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# On an identity in law obtained by A. Földes and P. Révész

by

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ABSTRACT. – Using jointly Ray-Knight theorem on Brownian local times, time reversal, and the Ciesielski-Taylor identity in law, another identity in law by A. Földes and P. Révész is recovered, and generalized.

*Key words :* Brownian local times, time reversal.

RÉSUMÉ. – En utilisant conjointement le théorème de Ray-Knight sur les temps locaux browniens, un résultat classique de retournement du temps, et l'identité en loi de Ciesielski-Taylor, on retrouve et on généralise une identité en loi obtenue par A. Földes et P. Révész.

## 1. THE FÖLDES-RÉVÉSZ IDENTITY

In their paper [2], A. Földes and P. Révész prove that, for $r > q$:

$$\int_0^\infty dy \, 1_{(0 < L(y, T_r) < q)} \overset{(\text{law})}{=} T_{\sqrt{q}}(R_2) \tag{1}$$

*Classification A.M.S. :* Primary: 60 J 65, 60 J 55. Secondary: 60 H 55, 60 G 44.

where, on the left-hand side, $L(y, T_r)$ denotes the local time at level $y$, up to time $T_r \equiv \inf\{t : L(0, t) > r\}$, of Brownian motion starting at 0, and, on the right-hand side, $T_{\sqrt{q}}(R_2)$ denotes the first hitting time of $\sqrt{q}$ by $R_2$, a two-dimensional Bessel process starting from 0.

In fact, in [2], it is shown that the Laplace transform, in $\lambda$, of the left-hand side of (1) is:

$$\frac{1}{I_0(\sqrt{2\lambda q})},$$

where $I_0$ is the modified Bessel function, with index 0, but it is well-known that this is the Laplace transform of $T_{\sqrt{q}}(R_2)$ (*see* Kent [3], for example).

In the sequel, we shall write $T_a(R_\delta)$ for the first hitting time of $a$ by a Bessel process of dimension $\delta$ starting from 0, and $\mathrm{BESQ}_r^\delta$ shall denote the square, starting at $r$, of a Bessel process with dimension $\delta$.

Here is a quick proof of (1).

*a*) From Ray-Knight's theorem on Brownian local times, we know that:

$$\int_0^\infty dy\, 1_{(0 < L(y, T_r) < q)} \overset{\text{(law)}}{=} \int_0^{T_0} dy\, 1_{(Y_y < q)},$$

where $(Y_y;\, y \geqq 0)$ denotes a $\mathrm{BESQ}_r^0$, and $T_0 = \inf\{y : Y_y = 0\}$. By the strong Markov property, we may as well assume that $Y_0 = q$, hence the explanation of the fact that the right-hand side of (1) does not depend on $r$, for $r \geqq q$.

*b*) Using time-reversal for Bessel processes (Getoor-Sharpe [7]; *see* also e. g. Revuz-Yor [8], Chapt. XI, Exercise (1.23), or Yor [9], formula (4.*c*), for another application), we now obtain:

$$\int_0^{T_0} dy\, 1_{(Y_y < q)} \overset{\text{(law)}}{=} \int_0^{\hat{L}_q} dy\, 1_{(\hat{Y}_y < q)} = \int_0^\infty dy\, 1_{(\hat{Y}_y < q)}$$

where $(\hat{Y}_y,\, y \geqq 0)$ is a $\mathrm{BESQ}_0^4$ and $\hat{L}_q = \sup\{y : \hat{Y}_y = q\}$.

*c*) The Ciesielski-Taylor identity in law (*see* [1], [6] for example) tells us that:

$$\int_0^\infty dy\, 1_{(\hat{Y}_y < q)} \overset{\text{(law)}}{=} T_{\sqrt{q}}(R_2),$$

which ends the proof of (1).

## 2. A GENERALISATION

Let $(B_t,\, t \geqq 0)$ denote Brownian motion starting from 0, and for convenience, we denote now by $(l_t,\, t \geqq 0)$ its local time at 0, instead of

$(L(0, t), t \geqq 0)$. The process $(X_t := |B_t| - \mu l_t, t \geqq 0)$ is, in the case $\mu = 1$, a Brownian motion (as seen from Tanaka's formula, for instance), and in any case, it is a process which possesses a number of very interesting properties. We state two of those.

