ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 29, nº 1 (1993), p. 105-117
<http://www.numdam.org/item?id=AIHPB_1993__29_1_105_0>

Ann. Inst. Henri Poincaré,

Vol. 29, n° 1, 1993, p. 105-117. *Probabilités et Statistiques*

On subinvariant measures for positive operators in L_1

by

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ABSTRACT. — We study the problem of existence of a strictly positive invariant element for a positive operator T on L_1, and show its equivalence with a ratio limit theorem for T*.

THEOREM A. — *Let* T *be a positive operator on* L_1 *satisfying*

(∗) *For* $0 \neq f \in L_\infty^+$ *there exists* $\nu \in L_\infty^*$ *with* $0 \leqq \nu = T^{**} \nu$ *and* $\nu(f) > 0$.

Then there exists $u \in L_1^+$, $u > 0$ a.e. and $T u \leqq u$. If $\sum_{n=0}^{\infty} T^{*n} 1 \equiv \infty$, then $T u = u$, and (∗) is also a necessary condition.

THEOREM B. — *Let* T *be a Cesàro-bounded positive operator in* L_1. *Then* (∗) *is equivalent to* $\overline{(I - T^*) L_\infty} \cap L_\infty^+ = \{0\}$.

RÉSUMÉ. — Nous étudions le problème de l'existence d'un élément strictement positif et sous-invariant pour un opérateur positif T sur L_1, et démontrons l'équivalence avec un théorème limite quotient pour T*.

(1) This research was started by the first two authors twenty years ago, but for various reasons was not completed prior to Shlomo Horowitz's untimely death in 1978. The other authors dedicate this paper to his memory.

THÉORÈME A. — *Soit* T *un opérateur positif sur* L_1, *satisfaisant la condition*

(*) *Pour toute* $0 \neq f \in L_\infty^+$ *il existe* $\nu \in L_\infty^*$ *satisfaisant* $0 \leqq \nu = T^{**}\nu$ *et* $\nu(f) > 0$.

Alors il existe $u \in L_1^+$, $u > 0$ p. p. et $Tu \leqq u$. Si $\sum_{n=0}^{\infty} T^{*n} 1 \equiv \infty$, on a $Tu = u$, et la condition (*) est aussi nécessaire.

THÉORÈME B. — *Soit* T *un opérateur positif Cesàro-borné dans* L_1. *Alors* (*) *est équivalente à* $\overline{(I - T^*) L_\infty} \cap L_\infty^+ = \{0\}$.

Birkhoff's individual ergodic theorem was originally proved for measure preserving transformations, and the study of necessary and sufficient conditions for the existence of an equivalent finite invariant measure was soon started by E. Hopf in 1932. The study continued later as finding strictly positive functions in $L_1(m)$, invariant under a given positive contraction T, and it turned out that the conditions are in terms of the dual operator T*. *See* [K], pp. 142-145, for the results and historical remarks. One of the necessary and sufficient conditions, introduced by Brunel [B_1], is $\overline{(I - T^*) L_\infty} \cap L_\infty^+ = \{0\}$. Horowitz [H] gave a proof of sufficiency without using weakly wandering functions.

Sucheston [S] started the study of power-bounded positive operators in L_1. The ratio limit theorem for such operators was studied in [I-Mo]. Derriennic and Lin [DL] extended Sucheston's decomposition to Cesàro bounded $\left(\sup_n \| M_n(T) \| < \infty, \text{ where } M_n(T) = \frac{1}{n} \sum_{k=0}^{n-1} T^k\right)$ positive operators on $L_1(m)$, and obtained some necessary and sufficient conditions for the existence of $u > 0$ a. e., $u \in L_1$ with $Tu = u$. Some of these conditions were extended to Banach lattices; for the latest results and references, *see* [BS].

Sato [Sa] proved that for a Cesàro bounded positive operator on $L_1(m)$, a. e. convergence of $M_n(T^*) f$ for every $f \in L_\infty$ is equivalent to the L_1-norm convergence of $M_n(T) v$ for every $v \in L_1$. If, in that case, $\lim M_n(T^*) 1 \not\equiv 0$, we have non-zero positive fixed points (invariant measures) for T. If there exists $u \in L_1^+$ with $u > 0$ a. e. and $Tu \leqq u$, then $M_n(T^*) f$ converges a. e. for every $f \in L_\infty$ (or even with $\int |f| u\, dm < \infty$) [DL].

Ornstein [O] studied, for abstract positive operators, the relationship between invariant measures and the ratio limit theorem, using a method of "modifications" (*see* also [B_2]).

The following proposition follows from ideas in [O].

PROPOSITION 1. – *Let* T *be a positive operator on* $L_1(X, \Sigma, m)$. *Then the following are equivalent:*

(i) *There exists* $u \in L_1^+(m)$, $u>0$ *a.e. and* $Tu \leqq u$.

(ii) *For every* $f, g \in L_\infty^+(m)$, *the ratios* $\sum_{k=0}^{n} T^{*k} f(x) / \sum_{k=0}^{n} T^{*k} g(x)$ *converge* (*to a finite limit*) *a.e. on the set* $S_g = \left\{ x : \sum_{k=0}^{\infty} T^{*k} g(x) > 0 \right\}$, *and* $T^{*n+1} f(x) / \sum_{k=0}^{n} T^{*k} 1(x) \to 0$ *a.e.*

Furthermore, if (ii) [*or* (i)] *is satisfied and* $\sum_{k=0}^{\infty} T^{*k} 1 = \infty$ *a.e., then we have* $Tu = u$.

