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Spitzer's condition for random walks and Lévy Processes

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ABSTRACT. – Spitzer's condition holds for a random walk S if the probabilities $\rho_n = P\{S_n > 0\}$ converge in Césaro mean to ρ, and for a Lévy process X at ∞ (at 0, respectively) if $t^{-1} \int_0^t \rho(s)ds \to \rho$ as $t \to \infty(0)$, where $\rho(s) = P\{X_s > 0\}$. It has been shown in Doney [4] that if $0 < \rho < 1$ then this happens for a random walk if and only if ρ_n converges to ρ. We show here that this result extends to the cases $\rho = 0$ and $\rho = 1$, and also that Spitzer's condition holds for a Lévy process at $\infty(0)$ if and only if $\rho(t) \to \rho$ as $t \to \infty(0)$.

Key words: Stable laws, ladder variables, arc-sine law, local limit theorems, Wiener-Hopf factorisation.

RÉSUMÉ. – Une marche aléatoire S vérifie la condition de Spitzer si $\rho_n = P\{S_n > 0\}$ converge en moyenne de Césaro vers ρ. Doney [4] a établi que pour $0 < \rho < 1$, ceci a lieu si et seulement si ρ_n converge vers ρ. On montre ici que cette équivalence reste vraie pour $\rho = 0$ et $\rho = 1$. Pour un processus de Lévy X, l'analogue de la condition de Spitzer à l'infini (respectivement à l'origine) est que $t^{-1} \int_0^t \rho(s)ds \to \rho$ quand $t \to \infty$ (0), où $\rho(s) = P\{X_s > 0\}$. On prouve que cette condition est satisfaite si et seulement si $\rho(t) \to \rho$ quand $t \to \infty$ (0).

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1. INTRODUCTION AND RESULTS

Spitzer's condition holds for a random walk $S = (S_m, m \geq 0)$ if

$$\frac{1}{n}\sum_{1}^{n} P\{S_m > 0\} \to \rho \text{ as } n \to \infty. \tag{1}$$

This condition plays a key role in fluctuation theory, particularly when $\rho \in (0,1)$. It is then necessary and sufficient for the generalized Arc-sine theorem to hold for the proportion of time spent in the positive half-line by the random walk, and also for first upwards passage times $\min\{n : S_n > x\}$ $(x \geq 0)$ to be in the domain of attraction of a stable law of index ρ. The cases $\rho = 0$ and $\rho = 1$ can also arise in (1), but are of a somewhat different nature; for example in these cases (1) is equivalent to the proportion of time spent in the positive half-line having a limiting distribution which is degenerate (at 0 and 1 respectively).

A question about Spitzer's condition which has puzzled probabilists since Spitzer's original paper [8] is whether or not (1) is equivalent to the apparently stronger statement

$$P\{S_n > 0\} \to \rho \text{ as } n \to \infty. \tag{2}$$

By exploiting a local limit theorem and an appropriate formula for $P\{S_n > 0\}$, Doney [4] has given an affirmative answer to this question when $\rho \in (0,1)$. Harry Kesten has pointed out that the result is still valid in the cases $\rho = 0, 1$ and produced a proof of a quite different nature. He has also kindly agreed to our giving his proof here, so that we can now state the complete result:

THEOREM 1. – *For any random walk S and for any $0 \leq \rho \leq 1$, the statements* (3) *and* (4) *are equivalent.*

However the main purpose of this note is to discuss analogous questions for a Lévy process $X = (X_t, t \geq 0)$. There are actually two questions, because the obvious analogues of (1) and (2), which are

$$\frac{1}{t}\int_0^t P\{X_s > 0\} ds \to \rho \tag{3}$$

and

$$P\{X_t > 0\} \to \rho \tag{4}$$

respectively, make sense both as $t \to \infty$ and as $t \to 0+$. The fact that (3) is necessary and sufficient for the Arc-Sine theorem to hold is well-known (recent proofs of this old result are available in Getoor and Sharpe [7] and Theorem VI.14 in [1]; *see* also [2]). The importance of the "large t" version of (3) for the tail behaviour of first passage times $\inf\{t : X_t > x\}$ $(x \geq 0)$ can be found in Theorem VI.18 of [1]. Again (4) looks obviously stronger that (3); nonetheless the following analogue of Theorem 1 holds.

