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by

Serguei M. KOZLOV †
U.F.R. M.I.M., U.R.A. 225 C.N.R.S.,
Université de Provence Aix-Marseille-I,
3, place Victor-Hugo, 13331 Marseille Cedex, France

and

Andrei L. PIATNITSKI
Lebedev Physical Institute,
Leninski prospect 53, Moscow 117333, Russia

ABSTRACT. – The asymptotic behaviour of effective diffusion for a parabolic operator in \( \mathbb{R}^n \) with smooth periodic potential and small periodic initial diffusion is studied. We obtain logarithmic asymptotics of effective diffusion with respect to the initial diffusion. The answer is given in terms of auxiliary variational problem on the torus, which consists in minimizing a length in certain metric of the curves passing through the maximal point of the potential and having given homological class. The paper generalizes our previous result [3] where the case of piecewise constant potential was investigated.

Key words: Homogenization, large deviations, effective diffusion.

RÉSUMÉ. – On considère le comportement asymptotique de la diffusion effective pour une équation parabolique avec un potentiel périodique,

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lorsque le coefficient de diffusion initial tend vers zéro. On trouve cet asymptotique au sens logarithmique en termes de problèmes variationnels liés au potentiel donné.

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**INTRODUCTION**

In the paper we study a diffusion model of biological particles assuming that the life conditions and diffusive properties of the medium are described by periodic functions. To investigate the behaviour of a population of such particles for large time one can apply the methods of automodel homogenization theory of parabolic equations. As it follows from [1], [2], the long time behaviour of the population could be described in terms of so-called effective diffusion which in our case depends on the initial diffusivity and the potential. The goal of this work is to obtain the asymptotics of the effective diffusion as the initial diffusion goes to zero. Here we adopt the terminology and the main notations from [3] where the similar problem was solved for special case when the potential is piecewise constant and the initial diffusion doesn’t depend on the point of medium.

The corresponding parabolic equation for the density $u(x, t)$ of the particles has the following form

$$\begin{aligned}
\frac{\partial}{\partial t} u &= \mu^2 \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} u + v(x) u \\
u|_{t=0} &= v_0(x),
\end{aligned}$$

(0.1)

where $v_0(x)$ is the initial density and $\mu$ is small positive parameter characterizing the initial diffusion of the particle, summation over repeated indices is omitted. From now on we suppose that the matrix $(a_{ij}(x))$ is periodic symmetric and uniformly elliptic and the potential $v(x)$ is periodic and has only one global maximum point on each period.

To describe the behaviour of $u(x, t)$ for large time let’s introduce the following eigenvalue problem

$$\left(\mu^2 \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + v(x)\right) p = \lambda p$$

(0.2)

and denote the first eigenvalue and eigenfunction of this problem simply by $\lambda$ and $p(x)$ respectively. We fix the choice of $p(x)$ by the normalisation

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condition \( \langle p \rangle = 1 \), where \( \langle \cdot \rangle \) means the average of a periodic function over the period. Then according to [2] for any positive \( \mu \) the asymptotic relation

\[
u(x, t) = \exp(\lambda t) p(x) \hat{u}(x, t) \left(1 + o(1)\right)
\]

(0.3)

holds in the region \( \{ (x, t) \mid x^2 < c_1 t + c_2 \} \) as \( t \to \infty \). The function \( \hat{u}(x, t) \) describing the diffusion properties of the solution \( u(x, t) \) satisfies the homogenized parabolic equation

\[
\frac{\partial}{\partial t} \hat{u} = \frac{\partial}{\partial x_i} \sigma_{ij} \frac{\partial}{\partial x_j} \hat{u}, \quad \hat{u} \big|_{t=0} = \langle p^2 \rangle v_0(x)
\]

(0.4)

whose matrix of constant coefficients \( (\sigma_{ij}) = (\sigma_{ij}(\mu)) \) is called the effective diffusion matrix. For reader convenience we outline here the method for constructing of \( \sigma_{ij} \). Namely let \( \psi_k, k = 1, 2, \ldots, n \), be periodic solutions of the equations

\[
\frac{\partial}{\partial x_i} \left( p^2(x) a_{ij}(x) \frac{\partial}{\partial x_j}(\psi_k + x_k) \right) = 0
\]

(0.5)

where \( x_k \) is \( k \)-th independent variable. Then the effective diffusion matrix \( (\sigma_{ij}) \) is defined by the formula

\[
\sigma(\mu) = \langle (I + \nabla \psi)^T p^2 a (I + \nabla \psi) \rangle / \langle p^2 \rangle,
\]

(0.6)

where \( \nabla \psi = \left( \frac{\partial}{\partial x_i} \psi_j \right) \), \( I \) is the unit matrix and the symbol \( ^T \) means transposition of the matrix, \( a = (a_{ij}) \).

The expression in the right hand side of (0.6) depends on \( \mu \) in explicit manner and the studying of this expression for small \( \mu \) is a complicated problem involving resolution of the singularly perturbed PDE. So we use another approach to find the asymptotics of \( \sigma(\mu) \) as \( \mu \to 0 \). Namely we’ll transform equation (0.1) to an equation without potential and then applying the rough estimates for effective diffusion replace this equation by another one of divergence form or equivalently with potential drift and isotropic diffusion. To that end we must control the ground state \( p(x) \) all over the torus as \( \mu \) goes to zero, we find it to be of order \( \exp(-W(x)/\mu) \) in logarithmic sense where \( W(x) \) is a distance from the point \( x \) to the maximum point in the certain metric on the torus. Here our analysis relies on Freidlin-Wentzel results [12] where analogues problem for the invariant measure for diffusion operator was considered. After that using the result of [4] we’ll show (and this is the main result of the paper) that effective diffusion \( \sigma(\mu) \) satisfies the following limiting relation

\[
\lim_{\mu \to 0} \frac{\mu \ln \sigma(\mu)}{} = -\Theta,
\]
where $\Theta$ is a positive matrix, whose coefficients will be given in terms of auxiliary variational problems. Here we point out that each its eigenvalue is the minimal length of the closed curve passing through the maximum point of the potential and having fixed homological class, in the metric constructed in terms of the potential and quadratic form of the principal part of the operator. These curves also define a sequence of invariant subspaces of $\Theta$. Detailed construction of limiting matrix is presented in Section 3.