Proof. – (i) ⇒ (ii). T^* is a contraction of $L_1(u\,dm)$, and the Chacon-Ornstein theorem and lemma apply.

(ii) ⇒ (i). By (ii), $\lim_{n \to \infty} \sum_{k=0}^{n} T^{*k} f \Big/ \sum_{k=0}^{n} T^{*k} 1$ exists a.e., and for $f \geqq 0$ the limit must be positive a.e. on $S_f = \left\{ \sum_{k=0}^{\infty} T^{*k} f > 0 \right\}$. we may and do assume m finite. Define for f bounded measurable

$$\mu(f) = \int \lim_n \left(\sum_{k=0}^{n} T^{*k} f / \sum_{k=0}^{n} T^{*k} 1 \right) dm = \lim_n \int \left(\sum_{k=0}^{n} T^{*k} f / \sum_{k=0}^{n} T^{*k} 1 \right) dm$$

which exists by the dominated convergence theorem. $\mu \geqq 0$ is a measure by the Vitali-Hahn-Saks theorem. Now $\left| T^{*n+1} f / \sum_{k=0}^{n} T^{*k} 1 \right| \leqq \| T^* |f| \|_\infty$. For $f \in L_\infty^+$,

$$\mu(T^* f) \leqq \mu(f) + \int \lim \left(T^{*n+1} f / \sum_{k=0}^{n} T^{*k} 1 \right) dm = \mu(f).$$

LEMMA 2. – *Let* X *be a compact Hausdorff space, and* Σ *the* σ-*algebra of Baire sets. Let* Q *be a positive operator on* C(X).

(i) *There exists a transition measure* $Q(x, A)$ *on* $X \times \Sigma$ *such that* $Qf(x) = \int f(y) Q(x, dy)$ *extends* Q *to* $B(X, \Sigma)$.

(ii) *If* $0 \leqq \nu \in C(X)^*$ *satisfies* $Q^* \nu = \nu$ ($\nu \neq 0$), *then* Q *can be extended to a linear contraction* S *of* $L_1(X, \Sigma, \nu)$.

Proof. — (i) is a standard result: $Q(x, A) = Q^* \delta_x(A)$ (*see* [H]).
(ii) For $f \in C(X)$ we have

$$\int Q f \, d\nu = \langle Q f, \nu \rangle = \langle f, Q^* \nu \rangle = \langle f, \nu \rangle = \int f \, d\nu$$

Standard arguments yield the result. Note that the dual of the $L_1(\nu)$ contraction S is the operator R on $L_\infty(\nu)$ defined by $R f = d(Q^*(f \nu))/d\nu$.

THEOREM 3. — *Let* T *be a positive operator on* $L_1(X, \Sigma, m)$. *Assume*
($*$) *For every* $0 \neq f \in L_\infty^+$ *there exists* $\nu \in L_\infty^*$ *satisfying* $0 \leqq \nu = T^{**} \nu$ *and* $\nu(f) > 0$. *Then:*

(i) *There exists* $u \in L_1^+(m)$, $u > 0$ a.e. and $T u \leqq u$.

(ii) *For every* $f \in L_\infty^+(m)$, $\sum_{k=0}^{n} T^{*k} f(x) / \sum_{k=0}^{n} T^{*k} 1(x)$ *converges a.e., and the limit is positive a.e. on* $\left\{ x : \sum_{k=0}^{\infty} T^{*k} f(x) > 0 \right\}$. *Moreover, the convergence holds if* $\int f u \, dm < \infty$.

Proof. — We use for the proof the following reductions, given in [H]: $L_\infty(X, \Sigma, m)$ is isometrically and order isomorphic to a space $C(\tilde{X})$ with $\tilde{X}$ a (extremally disconnected) compact Hausdorff space, the finite measure m corresponds to a measure $\tilde{m} \in C(\tilde{X})^*$, $\mu \in L_\infty^*(m)$ is a measure if and only if its representation $\tilde{\mu} \in C(\tilde{X})^*$ is absolutely continuous with respect to $\tilde{m}$, $L_\infty(\tilde{X}, \tilde{\Sigma}, \tilde{m})$ is isometrically order isomorphic to $L_\infty(X, \Sigma, m)$ ($\tilde{\Sigma}$ is the σ-algebra of Baire sets), and the isometry induces an isometry of $L_1(m)$ and $L_1(\tilde{m})$.

Let $\tilde{T}$ be the representation of T in $L_1(\tilde{m})$. Then $\tilde{T}$ also satisfies ($*$), and if $\tilde{u} \in L_1^+(\tilde{m})$ satisfies $\tilde{T} \tilde{u} \leqq \tilde{u}$, $T u \leqq u$. Also $u \sim m \Leftrightarrow \tilde{u} \sim \tilde{m}$. Furthermore, T^* is represented by a positive operator Q on $C(\tilde{X})$. The extension of Q to $B(\tilde{X}, \tilde{\Sigma})$ satisfies $Q \tilde{f} = \tilde{T}^* f = \tilde{m} - a.e.$

Thus, we may assume X to be a compact Hausdorff space, T^* given by extending a positive operator Q on $C(X)$, and $L_\infty^*(m) = C(X)^*$. The ideas now follow [H].

