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Additions and correction to "the bootstrap of the mean with arbitrary bootstrap sample size" (*)

by

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ABSTRACT. — Some inaccuracies in [2] are corrected and some additional results are presented. The bootstrap central limit theorem in the domain of attraction case is improved to include convergence of bootstrap moments. Self-normalized limit theorems for variables in the domain of attraction of a p-stable law are bootstrapped, thus freeing the bootstrap from the index p and the norming constants $\{b_n\}$. Simultations on the bootstrap of the self-normalized sums for a few values of p and n are also included.

RÉSUMÉ. — Nous corrigeons quelques inexactitudes de l'article [2] et nous présentons certains résultats complémentaires. Nous améliorons le théorème central limite « bootstrap » pour obtenir la convergence des moments « bootstrap ». Des théorèmes limites auto-normalisés pour des variables dans le domaines d'attraction d'une loi p-stable sont donnés sous forme bootstrap, ce qui libère le bootstrap de l'indice p et des constantes de normalisation (b_n). On présente aussi des simulations du bootstrap des sommes auto-normalisée pour quelques valeurs de p et n.

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1. INTRODUCTION

Remarks 2.3 and 2.4 in [2] are inaccurate, and we correct them in Section 2. We take the opportunity to broaden our previous study on the bootstrap of the mean [2] in two directions. Bickel and Freedman [3] observe that if $EX^2<\infty$, not only does the bootstrap CLT hold a.s. in the sense that e.g. $d_{BL_1}\left(\hat{\mathscr{L}}\left(\sum_{j=1}^{n}(X_{nj}-\bar{X}_n)/n^{1/2}\right), N(0, \operatorname{Var} X)\right)\to 0$ a.s. but that actually d_{BL_1} can be replaced by the Mallows distance d_2 which metrizes weak convergence plus convergence of the second moments. This can be strengthend to include convergence of exponential bootstrap moments even for different bootstrap sample sizes m_n, as long as $m_n \geqq cn$ for some $c>0$. Curiously enough, if $m_n/n\to 0$ then a.s. convergence of the p-th bootstrap moment for $p\geqq 2$ implies (is equivalent to) futher integrability of X, namely $\sum_{n=1}^{\infty} P\{|X|>m_n^{1/2-1/p}n^{1/p}\}<\infty$. The case $EX^2=\infty$ is also thoroughly examined.

In another direction, we look at the bootstrap of selfnormalized (Studentized) sums, in a sense expanding on Remark 2.3 of [2]. It is well known (e.g. Logan *et al.* [6]) that if X belongs to some domain of attraction with normings b_n and centers a_n then the random vectors $\left\{\left(b_n^{-1}\sum_{i=1}^{n}X_i-a_n,\ b_n^{-2}\sum_{i=1}^{n}X_i^2\right)\right\}_{n=1}^{\infty}$ converge in law. In particular, if X is in the domain of attraction of a p-stable random variable, $1<p\leqq 2$, then $\left\{\sum_{i=1}^{n}(X_i-EX)\Big/\left(\sum_{i=1}^{n}X_i^2\right)^{1/2}\right\}$ converges in law. (It is irrelevant whether one takes $\left(\sum_{i=1}^{n}X_i^2\right)^{1/2}$ or $\left(\sum_{i=1}^{n}(X_i-\bar{X}_n)^2\right)^{1/2}$: *see* e.g. [6].) We show that if $m_n/n\to 0$ then the bootstrap of this statistic,

$$\left\{\sum_{i=1}^{m_n}(X_{ni}-\bar{X}_n)\Big/\left(\sum_{i=1}^{n}X_{ni}^2\right)^{1/2}\right\}$$

converges weakly in probability to the same limit as the original for all $1<p\leqq 2$ and all possible norming sequences $\{b_n\}$. This suggests a procedure for constructing bootstrap confidence intervals for the mean which is robust against integrability properties. Some simulations in the infinite variance case are included.

2. CORRECTIONS TO [2]

Remark 2.3 in [2] on random normings for the bootstrap CLT with normal limit refers only to the case $EX^2=\infty$, although this is not explicitly stated there, and the norming (2.20) is only valid for $m_n<cn$ for some $c<\infty$. Under these constraints, the remark is correct. The normings described there can be modified to hold simultaneously for $EX^2=\infty$ and $EX^2<\infty$ as follows:

$$\hat{a}_n(\omega)=\left[(m_n/n)\sum_{i=1}^{n}(X_i-\bar{X}_n)^2\right]^{1/2}\qquad \text{if}\quad m_n\geqq n,$$

and $\hat{a}_n(\omega)=$ average over all the $\binom{n}{m_n}$ combinations $1\leqq j_1<\ldots<j_{m_n}\leqq n$ *of*

$$\left[\sum_{i=1}^{m_n}\left(X_{j_i}-m_n^{-1}\sum_{i=1}^{m_n}X_{j_i}\right)^2\right]^{1/2}\qquad \text{if}\quad m_n\leqq n.$$

[This replaces equation (2.19).] For $m_n/n\to 0$ one may as well take $\hat{a}_n(\omega)$ to be the average of

$$\left[\sum_{i=km_n+1}^{i=(k+1)\,m_n}\left(X_i-m_n^{-1}\sum_{i=km_n+1}^{i=(k+1)\,m_n}X_i\right)^2\right]^{1/2},\qquad k=1,\ \ldots,\ [n/m_n].$$

Moreover, for $m_n\leqq cn$, $c<\infty$, another possible norming is $\hat{a}_n^{\omega}(\omega')=\left[\sum_{j=1}^{m_n}(X_{n,j}^{w}(\omega')-\bar{X}_n^{\omega})^2\right]^{1/2}$. [This replaces equation (2.20).] The proofs are as indicated in [2] using convergence of the sequence $\left\{\sum_{i=1}^{n}(X_i-EX)^2/b_n^2\right\}$ instead of $\left\{\sum_{i=1}^{n}X_i^2/b_n^2\right\}$.