Similar problem for a diffusive particle in the presence of a vector field was largely discussed in the physical and mathematical literature (see [5], [6] for incompressible vector field, [7] for general discussion and [8] where different from [4] approach to the case of potential vector field is presented). We also mention paper [10] where more realistic case of Fokker-Plank equation was treated.

1. The asymptotic properties of periodic eigenvalue problem

In this paragraph we study the asymptotic behaviour of the first eigenvalue and eigenfunction of (0.2) for small $\mu$. Without loss of generality we'll suppose that $\max v(x) = 0$ and the set of its maximum points $\{ x \in R^n | v(x) = 0 \}$ coincides with integer lattice $Z^n$.

**Proposition 1.** - The first eigenvalues $\lambda$ satisfies the estimates

$$0 \leq -\lambda \leq c \mu$$

(1.1)

where the constant $c$ doesn’t depend on $\mu$.

**Proof.** – The first inequality (1.1) is simple consequence of the following variational representation for $\lambda$

$$-\lambda = \mu^2 \inf_{\|u\|_{L^2(T^n)} = 1} \left\{ \int_{T^n} a_{ij}(x) \frac{\partial}{\partial x_i} u \frac{\partial}{\partial x_j} u \, dx \right\}$$

$$+ \frac{1}{\mu^2} \int_{T^n} |v(x)| u^2 \, dx$$

(1.2)

where $T^n = R^n / Z^n$ is standard $n$-dimensional torus. To prove the inequality from above let us introduce the test function

$$u(x) = c \mu^{-n/4} \exp \left( - |x|^2 / \mu \right) \chi(|x|)$$
where the normalizing constant $c$ is used to satisfy the relation $\|u\|_{L^2(T^n)} = 1$ and $\chi(|x|)$ is smooth positive cutoff, $\chi = 1$ for $|x| < \frac{1}{8}$ and $\chi = 0$ for $|x| > \frac{1}{4}$. Under such choice of $\chi$ the constant $c$ is uniformly in $\mu$ positive and bounded. We have, substituting this test function into (1.2)

$$-\lambda/\mu^2 \leq \int_{T^n} a_{ij}(x) \frac{\partial}{\partial x_i} u \frac{\partial}{\partial x_j} u \, dx + \frac{1}{\mu^2} \int_{T^n} |v(x)| u^2 \, dx$$

$$\leq c \mu^{-\frac{n}{2}} \left( \frac{4}{\mu^2} \int_{T^n} a_{ij}(x) x_i x_j e^{-|x|^2/\mu} e^{-|x|^2/\mu} \chi^2(|x|) \, dx 
+ \frac{4}{\mu} \int_{T^n} a_{ij}(x) x_i e^{-2|x|^2/\mu} \frac{\partial}{\partial x_j} \chi^2(|x|) \, dx 
+ \int_{T^n} a_{ij}(x) e^{-2|x|^2/\mu} \frac{\partial}{\partial x_i} \chi(|x|) \frac{\partial}{\partial x_j} \chi(|x|) \, dx 
+ \frac{1}{\mu^2} \int_{T^n} |v(x)| e^{-2|x|^2/\mu} \chi^2(|x|) \, dx \right).$$

The second and third integrals here are bounded because corresponding derivatives of $\chi(|x|)$ are equal to zero in the domain $\{x : |x| < \frac{1}{8}\}$. Then taking into account our assumptions we obtain the inequality $|v(x)| \leq c_1|x|^2$ for some positive constant $c_1$. This yields

$$-\lambda/\mu^2 \leq c + c^2 \mu^{-\frac{n}{2}} \left( \frac{4}{\mu^2} \int_{T^n} a_{ij}(x) x_i x_j e^{-2|x|^2/\mu} \chi^2(|x|) \, dx 
+ \frac{c_1}{\mu^2} \int_{T^n} x^2 e^{-2|x|^2/\mu} \chi^2(|x|) \, dx \right)$$

$$\leq c + \frac{4c^2}{\mu} \int_{R^n} a_{ij}(\sqrt{\mu} y) e^{-2|y|^2} y_i y_j dy + \frac{c_1 c^2}{\mu} \int_{R^n} y^2 e^{-2|y|^2} \, dx \leq c + \frac{c_2}{\mu}.$$ 

Now let’s establish the main properties of the eigenfunction $u_0(x)$ which solves (1.2).

**Proposition 2.** For any $\mu > 0$

$$1 \leq \max_{T^n} u_0(x) \leq c \mu^{-n/2} \quad (1.3)$$

**Proof.** The left inequality immediately follows from the relation $\|u_0\|_{L^2(T^n)} = 1$. The right one can be derived from standard estimates for
elliptic equations. Indeed, in rescaled coordinates \( y = \frac{x}{\mu} \) the equation (0.2) takes the sight

\[
\frac{\partial}{\partial y_i} \alpha_{ij} (\mu y) \frac{\partial}{\partial y_j} u_0 + v (\mu y) u_0 = \lambda u_0
\]  

(1.4)

while \( L^2 \)-norm of \( u_0 (\mu y) \) over the period is equal to \( \mu^{-n/2} \). According to Proposition 1 the coefficients of (1.4) are uniformly in \( \mu \) bounded so by the internal estimates for the solutions of elliptic equations [11] we obtain the second inequality (1.3).

