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A note on parabolic convexity and heat conduction

by

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ABSTRACT. – We introduce the notion of parabolic convexity and show its interplay with heat conduction. The mathematical method is based on Brownian motion and the Ehrhard inequality [8].

RÉSUMÉ. – Nous introduisons la notion de convexité parabolique et nous démontrons son interaction avec la conduite de la chaleur. La méthode mathématique est basée sur le mouvement brownien et l'inégalité de Ehrhard [8].

1. INTRODUCTION

A subset E of $\mathbb{R}_+ \times \mathbb{R}^n$ is said to be parabolically convex, if for any $\zeta_0 = (t_0, x_0)$ and $\zeta_1 = (t_1, x_1)$ belonging to E,

$$(((1-\theta)\sqrt{t_0} + \theta\sqrt{t_1})^2, (1-\theta)x_0 + \theta x_1) \in E, \quad \text{all} \quad 0 \leq \theta \leq 1$$

or, stated equivalently,

$$t_0 \neq t_1 \Rightarrow \left(t, \frac{\sqrt{t_1} - \sqrt{t}}{\sqrt{t_1} - \sqrt{t_0}} x_0 + \frac{\sqrt{t} - \sqrt{t_0}}{\sqrt{t_1} - \sqrt{t_0}} x_1\right) \in E, \quad \text{all} \quad t \in [t_0, t_1]$$

and

$$t_0 = t_1 \Rightarrow \{t_0\} \times [x_0, x_1] \subseteq E.$$

The purpose of this paper is to show some nice properties of parabolically convex sets in connection with heat conduction. As far as we know, the notion of parabolic convexity has not been stressed on earlier, at least not explicitly.

In what follows, D stands for a domain in $\mathbb{R} \times \mathbb{R}^n$ and $p : D \times D \to [0, +\infty[$ is the Green function of the heat operator in D equipped with the Dirichlet boundary condition zero (for details, *see* Watson [12]). Given $\zeta_0 = (t_0, x_0) \in D$ and $r \geq 0$, the set

$$B(\zeta_0, r) = \{\zeta \in D; p(\zeta, \zeta_0) > r\}$$

is called a heat ball in D with centre at ζ_0 (*cf.* Bauer [1] and Watson [13]). For the sake of comparison, recall that, if M is a Greenian domain in $\mathbb{R}^n$ and if $g : M \times M \to [0, +\infty]$ denotes the Green function of the Laplace operator in M equipped with the Dirichlet boundary condition zero, then given $x_0 \in M$ and $r \geq 0$, the set $\{x \in M; g(x, x_0) > r\}$ is called a harmonic ball in M with centre at x_0 [1].

The starting point of his paper is a rather old theorem by Gabriel [6], stating that harmonic balls in convex regions are convex (for a probabilistic proof, *see* Borell [3]). But remarkably enough, it still seems to be unknown whether heat balls in convex domains must be convex or not. Note, however, that if B is a heat ball in D, then each section $B \cap \{t = \tau\}$ is convex as soon as every section $D \cap \{t = \tau\}$ is convex (Borell [2]). The main purpose of this paper is to show the following

THEOREM 1.1. – *If the set $D \cap \{t > 0\}$ is parabolically convex, then any heat ball in D with its centre in $D \cap \{t = 0\}$ is parabolically convex.*

The proof of Theorem 1.1 is based on Brownian motion and Brunn-Minkowski inequalities of Gaussian measures as in the papers [2] and [3]. A similar line of reasoning may be found in two early papers by Brascamp and Lieb ([5], [6]). For additional information, *see* Hörmander's book [10].

2. SIMPLE PROPERTIES OF PARABOLICALLY CONVEX SETS

A family $(A_s)_{s>0}$ of subsets of $\mathbb{R}^n$ is said to be concave if

$$A_{(1-\theta)s_0+\theta s_1} \supseteq (1-\theta)A_{s_0} + \theta A_{s_1}$$

for all $s_0, s_1 > 0$ and all $0 \leq \theta \leq 1$.