CLAIM 1. — *For every* $f \in B(X, \Sigma)$, $\sum_{k=0}^{n-1} T^{*k} f / \sum_{k=0}^{n-1} T^{*k} 1$ *converges* $m - a.e.$

We may assume $f \geqq 0$. Since $T^{*k} f = Q^k f$ a.e., we define

$$E = \left\{ x : \lim_{n \to \infty} M_n(Q) f(x) / M_n(Q) 1(x) \text{ exists} \right\}.$$

Assume $m(E^c) > 0$. By the assumption of the theorem, there exists $0 \leqq \nu \in C(X)^*$ with $Q^* \nu = \nu$ and $\nu(E^c) > 0$. By lemma 2, Q can be extended to a contraction S of $L_1(\nu)$. By the Chacon-Ornstein theorem, $M_n(Q)f/M_n(Q)1 = M_n(S)f/M_n(S)1$ (ν a. e.) converges $\nu - a.e.$ Hence $\nu(E^c) = 0$ which is a contradiction, showing $m(E^c) = 0$.

CLAIM 2. – *For every* $0 \leqq f \in B(X, \Sigma)$ *with* $\int f\,dm > 0$ *we have* $\lim_{n \to \infty} \sum_{k=0}^{n-1} T^{*k} f / \sum_{k=0}^{n-1} T^{*k} 1 > 0$ *m - a. e. on* $\{f > 0\}$.

Let $B = \{x : \lim_{n \to \infty} M_n(Q) f(x)/M_n(Q) 1(x) = 0\}$ and let $A = \{f > 0\}$. Assume $m(A \cap B) > 0$. Then by our assumption, there exists $0 \leqq \nu \in C(X)^*$ with $Q^* \nu = \nu$ and $\nu(A \cap B) > 0$. Again, by lemma 2, we use the Chacon-Ornstein theorem to obtain $\lim_{n \to \infty} M_n(Q) f(x)/M_n(Q) 1(x) > 0$ ν - a. e. on $A = \{f > 0\}$, since $\nu(A) \geqq \nu(A \cap B) > 0$ implies $\int f\,d\nu > 0$. But then $\nu(A \cap B) = 0$, a contradiction. Hence $m(A \cap B) = 0$.

CLAIM 3. – *For any* $0 \leqq f \in B(X, \Sigma)$ *there exists* α *such that*

$$T^{*n} f(x) / \sum_{k=0}^{n-1} T^{*k} 1(x) \leqq \alpha \ m\text{-a. e.}$$

and $T^{*n} f(x) / \sum_{k=0}^{n-1} T^{*k} 1(x) \to 0$ m - a. e.

$$T^{*n} f \leqq \sum_{k=0}^{n-1} T^{*k}(T^* f) \leqq \sum_{k=0}^{n-1} T^{*k} 1 (\|T^* f\|_\infty)$$

so $\alpha = \|T^* f\|_\infty$.

Let $F = \{x : \lim_{n \to \infty} Q^n f(x) / \sum_{k=0}^{n-1} Q^k 1(x) = 0\}$.

Assume $m(F^c) > 0$, so again there exists $0 \leqq \nu \in C(X)^*$ with $Q^* \nu = \nu$ and $\nu(F^c) > 0$. The Chacon-Ornstein lemma ([K], p. 12) applied to Q as a contraction in $L_1(\nu)$ yields $Q^n f(x) / \sum_{k=0}^{n-1} Q^k 1(x) \to 0$ ν - a. e., so $\nu(F^c) = 0$, a contradiction. Hence $m(F^c) = 0$.

Now claims 1, 2 and 3 allow us to deduce, by proposition 1, that there exists $u \in L_1^+$, $u > 0$ a. e., satisfying $T u \leqq u$.

COROLLARY 4. — *Let* T *be a positive operator on* $L_1(X, \Sigma, m)$. *Assume that* $\sum_{n=0}^{\infty} T^{*n} 1 = \infty$ *a.e. Then the following are equivalent:*

(i) *There exists* $u \in L_1$, $u > 0$ *a.e. and* $Tu = u$.

(ii) *For every* $f \in L_\infty^+$, $\sum_{k=0}^{n} T^{*k} f / \sum_{k=0}^{n} T^{*k} 1$ *converges a.e., with the limit positive a.e. on* $\left\{ \sum_{k=0}^{\infty} T^{*k} f > 0 \right\}$, *and* $T^{*n+1} f / \sum_{k=0}^{n} T^{*k} 1 \to 0$ *a.e.*

(iii) *For every* $0 \neq f \in L_\infty^+$ *there exists* $\nu \in L_\infty^*$ *with* $0 \leqq \nu = T^{**} \nu$ *and* $\nu(f) > 0$.

Proof. — (i)$\Leftrightarrow$(ii) by proposition 1. (iii)$\Rightarrow$(ii) by theorem 3, and (i)$\Rightarrow$(iii) by taking $d\nu = u\,dm$.

Remarks. — 1. The assumption $\sum_{n=0}^{\infty} T^{*n} 1 = \infty$ a.e. is used only for proving (ii)$\Rightarrow$(i).

2. If T is a Cesàro bounded positive operator satisfying (i), then $\sum_{n=0}^{\infty} T^{*n} 1 < \infty$ a.e. on Z (of the Sucheston Y+Z decomposition), by theorem 3.1 of [DL]. Examples with $m(Z) > 0$ as in [DL] show that the assumption of corollary 4 is not necessary for (i).

