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Occupation times of compact sets by planar Brownian motion

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ABSTRACT. – Let K be a connected compact subset of $\mathbb{R}^2$. We consider the distribution of the occupation time (over the interval $[0, t]$) of K by a Brownian motion started at an arbitrary point in $\mathbb{R}^2$. We obtain an explicit formula for the distribution of the occupation time of a closed disc by a Brownian motion. Using this result, we then obtain some Tauberian asymptotic results for the general case. The main technique involves calculating the Itô excursion law for the BES(2) process.

Key words : Arc-sine law, occupation time, excursion theory.

RÉSUMÉ. – Soit $K \subset \mathbb{R}^2$ compact. On considère la distribution du temps de séjour (pendant l'intervalle $[0, t]$) dans K par un mouvement brownien issu d'un point arbitraire dans $\mathbb{R}^2$. On obtient une formule explicite pour la distribution du temps de séjour dans un disque fermé, que l'on utilise alors pour obtenir quelques résultats asymptotiques taubériens pour le cas général. La technique principale consiste à calculer la loi d'Itô des excursions pour le processus BES(2).

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let K be a compact subset of $\mathbb{R}^2$ with non-empty interior; such a set will be referred to as a compact region in $\mathbb{R}^2$. Consider the problem of finding the distribution of the occupation time of K during the interval $[0, t]$ by a Brownian motion started at an arbitrary point in $\mathbb{R}^2$. Of course, one cannot hope to obtain any explicit results without making some specific assumptions about K. We first take K to be a closed disc.

Let $B=(X, Y)$ be a Brownian motion in $\mathbb{R}^2$ started at 0 and let D_R denote the closed disc of radius R, $\{z:|z|\leqq R\}$. For fixed z let $U_t(z; R)$ and $V_t(z)$ denote the occupation times

$$U_t(z; R)=\int_0^t 1_{D_R}(B_s+z)\,ds, \qquad V_t(z)=\int_0^t 1_K(B_s+z)\,ds. \tag{1.1}$$

(In three or more dimensions, U_∞, $V_\infty<\infty$ almost surely and the distribution of U_∞ is given by the Ciesielski-Taylor theorem).

The occupation time $V_t(z)$ is related to the following problem concerned with heat-flow with cooling. Suppose that an infinite plate ($\mathbb{R}^2$) is initially at temperature 1 and that the plate is insulated everywhere except for a "hole" shaped like K where the plate is exposed to an environment kept at constant temperature 0. The rate of heat loss due to cooling is proportional to the difference in temperature between the plate and its surrounding environment. Let $\alpha>0$ denote the constant of proportionality and let $u(t, z)$ denote the temperature at the point z at time t. Then u satisfies

$$\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u-\alpha\, 1_K(z)\, u,$$

$$u(0, z)=1.$$

But this problem can be reformulated probabilistically: according to Kac's formula [*see* Itô and McKean (1965), §2.6], $u(t, z)=\mathbb{E}[e^{-\alpha V_t(z)}]$.

The main results in this paper are the following.

THEOREM 1. — *Fix* $\alpha, \lambda>0$ *and let* T *be an exponential random variable of rate* λ, *independent of* B. *If* $z\in D_R$, $|z|=r\leqq R$, *the distribution of* $U_t(z; R)$ *is specified by*

$$\mathbb{E}[\exp\{-\alpha U_T(z; \mathbb{R})\}]=\frac{\theta I_0(r\gamma)K_1(R\theta)}{\gamma I_1(R\gamma)K_0(R\theta)+\theta I_0(R\gamma)K_1(R\theta)}$$
$$+\left(\frac{\theta}{\gamma}\right)^2\left\{1-\frac{I_0(r\gamma)}{I_0(R\gamma)}+\frac{\gamma I_0(r\gamma)I_1(R\gamma)K_0(R\theta)}{I_0(R\gamma)[\gamma I_1(R\gamma)K_0(R\theta)+\theta I_0(R\gamma)K_1(R\theta)]}\right\}, \tag{1.2a}$$

where $\gamma=\sqrt{2(\lambda+\alpha)}$, $\theta=\sqrt{2\lambda}$ *and* I_ν *and* K_ν *are modified Bessel functions.*