THEOREM 2. – *For any Lévy process X and for any $0 \leq \rho \leq 1$, the statements* (3) *and* (4) *are equivalent (as $t \to \infty$, or as $t \to 0+$).*

This paper is organized as follows. Section 2 presents Kesten's argument to prove Theorem 1 in the case $\rho = 0, 1$, and shows how Theorem 2 for large times can be reduced to Theorem 1 by looking at the random walk $(X_n, n \in \mathbb{N})$ (this trick is clearly not available for small times). Theorem 2 is established for small times in Section 3. We treat the case $\rho = 0, 1$, and then give two different proofs for $0 < \rho < 1$. The first is the simplest; it is based on a duality identity for the ladder time processes and does not use any local limit theorem. The second is essentially an adaptation of Doney's method for random walks; in particular it requires a version of the local limit theorem for small times. Though this is more involved than the first argument, it relies on some not so well-known facts on Lévy processes which may be of independent interest.

2. LARGE TIMES

2.1. The case $\rho = 0, 1$ in Theorem 1

Suppose that $S = (S_n, n \geq 0)$ is any random walk with $S_0 \equiv 0$. The proof of Theorem 1 for $\rho = 1$ hinges on the following observation, which may be of more general interest.

LEMMA 1. – (i) *Suppose that n is such that $\rho_n = P\{S_n > 0\} \in (0, 1)$, and for any $0 < \epsilon < 1$ write $\delta = \delta(\epsilon, n) = (1 - \rho_n)/\epsilon\rho_n$. Then for any fixed integer $r > 2$*

$$P\{S_m > 0 \text{ for } 2n \leq m \leq rn\} \geq (1 - \delta)(1 - \epsilon)^r \rho_n^{r+1}. \tag{5}$$

(ii) *If $\{n_i, i \geq 1\}$ is a subsequence of the integers such that $\rho(n_i) < 1$, $\rho(n_i) \to 1$, then*

$$P\{S_m > 0 \text{ for } 2n_i \leq m \leq rn_i\} \to 1.$$

Proof. – (i) We define $q_n = q_n(\epsilon)$ to be the conditional ϵ-quantile of $I_n = \min_{j \leq n} S_j$, given $S_n > 0$, so that $q_n < 0$ and

$$P\{I_n < q_n | S_n > 0\} < \epsilon, \quad P\{I_n \leq q_n | S_n > 0\} \geq \epsilon.$$

In particular,

$$\epsilon P\{S_n > 0\} \le P\{I_n \le q_n\}. \tag{6}$$

Also, writing $\sigma(A)$ for the time that S first enters A, we have

$$\begin{aligned} P\{S_n \le 0\} &\ge P\{S_n \le 0, I_n \le q_n\} \\ &\ge \sum_{\ell \le n} P\{\sigma(-\infty, q_n] = \ell\} P\{S_{n-\ell} \le -q_n\} \\ &\ge P\{I_n \le q_n\} \min_{\ell \le n} P\{S_\ell \le -q_n\}. \end{aligned}$$

Using (6), this gives

$$\min_{\ell \le n} P\{S_\ell \le -q_n\} \le (1-\rho_n)/\epsilon\rho_n = \delta,$$

so there exists $\ell = \ell(\epsilon, n) \le n$ such that $P\{S_\ell > -q_n\} \ge 1 - \delta$.

Now consider the event

$$S_n > 0, S_{n+\ell} - S_n > -q_n, S_{(s+1)n+\ell} - S_{sn+\ell} > 0,$$

$$\min_{j \le n}\{S_{sn+\ell+j} - S_{sn+\ell}\} \ge q_n, \qquad s = 1, 2, \ldots, r.$$

On this event we clearly have $S_m > 0$ for $n + \ell \le m \le rn + \ell$, and recalling that $\ell \le n$ the estimate (5) follows.

(ii) Just observe that $\delta(\epsilon, n_i) \to 0$ as $i \to \infty$. □

It is now easy to establish Theorem 1 for $\rho = 0, 1$. Suppose that $n^{-1}\sum_1^n \rho_m \to 1$; then either $\rho_n = 1$ for one (and then all) $n > 0$, or $\rho_n < 1$ for all n. It is easy to see that in the latter case there exists a subsequence $\{n_i, i \ge 1\}$ such that $\rho(n_i) \to 1$ and $n_{i+1}/n_i \le 2$ for all i. Now for all large n we can find i such that $4n_i \ge n \ge 2n_i$, and by (ii) of Lemma 1, it follows that $\rho_n \to 1$. Of course if $n^{-1}\sum_1^n \rho_m \to 0$ the result follows by considering $-S$, it being well known that $P\{S_n = 0\} \to 0$ (*see* e.g. Equation (5) in Spitzer [9] on page 72).