Denote by \( x^0_\mu \) the maximum point of \( u_0 (x) \).

**Proposition 3.** - The maximum point \( x^0_\mu \) goes to zero as \( \mu \to 0 \).

**Proof.** - According to maximum principle [11] \( x^0_\mu \) is located inside the domain \( \{ x \in T^n : v (x) - \lambda > 0 \} \). By Proposition 1 we find \( -v (x^0_\mu) < -\lambda < c \mu \) and required statement is the consequence of our assumption about uniqueness of maximum point of \( v (x) \) on \( T^n \).

Now let \( \delta \) be an arbitrary positive number. We set \( Q_\delta = \{ x : |x| < \delta \} \). Our next aim is to estimate \( u_0 (x) \) at the boundary \( \partial Q_\delta \) from below.

**Proposition 4.** - There exist positive \( c_0 \) and \( c_1 \) such that

\[
u_0 (x)_{|\partial Q_\delta} \geq c_1 e^{-c_0 \delta / \mu}.
\]

**Proof.** - Applying Harnack inequality [11] to the solution \( u_0 (\mu y) \) of (1.4) one can obtain that

\[
c^{-1} \leq u_0 (\mu y_1)/u_0 (\mu y_2) \leq c
\]

(1.5)

for any \( y_1, y_2 \) such that \( |y_1 - y_2| \leq 1 \), where the constant \( c \geq 1 \) doesn’t depend on \( \mu \). In coordinates \( x = \mu y \) we find

\[
c^{-1} \leq u_0 (x_1)/u_0 (x_2) \leq c
\]

(1.6)

for any \( x_1, x_2 \) such that \( |x_1 - x_2| < \mu \). By Proposition 3 \( x^0_\mu \in Q_\delta \) for sufficiently small \( \mu \) therefore the distance between \( x^0_\mu \) and any point \( x \) of \( \partial Q_\delta \) is less than \( 2 \delta \). Connecting \( x^0_\mu \) with an arbitrary point of \( \partial Q_\delta \) by the sequence of \( \lfloor 2 \delta / \mu \rfloor \) points \( (\lfloor . \rfloor \) is the integer part) lying on the same line and iterating (1.6) we find

\[
u_0 (x)/u_0 (x^0_\mu) \geq c^{-\lfloor 2\delta / \mu \rfloor} \geq c_1 e^{2\delta \ln c / \mu} = c_1 e^{-c_0 \delta / \mu}
\]

for any \( x \in \partial Q_\delta \). To complete the proof it suffices to note that \( u_0 (x^0_\mu) \geq 1 \).
Remark 1. – One can easily establish the exact asymptotics of $\lambda$, which is $\lambda = c \mu (1 + O (\mu))$ for the potential with nondegenerate maximum point, but the result of the Proposition 1 is sufficient for our purposes.

Now from Propositions 2, 4 we have
\[
\int_{T^n} u_0(x) \, dx \geq \mu^n
\]
so for
\[
p(x) = \frac{u_0}{\langle u_0(x) \rangle}
\]
we have the following

**Proposition 5.** – The function $p(x)$ satisfies the estimates:
\[
\max_{T^n} p(x) \leq c_2 \mu \frac{-3n}{2}, \quad p(x)|_{\partial Q_i} \geq c_3 e^{-c_4 \delta / \mu}.
\]

Now we are going to describe the logarithmic asymptotics of $p(x)$. For this purpose we define the function $\bar{W}(x)$ as the solution of the following variational problem
\[
\bar{W}^2(x) = \inf_{x(t) \in T^n} \int_0^1 (-v(x(t))) a^{ij}(x(t)) \dot{x}_i(t) \dot{x}_j(t) \, dt
\]
where $(a^{ij}(x)) = (a_{ij}(x))^{-1}$ and inf is taken over all smooth paths connecting $x$ with 0. We will also use the function $W(x) = \bar{W}(x)/2$ which is more convenient in probabilistic interpretation.

**Lemma 1.** – Uniformly in $x \in T^n$
\[
\lim_{\mu \to 0} \mu \ln p(x) = -2W(x) = -\bar{W}(x).
\]

**Proof.** – Let’s fix arbitrary $\delta > 0$ and devide the equation (0.1) by the function $(-\mu v(x))$ in the domain $T^n \setminus Q_\delta$:
\[
-\frac{a_{ij}(x)}{v(x)} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} p(x) - \mu \frac{1}{v(x)} \left( \frac{\partial}{\partial x_i} a_{ij}(x) \right) \frac{\partial}{\partial x_j} p(x)
\]
\[
- \frac{1}{\mu} \left( \frac{-v(x) + \lambda}{v(x)} \right) p(x) = 0.
\]
Denote by $\xi_t^x$ the diffusion process issuing from the point $x$ and corresponding to the operator
\[
B_\mu = -\frac{a_{ij}(x)}{v(x)} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} - \mu \frac{1}{v(x)} \left( \frac{\partial}{\partial x_i} a_{ij}(x) \right) \frac{\partial}{\partial x_j}.
\]
Let $\tau(x)$ be the exit time from $T^n \setminus Q_\delta$. Then the solution $p(x)$, $x \in T^n \setminus Q_\delta$ can be represented in the following probabilistic form (see [12]):

$$p(x) = M \left[ \exp \left( -\frac{1}{\mu} \int_0^{\tau(x)} \frac{v(\xi^x_s) - \lambda}{v(\xi^x_s)} \, ds \right) p(\xi^x_{\tau(x)}) \right]$$

$$= M \left[ \exp \left( -\frac{\tau(x)}{\mu} + \frac{\lambda}{\mu} \int_0^{\tau(x)} \frac{ds}{v(\xi^x_s)} \right) p(\xi^x_{\tau(x)}) \right]$$

$$= M \left[ \exp \left( -\frac{\tau(x)}{\mu} (1 + O(\mu)) \right) p(\xi^x_{\tau(x)}) \right].$$