If $z \notin D_R$, $|z| = r > R$, *the distribution of* $U_t(z;\ R)$ *is specified by*

$$\mathbb{E}[\exp\{-\alpha U_T(z;\ R)\}]$$
$$= 1 - \frac{K_0(r\theta)}{K_0(R\theta)} + \frac{\theta I_0(R\gamma) K_0(r\theta)(K_1(R\theta)}{K_0(R\theta)[\gamma I_1(R\gamma) K_0(R\theta) + \theta I_0(R\gamma) K_1(R\theta)]}$$
$$+ \left(\frac{\theta}{\gamma}\right)^2 \frac{\gamma I_1(R\gamma) K_0(r\theta)}{\gamma I_1(R\gamma) K_0(R\theta) + \theta I_0(R\gamma) K_1(R\theta)}. \quad (1.2b)$$

The proof of Theorem 1 occupies Section 2. We shall use methods of excursion theory which are inspired by the excursion proof of the arc-sine law for Brownian motion [*see* Rogers and Williams (1987), § VI.53].

Although the formulae (1.2*a*, *b*) are very complicated functions of λ and α and it is not possible to invert the Laplace transform, we can nevertheless deduce some interesting asymptotic results from Theorem 1.

COROLLARY 1. — *Fix* $\alpha > 0$. *Then as* $t \to \infty$, *the following asymptotic result holds for all* $z \in \mathbb{R}^2$:

$$\mathbb{E}[\exp\{-\alpha U_t(z;\ R)\}] \sim \frac{c}{\log t}$$

where

$$c = \frac{2 I_0((|z| \wedge R)\sqrt{2\alpha})}{R\sqrt{2\alpha}\, I_1(R\sqrt{2\alpha})} + 2\log\left(\frac{|z|}{|z| \wedge R}\right).$$

Proof. — Let $z \in \mathbb{R}^2$ be arbitrary and define

$$\varphi_R(\lambda) = \int_0^\infty e^{-\lambda t}\, \mathbb{E}[\exp\{-\alpha U_t(z;\ R)\}]\, dt. \quad (1.3)$$

Using the following asymptotic properties of Bessel functions,

$$K_0(x) \sim \log 1/x, \qquad K_1(x) \sim 1/x \quad \text{as } x \downarrow 0,$$

it is easy to see from (1.2*a*, *b*) that as $\lambda \downarrow 0$,

$$\varphi_R(\lambda) \sim \frac{c}{\lambda \log 1/\lambda}. \quad (1.4)$$

(The constant c must be calculated from (1.2*a*) and (1.2*b*) separately; the formula for c given above is merely a compact way of writing the different values of c for the cases $|z| \leq R$ and $|z| > R$.) The function $1/\log\lambda$ is slowly varying (at infinity) and the function $t \mapsto \mathbb{E}[e^{-\alpha U_t}]$ is clearly monotone, so by Karamata's Tauberian theorem together with the monotone density theorem [*see* Bingham *et al.* (1987), Theorems 1.7.1 and 1.7.2] the result follows. □

We can deduce from the above results for the disc the following asymptotic result for the occupation time of a general compact region.

COROLLARY 2. — *Let* K *be a compact region in* $\mathbb{R}^2$ *and let* $z\in\mathbb{R}^2$. *Then for fixed* $\alpha>0$, *as* $t\to\infty$, *the following asymptotic result holds*:

$$\mathbb{E}[\exp\{-\alpha V_t(z)\}]\sim\frac{c(K)}{\log t}$$

for some constant $c(K)>0$ *depending on* z, α *and* K.

Proof. — Let $\psi(\lambda)=\int_0^\infty e^{-\lambda t}\,\mathbb{E}[\exp\{-\alpha V_t(z)\}]\,dt$. From the proof of Corollary 1 above, we see that it is sufficient to establish the analogue of (1.4), namely

$$\psi(\lambda)\sim\frac{c(K)}{\lambda\log 1/\lambda} \tag{1.5}$$

as $\lambda\downarrow 0$, or equivalently, the limit

$$\lim_{\lambda\downarrow 0}(\lambda\log 1/\lambda)\,\psi(\lambda) \tag{1.6}$$

exists and is positive.