The computations in Remark 2.4 of [2] are correct but they do not show what we say there. In fact, *in Theorem 2.2 the centering $\tilde{X}_n^{\omega}$ can be replaced by $\bar{X}_n^{\omega}$*. To see this note that if $m_n>cn$, then $a_n>c'b_n$ for some constant c' and therefore

$$P\left\{\sum_{i=1}^{n}I_{|X_i|\geqq a_n}\neq 0\right\}$$
$$=P\left\{\sum_{i=1}^{n}I_{|X_i|\geqq a_n}>\delta\right\}\leqq\delta^{-1}nP\{|X|>c'b_n\}\to 0\qquad\text{for}\quad 0<\delta<1.$$

This shows that $(m_n/na_n)\sum_{i=1}^{n}X_iI_{|X_i|>a_n}\to 0$ in probability and the equivalence between the centerings $\bar{X}_n$ and $\tilde{X}_n$ follows.

We also correct some minor misprints: on page 465, line 5, $\frac{m_n}{b_{m_n}} \mathrm{U}(b_{m_n})$ should be $\frac{m_n}{b_{m_n}^2} \mathrm{U}(b_{m_n})$; in (2.21) the sum should be for $i \leqq n'$ instead of $i \leqq m_n$; on page 475, lines 9 and 13, $\bar{a}_n$ and $\bar{p}_n$ should be replaced by $\bar{a}_k$ and $\bar{p}_k$; finally in the statement of Theorem 3.4, the constant c in $m_n/m_{2n} \geqq c$ should be strictly positive.

3. CONVERGENCE OF MOMENTS

The bootstrap in probability of the mean in the domains of attraction case (Theorem 2.2 and Corollary 2.6 in [2]) can be strengthned to include convergence in probability of bootstrap moments, even exponential in the normal case. Weak convergence together with convergence of the r-th absolute moment is metrizable (Mallows-Wassertein distances; *see* e.g. Bickel and Freedman [3]). We will call d_r any distance metrizing this convergence.

The following theorem improves on Theorem 2.1 of [3]; we only state it for real random variables but it is obvious that it extends to random vectors in $\mathbf{R}^k$, $k < \infty$.

3.1. THEOREM. – (*a*) *If* $\mathrm{EX}^2 < \infty$ *and* $m_n/n \geqq c > 0$ *then for all* $t > 0$

$$(3.1) \qquad \hat{\mathrm{E}} \exp\left\{ t \sum_{i=1}^{m_n} (\mathrm{X}_{nj} - \bar{\mathrm{X}}_n)/m_n^{1/2} \right\} \to \mathrm{E}\, e^{tg} \quad \text{a.s.}$$

where g *is* $\mathrm{N}(0, \mathrm{Var}\,\mathrm{X})$. *In particular*

$$(3.2) \qquad d_p\left[\mathscr{L}\left(\sum_{i=1}^{m_n} (\mathrm{X}_{nj} - \bar{\mathrm{X}}_n)/m_n^{1/2} \right), \mathrm{N}(0, \mathrm{Var}\,\mathrm{X}) \right] \to 0 \quad \text{a.s. for all } p > 0.$$

(*b*) *If* X *is in the domain of attraction of a normal law with norming constants* $b_n \nearrow \infty$, *that is* $\mathscr{L}\left(\sum_{j=1}^{n} (\mathrm{X}_j - \mathrm{EX})/b_n \right) \to_w \mathrm{N}(0, 1)$, *and if* $m_n/n \geqq c > 0$ *and* $a_n = b_n (m_n/n)^{1/2}$, *then*

$$(3.1)' \qquad \hat{\mathrm{E}} \exp\left\{ t \sum_{i=1}^{m_n} (\mathrm{X}_{nj} - \bar{\mathrm{X}}_n)/a_n \right\} \to \mathrm{E}\, e^{tg} \quad \textit{in probability},$$

where g *is* $\mathrm{N}(0, 1)$. *In particular*

$$(3.2)' \qquad d_p\left[\hat{\mathscr{L}}\left(\sum_{i=1}^{m_n} (\mathrm{X}_{nj} - \bar{\mathrm{X}}_n)/a_n \right), \mathrm{N}(0, 1) \right] \to 0 \quad \textit{in probability for all } p > 0.$$