The last equality here follows from Proposition 1 which states that $\lambda/\mu$ is bounded. We also use the fact $\xi^x_s \in T^n \setminus Q_\delta$ for $s < \tau(x)$ and uniqueness of the maximum point. The function $O(\mu)$ depends on $\delta$. By Proposition 5

$$c_1 e^{-c\delta/\mu} \leq p(x)|_{\partial Q_\delta} \leq c_2 \mu^{-3n/2}$$

Therefore

$$c_1 e^{-c\delta/\mu} M e^{-\frac{\tau(x)}{\mu} (1+O(\mu))} \leq p(x) \leq c_2 \mu^{-3n/2} M e^{-\frac{\tau(x)}{\mu} (1+O(\mu))}$$

for $x \in T^n \setminus Q_\delta$. Now let $\delta_1$ be positive sufficiently small number. We set $\delta_0 = \inf_{\partial Q_{\delta_1}} W(x)$ and suppose that $\delta$ satisfies the relation $\delta < \delta_0^2$. For $x \in T^n \setminus Q_\delta$ we define the function $W_\delta(x)$ as follows

$$4W^2_\delta(x) = \inf_{\{x(t) : x(0) = x, x(1) \in \partial Q_\delta \}} \int_0^1 (-v(x(t))) a^{ij}(x(t)) \dot{x}_i(t) \dot{x}_j(t) \, dt$$

According to the definitions of $W(x)$ and $W_\delta(x)$ and relation for $\delta$ and $\delta_1$ we have for $x \in T^n \setminus Q_{\delta_1}$

$$0 \leq W(x) - W_\delta(x) \leq c \delta \quad (1.8)$$

$$W(x) \geq \delta_0 \quad (1.9)$$

First we are going to estimate $M e^{-\frac{\tau(x)}{\mu} (1+O(\mu))}$ from below. For this purpose let’s construct the smooth path $\tilde{\varphi}(t)$, $\tilde{\varphi}(0) = x \in T^n \setminus Q_\delta$, $|\tilde{\varphi}(1)| = \delta/2$, such that

$$\frac{1}{4} \int_0^1 (-v(\tilde{\varphi}(t))) a^{ij}(\tilde{\varphi}(t)) \dot{\tilde{\varphi}}_i(t) \dot{\tilde{\varphi}}_j(t) \, dt \leq W^2(x).$$

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Then the function \( \varphi (t) = \hat{\varphi} \left( \frac{t}{W(x)} \right) \) satisfies the following relations:

\[
\frac{1}{4} \int_0^W (v (\varphi (t))) a_{i j} (\varphi (t)) \dot{\varphi}_i (t) \dot{\varphi}_j (t) dt \leq W(x),
\]

\[
\varphi (0) = x, \quad |\varphi (W(x))| = \delta /2.
\]

According to [12, chapter 3] this implies

\[
P \{ \rho_0, W(x) (\xi^x_t, \varphi (t)) < \delta /4 \} \geq c \exp \left( -\frac{W(x) + \delta}{\mu} \right)
\]

where \( \rho_0, W(x) (\xi^x_t, \varphi (t)) = \sup_{0 \leq t \leq W(x)} |\xi^x_t - \varphi (t)|. \) Hence taking into account the implication \( \{ \rho_0, W(x) (\xi^x_t, \varphi (t)) < \delta /4 \} \subset \{ \tau (x) \leq W(x) \} \)

we have

\[
P \{ \tau (x) \leq W(x) \} \geq c \exp \left( -\frac{W(x) + \delta}{\mu} \right)
\]

At last by Chebysheff inequality

\[
M e^{-\frac{\tau(x)}{\mu} (1+O(\mu))} \geq P \{ \tau (x) \leq W(x) \} e^{-\frac{W(x)}{\mu} (1+O(\mu))}
\]

\[
\geq e^{-\frac{W(x)}{\mu} (1+O(\mu))} e^{-\frac{W(x)+\delta}{\mu} (1+O(\mu))}
\]

\[
\geq c(\delta) e^{-2\frac{W(x)+\delta}{\mu}} \quad (1.10)
\]

To prove a similar upper bound we rewrite the quantity \( M e^{-\frac{\tau(x)}{\mu} (1+O(\mu))} \)

in the following form

\[
M e^{-\frac{\tau(x)}{\mu} (1+O(\mu))} = M \{ \tau (x) < 3 \max W \} e^{-\frac{\tau(x)}{\mu} (1+O(\mu))}
\]

\[
+ \chi_{\{ \tau (x) \geq 3 \max W \}} e^{-\frac{\tau(x)}{\mu} (1+O(\mu))}
\]

\[
\leq e^{3 \max W} M e^{-\tau(x)/\mu} + ce^{-2 \max W/\mu}
\]

\[
= c_1 M e^{-\tau(x)/\mu} + ce^{-2 \max W/\mu}
\]

\[
= c_1 \int_0^\infty e^{-s/\mu} d\theta (s) + ce^{-2 \max W/\mu}
\]

\[
= \frac{c_1}{\mu} \int_0^\infty \theta (s) e^{-s/\mu} ds + ce^{-2 \max W/\mu}
\]