2.2. The case $t \to \infty$ in Theorem 2

Let $X = (X_t, t \ge 0)$ be any real-valued Lévy process and put $\rho(t) = P\{X_t > 0\}$. The key result for the 'large t' case is

LEMMA 2. – *It holds that*

$$\lim_{n \to \infty} \sup_{0 \le t \le 1} |\rho(n+t) - \rho(n)| = 0.$$

Proof. – Let $M_n = \sup_{0\leq t\leq 1}|X_{n+t} - X_n|$ and, given $\varepsilon > 0$, choose K_ε such that $P\{M_n > K_\varepsilon\} \leq \varepsilon$. Then for any $t \in [0,1]$

$$X_n > 0, M_n \leq K_\varepsilon \Rightarrow X_{n+t} \geq -K_\varepsilon,$$

and hence

$$\rho(n+t) + P\{-K_\varepsilon \leq X_{n+t} \leq 0\} \geq P\{X_n > 0, M_n \leq K_\varepsilon\} \geq (1-\varepsilon)\rho(n).$$

Now it is known that the second term on the left converges to zero uniformly in t as $n \to \infty$ (see e.g. Lemma 2.5 of Getoor and Sharpe [7]), so we deduce that

$$\liminf_{n\to\infty} \inf_{t\in[0,1]} \{\rho(n+t) - \rho(n)\} \geq 0.$$

The result follows by applying the same argument to $-X$. □

The "large t" case in Theorem 2 is now straightforward. More precisely, consider the random walk $S = (S_n = X_n, n \in \mathbb{N})$. It is immediate from Lemma 2 that (3) holds as $t \to \infty$ if and only if (1) holds, and that (4) is equivalent to (2). Hence Theorem 2 follows from Theorem 1, which was proved for $\rho \in (0, 1)$ in Doney [4] and in Section 2.1 for $\rho = 0, 1$.

3. SMALL TIMES

The purpose of this section is to prove Theorem 2 when $t \to 0+$. The case when the Lévy process $X = (X_t, t \geq 0)$ is a compound Poisson process with drift is degenerate from the viewpoint of Theorem 2 (because we are working with small times), and will be implicitly excluded in the sequel. In particular, this implies that $P\{X_t = 0\} = 0$ for all $t > 0$, and that the mapping $t \to \rho(t) = P\{X_t > 0\}$ is continuous on $t \in (0, \infty)$ (because X is continuous in probability).

3.1. The case $\rho = 0, 1$

The argument relies on a simple measure theoretic fact.

LEMMA 3. – *Let $B \subset [0, \infty)$ be measurable set such that*

$$\lim_{t\to 0+} t^{-1} m(B \cap [0,t]) = 1,$$

where m denotes Lebesgue measure. Then $B + B \supset (0, \epsilon)$ for some $\epsilon > 0$.

Proof. – Pick $T > 0$ such that $t^{-1}m(B \cap [0,t]) > 3/4$ for all $t \leq T$. Then

$$m(B \cap [t, 2t]) \geq \frac{1}{2}t \text{ for all } t < \frac{1}{2}T. \tag{7}$$

Suppose now that there exists $t < \frac{1}{2}T$ such that $2t \notin B + B$. Then for every $s \in [0,t] \cap B$, $2t - s \in B^c \cap [t, 2t]$ and therefore

$$\begin{aligned} m(B \cap [t, 2t]) &= t - m(B^c \cap [t, 2t]) \\ &\leq t - m(2t - B \cap [0, t]) \\ &\leq t - m(B \cap [0, t]) < \frac{1}{4}t, \end{aligned}$$

and this contradicts (7). □

We are now able to complete the proof of Theorem 2 (as $t \to 0+$) for $\rho = 0, 1$. Obviously it suffices to consider the case $\rho = 1$, so assume $t^{-1}\int_0^t \rho(s)ds \to 1$, and for $\delta \in (0,1)$ consider $B = \{t : \rho(t) \geq \delta\}$. Then B satisfies the hypothesis of Lemma 3 and we have that $B + B \supset (0, \epsilon)$ for some $\epsilon > 0$. For any $t \in (0, \epsilon)$ choose $s \in (0,t) \cap B$ with $t - s \in B$, so that $\rho(s) \geq \delta$ and $\rho(t-s) \geq \delta$. But by the Markov property

$$\rho(t) = P\{X_t > 0\} \geq P\{X_s > 0\}P\{X_{t-s} > 0\} \geq \delta^2.$$

Since δ can be chosen arbitrarily close to 1, we conclude that $\lim_{t \to 0+} \rho(t) = 1$.