From now on, let $H_n = \mathbb{R}_+ \times \mathbb{R}^n$. If $E \subseteq H_n$, we set $E(t) = \{x \in \mathbb{R}^n; (t,x) \in E\}, t > 0$. Moreover, we define a bijection $(s, y) = \psi(t, x)$ of H_n onto H_n by setting

$$\begin{cases} s = 1/\sqrt{t} \\ y = x/\sqrt{t}. \end{cases}$$

THEOREM 2.1. – *Let $E \subseteq H_n$. The following assertions are equivalent:*

(i) *E is parabolically convex.*

(ii) *The family $(sE(s^{-2}))_{s>0}$ is concave.*

(iii) *The set $\psi(E)$ is convex.*

Proof. – (i) $\Rightarrow$ (ii): Let $s_0 \geq s_1 > 0$ and put $s_\theta = (1-\theta)s_0 + \theta s_1$, where $0 \leq \theta \leq 1$ is fixed. Moreover, we choose $x_0 \in E(s_0^{-2})$ and $x_1 \in E(s_1^{-2})$ arbitrarily and shall prove that

$$s_\theta^{-1}((1-\theta)s_0 x_0 + \theta s_1 x_1) \in E(s_\theta^{-2}).$$

The special case $s_0 = s_1$ is trivial since $E(t)$ is convex for every $t > 0$. Therefore, assume $s_0 > s_1$ and consider the curve $(t, x(t)), t_0 \leq t \leq t_1$, where $t_0 = s_0^{-2}$ and $t_1 = s_1^{-2}$, and

$$x(t) = \frac{\sqrt{t_1} - \sqrt{t}}{\sqrt{t_1} - \sqrt{t_0}} x_0 + \frac{\sqrt{t} - \sqrt{t_0}}{\sqrt{t_1} - \sqrt{t_0}} x_1.$$

Note that $x(s_\theta^{-2}) \in E(s_\theta^{-2})$ since E is parabolically convex. Moreover,

$$x(s_\theta^{-2}) = \frac{s_1^{-1} - s_\theta^{-1}}{s_1^{-1} - s_0^{-1}} x_0 + \frac{s_\theta^{-1} - s_0^{-1}}{s_1^{-1} - s_0^{-1}} x_1,$$

where

$$\begin{aligned} \frac{s_1^{-1} - s_\theta^{-1}}{s_1^{-1} - s_0^{-1}} &= s_\theta^{-1} \frac{s_\theta/s_1 - 1}{s_1^{-1} - s_0^{-1}} \\ &= s_\theta^{-1} \frac{(1-\theta)s_0/s_1 - (1-\theta)}{s_1^{-1} - s_0^{-1}} = s_\theta^{-1}(1-\theta)s_0 \end{aligned}$$

and, in a similar way,

$$\frac{s_\theta^{-1} - s_0^{-1}}{s_1^{-1} - s_0^{-1}} = s_\theta^{-1}(1-\theta)s_1.$$

Accordingly, from these equations,

$$s_\theta^{-1}((1-\theta)s_0x_0+\theta s_1x_1)=x(s_\theta^{-2})\in E(s_\theta^{-2}).$$

(ii) $\Rightarrow$ (i): The proof is, again, simple and it is omitted here.
(ii) $\Leftrightarrow$ (iii): The equivalence follows at once from the equality

$$\psi(E)=\{(s,y)\in H_n; y\in sE(s^{-2})\}.$$

This completes our proof of Theorem 2.1.