Proof. – Let us recall that convexity of $f(x)=e^{tx}$ implies $\mathrm{E}\,e^{t(\mathrm{X}+\mathrm{Y})}\leqq(\mathrm{E}\,e^{2t\mathrm{X}}+\mathrm{E}\,e^{2t\mathrm{Y}})/2$ for any rv's X and Y, and that if X and Y are independent and Y is centered then $\mathrm{E}\,e^{t(\mathrm{X}+\mathrm{Y})}\geqq\mathrm{E}\,e^{t\mathrm{X}}$. Moreover if $\{\varepsilon_i\}$ is a Rademacher sequence then $\mathrm{E}\,e^{\sum_{i=1}^{n} a_i\varepsilon_i}\leqq e^{\sum_{i=1}^{n} a_i^2/2}$ (since $\mathrm{E}\,e^{a\varepsilon}\leqq e^{a^2/2}$). To prove (*b*) we take a Rademacher sequence $\{\varepsilon_i\}$ independent of $\{X_{nj}\}$ and a copy $\{\mathrm{X}'_{nj}\}$ of $\{\mathrm{X}_{nj}\}$ independent of the rest of the variables. Then we have, for each $\omega\in\Omega$ (which we omit),

$$(3.3)\quad \hat{\mathrm{E}}\,e^{t\sum_{j=1}^{m_n}(\mathrm{X}_{nj}-\bar{\mathrm{X}}_n)/a_n}\leqq\hat{\mathrm{E}}\,e^{t\sum_{j=1}^{m_n}\varepsilon_j(\mathrm{X}_{nj}-\mathrm{X}'_{nj})/a_n}\leqq\hat{\mathrm{E}}\,e^{2t\sum_{j=1}^{m_n}\varepsilon_j\mathrm{X}_{nj}/a_n}$$

$$\leqq\hat{\mathrm{E}}\,e^{2t^2\sum_{j=1}^{m_n}\mathrm{X}_{nj}^2/a_n^2}=\left[n^{-1}\sum_{i=1}^{n}\exp(2t^2\mathrm{X}_i^2/a_n^2)\right]^{m^n}.$$

Since $\max_{i\leqq n}\mathrm{X}_i^2/a_n^2\to 0$ in probability, $n^{-1}\sum_{i=1}^{n}\exp(2t^2\mathrm{X}_i^2/a_n^2)\to 1$ in probability. Therefore the logarithm of the last term in (3.3) is asymptotic to $(m_n/n)\sum_{i=1}^{n}(e^{2t^2\mathrm{X}_i^2/a_n^2}-1)$ which in turn is asymptotic to

$$(m_n/n)\sum_{i=1}^{n}2t^2\mathrm{X}_i^2/a_n^2=2t^2b_n^{-2}\sum_{i=1}^{n}\mathrm{X}_i^2\to 2t^2$$

in probability. Hence, for all t, the sequence $\left\{\hat{\mathrm{E}}\exp\left(t\sum_{i=1}^{m_n}(\mathrm{X}_{nj}-\bar{\mathrm{X}}_n)/a_n\right)\right\}_{n-1}^{\infty}$ is stochastically bounded. Let $\mathrm{V}_n=\sum_{j=1}^{m_n}(\mathrm{X}_{nj}-\bar{\mathrm{X}}_n)/a_n$. We have

$$\begin{aligned}\mathrm{P}\{|\hat{\mathrm{E}}\,e^{t\mathrm{V}_n}-\mathrm{E}\,e^{tg}|>\varepsilon\}\\ \leqq\mathrm{P}\{|\hat{\mathrm{E}}\exp t(\mathrm{V}_n\wedge c)-\mathrm{E}\exp t(g\wedge c)]|>\varepsilon/2\}\\ +2\mathrm{P}\{e^{-tc}\hat{\mathrm{E}}\exp(2t\mathrm{V}_n)>\varepsilon/2-e^{-tc}\mathrm{E}\exp(2tg)\}\end{aligned}$$

for any c. The first probability tends to zero by weak convergence in probability of V_n to g, for all c, and the second tends to zero uniformly in n as $c\to\infty$ by stochastic boundedness of $\{\hat{\mathrm{E}}\exp(2t\mathrm{V}_n)\}$. This proves (*b*). For (*a*) we just notice that the above arguments with $a_n=m_n^{1/2}$ and $b_n=n^{1/2}$, give a. s. boundedness of the sequence $\{\hat{\mathrm{E}}\exp(2t\mathrm{V}_n)\}$ because $\sum_{i=1}^{n}\mathrm{X}_i^2/b_n^2\to\mathrm{EX}^2$ a. s. and $\max_{i\leqq n}\mathrm{X}_i^2/a_n^2\to 0$ a. s. □

3.2. THEOREM. – *If for $m_n \nearrow \infty$*

$$\hat{\mathscr{L}}\left(m_n^{-1/2} \sum_{j=1}^{m_n} (X_{nj}^{\omega} - c_j(\omega))\right) \to_w N(0, 1) \quad a.s. \tag{3.4}$$

then

(3.5)

$$EX^2 < \infty \qquad \text{and} \qquad d_2\left(\hat{\mathscr{L}}\left(m_n^{-1/2} \sum_{i=1}^{m_n} (X_{ni} - \bar{X}_n)\right), N(0, 1)\right) \to 0 \quad a.s.$$

Proof. – We have by the converse CLT that

$$n^{-1} \sum_{i=1}^{n} X_i^2 I_{|X_i| \leqq m_n^{1/2}} - \left(n^{-1} \sum_{i=1}^{n} X_i I_{|X_i| \leqq m_n^{1/2}}\right)^2 \to 1 \quad \text{a.s.}$$