3. The equivalence of the conditions of corollary 4 is still true if, instead of $\sum_{n=0}^{\infty} T^{*n} 1 = \infty$ a.e., we assume that

$$\textit{for every } f \in L_\infty^+ \quad \textit{with } f \not\equiv 0, \qquad m\left(\left\{ \sum_{n=0}^{\infty} T^{*n} f = \infty \right\} \right) > 0$$

We have to show only (ii)$\Rightarrow$(i). Let $A = \left\{ \sum_{n=0}^{\infty} T^{*n} 1 = \infty \right\}$. By (ii) $\left\{ \sum_{n=0}^{\infty} T^{*n} f = \infty \right\} \subset A$ for any $f \in L_\infty^+$. In the proof of proposition 1, we define $\mu(f)$ by integrating *over* A, and obtain $\mu \sim m$ and invariant.

4. If T is Cesàro bounded and has an equivalent finite invariant measure, then $m(Y) > 0$, and $\sum_{n=0}^{\infty} T^{*n} 1 = \infty$ a.e. on Y. The next example shows that without any boundedness assumption, we may have $\sum_{n=0}^{\infty} T^{*k} 1 < \infty$ a.e.

EXAMPLE. – On $\mathbf{Z}$ we take the counting measure, and for $f\in l_1$ we define $(\mathrm{T}f)(n)=f(n+1)w(n)$, where $w(n)=2$ for $n>0$, $w(n)=1/2$ for $n<0$, and $w(0)=1$. Then $\|\mathrm{T}\|=2$. We easily compute $(\mathrm{T}^*h)(n)=h(n-1)w(n-1)$ for $h\in l_\infty$. Define $u\in l_1$ by $u(0)=1$, $u(n)=\frac{1}{2^{n-1}}$ for $n>0$, and $u(n)=\frac{1}{2^{|n|}}$ for $n<0$.

It is easy to check that $\mathrm{T}u=u$. $\mathrm{T}^*1(n)=\mathrm{w}(\mathrm{n}-1)$, and, by induction, $\mathrm{T}^{*k}1(n)=\prod_{j=1}^{k}w(n-j)$. For $n\leqq 0$, $\mathrm{T}^{*k}1(n)=\frac{1}{2^k}$, and $\sum_{k=0}^{\infty}\mathrm{T}^{*k}1(n)=2$. For $n\geqq 1$, when $k>n$, $\mathrm{T}^{*k}1(n)=\frac{2^{n-1}}{2^{k-n}}=\frac{1}{2^{k+1-2n}}$, so again $\sum_{k=0}^{\infty}\mathrm{T}^{*k}1(n)<\infty$.

We now study Cesàro bounded operators. The following result was overlooked in [DL].

LEMMA 5. – *Let* T *be a Cesàro bounded positive operator in* $\mathrm{L}_1(\mathrm{X},\Sigma,m)$. *Then* $\|\mathrm{T}^n/n\|\to 0$.

Proof. – The estimate in the proof of theorem 2.1 (v) of [DL] yields, for any $f\in\mathrm{L}_1$,

$$\|\mathrm{T}^n f/n\|_1\leqq\|\mathrm{T}^n|f|/n\|\leqq\mathrm{K}^2\|\mathrm{T}\|\,\|f\|_1/\sum_{k=1}^{n}\frac{1}{k}$$

where $\mathrm{K}=\sup_n\|\mathrm{M}_n(\mathrm{T})\|$.

If condition (*) of theorem 3 holds, there exists a finite subinvariant measure μ equivalent to m. Then T^* is a contraction of $\mathrm{L}_1(\mu)$, and $\mathrm{M}_n(\mathrm{T}^*)f$ converges a.e. for $f\in\mathrm{L}_1(\mu)$, and, in particular, for any $f\in\mathrm{L}_\infty$ (by [DL], theorem 3.1 and corollary 3.1, where only subinvariance is needed.) Also, if $m(\mathrm{Y})>0$ (Where $\mathrm{X}=\mathrm{Y}+\mathrm{Z}$ is the Sucheston decomposition), $\mathrm{T}u=u$ on $\mathrm{Y}\,(u=d\mu/dm)$. However, the convergence of $\mathrm{M}_n(\mathrm{T}^*)f$ does not seem to yield directly an *equivalent* invariant measure, even if $m(\mathrm{Y})>0$. On the other hand, the existence of an equivalent finite invariant measure implies $m(\mathrm{Y})>0$.

THEOREM 6. – *Let* T *be a Cesàro bounded positive operator on* $\mathrm{L}_1(\mathrm{X},\Sigma,m)$. *Then the following are equivalent:*

(i) $\overline{(\mathrm{I}-\mathrm{T}^*)\mathrm{L}_\infty}\cap\mathrm{L}_\infty^+=\{0\}$.

(ii) $\limsup_{n\to\infty}\|\mathrm{M}_n(\mathrm{T}^*)1_\mathrm{A}\|_\infty>0$ *for any* $\mathrm{A}\in\Sigma$ *with* $m(\mathrm{A})>0$.

(iii) *For every* $0\neq f\in\mathrm{L}_\infty^+$ *there exists* $\nu\in\mathrm{L}_\infty^*$, $0\leqq\nu=\mathrm{T}^{**}\nu$, *with* $\nu(f)>0$.

Proof. – (i)$\Leftrightarrow$(ii) by the mean ergodic theorem ([K], p. 73) applied to T^*, since $\|\mathrm{T}^{*n}/n\|\to 0$ (lemma 5). (iii)$\Rightarrow$(ii) is obvious.