We can find $0<r<R$ such that $D_r\subset K\subset D_R$, so that $U_t(z;r)\leqq V_t(z)\leqq U_t(z;R)$ for all t and hence

$$\varphi_R(\lambda)\leqq\psi(\lambda)\leqq\varphi_r(\lambda)$$

for all $\lambda>0$. (The functions φ_r and φ_R are as defined at (1.3).) The asymptotic result (1.4) therefore implies that there exist positive constants c_1 and c_2 such that

$$c_1\leqq(\lambda\log 1/\lambda)\,\psi(\lambda)\leqq c_2 \tag{1.7}$$

for all $\lambda>0$ sufficiently small.

Because $\lambda\mapsto\psi(\lambda)$ is the Laplace transform of a positive function, it is a completely monotonic function (*i.e.* a non-negative function whose even derivatives are all non-negative and whose odd derivatives are all non-positive). Therefore the function ψ is analytic in the (open) right half-plane $\{z=x+iy:x>0\}$ [*see* Widder (1941), Chapter IV, Theorem 3 *a*] and so $(\lambda\log 1/\lambda)\,\psi(\lambda)$ is also analytic in the right half-plane, provided we take the principal branch of the log function. Furthermore, for $\beta\in\mathbb{R}$ and $\lambda>0$,

$$\left|\int_0^\infty e^{-(\lambda+i\beta)t}\,\mathbb{E}[e^{-\alpha V_t}]\,dt\right|\leqq\int_0^\infty e^{-\lambda t}\,\mathbb{E}[e^{-\alpha V_t}]\,dt$$

and also, since $\mathbb{E}[e^{-\alpha V_t}]\downarrow 0$ as $t\to\infty$, the integral $\int_0^\infty e^{-i\beta t}\,\mathbb{E}[e^{-\alpha V_t}]\,dt$ exists as a Riemann integral. Hence from (1.7) we see that the function $(\lambda\log 1/\lambda)\,\psi(\lambda)$ is actually bounded in a neighbourhood of 0 in the right half-plane, not just for sufficiently small real $\lambda > 0$. But since the transformation $\lambda \mapsto 1/\lambda$ maps the right half-plane to itself, the function $\lambda \mapsto (\lambda^{-1} \log \lambda)\, \psi(1/\lambda)$ is analytic in the right half-plane and bounded in a neighbourhood of infinity, and so the worst that can happen is that it has a removable singularity at infinity. This shows in particular that the limit

$$\lim_{\lambda \to \infty} (\lambda^{-1} \log \lambda)\, \psi(1/\lambda)$$

exists, which implies (1.5). □

The constants c_1 and c_2 in (1.7) are given by the asymptotic result of Corollary 1 and since the constant $c(\mathrm{K})$ involved in the asymptotic result of Corollary 2 lies between c_1 and c_2, it is possible to obtain upper and lower bounds for $c(\mathrm{K})$ in terms of z, the diameter of K and other quantities depending on the geometry of K.

The preceding two results concern the large-time behaviour of V_t and as such are related to Birkhoff's ergodic theorem for planar Brownian motion, which states the following: let f be such that $\gamma \equiv \int_{\mathbb{R}^2} f(x)\, dx < \infty$ and let τ_n be the successive hitting times by B_t of the unit circle via the circle of radius 2, then $\mathbb{P}^0$-a.s.

$$\lim_{n \to \infty} \frac{1}{n} \int_0^{\tau_n} f(\mathrm{B}_s)\, ds = \frac{\gamma \log 2}{\pi}.$$

[*See* §7.17 of Itô and McKean (1965).] Specializing to the occupation time of the disc, we have $\mathrm{U}_{\tau_n}(0;\, \mathrm{R})/n \to \mathrm{R}^2 \log 2$ almost surely.