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where \( \theta(s) \) is the distribution function of \( \tau(x) \). Let's divide the last integral into three parts:

\[
M e^{-\frac{\tau(x)}{\mu} (1+O(\mu))} \leq \frac{c_1}{\mu} \left( \int_0^{W(x)/2} \theta(s) e^{-s/\mu} ds + \int_{W(x)/2}^{2W(x)} \theta(s) e^{-s/\mu} ds + \int_{2W(x)}^{\infty} \theta(s) e^{-s/\mu} ds \right) + ce^{-2\max W/\mu}
\]

(1.11)

The last term in brackets is evidently less than \( \exp(-2W(x)/\mu) \). Then from [12, chapter 4] it follows that uniformly in \( s \in [\delta_0, 2 \max W] \) and \( x \in T^n \backslash Q_{\delta_1} \)

\[
\theta(s) \leq c(\delta) e^{-\frac{W^2(x) - c_1 \delta}{\mu s}} \leq c(\delta) e^{-\frac{W^2(x) - c_\delta}{\mu s}}
\]

where we also use (1.8). Thus for any \( s \leq W(x)/2 \)

\[
\theta(s) \leq \theta(W(x)/2) \leq c(\delta) e^{-\frac{2W(x) - c_\delta}{\mu}}.
\]

So the first integral in (1.11) is not greater than \( c(\delta) e^{-2\frac{W(x) - c_\delta}{\mu}} \). Let's estimate the second one

\[
\int_{W(x)/2}^{2W(x)} \theta(s) e^{-s/\mu} ds \leq \int_{W(x)/2}^{2W(x)} c(\delta) e^{-\frac{W^2(x) - c_\delta}{\mu s}} e^{-s/\mu} ds
\]

\[
= c(\delta) \int_{W(x)/2}^{2W(x)} c_\delta e^{\frac{c_\delta}{s}} e^{-\frac{1}{\mu} \left( \frac{W^2(x)}{s} + s \right)} ds
\]

\[
\leq c(\delta) e^{c_\delta/\mu} \int_{W(x)/2}^{2W(x)} e^{-\frac{1}{\mu \min s} \left( \frac{W^2(x)}{s} + s \right)} ds
\]

\[
= c(\delta) e^{c_\delta/\mu} \int_{W(x)/2}^{2W(x)} e^{-2W(x)/\mu} ds
\]

\[
\leq c(\delta) e^{-2\frac{W(x) - c_\delta}{\mu}}.
\]
Combining the estimates for all terms in (1.11) we find that
\[ M e^{-\frac{\tau(x)}{\mu} (1 + O(\mu))} \leq c(\delta) e^{-2 \frac{W(x) - c\delta}{\mu}} \]
uniformly in \( x \in T^n \setminus Q_{\delta_1} \). By (1.9), (1.10) and last inequality we obtain for \( x \in T^n \setminus Q_{\delta_1} \)
\[ \lim_{\mu \to 0} \mu \ln p(x) = -\bar{W}(x) \]
uniformly over this set. Then according to the definition of \( W(x) \) and Propositions 5 the following relations
\[ 0 \leq \bar{W}(x) \leq c \delta_1, \]
\[ c e^{-c\delta_1/\mu} \leq p(x) \leq c \mu^{-3n/2} \]
hold uniformly in \( x \in Q_{\delta_1} \). Hence
\[ \max_{Q_{\delta_1}} |\mu \ln p(x) - \bar{W}(x)| \leq c \delta_1. \]
With (1.12) this implies
\[ -\delta_1 - \bar{W}(x) \leq \liminf_{\mu \to 0} \mu \ln p(x) \leq \limsup_{\mu \to 0} \mu \ln p(x) \leq \delta_1 - \bar{W}(x) \]
uniformly on \( T^n \) and since \( \delta_1 \) is arbitrary number the lemma is proved.

2. REDUCTION TO THE EQUATION WITHOUT POTENTIAL

This section is devoted to a transformation of (0.1) to the equation without potential. For this aim we introduce new unknown function \( q(x, t) \) by the relation \( u(x, t) = e^{\lambda t} p(x) q(x, t) \). Substituting this expression into (0.1) instead of \( u(x, t) \) and taking into account equation (0.2) we find the equation for the function \( q(x, t) \):
\[ p(x) \frac{\partial}{\partial t} q(x, t) = p(x) \mu^2 \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} q(x, t) + 2 \mu^2 a_{ij}(x) \left( \frac{\partial}{\partial x_i} p(x) \right) \frac{\partial}{\partial x_j} q(x, t) \]

Let's multiply the last equation by \( p(x) \):
\[ p^2(x) \frac{\partial}{\partial t} q(x, t) = p^2(x) \mu^2 \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} q(x, t) + 2 p(x) \mu^2 a_{ij}(x) \left( \frac{\partial}{\partial x_i} \ln p(x) \right) \frac{\partial}{\partial x_j} q(x, t) \]
After simple transformation we have

\[ p^2(x) \frac{\partial}{\partial t} q(x, t) = \mu^2 \frac{\partial}{\partial x_i} p^2(x) a_{ij}(x) \frac{\partial}{\partial x_j} q(x, t) \]

\[ q|_{t=0} = v_0(x)p^{-1}(x) \]