COROLLARY 2.1. – *Let M be a domain in $\mathbb{R}^n$. Then the set $\mathbb{R}_+\times M$ is parabolically convex if and only if M is convex.*

The reader should note that, if a set $E\subseteq H_n$ is parabolically convex, the sets $(a,0)+E, a>0$, need not be parabolically convex. Indeed, if so, the set E must necessarily be convex since the curvature of the curve

$$\left(t,\frac{\sqrt{t_1+a}-\sqrt{t}}{\sqrt{t_1+a}-\sqrt{t_0+a}}x_0+\frac{\sqrt{t}-\sqrt{t_0+a}}{\sqrt{t_1+a}-\sqrt{t_0+a}}x_1\right), a+t_0\le t\le a+t_1,$$

tends to zero uniformly as a tends to plus infinity.

3. THE MAIN RESULT

From now on, the function

$$\Phi(x)=\int_{-\infty}^{x}\exp(-\lambda^2/2)d\lambda/\sqrt{2\pi},\ x\in[-\infty,+\infty]$$

denotes the distribution function of a $N(0;1)$-distributed random variable and we let $\Phi^{-1}:[0,1]\to[-\infty,+\infty]$ be its inverse function.

THEOREM 3.1. – *Let D be a domain in $\mathbb{R}\times\mathbb{R}^n$ such that the set $D^+=D\cap\{t>0\}$ is parabolically convex. Moreover, assume $A\subseteq\mathbb{R}^n$ is a non-empty convex domain such that $\{0\}\times A\subseteq D\cap\{t=0\}$ and define*

$$u(\zeta)=\int_A p(\zeta,(0,y))dy,\ \zeta\in D^+.$$

Then the function $\Phi^{-1}\circ u\circ\psi^{-1}$ is concave in $\psi(D^+)$, and, especially, the level sets $\{u>r\}, r\ge 0$, are parabolically convex. Moreover, if

$T = \sup\{t; D^+(t) \neq \phi\}$, *then* $\lim_{\zeta \to \zeta_0} u(\zeta) = 0$ *for any* $\zeta_0 = (t_0, x_0) \in \partial D$ *with* $0 < t_0 < T$.

Proof. – Let $\beta = (\beta(t))_{t \geq 0}$ denote the normalized Brownian motion in $\mathbb{R}^n$ and suppose $\mu = \mathbb{P}_\beta$ is Wiener measure on $C(\bar{\mathbb{R}}_+; \mathbb{R}^n)$, the space of all continuous mappings of $\bar{\mathbb{R}}_+$ into $\mathbb{R}^n$ equipped with the topology of uniform convergence on compacts. To prove Theorem 3.1, we will make use of the Ehrhard inequality [8] of the Brunn-Minkowski type, stating that

$$\Phi^{-1}(\mu((1-\theta)B_0 + \theta B_1)) \geq (1-\theta)\Phi^{-1}(\mu(B_0)) + \theta\Phi^{-1}(\mu(B_1))$$

for every $0 \leq \theta \leq 1$ and every convex Borel sets B_0 and B_1 in $C(\bar{\mathbb{R}}_+; \mathbb{R}^n)$. To this end, we represent u in terms of Brownian motion (*see* Doob [4]) and have

$$u(t,x) = \mathbb{P}[\beta(t) \in A - x, \beta(\tau) \in D^+(t-\tau) - x, \text{ all } \quad 0 < \tau < t],$$

or, expressed slightly differently,

$$u(t,x) = \mathbb{P}[\beta(t) \in A - x, \beta(t\tau) \in D^+(t(1-\tau)) - x, \text{ all } \quad 0 < \tau < 1].$$

Moreover, since the processes $(\beta(t\tau))_{\tau \geq 0}$ and $(\sqrt{t}\beta(\tau))_{\tau \geq 0}$, have the same probability laws, we conclude that

$$\begin{aligned} u(t,x) = \mathbb{P}[\beta(1) \in (A-x)/\sqrt{t}, \beta(\tau) \\ \in (D^+(t(1-\tau)) - x)/\sqrt{t}, \text{ all } \quad 0 < \tau < 1]. \end{aligned}$$

Thus, if $(s,y) = \psi(t,x)$, that is, $t = s^{-2}$ and $x = y/s$, then

$$\begin{aligned} u(\psi^{-1}(s,y)) = \mathbb{P}[\beta(1) \in sA - y, \beta(\tau) \\ \in sD^+((1-\tau)s^{-2}) - y, \text{ all } \quad 0 < \tau < 1]. \end{aligned}$$

Now applying the Ehrhard inequality and Theorem 2.1, we conclude that the function $\Phi^{-1} \circ u \circ \psi^{-1}$ is concave, if the set D^+ is parabolically convex.