It is easy to see from the asymptotic behaviour of I_ν and K_ν at infinity that as $\lambda \to \infty$, $\varphi_\mathrm{R}(\lambda) \sim 1/\lambda$, which is as expected since $\mathbb{E}[e^{-\alpha \mathrm{U}_t(z)}] \uparrow 1$ as $t \downarrow 0$. For a point z on the boundary of the disc, one could of course use the asymptotic behaviour of $\varphi_\mathrm{R}(\lambda) - 1/\lambda$ as $\lambda \to \infty$ to obtain the leading-term asymptotics of $\mathbb{E}[e^{-\alpha \mathrm{U}_t(z)}]$ for small t, but there is an easier method because this behaviour is determined by the behaviour of $\mathbb{E}[\mathrm{U}_t]$ for small t and since there are other well-known methods for determining such asymptotics, we shall not pursue them here. One can also differentiate (1.2 *a*, *b*) to obtain explicit formulae for the Laplace transforms of moments of U_t. These are still very complicated functions of λ and one can only hope to get asymptotic results for large t; but these too are already known by other means [see for example, Darling and Kac (1957).]

Theorem 1 may be regarded as a generalization of the classical arc-sine law for one-dimensional Brownian motion. Note that $\mathrm{U}_t(z;\, \mathrm{R})$ as defined in (1.1) may be written as

$$\mathrm{U}_t(r;\, \mathrm{R}) = \int_0^t 1_{[0,\, \mathrm{R}]}(\rho_s)\, ds \tag{1.8}$$

where ρ is a BES(2) process started at $r>0$. Theorem 1 is therefore an analogue of the arc-sine law for the two-dimensional Bessel process. Similar analogues of the arc-sine law for one-dimensional time-homogeneous diffusions have been obtained by Truman and Williams (1990), assuming that the drift is a bounded C^1 function and that the speed measure has an L^1 density. The Bessel process does not satisfy these assumptions, which enabled Truman and Williams to use Kac's formula to obtain the result of Theorem 1. (However, we shall see later that the formula (1.2 *a*) is exactly identical to the formula Truman and Williams (1990) obtained using Kac's theorem.)

2. PROOF OF THEOREM 1

Throughout, we use the representation (1.8) of U_t in terms of a BES(2) process. Let ρ be a BES(2) process started at R. We regard R as fixed and we write $U_t(r) \equiv U_t(r; R)$ for convenience. We begin by developing some excursion theory needed in the proof. For a more detailed treatment of the ideas of excursion theory used in the sequel, we refer the reader to the relevant sections in Chapter VI of Rogers and Williams (1987).

Consider the excursions of ρ from R. The key to excursion theory is the Itô excursion measure n, which is a σ-finite measure on the canonical space of excursions $U \equiv \{\text{continuous } f: \mathbb{R}^+ \to \mathbb{R}^+ \text{ s.t. } f^{-1}(\mathbb{R}^+ \setminus \{R\}) = (0, \zeta) \text{ for some } \zeta\}$. The essential idea behind the excursion measure n – which is an immediate consequence of the celebrated theorem of Itô [Rogers and Williams (1987), Theorem VI.47.6] – is that if $A_1, A_2, \ldots, A_k$ are disjoint (Borel) sets of excursions in U with $n(A_i) < \infty$ for $i = 1, 2, \ldots, k$, and if $N_t(A_i)$ is the number of excursions in A_i by the time ρ as accumulated *local time t* at R, then under the canonical measure $\mathbb{P}^R$ associated with the process ρ, $N_t(A_1), \ldots, N_t(A_k)$ are independent Poisson processes with rates $n(A_1), \ldots, n(A_k)$. In particular, the local time when the first excursion in A_i occurs is exponentially distributed with rate $n(A_i)$. Furthermore, the probability that the first excursion in $A = \bigcup_{i=1}^{k} A_i$ belongs to A_i is $n(A_i)/n(A)$. Next, define a family of measures $(n_t)_{t \geqq 0}$ on the state space $\mathbb{R}^+$ of ρ by

$$n_t(B) = n(\{f \in U : f(t) \in B,\ t < \zeta\})$$

for Borel sets B, where ζ is the lifetime of the excursion f. The family (n_t) is an entrance law for the semigroup associated with the BES(2) process killed at R, and the Itô excursion measure n is completely specified by the family (n_t) [*see* Rogers and Williams (1987), Theorem VI.48.1].