This equation can be studied by the methods of homogenization theory. The homogenized equation takes the sight (see [2])

\[ \langle p^2 \rangle \frac{\partial}{\partial t} \hat{q}(x, t) = \mu^2 \frac{\partial}{\partial x_i} \hat{a}_{ij} \frac{\partial}{\partial x_j} \hat{q}(x, t) \]

\[ \hat{q}|_{t=0} = v_0(x)/\langle p^2 \rangle \]

(2.1)

where \( \hat{a}_{ij} \) is effective diffusion matrix for the elliptic part of the equations. This matrix differs from \( \sigma_{ij} \) only by factor \( \langle p^2 \rangle \) [see (0.4)-(0.6)]. Using the equality \( \langle p \rangle = 1 \) and Proposition 5 it is easy to see that

\[ 1 \leq \langle p^2 \rangle \leq c \mu^{-3n}. \]

Thus it is enough to study the asymptotic behaviour of the effective diffusion matrix for the elliptic operator of the form \( \frac{\partial}{\partial x_i} p^2(x) a_{ij}(x) \frac{\partial}{\partial x_j} \).

3. ASYMPTOTICS OF EFFECTIVE DIFFUSION

In order to study the properties of the function \( \bar{W}(x) \) more carefully we define the function \( W(x, \delta) \) as follows

\[ W^2(x, \delta) = \inf_{\{x(t), x(0)=x, x(1)=0\}} \int_0^1 (-v(x) + \delta) a^{ij}(x) \dot{x}_i \dot{x}_j dt \]

The following statement is a simple consequence of the definitions for \( \bar{W}(x) \) and \( W(x, \delta) \).

**Proposition 6.** - Uniformly in \( x \in T^n \)

\[ \lim_{\delta \to 0} W(x, \delta) = \bar{W}(x). \]

**Proposition 7.** - \( W(x) \) is Lipschitz function on \( T^n \).
Proof. For any $\delta > 0$ the function $W(x, \delta)$ is the distance of $x$ from 0 in the riemannian metrics $((-v(x) + \delta) a_{ij}(x))$ on $T^n$. Therefore $W(x, \delta)$ is Lipschitz function and by triangle axiom $|W(x, \delta) - W(y, \delta)| \leq \max\limits_{T^n} ((-v(x) + \delta) \nu(x))|x - y|$ where the notation $\nu(x)$ is used for the maximal eigenvalue of the matrix $a_{ij}(x)$. Passing to the limit in both sides of this inequality as $\delta \to 0$ and considering Proposition 6 we obtain

$$|\tilde{W}(x) - \tilde{W}(y)| \leq \max\limits_{T^n} ((-v(x) \nu(x))|x - y|.$$  

PROPOSITION 8. The only local minimum point of $\tilde{W}(x)$ on $T^n$ is the origin.

Proof. Let's suppose that there exists another local minimum point say $x_0 \neq 0$ of $\tilde{W}(x)$ on $T^n$. Then for some $\rho > 0$ the inequality $\tilde{W}(x_0) < \min\limits_{|x - x_0| = \rho} \tilde{W}(x)$ holds. According to Proposition 6 for sufficiently small $\delta$ the same inequality holds for the functions $W(x, \delta)$. This contradicts the triangle axiom. The proposition is proved.

Now having the function $\tilde{W}(x)$ continued by periodicity to all $R^n$, we construct the matrix $A_{\tilde{W}}$ as follows. Let $\beta_1$ be the solution of the following variational problem

$$\beta_1 = \inf\limits_{i \in Z^n \setminus 0} F(i), \quad F(i) = \inf\limits_{\{x(t), x(0) = 0, x(1) = i\}} \sup\limits_{t} \tilde{W}(x(t))$$

where $\inf$ is taken over all smooth paths connecting 0 with $i \in Z^n \setminus 0$. For every sequence $\{i_k\}_{k=1}^{\infty}, i_k \in Z^n \setminus 0$, denote by $\Lambda(\{i_k\})$ the set of the limiting points for the normalized sequence $\{i_k/|i_k|\}$. Then we define $\Lambda_{\beta_1}$ as a union of $\Lambda(\{i_k\})$ over all the sequences $\{i_k\}$ satisfying the relation $\lim\limits_{k \to \infty} F(i_k) = \beta_1$. According to the definition of $\beta_1$ and compactness of the unit sphere the set $\Lambda_{\beta_1}$ is closed and not empty. We also set $\Gamma_{\beta_1} = \{x \in \mathbb{R}^n : x/|x| \in \Lambda_{\beta_1}\}$.

PROPOSITION 9. $\Gamma_{\beta_1}$ is a linear subspace of $\mathbb{R}^n$.

Proof. By periodicity of $W$ it is easy to see that $\Gamma_{\beta_1}$ is symmetric with respect to the origin. Suppose that $\Lambda_{\beta_1}$ contains two linearly independent vectors $z_1$ and $z_2$. Then there exist two sequences $\{i_k^1\}$ and $\{i_k^2\}$, $F(i_k^1) \to \beta_1, j = 1, 2$, such that $(i_k^j/|i_k^j|) \to z_j$. The inequality $F(t^1 i_1 + t^2 i_2) \leq \max(F(i_1), F(i_2))$ is evidently true for any integer $t^1$ and $t^2, |t^1| + |t^2| > 0$, therefore any sequence of the form $(t^1_k i^1_k + t^2_k i^2_k), |t^1_k| + |t^2_k| > 0$, satisfies the relation $\lim\limits_{k \to \infty} F(t^1_k i^1_k + t^2_k i^2_k) = \beta_1$. At last...
arbitrary unit vector from the span of \( z_1 \) and \( z_2 \) can be approximated by the sequence of the form \((l_k^1 i_k^1 + l_k^2 i_k^2) / (\|l_k^1 i_k^1 + l_k^2 i_k^2\|)\) with integer \( l_k^1 \) and \( l_k^2 \).

