statistics and Applied Probability, University of Alberta, 1989.
[7] H. Föllmer, Random fields and diffusion processes, École d'Été de Probabilités de Saint-Flour XV-XVII, 1985-87, *Lect. Notes Math.*, Vol. **1362**, Springer-Verlag, Berlin.
[8] U. G. Haussman and E. Pardoux, Time reversal of diffusions, *Ann. Prob.*, Vol. **14**, 1986, pp. 1188-1205.
[9] J. Jacod and P. Protter, *Time reversal on Lévy processes*, to appear.
[10] T. Jeulin and M. Yor, Inégalité de Hardy, semi-martingales et faux-amis, Séminaire de Probabilités XIII, *Lect. Notes Math.*, Vol. **721**, 1979, pp. 332-359, Springer-Verlag, Berlin.
[11] E. Nelson, *Dynamical theories of Brownian motion*, Princeton University Press, 1967.
[12] C. Stricker, Une Characterization des quasimartingales, Séminaire de Probabilités IX, *Lect. Notes Math.*, Vol. **465**, 1975, pp. 420-424, Springer-Verlag, Berlin.

(*Manuscript received September* 4, 1989)
(*In revised form December* 11, 1989.)