(i)$\Rightarrow$(iii): By the separation theorem, there exists $\eta\in\mathrm{L}_\infty^*$ with $\eta(f)\neq 0$ and $\mathrm{T}^{**}\eta=\eta$. We need a *positive* functional, so let $\mu=|\eta|$. Then

$T^{**}\mu \geqq \mu$, and $\mu(f) \geqq |\eta(f)| > 0$. Let $\nu = \lim \uparrow T^{**n}\mu$. Since T^{**} is Cesàro bounded, $\|\nu\| \leqq K\|\mu\|$. $T^{**}\nu = \nu$ and $\nu(f) > 0$ are obvious.

LEMMA 7. — *Let* T *be a linear operator on* $L_1(m)$, *and assume* $\overline{(I-T^*)L_\infty} \cap L_\infty^+ = \{0\}$. *Define* $\rho(f) = \inf\{\alpha : f \leqq \alpha + (I-T^*)\Phi$ *for some* $\Phi \in L_\infty\}$. *Then:*

1. $\rho(\lambda) = \lambda$ *for every real number* λ.
2. $\rho(f+g) \leqq \rho(f) + \rho(g)$, *and* $\rho(\lambda f) = \lambda\rho(f)$ *for* $\lambda \geqq 0$.
3. $\rho(f+\lambda) = \rho(f) + \lambda$ *for any real* λ.
4. $f \geqq g \Rightarrow \rho(f) \geqq \rho(g)$.
5. $\rho(T^* f) = \rho(f)$, *and* $\rho(T^* f - f) = \rho(f - T^* f) = 0$.
6. *If* $\nu \in L_\infty^*$ *satisfies* $\nu(f) \leqq \rho(f)$ *for every* $f \in L_\infty$, *then* $\nu \geqq 0$, $T^{**}\nu = \nu$, *and* $\nu(1) = 1$.
7. $\rho(f) = \sup\{\nu(f) : T^{**}\nu = \nu \geqq 0, \nu \in L_\infty^*$ *with* $\nu(1) = 1\}$.

If, in addition, T *is positive and Cesàro bounded, then* $\rho(f) > 0$ *for* $0 \neq f \in L_\infty^+$.

Proof. — 1. Putting $\alpha = \lambda$ and $\Phi = 0$ we obtain $\rho(\lambda) \leqq \lambda$. If $\rho(\lambda) < \lambda$, there exist $\varepsilon > 0$ and $\Phi \in L_\infty$ such that $\lambda \leqq \lambda - \varepsilon + (I-T^*)\Phi$. Then $(I-T^*)\Phi \geqq \varepsilon$, contradicting the assumption $\overline{(I-T^*)L_\infty} \cap L_\infty^+ = \{0\}$.

2. For $\varepsilon > 0$, let $\Phi_1, \Phi_2 \in L_\infty$ such that $f \leqq \rho(f) + \varepsilon + (I-T^*)\Phi_1$, $g \leqq \rho(g) + \varepsilon + (I-T^*)\Phi_2$. Then

$$f + g \leqq \rho(f) + \rho(g) + 2\varepsilon + (I-T^*)(\Phi_1 + \Phi_2)$$

Hence $\rho(f+g) \leqq \rho(f) + \rho(g)$. Similarly, we obtain $\rho(\lambda f) = \lambda\rho(f)$.

3. Is proved as above, and 4. is easy.

5. The first part follows from

$$T^* f \leqq \alpha + (I-T^*)(\Phi - f) \quad \Leftrightarrow \quad f \leqq \alpha + (I-T^*)\Phi$$

For the second part, $\alpha = 0$ and $\Phi = f$ yield $\rho(f - T^* f) \leqq 0$, and $\rho(f - T^* f) < 0$ yields a contradiction as in (1). Similiarly, $\rho(T^* f - f) = 0$.

6. Let $\nu(f) \leqq \rho(f)$ for every $f \in L_\infty$. Then $\nu(T^* f - f) \leqq 0$ and $\nu(f - T^* f) \leqq 0$, so $T^{**}\nu = \nu$. For $f \geqq 0$, we have

$$-\nu(f) = \nu(-f) \leqq \rho(-f) \leqq \rho(0) = 0$$

so $\nu \geqq 0$. Finally, $\nu(1) \leqq 1$ and $-\nu(1) = \nu(-1) \leqq \rho(-1)$ imply $\nu(1) = 1$.

7. Follows from 6. and a general consequence of the Hahn-Banach theorem ([M], p. 272). [The inequality $\nu(f) \leqq \rho(f)$ for all the norm one positive invariant functionals is immediate].

The final assertion follows from theorem 6. ■

Our next results will first require the following technical lemma.

LEMMA 8. — *Let* $\mathscr{F}$ *be a family of functions in* L_∞, $0 \leqq f \leqq 1$, *satisfying:*

(i) *if* $f_1, f_2 \in \mathscr{F}$ *and* $f_1 + f_2 \leqq 1$, *then* $f_1 + f_2 \in \mathscr{F}$.

(ii) *For each* $0 \neq g \in L_\infty^+$ *there exists* $0 \neq f \leqq g$ *such that* $f \in \mathscr{F}$.

Then there exists an increasing sequence $\{f_n\} \subset \mathscr{F}$ *such that* $f_n \nearrow 1$.