The best way to calculate $n_t(dx)$ is via its Laplace transform

$$n_\lambda(x)=\int_0^\infty e^{-\lambda t}\, n_t(x)\, dt.$$

(There is no scope for confusion in the notation here as we shall follow the convention of using Roman subscripts for the entrance law and Greek subscripts for its Laplace transform.) The measure n_λ admits a probabilistic interpretation in the context of marked excursions. If we mark the time axis $[0, \infty)$ with points of a Poisson process of rate λ, independent of ρ, then some excursions will contain a mark. (Strictly speaking, we need to incorporate the Poisson process of marks into the Poisson process of excursions and define a process on a canonical space of marked excursions. See section VI.49 of Rogers and Williams (1987) for the precise details.) An important observation is that we can either first mark the time axis with points of an independent Poisson process of rate λ and then decompose the path of ρ into (marked) excursions from R, or we can first decompose the path of ρ into (unmarked) excursions and then mark each excursion independently with an independent Poisson process of rate λ – the result is the same. A precise formulation of this obvious result can be found in Theorem VI.49.2 of Rogers and Williams (1987). One consequence of this is that

$$\lambda\, n_\lambda(\mathbb{R}^+)=\lambda\int_0^\infty e^{-\lambda t}\, n_t(\mathbb{R}^+)\, dt$$

is the n-measure (i.e. local time rate) of marked excursions and $\lambda\, n_\lambda([\mathrm{R}, \infty])$, $\lambda\, n_\lambda([0, \mathrm{R}])$ are respectively the n-measures of the "up" and "down" excursions from R which contain a mark. Moreover, for a Borel subset $\mathrm{B}\subset\mathbb{R}^+$,

$$\lambda\int_0^\infty e^{-\lambda t}\,\mathbb{P}^{\mathrm{R}}(\rho_t\in \mathrm{B})\, dt=\lambda\,(\mathrm{R}_\lambda 1_{\mathrm{B}})(\mathrm{R})=\frac{\lambda n_\lambda(\mathrm{B})}{\lambda n_\lambda(\mathbb{R}^+)}, \tag{2.1}$$

where R_λ is the resolvent operator. Thus heuristically speaking, $\lambda\, n_\lambda(\mathrm{B})$ is the n-measure of excursions of which are in the set B at the time of the first mark.

To prove Theorem 1, we first suppose that the Brownian motion B is started at a point z on the circumference of the disc: $|z|=r=\mathrm{R}$. Fix α, $\lambda>0$. Mark the time axis $[0, \infty)$ with red points according to a Poisson process of rate λ, independent of ρ. Also, mark $[0, \infty)$ with blue points according to another Poisson process, independent of the first, at rate $\alpha\, 1_{[0,\mathrm{R}]}(\rho_t)$. Thus, conditional on ρ, the number of blue points in the time interval $[0, t]$ has a Poisson distribution with mean $\alpha\, \mathrm{U}_t(\mathrm{R})$ and the probability that there are no blue points in $[0, t]$ is $\mathbb{E}^r[e^{-\alpha U_t(R)}]$. Hence

$$\mathbb{E}^r[e^{-\alpha U_T}(R)] = \int_0^\infty \lambda e^{-\lambda t}\, \mathbb{E}^r[e^{-\alpha U_t(R)}]\, dt$$
$$= \mathbb{P}^r(\text{first red point appears before the first blue point}). \qquad (2.2)$$

The problem now is that an excursion of ρ into the interval $[0, R]$ could well contain both a red and a blue point. To get around this problem, we use the following trick which is also used in the proof of the arc-sine law in Rogers and Williams (1987). The red and blue marks could also be produced by the following alternative method. Firstly, mark $[0, \infty)$ with red points, not at constant rate λ, but at rate $\lambda\, 1_{(R, \infty)}(\rho_t)$ and independently (conditional on ρ) mark the time axis with green points at rate $(\lambda+\alpha)\, 1_{[0, R]}(\rho_t)$. Next, recolour each green point, independently of other green points, red with probability $\lambda/(\lambda+\alpha)$ and blue with probability $\alpha/(\lambda+\alpha)$. Since before recolouring, a red point can only occur during an excursion by ρ into (R, ∞) and a green point can only occur during an excursion into $[0, R]$, no excursion can contain both a red point and a green point. We also have

$$\begin{aligned} &\mathbb{P}\ (\text{first red point appears before the first blue point})\\ &= \mathbb{P}\ (\text{first red point appears before the first green point})\\ &\quad + \mathbb{P}\ (\text{first green pt. appears before first red pt. and is recoloured red})\\ &= \mathbb{P}\ (\text{1st red pt. before 1st green pt.})\\ &\quad + \frac{\lambda}{\lambda+\alpha}\, \mathbb{P}\ (\text{1st green pt. before 1st red pt.}) \end{aligned} \qquad (2.3)$$

It is now a matter of computing the two probabilities in (2.3), which we do using excursion theory.