THEOREM 1 ([4]; *see* also Chapter 8 of [5]). — *We have*

$$\int_0^1 ds\, 1_{(X_s \leqq 0)} \overset{\text{(law)}}{=} Z_{(1/2), (1/2\mu)},$$

*where* $Z_{a, b}$ *denotes a beta variable with parameters* $a$ *and* $b$, i.e.

$$P(Z_{a, b} \in dt) = \frac{t^{a-1}(1-t)^{b-1}\, dt}{B(a, b)} \quad (0 < t < 1).$$

THEOREM 2 (*see* Chapter 9 of [5]). — *Let* $(l_t^\mu, t \geqq 0)$ *be the local time at* 0 *of the process* $(X_t, t \geqq 0)$, *and* $\tau_r^\mu := \inf\{t : l_t^\mu > r\}$. *Then, for fixed* $r > 0$, *the processes* $(l_{\tau_r^\mu}^x(X); x \geqq 0)$ *and* $(l_{\tau_r^\mu}^{-x}(X); x \geqq 0)$ *are independent, and their respective distributions are* $Q_r^0$, *and* $Q_r^{2-(2/\mu)}$, *where* $Q_r^\delta$ *denotes the law of the square of a* $\delta$*-dimensional Bessel process, starting from* $r$ *and absorbed at* 0.

We now prove the following

THEOREM 3. — *Let* $r > q$. *Then, we have*

$$\int_{-\infty}^0 dy\, 1_{(0 < l_{\tau_r^\mu}^y(X) < q)} \overset{\text{(law)}}{=} T_{\sqrt{q}}(R_{2/\mu}). \tag{2}$$

*Proof.* — Following the same sequence of arguments as in the first paragraph, we find with the help of Theorem 2, that the left-hand side of (2) is equal in law, to: $\int_0^\infty dy\, 1_{(\hat{Y}_y \leqq q)}$, where $(\hat{Y}_y, y \geqq 0)$ is a $BESQ_0^{2+(2/\mu)}$. Then, the Ciesielski-Taylor identity in law tells us that:

$$\int_0^\infty dy\, 1_{(\hat{Y}_y \leqq q)} \overset{\text{(law)}}{=} T_{\sqrt{q}}(R_{2/\mu}). \quad \square$$

## REFERENCES

[1] Z. CIESIELSKI and S. J. TAYLOR, First Passage Time and Sojourn Density for Brownian Motion in Space and the Exact Hausdorff Measure of the Sample Path, *Trans. Amer. Math. Soc.*, Vol. **103**, 1962, pp. 434-450.

[2] A. FÖLDES and P. RÉVÉSZ, On hardly visited Points of the Brownian Motion, *Prob. Th. and Rel. Fields*, Vol. **91**, 1992, pp. 71-80.

[3] J. KENT, Some probabilistic properties of Bessel functions, *The Annals of Prob.*, Vol. **6**, (5), 1978, pp. 760-770.
[4] F. PETIT, *Thèse*, Université Paris-VII, Presented on February 15th, 1992.
[5] M. YOR, Some aspects of Brownian motion, *Course given in ETH Zürich*, Fall 1991.
[6] M. YOR, Une explication du théorème de Ciesielski-Taylor, *Ann. Inst. Henri Poincaré*, Vol. **27**, n° 2, 1991, pp. 201-213.
[7] R. K. GETOOR and M. J. SHARPE, Excursions of Brownian motion and Bessel processes, *Zeitschrift für Wahr.*, t. 47, 1979, pp. 83-106.
[8] D. REVUZ and M. YOR, *Continuous martingales and Brownian motion*, Springer, 1991.
[9] M. YOR, A propos de l'inverse du mouvement brownien dans $\mathbb{R}^n$ $(n \geqq 3)$, *Ann. Inst. Henri Poincaré*, Vol. **21**, n° 1, 1984, pp. 27-38.

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