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ABSTRACT. — Using jointly Ray-Knight theorem on Brownian local times, time reversal, and the Ciesielski-Taylor identity in law, another identity in law by A. Földes and P. Révész is recovered, and generalized.

Key words: Brownian local times, time reversal.

1. THE FÖLDES-RÉVÉSZ IDENTITY

In their paper [2], A. Földes and P. Révész prove that, for $r > q$:

\[ \int_{0}^{\infty} dy \ 1_{0 < L(y, T_{r}) < q}^{\text{law}} = T_{\sqrt{q}}(R_{2}) \]  

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where, on the left-hand side, \( L(y, T_r) \) denotes the local time at level \( y \), up to time \( T_r \equiv \inf \{ t : L(0, t) > r \} \), of Brownian motion starting at 0, and, on the right-hand side, \( T_q(R_2) \) denotes the first hitting time of \( \sqrt{q} \) by \( R_2 \), a two-dimensional Bessel process starting from 0.

In fact, in [2], it is shown that the Laplace transform, in \( \lambda \), of the left-hand side of (1) is:

\[
\frac{1}{I_0(\sqrt{2\lambda q})},
\]

where \( I_0 \) is the modified Bessel function, with index 0, but it is well-known that this is the Laplace transform of \( T_q(R_2) \) (see Kent [3], for example).

In the sequel, we shall write \( T_q(R_\delta) \) for the first hitting time of \( a \) by a Bessel process of dimension \( \delta \) starting from 0, and \( \text{BESQ}_\delta \) shall denote the square, starting at \( r \), of a Bessel process with dimension \( \delta \).

Here is a quick proof of (1).

\( a) \) From Ray-Knight's theorem on Brownian local times, we know that:

\[
\int_0^\infty dy \mathbb{1}_{(0 < L(y, T_r) < q)}^{(\text{law})} = \int_0^{T_0} dy \mathbb{1}_{(Y_y < q)},
\]

where \( (Y_y; y \geq 0) \) denotes a \( \text{BESQ}_0 \), and \( T_0 = \inf \{ y : Y_y = 0 \} \). By the strong Markov property, we may as well assume that \( Y_0 = q \), hence the explanation of the fact that the right-hand side of (1) does not depend on \( r \), for \( r \geq q \).

\( b) \) Using time-reversal for Bessel processes (Getoor-Sharpe [7]; see also e.g. Revuz-Yor [8], Chapt. XI, Exercise (1.23), or Yor [9], formula (4.c), for another application), we now obtain:

\[
\int_0^{T_0} dy \mathbb{1}_{(Y_y < q)}^{(\text{law})} = \int_0^{\hat{T}_q} dy \mathbb{1}_{(\hat{Y}_y < q)} = \int_0^{\infty} dy \mathbb{1}_{(\hat{Y}_y < q)},
\]

where \( (\hat{Y}_y; y \geq 0) \) is a \( \text{BESQ}_0 \) and \( \hat{T}_q = \sup \{ y : \hat{Y}_y = q \} \).

\( c) \) The Ciesielski-Taylor identity in law (see [1], [6] for example) tells us that:

\[
\int_0^{\infty} dy \mathbb{1}_{(\hat{Y}_y < q)}^{(\text{law})} = T_\sqrt{q}(R_2),
\]

which ends the proof of (1).

2. A GENERALISATION

Let \( (B_t, t \geq 0) \) denote Brownian motion starting from 0, and for convenience, we denote now by \( (l_t, t \geq 0) \) its local time at 0, instead of
(L(0, t), \ t \geq 0). The process \((X_t := |B_t| - \mu l_t, \ t \geq 0)\) is, in the case \(\mu = 1\), a Brownian motion (as seen from Tanaka’s formula, for instance), and in any case, it is a process which possesses a number of very interesting properties. We state two of those.

**Theorem 1** ([4]; see also Chapter 8 of [5]). We have

\[
\int_0^1 ds 1_{(X_s \leq 0)} \overset{(\text{law})}{=} Z_{(1/2), (1/2 \mu)},
\]

where \(Z_{a, b}\) denotes a beta variable with parameters \(a\) and \(b\), i.e.

\[
P(Z_{a, b} \in dt) = \frac{t^{a-1} (1-t)^{b-1} dt}{B(a, b)} \quad (0 < t < 1).
\]

**Theorem 2** (see Chapter 9 of [5]). Let \((l^p_t, \ t \geq 0)\) be the local time at 0 of the process \((X_t, \ t \geq 0)\), and \(\tau^p_r := \inf \{ t : l^p_t > r \}\). Then, for fixed \(r > 0\), the processes \((l^{	ext{B}}_p(X), \ x \geq 0)\) and \((l^{	ext{B}}_r(X), \ x \geq 0)\) are independent, and their respective distributions are \(Q_0^r\), and \(Q_r^{-2/(2 \mu)}\), where \(Q_0^r\) denotes the law of the square of a \(\delta\)-dimensional Bessel process, starting from \(r\) and absorbed at 0.

We now prove the following

**Theorem 3.** Let \(r > q\). Then, we have

\[
\int_{-\infty}^0 dy 1_{(0 < \tilde{v}_q^r(X) < q)} \overset{(\text{law})}{=} T_q^{\sqrt{\mu}}(R_{2/\mu}). \quad (2)
\]

**Proof.** Following the same sequence of arguments as in the first paragraph, we find with the help of Theorem 2, that the left-hand side of (2) is equal in law, to:

\[
\int_0^\infty dy 1_{(\hat{y}, y \geq 0)} \text{ where } (\hat{y}, y \geq 0) \text{ is a BESQ}_{0}^{2+(2/\mu)}.
\]

Then, the Ciesielski-Taylor identity in law tells us that:

\[
\int_0^\infty dy 1_{(\hat{y}, y \geq 0)} \overset{(\text{law})}{=} T_q^{\sqrt{\mu}}(R_{2/\mu}). \quad \square
\]

**References**


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