The transition density function $p_t(x, y)$ for ρ is

$$p_t(x, y) = \frac{y}{t} \exp\left(-\frac{x^2+y^2}{2t}\right) I_0\left(\frac{xy}{t}\right), \qquad x>0, \qquad t>0$$

[*see* Revuz and Yor (1991), p. 415]. From this, we obtain the resolvent density $r_\lambda(x, y)$ [*see* Erdélyi (1954), §4.17 Eq. (4)]

$$r_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_t(x, y)\, dt = 2y\, I_0((x \wedge y)\theta)\, K_0((x \vee y)\theta) \qquad (2.4)$$

(recall that $\theta = \sqrt{2\lambda}$.) Letting $y \to x$ in the above shows that

$$r_\lambda(R, R) = \mathbb{E}^R\left[\int_0^\infty e^{-\lambda t}\, dL_t^R\right] = 2R\, I_0(R\theta)\, K_0(R\theta),$$

where L_t^R is the local time spent at R by time t. (The local time is defined only up to constant multiples but the exact normalization is of no importance here; the identity $r_\lambda(R, R) = \int_0^\infty \lambda e^{-\lambda t} \mathbb{E}^R [L_t^R] \, dt$ in (2.4) holds if L_t^R is given by Tanaka's formula, the so-called semi-martingale normalization of local time).

We first need to calculate $\lambda n_\lambda(\mathbb{R}^+)$, the n-measure of excursions which contain a mark (from a Poisson process of constant rate λ). If T is the time of the first mark, then L_T^R has an exponential distribution of rate $\lambda n_\lambda(\mathbb{R}^+)$. Therefore, with $\theta = \sqrt{2\lambda}$,

$$(\lambda n_\lambda(\mathbb{R}^+))^{-1} = \mathbb{E}[L_T^R] = \mathbb{E}^R \left[\int_0^\infty e^{-\lambda t} \, d L_t^R \right] = r_\lambda(R, R) = 2 R I_0(R\theta) K_0(R\theta). \tag{2.5}$$

Next, we can use (2.1) and (2.5) to obtain a formula for the density of $n_\lambda(dx) = n_\lambda(x) \, dx$:

$$n_\lambda(x) = \frac{r_\lambda(R, x)}{\tau_\lambda(R, R)} = \frac{x I_0((x \wedge R)\theta) K_0((x \vee R)\theta)}{R I_0(R\theta) K_0(R\theta)}. \tag{2.6}$$

Putting $\beta = \lambda + \alpha$, the n-measure of excursions which contain at least one green mark – which is the same as the n-measure of β-marked excursions into [0, R] – is given by $\beta n_\beta([0, R])$, as explained earlier. From (2.6) we have

$$\beta n_\beta([0, R]) = \frac{\beta}{R K_0(R\gamma) I_0(R\gamma)} \int_0^R x K_0(R\gamma) I_0(x\gamma) \, dx = \frac{\beta}{R I_0(R\gamma)} \int_0^R x I_0(x\gamma) \, dx, \tag{2.7}$$

where we have put $\gamma = \sqrt{2\beta}$. We now proceed to calculate the integral in (2.7) (¹).

It is known that the Bessel function J_ν satisfies the following identity:

$$J_{\mu+\nu+1}(z) = \frac{z^{\nu+1}}{2^\nu \Gamma(\nu+1)} \int_0^{\pi/2} J_\mu(z \sin u) (\sin u)^{\mu+1} (\cos u)^{2\nu+1} \, du \tag{2.8}$$

[*see* Watson (1952), § 12.11, Eq. (1)]. Putting $\mu = \nu = 0$ in (2.8) gives

$$J_1(z) = z \int_0^{\pi/2} J_0(z \sin u) \sin u \cos u \, du = z \int_0^1 x J_0(zx) \, dx. \tag{2.9}$$

(¹) This, and other similar computations elsewhere in the paper, can be quickly verified using a suitable computer package for symbolic computations, such as Mathematica.