**Proposition 10.** There exists \( \delta_0 > 0 \) such that the inequality \( F(i) \geq \beta_1 + \delta_0 \) holds for all \( i \in Z^n \setminus \Gamma_{\beta_1} \).

**Proof.** Let us suppose that the statement is false. Then one can find the sequence \( \{i_k\} \), \( i_k \in Z^n \setminus \Gamma_{\beta_1} \), to satisfy the equality \( \lim_{k \to \infty} F(i_k) = \beta_1 \). Denote by \( i_k' \) the projection of \( i_k \) onto \( \Gamma_{\beta_1} \). By definition of \( \Gamma_{\beta_1} \), for each \( k \) there exists the sequence \( \{i_{km}\}_{m=1}^\infty \) of integer vectors such that \( F(i_{km}) \to \beta_1 \) and \( (i_{km}/|i_{km}|) \to (i_k'/|i_k'|) \) as \( m \to \infty \). This implies the convergence \( (i_k - \frac{|i_k'|}{|i_{km}|} i_{km}) \to (i_k - i_k') \) as \( m \to \infty \). Approximating \( (|i_k'|/|i_{km}|) \) by the rational numbers we obtain the sequence \( \{j_{km}\}_{m=1}^\infty \), \( j_{km} \in Z^n \setminus 0 \) satisfying the following relations \( F(j_{km}) \leq F(i_k) \) and \( \lim_{m \to \infty} (j_{km}/|j_{km}|) = (i_k - i_k')/|i_k - i_k'|. \) Under the proper choice of \( m = m(k) \) any limiting point of the sequence \( \{j_{km(k)}/|j_{km(k)}|\} \) is orthogonal to \( \Gamma_{\beta_1} \). On the other hand \( F(j_{km(k)}) \to \beta_1 \), so this limiting point lies in \( \Gamma_{\beta_1} \).

**Proposition 11.** In \( \Gamma_{\beta_1} \) there exists a basis of integer vectors.

**Proof.** It is enough to choose arbitrary orthonormal basis \( z_1, \ldots, z_k \) in \( \Gamma_{\beta_1} \), \( k = \dim \Gamma_{\beta_1} \), and approximate each \( z_i \) by the sequence \( \{z_{im}/|z_{im}|\} \) such that \( F(z_{im}) \to \beta_1 \) as \( m \to \infty \). Indeed for sufficiently large \( m \) vectors \( z_{1m}, \ldots, z_{km} \) are linearly independent and by Proposition 10 lie in \( \Gamma_{\beta_1} \). The proposition is proved.

Now we set \( \beta_2 = \inf_{i \in Z^n \setminus \Gamma_{\beta_1}} F(i) \). According to Proposition 10 \( \beta_2 > \beta_1 \).

Like \( \Lambda_{\beta_1} \) above \( \Lambda_{\beta_2} \) is defined as a union of \( \Lambda(\{i_k\}) \) over all the sequences \( \{i_k\}, i_k \in Z^n \setminus 0 \), such that \( \limsup_{k \to \infty} F(i_k) \leq \beta_2 \). Let \( \Gamma_{\leq \beta_2} = \{x \in R^n : \frac{x}{|x|} \in \Lambda_{\beta_2}\} \) and let \( \Gamma_{\beta_2} \) be the orthocomplement to \( \Gamma_{\beta_1} \) in \( \Gamma_{\leq \beta_2} \). The following assertion can be proved in the same way as Propositions 9, 10 and 11.

**Proposition 12.** \( \Gamma_{\leq \beta_2} \) is the linear subspace of \( R^n \). There exists \( \delta_0 > 0 \) such that \( F(i) > \beta_2 + \delta_0 \) for any \( i \in Z^n \setminus \Gamma_{\leq \beta_2} \). In \( \Gamma_{\beta_2} \) there exists the basis of integer vectors.

The next step gives \( \beta_3, \Gamma_{\leq \beta_3} \) and \( \Gamma_{\beta_3} \) and so on. Continuing the process we find \( \beta_1, \ldots, \beta_s \) and \( \Gamma_{\beta_1}, \ldots, \Gamma_{\beta_s}, \Gamma_{\beta_1} \oplus \ldots \oplus \Gamma_{\beta_s} = R^n \), where \( 1 \leq s \leq n \). Let \( z_1, \ldots, z_n \) be the orthonormal basis in \( R^n \) consisting of the bases of \( \Gamma_{\beta_1}, \ldots, \Gamma_{\beta_s} \). We introduce the symmetric operator \( A_W \) to be

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diagonal in the basis $z_1, \ldots, z_n$ with eigenvalues $\beta_i$ in the corresponding subspaces $\Gamma_{\beta_i}$.

Let us now consider the matrix $A_W$ as a function of $W(\cdot)$.