3.2. A first proof for the case $0 < \rho < 1$

Recall the ladder time process L^{-1} (that is the inverse local time at the supremum), is a subordinator. According to a formula due to Fristedt [6], its Laplace exponent Φ is given by

$$\Phi(q) = \exp\left\{\int_0^\infty \left(e^{-t} - e^{-qt}\right)t^{-1}\rho(t)dt\right\}, \qquad q \geq 0. \tag{8}$$

See for instance Corollary VI.10 in [1]. The Laplace exponent $\widehat{\Phi}$ corresponding to the dual Lévy process $\widehat{X} = -X$ then satisfies

$$\Phi(q)\widehat{\Phi}(q) = q$$

(this follows from the Frullani integral, *see* e.g. Equation VI.3 in [1]). We deduce from (8) that

$$\int_0^\infty e^{-qt}\rho(t)dt = \Phi'(q)/\Phi(q) = \Phi'(q)\widehat{\Phi}(q)/q. \tag{9}$$

Suppose now that (3) holds as $t \to 0+$. By Theorem VI.14 in [1], this implies that Φ is regularly varying at ∞ with index ρ, and also that $\widehat{\Phi}$ is regularly varying at ∞ with index $1-\rho$. Because Φ and $\widehat{\Phi}$ are Laplace exponents of subordinators with zero drift, we obtain from the Lévy-Khintchine formula that

$$\Phi'(q) = \int_{(0,\infty)} e^{-qx} x d(-T(x)), \qquad \widehat{\Phi}(q)/q = \int_0^\infty e^{-qx} \widehat{T}(x) dx,$$

where T (respectively, $\widehat{T}$) is the tail of the Lévy measure of the ladder time process of X (respectively, of $\widehat{X}$). We now get from (9)

$$\rho(t) = \int_{(0,t)} \widehat{T}(t-s) s d(-T(s)) \qquad \text{for a.e. } t > 0. \tag{10}$$

By a change of variables, the right-hand-side can be re-written as

$$t \int_{(0,1)} \widehat{T}(t(1-u)) u d(-T(tu)) = \int_{(0,1)} \frac{\widehat{T}(t(1-u))}{\widehat{\Phi}(1/t)} u d\left(-\frac{T(tu)}{\Phi(1/t)}\right).$$

Now, apply a Tauberian theorem, the monotone density theorem and the uniform convergence theorem (*see* Theorems 1.7.1, 1.7.2 and 1.5.2 in [3]). For every fixed $\varepsilon \in (0,1)$, we have uniformly on $u \in [\varepsilon, 1-\varepsilon]$ as $t \to 0+$:

$$\frac{T(tu)}{\Phi(1/t)} \to \frac{u^{-\rho}}{\Gamma(1-\rho)} \quad , \quad \frac{\widehat{T}(t(1-u))}{\widehat{\Phi}(1/t)} \to \frac{(1-u)^{-\rho}}{\Gamma(\rho)}.$$

Recall $\rho(t)$ depends continuously on $t > 0$. We deduce from (10) that

$$\liminf_{t \to 0+} \rho(t) \geq \frac{\rho}{\Gamma(\rho)\Gamma(1-\rho)} \int_\varepsilon^{1-\varepsilon} (1-u)^{\rho-1} u^{-\rho} du,$$

and as ε can be picked arbitrarily small, $\liminf_{t\to 0+} \rho(t) \geq \rho$. The same argument for the dual process gives $\liminf_{t\to 0+} P\{X_t < 0\} \geq 1-\rho$, which completes the proof.