Then if $EX^2 = \infty$, by inequality (2.7) in [2] this reduces to $n^{-1} \sum_{i=1}^{n} X_i^2 I_{|X_i| \leqq m_n^{1/2}} \to 1$ a.s. which implies, by the law of large numbers, $\sup_{c>0} EX^2 I_{|X| \leqq c} \leqq 1$ *i.e.* $EX^2 \leqq 1$, contradiction. Thus, $EX^2 < \infty$. Then

$$\hat{\mathscr{L}}\left(\sum_{i=1}^{m_n} (X_{ni} - \bar{X}_n)/m_n^{1/2}\right) \to_w N(0, 1) \quad \text{a.s.}$$

and, since

$$\hat{E}\left(\sum_{i=1}^{m_n} (X_{ni} - \bar{X}_n)/m_n^{1/2}\right)^2 = n^{-1} \sum_{i=1}^{n} X_i^2 - (\bar{X}_n)^2 \to 1 \quad \text{a.s.},$$

the result follows. □

3.3. THEOREM. – *For any $p \geqq 2$ and $m_n \nearrow \infty$, consider*

(i) $EX^2 < \infty$;

(ii) $\sum_{i=1}^{\infty} P\{|X| > m^{1/2 - 1/p} n^{1/p}\} < \infty$;

(iii) $d_p\left(\hat{\mathscr{L}}\left(m_n^{-1/2} \sum_{i=1}^{m_n} (X_{ni} - \bar{X}_n)\right), N(0,1)\right) \to 0$ *a.s.*

Then (i) *and* (ii) *together are equivalent to* (iii).

Proof. – Suppose (iii) holds. Then $EX^2 < \infty$ by Theorem 3.2. From randomization by a Rademacher sequence independent of $\{X_{nj}\}$, convexity of $y=|x|^p$, $p \geqq 1$, and Kinchin's inequality (e. g. [1], p. 176) we obtain

$$2\hat{\mathrm{E}}\left|m_n^{-1/2}\sum_{j=1}^{m_n}(\mathrm{X}_{nj}-\bar{\mathrm{X}}_n)\right|^p \geqq \hat{\mathrm{E}}\left|m_n^{-1/2}\sum_{j=1}^{m_n}\varepsilon_i(\mathrm{X}_{nj}-\bar{\mathrm{X}}_n)\right|^p$$
$$\geqq c_p\,\hat{\mathrm{E}}\left|m_n^{-1}\sum_{j=1}^{m_n}(\mathrm{X}_{nj}-\bar{\mathrm{X}}_n)^2\right|^{p/2}$$

for some $c_p>0$. Therefore, by (iii), there is $c<\infty$ such that $\limsup_{n\to\infty}\hat{\mathrm{E}}\left|m_n^{-1}\sum_{i=1}^{m_n}\mathrm{X}_{nj}^2\right|^{p/2} \leqq c$ a. s. (since $\bar{\mathrm{X}}_n \to 0$ a. s.).

Since

$$\hat{\mathrm{E}}\left|m_n^{-1}\sum_{j=1}^{m_n}\mathrm{X}_{nj}^2\right|^{p/2} \geqq \hat{\mathrm{E}}\,m_n^{-p/2}\sum_{j=1}^{m_n}|\mathrm{X}_{nj}|^p = m_n^{1-p/2}\,n^{-1}\sum_{i=1}^{n}|\mathrm{X}_i|^p$$

we have $\limsup_{n\to\infty} n^{-1} m_n^{1-p/2}\sum_{i=1}^{n}|\mathrm{X}_i|^p \leqq c$ a. s. Then, by Feller's theorem in e. g. Stout [6], p. 132, we have either $\mathrm{E}|\mathrm{X}|^p<\infty$ or $\sum_{n=1}^{\infty}\mathrm{P}\{|\mathrm{X}|>n^{1/p}m_n^{1/2-1/p}\}<\infty$, hence $\sum_{n=1}^{\infty}\mathrm{P}\{|\mathrm{X}|>n^{1/p}m_n^{1/2-1/p}\}<\infty$.

Suppose now that (i) and (ii) hold. Then by uniform integrability (e. g. [1], Exercise 13, p. 69) the proof of (iii) reduces to showing:

(*a*) $\lim_{t\to\infty}\sup_n (m_n^{1-p/2}/n)\sum_{i=1}^{n}|\mathrm{X}_i|^p\,\mathrm{I}_{|\mathrm{X}_i|\geqq t m_n^{1/2}}=0$ a. s. and

(*b*) $(m_n/n)\sum_{i=1}^{n}\mathrm{X}_i\,\mathrm{I}_{|\mathrm{X}_i|\geqq m_n^{1/2}} \to 0$ a. s.

Now condition (ii) implies $(m_n^{1-p/2}/n)\sum_{i=1}^{n}|\mathrm{X}_i|^p \to 0$ a. s. again by Feller's theorem (the case $\mathrm{E}|\mathrm{X}|^p<\infty$ is obvious). So condition (*a*) holds. As for (*b*) we note

$$\left|(m_n^{1/2}/n)\sum_{i=1}^{n}\mathrm{X}_i\,\mathrm{I}_{|\mathrm{X}_i|\geqq m_n^{1/2}}\right| \leqq n^{-1}\sum_{i=1}^{n}\mathrm{X}_i^2\,\mathrm{I}_{|\mathrm{X}_i|\geqq m_n^{1/2}} \to 0 \quad \text{a. s.}$$

by the law of large numbers. □

3.4. *Remark.* – If $m_n/n \to 0$ then the proof of the Theorem 3.1 shows that the condition $\sum_{i=1}^{\infty}\mathrm{P}\{|\mathrm{X}|>m_n^{1/2}\}<\infty$ [*i.e.* $p=\infty$ in condition (ii) of

Theorem 3.3] implies

$$\hat{\mathrm{E}}\exp\left\{t\sum_{i=1}^{m_n}(\mathrm{X}_{nj}-\bar{\mathrm{X}}_n)/m_n^{1/2}\right\}\to \mathrm{E}\,e^{tg}\quad \text{a.s.}$$

for all $t\in\mathbf{R}$ but we do not know if the converse holds.