**Lemma 2.** $A_W$ is the continuous monotonic function of $W$ in the functional space $C(T^n)$.

**Proof.** To prove continuity let us fix arbitrary continuous function $W_0(x)$ on $T^n$ and construct corresponding $\beta_1, \ldots, \beta_n$ and $\Gamma_\beta_1, \ldots, \Gamma_\beta_n$. If we set $\kappa = \frac{1}{10} \min_{1 \leq i < s} (\beta_{i+1} - \beta_i)$ then for any $W(x) \in \{ W(x) \in C(T^n) : |W_0 - W|_{C(T^n)} < \delta < \kappa \}$ and for any $i \in \mathbb{Z}^n \setminus 0$ we evidently have $|F_{W_0}(i) - F_W(i)| < \delta$ where indexes $W$ and $W_0$ are used to indicate the function $F(i)$ defined as above for $W$ and $W_0$ respectively. Now let $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$ and $e_1, \ldots, e_n$ be the eigenvalue and eigenvectors of $A_W$ respectively. It is easy to see that $|\limsup_{k \to \infty} F_{W_0}(i_k) - \beta_1| < \delta$ for any sequence $\{i_k\}$ satisfying the relation $\lim_{k \to \infty} F_{W_0}(i_k) = \beta_1$ and that $|F_W(i) - \beta_1| \geq 2 \kappa$ for $i \in \mathbb{Z}^n \setminus \Gamma_\beta_1$. This means that $|\gamma_i - \beta_1| < \delta$ for $i = 1, 2, \ldots, k_1$, $k_1 = \dim \Gamma_\beta_1$, and a span of $e_1, \ldots, e_{k_1}$ coincides with $\Gamma_\beta_1$. Similarly $|\gamma_{k_1+i} - \beta_2| < \delta$ for $i = 1, 2, \ldots, k_2$, $k_2 = \dim \Gamma_\beta_2$ and a span of $e_{k_1+1}, \ldots, e_{k_1+k_2}$ coincides with $\Gamma_\beta_2$ and so on. As a result we obtain $|A_{W_0} - A_W| < \delta$ in appropriate matrix norm. Monotonicity can be proved in the similar way.

The next statement is the main result of the paper.

**Theorem 1.** Effective diffusion matrix $\sigma(\mu)$ satisfies the following limiting relation

$$\lim_{\mu \to 0} \mu \ln(\sigma(\mu)) = -2 A_W.$$

**Proof.** As was mentionend in Section 2 it suffices to find the asymptotics of homogenized matrix for the operator $\frac{\partial}{\partial x_i} p^2(x) a_{ij}(x) \frac{\partial}{\partial x_j}$. For this purpose we approximate $W(x)$ on $T^n$ by the smooth function $W_\delta(x)$ with finite number of degenerate points and with the only global minimum point at the origin in such a way that the estimate $|W - W_\delta|_{C(T^n)} < \delta/3$ holds. Under this choice of $W_\delta$ we have for sufficiently small $\mu$ the following matrix inequality

$$e^{-\frac{\mu}{\sigma}(x) + 2\delta} \leq p^2(x) a_{ij}(x) \leq e^{-\frac{\mu}{\sigma}(x) + 2\delta} \quad (3.1)$$

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Further the homogenized operators keep the order relation of original operators [1] hence (3.1) implies the inequality

$$e^{-2\delta \mu \hat{a}_{ij}} \leq \hat{a}_{ij} (\mu) \leq e^{2\delta \mu \hat{a}_{ij}}$$

(3.2)

here $\hat{a}_{ij}$ is the homogenized matrix of the operator

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} e^{-W_\delta (x)} \mu \frac{\partial}{\partial x_j}.$$

The properties of $W_\delta (x)$ allow us to apply the results of [8], [4] to find the asymptotics of $\hat{a}_{ij}$. This yields the following relation

$$\lim_{\mu \to \infty} \mu \ln \hat{a}_{ij} = -2 A_{W_\delta}.$$

(3.3)

Taking into account (3.2) we have

$$-2 \delta I - 2 A_{W_\delta} \leq \lim_{\mu \to 0} \inf \mu \ln \hat{a}_{ij} \leq \lim_{\mu \to 0} \sup \mu \ln \hat{a}_{ij} \leq 2 \delta I - 2 A_{W_\delta}.$$

At last by Lemma 2 and choice of $W_\delta (x)$ the matrix $A_{W_\delta}$ tends to $A_W$ as $\delta \to 0$ thus

$$\lim_{\delta \to 0} \mu \ln \hat{a}_{ij} = \lim_{\mu \to 0} \mu \ln \sigma_{ij} = -2 A_W.$$

In conclusion let us derive some consequences from Theorem 1. Consider the symmetrical case when the matrix $a_{ij} (x)$ and the potential $v (x)$ are invariant with respect to any motion of $\mathbb{R}^n$ preserving the cube $[-1/2, 1/2]^n$. Then the matrix $a_W$ takes the form $a_W = \alpha_W I$ where $I$ is the unit matrix and regarding $\alpha_W$ Theorem 1 states that $\lim_{\mu \to \infty} \mu \ln \alpha_W$ is equal to the distance of 0 from $Z^n \setminus 0$ in the metric $(v (x) a^{ij} (x))$. This metric degenerates in the points of $Z^n$.

In our previous work [3] the similar asymptotics was found for the effective diffusion of the operator $(\mu^2 \Delta + v (x))$ with piecewise constant potential. The methods developed here allow these results to be generalized. Namely let us consider the equation of the form (0.1) with $v (x)$ given by the formula

$$v (x) = \begin{cases} 
1, & x \in \bigcup_{j \in Z^n} (Q + j) \\
0, & \text{otherwise}
\end{cases}$$

where $Q$ is simply connected domain with piecewise smooth boundary such that $Q$ and $(Q + j)$ don’t intersect for any $j \in Z^n$. Without loss of generality we can assume that $0 \in Q$. Then under the same assumption as above we have
THEOREM 2. \( \lim_{\mu \to 0} \mu \ln \sigma_{ij}(\mu) = -2A_W \) where \( W(x) \) is equal to the distance of \( x \) from \( Q \) on \( T^n \) in the metric \( (\alpha^{ij}(x)) \).

With appropriate simplifications the proof of the theorem is quite similar to the proof of Theorem 1 (see also [3]).

REFERENCES


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