For the modified Bessel functions we have $I_\nu(z)=e^{-i\nu\pi/2} J_\nu(iz)$ and substituting this in (2.9) yields

$$I_1(z)=z\int_0^1 x\, I_0(zx)\, dx$$

and hence

$$I_1(R\gamma)=\frac{\gamma}{R}\int_0^R x\, I_0(\gamma x)\, dx.$$

Upon substitution of this last integral into (2.7), we finally obtain the excursion measure (or local time rate) of green-marked excursions:

$$\beta n_\beta([0, R])=\frac{\beta}{\gamma}\frac{I_1(R\gamma)}{I_0(R\gamma)}=\frac{\gamma}{2}\frac{I_1(R\gamma)}{I_0(R\gamma)}. \tag{2.10}$$

Next, from (2.5) and (2.10) the local time rate of red marked excursions is given by

$$\begin{aligned}\lambda n_\lambda([R, \infty]) &= \lambda n_\lambda(\mathbb{R}^+)-\lambda n_\lambda([0, R])\\ &= \frac{1}{2 R I_0(R\theta) K_0(R\theta)}-\frac{\theta}{2}\frac{I_1(R\theta)}{I_0(R\theta)}.\end{aligned} \tag{2.11}$$

Using another well-known identity for Bessel functions [Watson (1952), §3.71, Eq. (20)],

$$I_0(z) K_1(z)+I_1(z) K_0(z)=1/z,$$

we can write

$$\begin{aligned}\frac{1}{2 R I_0(R\theta) K_0(R\theta)} &= \frac{\theta}{2 R\theta I_0(R\theta) K_0(R\theta)}\\ &= \frac{\theta}{2}\frac{I_0(R\theta) K_1(R\theta)+I_1(R\theta) K_0(R\theta)}{I_0(R\theta) K_0(R\theta)}\\ &= \frac{\theta}{2}\left(\frac{K_1(R\theta)}{K_0(R\theta)}+\frac{I_1(R\theta)}{I_0(R\theta)}\right).\end{aligned}$$

Substituting this into (2.11) then yields the excursion measure of red-marked excursions:

$$\lambda n_\lambda([R, \infty])=\frac{\theta}{2}\frac{K_1(R\theta)}{K_0(R\theta)}. \tag{2.12}$$

The local times at R at the time of the first green and the first red mark are exponential random variables whose rates are given respectively by (2.10) and (2.12). Moreover, since the Poisson process of excursions into [0, R] is independent of the excursions into [R, ∞), these are *independent* exponential random variables. Thus, for instance, the event that the first red mark appears before the first green mark is equivalent to the event that the local time at R when the first red mark appears is less than the local time at R when the first green mark appears. The probabilities in (2.3) can hence be calculated as follows:

$$\mathbb{P}(\text{1st red pt. before 1st green pt.}) = \frac{\lambda n_\lambda([\mathrm{R}, \infty])}{\lambda n_\lambda([\mathrm{R}, \infty]) + \beta n_\beta([0, \mathrm{R}])} \quad (2.13)$$

and

$$\mathbb{P}(\text{1st green pt. before 1st red pt.}) = \frac{\beta n_\beta([0, \mathrm{R}])}{\lambda n_\lambda([\mathrm{R}, \infty]) + \beta n_\beta([0, \mathrm{R}])}. \quad (2.14)$$

Substituting (2.10) and (2.12) into (2.13-2.14) and using (2.2) and (2.3) now yields the desired result (1.2*a*) in the case where $r = \mathrm{R}$.

Consider now the case that z is in the interior of the disc: $|z| = r < \mathrm{R}$. Define the hitting time $\tau_\mathrm{R} = \inf\{t : \rho_t = \mathrm{R}\}$. Next mark the time axis with red and green points and recolour the green points exactly as before, only this time the Bessel process ρ starts at $r < \mathrm{R}$. By the strong Markov property at τ_R we have