We conclude with the case $m_n/n\to 0$ and $\mathrm{EX}^2=\infty$.

3.5. THEOREM. – *If* X *is in the domain of attraction of a p-stable law* $0<p\leqq 2$, *that is*

$$\mathscr{L}\left(\sum_{i=1}^{n}(\mathrm{X}_i-\mathrm{EX\,I}_{|\mathrm{X}|\leqq\tau b_n})/b_n\right)\to_d \mathscr{L}(\theta)$$

where we can take $\tau=\infty$ *for* $1<p\leqq 2$ *and* $\tau=0$ *for* $0<p<1$, *and if* $m_n/n\to 0$, *then*

$$d_r\left[\hat{\mathscr{L}}\left(\sum_{j=1}^{m_n}\left(\mathrm{X}_{nj}-n^{-1}\sum_{i=1}^{n}\mathrm{X}_i\mathrm{I}_{|\mathrm{X}_i|\leqq\tau b_{m_n}}\right)\ b_{m_n}\right),\mathscr{L}(\theta)\right]\to 0\quad \textit{in probability},$$

for all $r\in(0,p)$.

Proof. – Given the bootstrap limit theorems 2.2 and 2.6 in [2], it suffices to show convergence in probability of the corresponding bootstrap moments. We only consider the case $1<p\leqq 2$ (the case $0<p\leqq 1$ is somewhat simpler). Let $1<r<p\leqq 2$. Let $\{\varepsilon_i\}$ be a Rademacher sequence independent of $\{\mathrm{X}_{ni}\}$. Then, using symmetrization and Khinchin's inequality we have

$$\begin{aligned}
&\hat{\mathrm{E}}\left|\sum_{j=1}^{m_n}(\mathrm{X}_{nj}-\bar{\mathrm{X}}_n)/b_{m_n}\right|^r\\
&\qquad\leqq c_r\left(\hat{\mathrm{E}}\left[\sum_{j=1}^{m_n}(\mathrm{X}_{nj}\mathrm{I}_{|\mathrm{X}_{nj}|\leqq b_{m_n}}-\hat{\mathrm{E}}\mathrm{X}_{nj}\mathrm{I}_{|\mathrm{X}_{nj}|\leqq b_{m_n}})/b_{m_n}\right]^2\right)^{r/2}\\
&\qquad\qquad+c_r\hat{\mathrm{E}}\left|\sum_{j=1}^{m_n}\varepsilon_j\mathrm{X}_{nj}\mathrm{I}_{|\mathrm{X}_{nj}|>b_{m_n}}/b_{m_n}\right|^r\\
&\quad\leqq c_r\left[(m_n/nb_{m_n}^2)\sum_{i=1}^{n}\mathrm{X}_i^2\mathrm{I}_{|\mathrm{X}_i|\leqq b_{m_n}}\right]^{r/2}+c_r'\hat{\mathrm{E}}\left(\sum_{j=1}^{m_n}\mathrm{X}_{nj}^2\mathrm{I}_{|\mathrm{X}_{nj}|>b_{m_n}}/b_{m_n}^2\right)^{r/2}\\
&\quad\leqq c_r\left[(m_n/nb_{m_n}^2)\sum_{i=1}^{n}\mathrm{X}_i^2\mathrm{I}_{|\mathrm{X}_i|\leqq b_{m_n}}\right]^{r/2}+c_r'(m_n/b_{m_n}^r)\hat{\mathrm{E}}|\mathrm{X}_{ni}|^r\mathrm{I}_{|\mathrm{X}_{ni}|\leqq b_{m_n}}
\end{aligned}$$

Each of these summands is bounded in probability because $(m_n/b_{m_n}^2)\mathrm{EX}^2\,\mathrm{I}_{|\mathrm{X}|\leqq b_{m_n}}$ converges to a constant and

$$\mathrm{E}\,(m_n/b_{m_n}^r)\,\hat{\mathrm{E}}\,|\mathrm{X}_{nj}|^r\,\mathrm{I}_{|\mathrm{X}_{nj}|>b_{m_n}}=(m_n/b_{m_n}^r)\,\mathrm{E}\,|\mathrm{X}|^r\,\mathrm{I}_{|\mathrm{X}|>b_{m_n}}$$

also converges to a constant by regular variation. Stochastics boundedness of the sequences $\left\{ \hat{\mathrm{E}} \left| \sum_{j=1}^{n} (\mathrm{X}_{nj} - \bar{\mathrm{X}}_n)/b_{m_n} \right|^r \right\}_{n=1}^{\infty}$, $r<p$, together with weak convergence in probability give the result. □

Theorem 3.5 is sharp. There are sequences m_n so that the conclusion of the theorem does not hold for $r=p$ (for $r=p<2$ the conclusion does not even make sense since $\mathrm{E}|\theta|^p = \infty$).