3.3. A second proof for the case $0 < \rho < 1$

A first goal is to invert Fristedt's formula, in order to express the one-dimensional distributions of the Lévy process in terms of the ladder process (L^{-1}, H), where L^{-1} is the ladder time process (that is the inverse local time at the supremum), and $H(t) = X_{L^{-1}(t)}$ $(t \geq 0)$ the ladder height process. The random walk version of this result is an important fluctuation identity in discrete time, which is known as the Baxter-Spitzer formula. We refer to Feller [5], Lemma 1 on page 605, for a proof based on the Wiener-Hopf factorization; *see* also Equation (9.3) on page 424.

LEMMA 4. – *We have the following identity between measures on* $(0,\infty) \times (0,\infty)$*;*

$$P\{X_t \in dx\}dt = t\int_0^\infty P\{L^{-1}(u) \in dt, H(u) \in dx\}u^{-1}du.$$

Proof. – We show that both sides have the same bivariate Laplace transform. Specifically, if $\kappa(\lambda,\mu)$ denotes the Laplace exponent of the bivariate subordinator (L^{-1}, H), put

$$D(\lambda,\mu) = \{\kappa(\lambda,\mu)\}^{-1}\frac{\partial}{\partial\lambda}\{\kappa(\lambda,\mu)\}.$$

Then for $\lambda > 1$ the Frullani integral gives

$$\begin{aligned} D(\lambda,\mu) &= \frac{\partial}{\partial\lambda}\log\{\kappa(\lambda,\mu)/\kappa(1,\mu)\} \\ &= \frac{\partial}{\partial\lambda}\left\{\int_0^\infty u^{-1}\{e^{-\kappa(1,\mu)} - e^{-\kappa(\lambda,\mu)}\}du\right\}. \end{aligned}$$

Writing the right-hand-side in terms of L^{-1} and H gives

$$\begin{aligned} D(\lambda,\mu) &= \frac{\partial}{\partial\lambda}\left\{\int_0^\infty u^{-1}\int_0^\infty\int_0^\infty [e^{-(t+\mu x)} - e^{-(\lambda t+\mu x)}]\right. \\ &\qquad \left.\times P\{L^{-1}(u) \in dt, H(u) \in dx\}du\right\} \\ &= \frac{\partial}{\partial\lambda}\left\{\int_{v=1}^\lambda\int_0^\infty\int_0^\infty\int_0^\infty e^{-(vt+\mu x)}tu^{-1}\right. \\ &\qquad \left.\times P\{L^{-1}(u) \in dt, H(u) \in dx\}dudv\right\} \\ &= \int_0^\infty\int_0^\infty e^{-(\lambda t+\mu x)}\int_{u=0}^\infty tu^{-1} \\ &\quad \times P\{L^{-1}(u) \in dt, H(u) \in dx\}du. \end{aligned} \tag{11}$$

On the other hand, Fristedt's formula for κ is (*see* Fristedt [6] or Corollary VI.10 in [1])

$$\log \kappa(\lambda, \mu) = \int_0^\infty t^{-1} dt \int_0^\infty \{e^{-t} - e^{-(\lambda t + \mu x)}\} P\{X_t \in dx\}.$$

Hence, for $\lambda > 1$,

$$\begin{aligned}\log\{\kappa(\lambda,\mu)/\kappa(1,\mu)\} &= \int_0^\infty t^{-1} dt \int_0^\infty (e^{-t} - e^{-\lambda t}) e^{-\mu x} P\{X_t \in dx\} \\ &= \int_0^\infty \int_0^\infty \int_1^\lambda e^{-tu} du e^{-\mu x} P\{X_t \in dx\} dt \\ &= \int_1^\lambda du \Big(\int_0^\infty \int_0^\infty e^{-(ut+\mu x)} P\{X_t \in dx\} dt \Big)\end{aligned}$$

so that

$$D(\lambda, \mu) = \int_0^\infty \int_0^\infty e^{-(\lambda t + \mu x)} P\{X_t \in dx\} dt. \tag{12}$$

Comparing (11) and (12), the result follows. □

We next give a local limit theorem which is more general than we need.