Proof. – $\mathscr{F}$ is non-empty by (ii). Let $\alpha_0=\sup\left\{\int f\,dm \mid f\in\mathscr{F}\right\}$. There exists a function $f_0\in\mathscr{F}$ such that $\int f_0\,dm\geqq\frac{a_0}{2}$. Let

$$\alpha_1=\sup\left\{\int f\,dm \mid f\leqq 1-f_0, f\in\mathscr{F}\right\};$$

it is clear that $0<\alpha_1\leqq\frac{\alpha_0}{2}$ and there exists a function $f_1\leqq 1-f_0, f_1\in\mathscr{F}$ and $\int f_1\,dm\geqq\frac{\alpha_1}{2}$. We define by induction a sequence of function $\{f_n\}\subset\mathscr{F}$ such that $\sum_{n=0}^{\infty} f_n\leqq 1$,

$$\sup\left\{\int f\,dm \mid f\leqq 1-\sum_{k=0}^{n-1} f_k, f\in\mathscr{F}\right\}=\alpha_n\leqq\frac{\alpha_0}{2^n} \quad \text{and} \quad \int f_n\,dm\geqq\frac{\alpha_n}{2}.$$

Assume that we have defined $\{f_k\}_{k=0}^{n}$ and $\left(\text{if } \sum_{k=0}^{n}\neq 1\right)$ define

$$\alpha_{n+1}=\sup\left\{\int f\,dm \mid f\leqq 1-\sum_{k=0}^{n} f_k, f\in\mathscr{F}\right\}.$$

it is clear that $0<\alpha_{n+1}\leqq\frac{\alpha_n}{2}\leqq\frac{\alpha_0}{2^{n+1}}$, and there exists a function $f_{n+1}\leqq 1-\sum_{k=0}^{n} f_k$, $f_{n+1}\in\mathscr{F}$ such that $\int f_{n+1}\,dm\geqq\frac{\alpha_{n+1}}{2}$. It is clear that $\sum_{k=0}^{n} f_k\in\mathscr{F}$, $\forall n$. If $f\in\mathscr{F}$ and $f\leqq 1-\sum_{k=0}^{\infty} f_k$ then $\int f\,dm\leqq\frac{\alpha_0}{2^n}$, $\forall n$, hence $f\equiv 0$. Thus $\sum_{k=0}^{\infty} f_k=1$ and the lemma is proved.

LEMMA 9. – *Let* T *be a linear operator on* $L_1(m)$ *such that* $(I-T^*)L_\infty\cap L_\infty^+=\{0\}$, *and assume that* $\rho(f_n)\to 1$ *whenever* $0\leqq f_n\leqq 1$ *satisfies* $f_n\nearrow 1$ *a.e. Then there exists* $u\in L_1^+$ *with* $Tu=u$, *and* $m\{u>0\}>0$.

Proof. – We will show the existence of a finite measure $\mu\neq 0$ such that $\int f\,d\mu\leqq\rho(f)$ for every $f\in L_\infty^+$, and obtain that $\int f\,d\mu\leqq\rho(f)$ for every $f\in L_\infty$ [by lemma 7, 3.], so that μ is invariant by lemma 7, 6. Let $\mathscr{M}$ be the set of positive measures which are dominated by

$\rho : \mu \in \mathcal{M} \Leftrightarrow \int f\, d\mu \leqq \rho(f)$, $\forall f \geqq 0$. Let us define a partial order in $\mathcal{M}$ as usual: $\mu_1 \leqq \mu_2 \Leftrightarrow \int f\, d\mu_1 \leqq \int f\, d\mu_2$, $\forall f \geqq 0$. Let $\{\mu_\alpha\}$ be a completely ordered subset of $\mathcal{M}$, and let $\gamma = \sup\{\mu_\alpha(X)\} \leqq 1$. Then there exists a sequence $\{\mu_n\} \subset \{\mu_\alpha\}$ such that $\mu_n(X) \nearrow \rho$. $\{\mu_n\}$ is an ordered sequence and $\mu = \lim_{n \to \infty} \mu_n$ is an upper bound for $\{\mu_\alpha\}$. Hence by Zorn's Lemma, $\mathcal{M}$ has maximal elements. Let μ be a maximal element of $\mathcal{M}$. If $\mu(X) < 1$, define $\tau = \rho - \mu$ and then: (i) $\tau(f+g) \leqq \tau(f) + \tau(g)$, (ii) $f_n \nearrow 1 \Rightarrow \tau(f_n) \to \tau(1)$, by our assumption.

CLAIM. — *If $\tau(1) > 0$, there exists a function $0 \leqq g \leqq 1$ and a positive number r such that for each $0 \leqq f \leqq g$, $r \int f\, dm \leqq \tau(f)$.*

If it is not true, then for every function $0 \leqq g \leqq 1$ and $0 < r < \tau(1)$ there exists a function $0 \leqq f \leqq g$ such that $r \int f\, dm \geqq \tau(f)$. Fix r, and define $\mathcal{F} = \{0 \leqq f \leqq 1 \mid r \int f\, dm \geqq \tau(f)\}$. If $f_1, f_2 \in \mathcal{F}$ and $f_1 + f_2 \leqq 1$ then

$$r \int (f_1 + f)_2\, dm = r \int f_1\, dm + r \int f_2\, dm \geqq \tau(f_1) + \tau(f_2) \geqq \tau(f_1 + f_2)$$

so $f_1 + f_2 \in \mathcal{F}$. We assumed that $\forall\, 0 \leqq g \leqq 1$, $\exists\, 0 \leqq f \leqq g$, $f \in \mathcal{F}$.