4. RANDOM NORMINGS FOR THE BOOTSTRAP OF THE MEAN IN GENERAL

If X is in the domain of attraction of the normal law, random normings in the bootstrap CLT have been discussed by several authors for $m_n = n$ (Bickel and Freedman [3] and others) and in [2] and in Section 1 above for any $\{m_n\}$. The normal case is easy to handle because $\left\{ \sum_{i=1}^{n} \mathrm{X}_i^2/b_n^2 \right\}$ converges in probability to a constant (a. s. if $\mathrm{EX}^2 < \infty$). If X is in the domain of attraction of a p-stable law, $1<p\leqq 2$ (the only values of p we will consider here), then

$$(4.1)\qquad \left\{ \sum_{i=1}^{n} (\mathrm{X}_i - \mathrm{EX}) \Big/ \left(\sum_{i=1}^{n} \mathrm{X}_i^2 \right)^{1/2} \right\}_{n=1}^{\infty}$$

still converges in law even though $\sum_{i=1}^{n} \mathrm{X}_i^2/\mathrm{b}_n^2$ does not converge in probability for $p \neq 2$. This limit theorem can be bootstrapped:

4.1. THEOREM. – *Let* X *be in the domain of attraction of a p-stable law,* $1<p\leqq 2$, *and let* $m_n/n \to 0$. *Then*

$$(4.2)\qquad w\text{-}\lim_{n\to\infty} \mathscr{L}\left[\sum_{j=1}^{m_n} (\mathrm{X}_{nj} - \bar{\mathrm{X}}_n) \Big/ \left(\sum_{j=1}^{m_n} \mathrm{X}_{nj}^2 \right)^{1/2} \right] = w\text{-}\lim_{n\to\infty} \mathscr{L}\left[\sum_{j=1}^{n} (\mathrm{X}_i - \mathrm{EX}) \Big/ \left(\sum_{i=1}^{n} \mathrm{X}_i^2 \right)^{1/2} \right]$$

in probability.

Proof. – The case $p=2$ has already been discussed above. So, let $1<p<2$. It is well known that the sequence (4.1) has a limit in law (Logan *et al.* [6], Csörgö and Horvath [4]), actually the sequence

$$(4.3)\qquad \left\{ \sum_{i=1}^{n} ((\mathrm{X}_i - \mathrm{EX})/b_n, \mathrm{X}_i^2/b_n^2) \right\}$$

converges in law to an infinitely divisible law in $\mathbb{R}^2$ without normal part. (4.2) will follow if we show that the sequence

$$\left\{ \sum_{i=1}^{n} ((X_{ni}-\bar{X}_n)/b_{m_n},\ X_{nj}^2/b_{m_n}^2) \right\} \tag{4.4}$$

converges weakly to the same limit as (4.3) in probability. The triangular array $\{(X_{nj}/b_{m_n}, X_{nj}^2/b_{m_n}^2), j \leq m_n, n \in \mathbf{N}\}$ is infinitesimal ω-a. s. ([2]). Hence, by the classical limit theory (e. g. [1]), proving that the limits of (4.3) and (4.4) coincide reduces to proving:

(i) $m_n \hat{P}\{(X_{n1}/b_{m_n}, X_{n1}^2/b_{m_n}^2) \in A\}$ converges in probability to

$$\nu(A) = \lim_{n \to \infty} n P\{(X/b_n, X^2/b_n^2) \in A\}$$

for all Borel sets A such that $0 \in (A^c)^0$ and $\nu(\delta A) = 0$;

(ii) for each $\delta > 0$ $m_n \hat{E} |(X_{n1}/b_{m_n}, X_{n1}^2/b_{m_n}^2)|^2 I_{|(X_{n1}/b_{m_n}, X_{n1}^2/b_{m_n}^2)| \leq \delta}$ converges in probability to some h_δ, with $h_\delta \to 0$ as $\delta \to 0$, where $|.|$ denotes any norm in $\mathbf{R}^2$; we will take $|(x, y)| = |x| \vee |y|$.

(iii)

$$(m_n/b_{m_n}) \hat{E} X_{n1} I_{|X_{n1}| > b_{m_n}} \to \lim_{n \to \infty} (n/b_n) E X I_{|X| > b_n}$$

in probability and

$$(m_n/b_{m_n}^2) \hat{E} X_{n1}^2 I_{|X_{n1}| \leq b_{m_n}} \to \lim_{n \to \infty} (n/b_n^2) E X^2 I_{|X| \leq b_n}$$

in probability.

[(i) ensures that the Lévy measures are the same, (ii) that the normal part of the limit is degenerate and (iii) that centering X_{ni} and not centering X_{ni}^2 in (4.4) have the same effect in the limit as centering X_i and not centering X_i^2 in (4.3)]. Note that an easy proof of weak convergence of (4.3) could be constructed along similar lines, that is, by checking that the triangular array $\{(X_i/b_n, X_i^2/b_n^2), i \leq n\}_{n=1}^{\infty}$ satisfies the classical conditions for the CLT.

Proof of (i). – We have

$$m_n \hat{P}\{(X_{n1}/b_{m_n}, X_{n1}^2/b_{m_n}^2) \in A\} = (m_n/n) \sum_{i=1}^{n} I_{(X_i/b_{m_n}, X_i^2/b_{m_n}^2) \in A}.$$

The expected value tends to $\nu(A)$ and the variance is dominated by

$$(m_n^2/n^2)\, n P\{(X/b_{m_n}, X^2/b_{m_n}^2) \in A\} \leq (\nu(A) + \varepsilon)(m_n/n)$$

for some $\varepsilon > 0$ and large n, which tends to zero.

Proof of (ii). – Only $\delta < 1$ needs to be considered. Then the sequence in (ii) is just $m_n b_{m_n}^{-2} \hat{E} |X_{n1}|^2 I_{|X_{n1}| \leq \delta b_{m_n}}$ and it is already proved in [2], pp. 469-470, that this sequence converges in probability for every $\delta > 0$ to the limit h_δ of its expected values $\{(m_n/b_{m_n}^2) U(\delta b_{m_n})\}$. Then $h_\delta \to 0$ as $\delta \to 0$ because X is in the domain of attraction of a p-stable law, $p<2$.