$$\begin{aligned}
&\mathbb{P}^r(\text{first red point before the first blue point})\\
&= \mathbb{P}^r(\text{1st red pt. before 1st green pt.})\\
&+ \frac{\lambda}{\lambda+\alpha}\mathbb{P}^r(\text{1st green pt. before 1st red pt.})\\
&= \mathbb{P}^r(\text{no green pts. before } \tau_\mathrm{R})\, \mathbb{P}^\mathrm{R}(\text{1st red pt. before 1st green pt.})\\
&+ \lambda(\lambda+\alpha)^{-1}[\mathbb{P}^r(\text{1st green pt. before } \tau_\mathrm{R})\\
&\quad + \mathbb{P}^r(\text{no green pts. before } \tau_\mathrm{R})\, \mathbb{P}^\mathrm{R}(\text{1st green pt. before 1st red pt.})].
\end{aligned} \quad (2.15)$$

It remains to calculate $\mathbb{P}^r$ (no green pts. before τ_R) $= \mathbb{E}^r[\exp\{-\beta\tau_\mathrm{R}\}]$ where $\beta = \lambda + \alpha$. But this can be done by a standard elementary argument as follows. First, by the Markov property, for $r < \mathrm{R} < y$ we have

$$p_t(r, y) = \int_0^t \mathbb{P}^r(\tau_\mathrm{R} \in du)\, p_{t-u}(\mathrm{R}, y).$$

Taking Laplace transforms and letting $y \downarrow \mathrm{R}$ then gives (using (2.4) for the resolvent density)

$$\mathbb{E}^r[\exp\{-\beta\tau_\mathrm{R}\}] = \frac{r_\lambda(r, \mathrm{R})}{r_\lambda(\mathrm{R}, \mathrm{R})} = \frac{\mathrm{I}_0((r \wedge \mathrm{R})\gamma)\,\mathrm{K}_0((r \vee \mathrm{R})\gamma)}{\mathrm{I}_0(\mathrm{R}\gamma)\,\mathrm{K}_0(\mathrm{R}\gamma)} = \frac{\mathrm{I}_0(r\gamma)}{\mathrm{I}_0(\mathrm{R}\gamma)} \quad (2.16)$$

(which of course is not a new result.) Putting (2.16) together with what we have already done at (2.13-2.14) and substituting into (2.15) gives the formula (1.2*a*).

The result (1.2*b*) for a starting point outside of D_R can be derived similarly, but this time using the fact that for $r>R$,

$$\mathbb{P}^r(\text{no red pts. before } \tau_R)=\mathbb{E}^r[\exp\{-\lambda\tau_R\}]=\frac{K_0(r\theta)}{K_0(R\theta)}.$$

This completes the proof of Theorem 1. □

We conclude this section with a few additional observations. For $x>0$, define $u(x):=\int_0^\infty e^{-\lambda t}\,\mathbb{E}[e^{-\alpha U_t(x)}]\,dt$. Next, introduce the notation

$$\partial_{R\pm} f(x)=\left.\frac{d}{dx}\right|_{x=R\pm} f(x)$$

for some function f. Formally applying the formula for $u(R)$ given in Truman and Williams (1990) [Eq. (61)] gives

$$u(R)=\frac{(\lambda+\alpha)^{-1}\,\partial_{R-}\,\mathbb{E}^x[\exp\{-(\lambda+\alpha)\tau_R\}]-\lambda^{-1}\,\partial_{R+}\,\mathbb{E}^x[\exp\{-\lambda\tau_R\}]}{\partial_{R-}\,\mathbb{E}^x[\exp\{-(\lambda+\alpha)\tau_R\}]-\partial_{R+}\,\mathbb{E}^x[\exp\{-\lambda\tau_R\}]}. \tag{2.17}$$

From (2.16) we have

$$\left.\begin{aligned}\partial_{R-}\,\mathbb{E}^x[\exp\{-(\lambda+\alpha)\tau_R\}]&=\gamma\frac{I_1(R\gamma)}{I_0(R\gamma)},\\ \partial_{R+}\,\mathbb{E}^x[\exp\{-\lambda\tau_R\}]&=-\theta\frac{K_1(R\theta)}{K_0(R\theta)}\end{aligned}\right\} \tag{2.18}$$

and substituting (2.18) into (2.17) shows that the formal result (2.17) indeed agrees with (1.2*a*) when $r=R$.

Finally, the method presented here can be used similarly to compute the occupation time distribution for an annulus. We leave the details to the reader.

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