Hence by lemma 8, there exists an increasing sequence $\{f_n\} \subset \mathcal{F}$ such that $f_n \nearrow 1$. But $1 = \lim_{n \to \infty} \int f_n\, dm \geqq 1/r \lim_{n \to \infty} \tau(f_n) = \dfrac{\tau(1)}{r} > 1$, a contradiction. So, the claim is proved.

It is clear that for the set $A = \{g > \delta\}$ ($\delta > 0$ small enough) we have $r \int_A f\, dm \leqq \tau(f)$, for each $0 \leqq f \in L_\infty$.

For $\mu' = \mu + rm I_A$, we have $\mu' \geqq \mu$ and $\mu' \in \mathcal{M}$, which contradicts the maximality of μ. Hence $\mu(X) = 1$.

Let $f \in L_\infty$ be any function. It can be found a number $\lambda \geqq 0$ such that $f + \lambda \geqq 0$; thus:

$$\rho(f) + \lambda = \rho(f + \lambda) \geqq \mu(f + \lambda) = \mu(f) + \lambda \Rightarrow \rho(f) \geqq \mu(f)$$

Hence μ is invariant by lemma 7, 6.

LEMMA 10. — *Let* T *be a Cesàro bounded positive operator on* $L_1(X, \Sigma, m)$. *Assume*

(∗∗) *If* $\|(\mathrm{I}-\mathrm{T}^*)\Phi_n\| \leqq \mathrm{K}$ *and* $\liminf_{n\to\infty}(\mathrm{I}-\mathrm{T}^*)\Phi_n \geqq 0$ *a.e., then* $\liminf_n(\mathrm{I}-\mathrm{T}^*)\Phi_n = 0$ *a.e.*

Then:

(i) $\overline{(\mathrm{I}-\mathrm{T}^*)\mathrm{L}_\infty} \cap \mathrm{L}_\infty^+ = \{0\}$.

(ii) *There exists* $u \in \mathrm{L}_1^+$, $u \not\equiv 0$, *such that* $\mathrm{T}u = u$.

(iii) $m(\mathrm{Y}) > 0$.

Proof. – (i) If $f \geqq 0$ and $f \in \overline{(\mathrm{I}-\mathrm{T}^*)\mathrm{L}_\infty}$, there exist functions $\Phi_n \in \mathrm{L}_\infty$ with $\|(\mathrm{I}-\mathrm{T}^*)\Phi_n - f\| \to 0$, so $\liminf(\mathrm{I}-\mathrm{T}^*)\Phi_n = f \geqq 0$, hence $f=0$. Clearly (ii) ⇒ (iii).

(ii) By (i), we can use the convex functional ρ defined in lemma 7, and $\rho(f) > 0$ for $0 \not\equiv f \in \mathrm{L}_\infty^+$. Let $0 \leqq f_n \leqq 1$ with $f_n \to 1$ a.e. Since ρ is order preserving, $\rho(f_n) \leqq 1$. We show that $\rho(f_n) \to 1$. If not, we have, by taking a suitable subsequence, $\lim \rho(f_n) < \alpha < 1$. Hence there exists (for n large enough) $\Phi_n \in \mathrm{L}_\infty$ with $f_n < \alpha + (\mathrm{I}-\mathrm{T}^*)\Phi_n$. Since T is positive,

$$\mathrm{M}_k(\mathrm{T}^*) f_n \leqq \alpha\, \mathrm{M}_k(\mathrm{T}^*)\, 1 + (\Phi_n - \mathrm{T}^{*k}\Phi_n)/k$$

Fix n. By lemma 5, we can choose k such that $\|\Phi_n - \mathrm{T}^{*k}\Phi_n\|/k < \frac{1}{n}$ so we obtain

$$f_n \leqq \alpha + [\mathrm{I} - \mathrm{M}_k(\mathrm{T}^*)](f_n - \alpha) + \frac{1}{n}$$

Now

$$\|[\mathrm{I}-\mathrm{M}_k(\mathrm{T}^*)](f_n-\alpha)\| \leqq 1 + \sup_j \|\mathrm{M}_j(\mathrm{T}^*)\|,$$

and $[\mathrm{I}-\mathrm{M}_k(\mathrm{T}^*)](f_n-\alpha) = (\mathrm{I}-\mathrm{T}^*)\Psi_n$. Thus $(\mathrm{I}-\mathrm{T}^*)\Psi_n \geqq f_n - \alpha - \frac{1}{n}$, so that $\liminf(\mathrm{I}-\mathrm{T}^*)\Psi_n \geqq 1-\alpha > 0$, contradicting (∗∗). Hence $\rho(f_n) \to 1$, and (ii) follows from lemma 9.

THEOREM 11. – *Let* T *be a Cesàro bounded positive operator on* $\mathrm{L}_1(\mathrm{X}, \Sigma, m)$. *Then the following are equivalent:*

(i) $(w^*\textit{-closure}\,(\mathrm{I}-\mathrm{T}^*)\mathrm{L}_\infty) \cap \mathrm{L}_\infty^+ = \{0\}$.

(ii) T *satisfies* (∗∗) *of lemma* 10.

(iii) *There exists* $u \in \mathrm{L}_1$, $u > 0$ *a.e., with* $\mathrm{T}u = u$.