Proof of (iii). – The second limit is already proved in [2] [*see* the proof of (ii) above]. The proof of the first limit is omitted in [2] altough it is used in Corollary 2.6 there. We give it here. Since

$$E(m_n/b_{m_n}) \hat{E} X_{n1} I_{|X_{n1}|>b_{m_n}} = (m_n/b_{m_n}) EXI_{|X|>b_{m_n}}$$

we only need to prove

$$E|(m_n/b_{m_n}) \hat{E} X_{n1} I_{|X_{n1}|>b_{m_n}} - (m_n/b_{m_n}) EXI_{|X|>b_{m_n}}|^r \to 0$$

for some $r>0$. We take $1<r<p$ and use symmetrization by a Rademacher sequence together with Khinchin's inequality to obtain (for suitable constants c and c')

$$E|(m_n/b_{m_n}) \hat{E} X_{n1} I_{|X_{n1}|>b_{m_n}} - (m_n/b_{m_n}) EXI_{|X|>b_{m_n}}|^r$$

$$\leqq c E \left| (m_n/nb_{m_n}) \sum_{i=1}^{n} \varepsilon_i X_i I_{|X_i|>b_{m_n}} \right|^r.$$

$$\leqq E \left| c' (m_n^2/n^2 b_{m_n}^2) \sum_{i=1}^{n} X_i^2 I_{|X_i|>b_{m_n}} \right|^{r/2}$$

$$\leqq c' (nm_n^r/n^r b_{m_n}^r) E|X|^r I_{|X|>b_{m_n}} = c' (m_n/n)^{r-1} (m_n/b_{m_n}^r) E|X|^r I_{|X|>b_{m_n}}.$$

Since $r-1>0$, $m_n/n \to 0$ and $\{(m_n/b_{m_n}^r) E|X|^r I_{|X|>b_{m_n}}\}$ converges by regular variation, (iii) follows. □

Theorem 4.1 may be useful if it is only known that X is in some domain of attraction. In that case one could take $\hat{t}_\alpha$ such that $\hat{P}\left\{\left|\sum_{i=1}^{m_n} (X_{ni}-\bar{X}_n)\right| \Big/ \left(\sum_{i=1}^{m_n} X_{ni}^2\right)^{1/2} > \hat{t}_\alpha\right\} \cong \alpha$ to obtain that

$$P\left\{|\bar{X}_n - EX| \Big/ \left(\sum_{i=1}^{n} X_i^2\right)^{1/2} > \hat{t}_\alpha\right\} \cong \alpha,$$

and $\hat{t}_\alpha$ is asymptotically correct in probability. (*See* Logan *et al.* [6] for properties of the limiting distributions of these sequences: the limits have densities which are Gaussian like at $\pm\infty$.) Of course m_n must be taken so that $m_n/n \to 0$. It is an open question what $\{m_n\}$ gives best results; some results in [2] seem to suggests that $m_n = n/(\log\log n)^{1+\delta}$ for some $\delta>0$ should not be a bad choice. We should also remark that $\sum_{i=1}^{n} X_i^2$ and $\sum_{i=1}^{m_n} X_{ni}^2$ can be replaced by $\sum_{i=1}^{n} (X_i - EX)^2$ and $\sum_{i=1}^{m_n} (X_{ni} - \bar{X}_n)^2$.

SIMULATIONS

The following simulations were performed. For each value of $p=1.1$, 1.5 and 1.9 and $n=50$ and 100, 1,000 samples of size n from the symetric distribution of F_p were drawn. Here F_p is the symmetric distribution $2F_p(-t)=t^{-1/p}$, $t>1$. These samples were used to compute, for each (n, p), the $\alpha=.90$, $.95$ and $.99$ sample quantiles of the statistic $S=\sum_{i=1}^{n} X_i \Big/ \left(\sum_{i=1}^{n} (X_i-\bar{X}_n)^2\right)^{1/2}$. These are t_α in the Tables bellow [one t-value for each choice of (n, p, α)]. They should be regarded as very good approximations of the true quantiles of S. From each of these samples, say $\mathbf{X}(n, p; i)=(X_1(n, p; i), \ldots, X_n(n, p; i))$, $i=1, \ldots, 1{,}000$, 1,000 bootstrap samples of size m_n were drawn, where $m_{50}=35$ and $m_{100}=65$ (*i.e.* m_n is slighty smaler than $n/\log\log n$), giving, for each n and p, 1,000 values of

$$\hat{S}(n, p)=\sum_{i=1}^{m_n} (X_{ni}-\bar{X}_n) \Big/ \left(\sum_{i=1}^{m_n} (X_{ni}-\bar{X}_{nn})^2\right)^{1/2}.$$

These values were used to compute the .90, .95 and .99 sample quantiles of $\hat{S}(n, p)$, $\hat{t}_{.95}(\mathbf{X}(n,p))$, $\hat{t}_{.90}(\mathbf{X}(n, p))$ and $\hat{t}_{.99}(\mathbf{X}(n, p))$. So, for each choice of (n, p, α), we obtained 1,000 independent replications of $\hat{t}_\alpha(\mathbf{X}(n, p))$ [one for each original sample $\mathbf{X}(n, p; i)$] and with these the distribution of $\hat{t}_\alpha(\hat{S}(n, p))$ was estimated. The Tables below show the median $m\hat{t}_\alpha$; the $.25$ and the $.75$ quantiles, $Q_1\hat{t}_\alpha$ and $Q_3\hat{t}_\alpha$ respectively; the mean $av\,\hat{t}_\alpha$ and the 10% trimmed mean $tav\,\hat{t}_\alpha$ of the distribution of $\hat{t}_\alpha$ for each n and p.