To prove the last part in Theorem 3.1, set $(s_0, y_0) = \psi(t_0, x_0)$. Since the set $\psi(D)$ is convex it is possible to find a non-zero vector $(a,b) \in \mathbb{R} \times \mathbb{R}^n$ and a $c \in \mathbb{R}$ such that

$$\begin{cases} as + b \cdot y \geq c, & (s,y) \in \psi(D) \\ as_0 + b \cdot y_0 = c. \end{cases}$$

Clearly, $D(t) \neq \emptyset$, $0 < t < T$, since $A \neq \emptyset$. Therefore, noting that $t_0 < T, b \neq 0$. Moreover,

$$u(t,x) \leq \mathbb{P}[a + b \cdot (x + \beta(\tau)) \geq c\sqrt{t-\tau}, \text{ all } \quad 0 < \tau < t],$$

and, consequently,

$$u(t,x) \leq \mathbb{P}\left[b \cdot \beta(\tau) \geq -c\frac{\tau}{\sqrt{t-\tau}+\sqrt{t}} + c\sqrt{t} - a - b \cdot x, \text{ all } \quad 0 < \tau < t\right].$$

Now remembering that

$$\underline{\lim}_{\tau \to 0^+} \frac{b \cdot \beta(\tau)}{(2\tau \ln \ln \tau^{-1})^{1/2}} = -|b|^2 \quad \text{a.s.}$$

by the law of the iterated logarithm for Brownian motion and noting that $c\sqrt{t_0} - a - bx_0 = 0$, we have that $\lim_{\zeta \to \zeta_0} u(\zeta) = 0$. This completes our proof of Theorem 3.1.

Example 3.1. – Suppose $D = \mathbb{R} \times \mathbb{R}^n$ and $A = \{x_n > 0\}$. Then

$$u(t,x) = \int_A \exp(-|x-y|^2/2t)dx/\sqrt{2\pi t}^n = \Phi(x_n/\sqrt{t}),\ (t,x) \in D^+,$$

and

$$\Phi^{-1}(u(\psi^{-1}(s,y))) = y_n.$$

Thus the function $\Phi^{-1} \circ u \circ \psi^{-1}$ is linear in this particular case. □

Example 3.2. – Suppose $D = \mathbb{R} \times B$, where B is a bounded convex domain in $\mathbb{R}^n$, and set

$$u(\zeta) = \int_B p(\zeta, (0,y))dy, \quad \zeta \in D^+,$$

and

$$v(x) = \int_0^\infty u(t,x)dt, \quad x \in B.$$

By Theorem 3.1, the level sets $\{u > r\}, r \geq 0$, are parabolically convex. Furthermore, it is well known that the level sets $\{v > r\}, r \geq 0$, are convex (Kawohl [11], Borell [4]). However, the level sets $\{u > r\}, r \geq 0$, need not be convex. To see this, let $k \in \mathbb{N}_+$ and choose $B = B_k = \{x \in \mathbb{R}^n; |x| < k, x_n > 0\}$. Now setting $u = u_k$, we have that $\lim_{k \to +\infty} u_k(t,x) = 2\Phi(x_n/\sqrt{t}) - 1$. Clearly, the set $\{(t,x) \in H_n; x_n > 0, 2\Phi(x_n/\sqrt{t}) - 1 > r\}$ is not convex for any $r > 0$, which proves the claim above. □

Finally, note that Theorem 1.1 is an immediate consequence of Theorem 3.1.

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