Proof. – (iii) ⇒ (i) ⇒ (ii) is easy. Note that (i) ⇒ (iii) can be proved directly by obtaining (as in theorem 6) that for every $0 \not\equiv f \in \mathrm{L}_\infty^+$ there exist $u \in \mathrm{L}_1^+$ with $\mathrm{T}u = u$ and $\langle u, f \rangle > 0$. Taking an invariant function with maximal support will yield (iii).

(ii) ⇒ (iii). We first show that if $m(\mathrm{A}) > 0$, then $m(\{\sum \mathrm{T}^{*n} 1_\mathrm{A} = \infty\}) > 0$.

Assume $\sum_{n=0}^{\infty} T^{*n} 1_A < \infty$ a.e., and let $B = \left\{ \sum_{n=0}^{\infty} T^{*n} 1_A > 0 \right\}$. Since $\varphi = \sum_{n=0}^{\infty} T^{*n} 1_A$ is finite a.e. and $T^* \varphi \leqq \varphi$, $L_\infty(B)$ is invariant under T^*, and if we define $S = I_B T$ on $L_1(B, m)$, we have that $S^* = T^* |_{L_\infty (B, m)}$. Clearly S is also Cesàro bounded. By property (ii) for T^*, we have that S^* also has the same property. By theorem 3, [which applies, in view of lemma 10(i) and theorem 6]

$$\sum_{n=0}^{N} S^{*n} 1_A / \sum_{n=0}^{N} S^{*n} 1 = I_B \left(\sum_{n=0}^{N} T^{*n} 1_A / \sum_{n=0}^{N} T^{*n} 1_B \right)$$

converges a.e. on B to a limit which is positive a.e. on A. Since by lemma 10, $m(Y(S)) > 0$, $m\left(\left\{ \sum_{n=0}^{\infty} S^{*n} 1_B = \infty \right\} \right) > 0$, and on that subset of B also $I_B \left(\sum_{n=0}^{\infty} T^{*n} 1_A \right) = \sum_{n=0}^{\infty} S^{*n} 1_A$ diverges, a contradiction to the assumption.

We now complete the proof. By theorem 3, there exists a finite subinvariant measure $\mu \sim m$, i.e. $d\mu = u\,dm$. By theorem 3.1 of [DL] (where only subinvariance is needed for the proof), T^* can be extended to a contraction of $L_1(\mu)$, which has as conservative part Y and dissipative part Z. By lemma 10, $m(Y) > 0$. Let $m(A) > 0$. Then $\sum_{n=0}^{\infty} T^{*n} 1_A < \infty$ a.e. on the dissipative part Z, so $\left\{ \sum_{n=0}^{\infty} T^{*n} 1_A = \infty \right\} \subset Y$ with non zero measure. On that set, by the Chacon-Ornstein theorem, $M_N(T^*) 1_A = e \sum_{n=0}^{N} T^{*n} 1_A / \sum_{n=0}^{N} T^{*n} e$ converges a.e. to a positive limit. Hence $m \{ \liminf_{n \to \infty} M_n(T^*) 1_A > 0 \} > 0$ if $m(A) > 0$. By theorem 3.2 of [DL], T has an equivalent finite invariant measure.

Remarks. — It is conjectured that the conditions of theorem 6 not only imply the existence of an equivalent subinvariant measure, but even the existence of an invariant one.

For the proof, we need to show that the conditions of theorem 6 imply $m(Y) > 0$ [or (**) of lemma 10], but our attempts have failed.

REFERENCES

[B_1] A. BRUNEL, Sur quelques problèmes de la théorie ergodique ponctuelle, *Thèse*, Univ. of Paris, 1966.

[B_2] A. BRUNEL, Sur les Sommes d'itérés d'un opérateur positif, Springer, *Lecture Notes in Math.*, Vol. **532**, 1976, pp. 19-34.

[BH] A. BRUNEL and S. HOROWITZ, *Invariant Measures for Positive Operators*, Unpublished, preliminary preprint, 1971.

[BS] A. BRUNEL and L. SUCHESTON, Sur l'existence d'éléments invariants dans les treillis de Banach, *C. R. Acad. Sci. Paris*, T. **300**, Series I, 1985, p. 59-62.

[DL] Y. DERRIENNIC and M. LIN, On Invariant Measures and Ergodic Theorems for Positive Operators, *J. Functional Analysis*, Vol. **13**, 1973, pp. 252-267.

[H] S. HOROWITZ, Transition Probabilites and Contractions of L_∞, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, Vol. **24**, 1972, pp. 263-274.

[I Mo] A. IONESCU-TULCEA and M. MORETZ, Ergodic Properties of Semi-Markovian Operators on the Z^1-Part, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, Vol. **13**, 1969, pp. 119-122.

[K] U. KRENGEL, *Ergodic Theorems*, De Gruyter, Berlin, New York, 1985.

[M] P. A. MEYER, *Probabilités et Potentiel*, Hermann, Paris, 1966.

[O] D. ORNSTEIN, The Sums of the Iterates of a Positive Operator, *Advances in Probability*, Vol. **2**, 1970, pp. 85-115.

[S] L. SUCHESTON, On the Ergodic Theorem for Positive Operators I, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, Vol. **8**, 1967, pp. 1-11.

[Sa] R. SATO, Positive Operators and the Ergodic Theorem, *Pacific J.*, Vol. **76**, 1978, pp. 215-219.

(*Manuscript received October* 9, 1991;
revised February 21, 1992.)