Note that the median of $\hat{t}_\alpha$ approximates t_α quite well and that the approximation of t_α by $\hat{t}_\alpha$ is acceptable at least 50% of times (actually more because the empirical distribution of $\hat{t}_\alpha$ is quite concentrated). Note however that the mean of $\hat{t}_\alpha$ is far off t_α, *particularly for* $p=1.1$: $\hat{t}_\alpha$ does take infrequent very large values which have a considerable effect on the mean (the trimmed mean is also quite close to $\hat{t}_\alpha$). The distribution of $\hat{t}_\alpha$ deserves thus further study. The results become better for larger p, and for each p fixed $m\hat{t}_\alpha$ is closer to t_α when $n=100$, as was to be expected. However the interquantile range $Q_3\hat{t}_\alpha-Q_1\hat{t}_\alpha$ is esentially the same for $n=50$ and for $n=100$; this suggests that the convergence of $\hat{t}_\alpha$ to t_α in probability takes place at a slow rate. These data do not show $\hat{t}_\alpha \to t_\alpha$ in pr. since m_n/n is too large. Analogous simulations were made for $S=\sum_{i=1}^{n} X_i \Big/ \left(\sum_{i=1}^{n} X_i^2\right)^{1/2}$, with similar results which we omit.

TABLES.

$p=1.9$

	$n=100$, $m=65$						$n=50$, $m=35$					
	t	$\mathrm{m}\,\hat{t}$	$Q_1\,\hat{t}$	$Q_3\,\hat{t}$	tav $\hat{t}$	ave $\hat{t}$	t	$\mathrm{m}\,\hat{t}$	$Q_1\,\hat{t}$	$Q_3\,\hat{t}$	tav $\hat{t}$	ave $\hat{t}$
$\alpha=.90$	1.32	1.35	0.91	1.77	1.34	1.42	1.32	1.34	0.97	1.79	1.38	1.39
.95	1.69	1.71	1.27	2.14	1.71	1.81	1.57	1.71	1.33	2.19	1.77	1.79
.99	2.40	2.33	1.90	2.83	2.37	2.48	2.21	2.37	1.90	2.96	2.46	2.51

$p=1.5$

	$n=100$, $m=65$						$n=50$, $m=35$					
	t	$\mathrm{m}\,\hat{t}$	$Q_1\,\hat{t}$	$Q_3\,\hat{t}$	tav $\hat{t}$	ave $\hat{t}$	t	$\mathrm{m}\,\hat{t}$	$Q_1\,\hat{t}$	$Q_3\,\hat{t}$	tav $\hat{t}$	ave $\hat{t}$
$\alpha=.90$	1.32	1.37	0.93	1.85	1.41	1.90	1.29	1.37	0.99	1.89	1.47	1.56
.95	1.70	1.70	1.28	2.25	1.78	2.34	1.59	1.74	1.34	2.33	1.86	2.00
.99	2.33	2.31	1.83	2.98	2.44	3.07	2.11	2.38	1.83	3.07	2.55	2.78

$p=1.1$

	$n=100$, $m=65$						$n=50$, $m=35$					
	t	$\mathrm{m}\,\hat{t}$	$Q_1\,\hat{t}$	$Q_3\,\hat{t}$	tav $\hat{t}$	ave $\hat{t}$	t	$\mathrm{m}\,\hat{t}$	$Q_1\,\hat{t}$	$Q_3\,\hat{t}$	tav $\hat{t}$	ave $\hat{t}$
$\alpha=.90$	1.32	1.39	0.97	2.17	1.65	10.78	1.27	1.40	1.03	2.21	1.81	2.47
.95	1.58	1.71	1.29	2.66	2.07	12.17	1.53	1.74	1.35	2.71	2.26	3.29
.99	2.15	2.25	1.72	3.52	2.82	14.29	2.00	2.33	1.78	3.61	3.04	5.00

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REFERENCES

[1] A. Araujo and E. Giné, *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, New York, 1980.

[2] M. A. Arcones and E. Giné, The Bootstrap of the Mean with Arbitrary Bootstrap Sample Size, *Ann. Inst. Henri Poincaré, Prob. Stat.*, Vol. 25, 1989, pp. 457-481.

[3] D. J. Bickel and D. A. Freedman, Some Assymptotic Theory for the Bootstrap, *Ann. Statist.*, Vol. **9**, 1981, p. 1196-1217.

[4] M. Csörgö and L. Horvath, Asymptotic Representation of Self-Normalized Sums, *Prob. Math. Statist.*, Vol. **9**, 1988, pp. 15-24.

[5] R. Le Page, M. Woodroofe and J. Zinn, Convergence to a Stable Distribution via Order Statistics, *Ann. Prob.*, Vol. **9**, 1981, pp. 624-632.

[6] B. F. Logan, C. L. Mallows, S. O. Rice and L. A. Shepp, Limit Distributions of Self Normalized Sums, *Ann. Prob.*, Vol. **1**, 1973, pp. 788-809.

[7] W. Stout, *Almost Sure Convergence*, Academic Press, New York, 1974.

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