PROPOSITION 1. – *Suppose that* $Y = (Y_t, t \geq 0)$ *is a real-valued Lévy process and there exists a measurable function* $r : (0, \infty) \to (0, \infty)$ *such that* $Y_t / r(t)$ *converges in distribution to some law which is not degenerate at a point as* $t \to 0+$. *Then*

(i) r *is regularly varying of index* $1/\alpha, 0 < \alpha \leq 2$, *and the limit distribution is strictly stable of index* α;

(ii) *for each* $t > 0$ Y_t *has an absolutely continuous distribution with continuous density function* $p_t(\cdot)$;

(iii) *uniformly for* $x \in \mathbb{R}$, $\lim_{t \to 0+} r(t) p_t(x r(t)) = p^{(\alpha)}(x)$, *where* $p^{(\alpha)}(\cdot)$ *is the continuous density of the limiting stable law.*

Proof. – (i) This is proved in exactly the same way as the corresponding result for $t \to \infty$.

(ii) If $\Psi(\lambda)$ denotes the characteristic exponent of Y, so that

$$E(\exp\{i\lambda Y_t\}) = \exp\{-t\Psi(\lambda)\}, \qquad t \geq 0, \quad \lambda \in \mathbb{R},$$

then we have $t\Psi(\lambda / r(t)) \to \Psi^{(\alpha)}(\lambda)$ as $t \to 0+$, where $\Psi^{(\alpha)}$ is the characteristic exponent of a strictly stable law of index α. Because we have excluded the degenerate case, $Re(\Psi(\lambda))$, the real part of the characteristic exponent (which is an even function of λ), is regularly varying of index α at $+\infty$. It follows that for each $t > 0, \exp -t\Psi(\cdot)$ is integrable over $\mathbb{R}$. Consequently (ii) follows by Fourier inversion, which also gives

$$r(t)p_t(xr(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp -\{i\lambda x + t\Psi(\lambda/r(t))\}d\lambda$$

and

$$p^{(\alpha)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp -\{i\lambda x + \Psi^{(\alpha)}(\lambda)\}d\lambda.$$

(iii) In view of the above formulae, it suffices to show that $|\exp -t\Psi(\lambda/r(t))| = \exp\{-tRe\Psi(\lambda/r(t))\}$ is dominated by an integrable function on $|\lambda| \geq K$ for some $K < \infty$ and all small enough λ. But this follows easily from Potter's bounds for regularly varying functions. (*See* Bingham *et al.* [3], Theorem 1.5.6.) □

We assume from now on that (3) holds as $t \to 0+$, and write $\Phi(\lambda) = \kappa(\lambda, 0)$ for the Laplace exponent of the subordinator L^{-1}. Then (*see* [1], Theorem VI.14) Φ is regularly varying at ∞ with index ρ. It follows that if we denote by a the inverse function of $1/\Phi(1/\cdot)$, then a is regularly varying with index $1/\rho$ and $L^{-1}(t)/a(t)$ converges in distribution to a non-negative stable law of index ρ as $t \to 0+$. In view of Proposition 1, L_t^{-1} has a continuous density which we denote by $g_t(\cdot)$, and $a(t)g_t(a(t)\cdot)$ converges uniformly to the continuous stable density, which we denote by $g^{(\rho)}(\cdot)$. Applying Lemma 4, we obtain the following expression for $\rho(t)$ that should be compared with (10).

$$\rho(t) = t\int_0^{\infty} g_u(t)u^{-1}du \qquad \text{for a.e. } t > 0. \tag{13}$$

We are now able to give an alternative proof of Theorem 2 for $0 < \rho < 1$ and $t \to 0+$. By a change of variable,

$$t\int_0^{\infty} g_u(t)u^{-1}du = t\int_0^{\infty} g_{su}(t)u^{-1}du,$$

for any $s > 0$. We now choose $s = 1/\Phi(1/t)$, so that $a(s) = t$, and note that

$$tg_{su}(t) = \frac{a(s)}{a(su)} \cdot a(su)g_{su}\left(a(su) \cdot \frac{a(s)}{a(su)}\right).$$

When $t \to 0+$, $s \to 0+$ and since a is regularly varying with index $1/\rho$, $a(s)/a(su)$ converges pointwise to $u^{-1/\rho}$. It then follows from Proposition 1 that

$$\lim_{t\to 0+} tg_{su}(t) = u^{-1/\rho}g^{(\rho)}(u^{-1/\rho}).$$

Recall that $\rho(t)$ depends continuously on $t > 0$, so that (13) and Fatou's lemma give

$$\liminf_{t\to 0+} \rho(t) \geq \int_0^\infty g^{(\rho)}(u^{-\frac{1}{\rho}})u^{-\frac{1}{\rho}-1}du = \rho \int_0^\infty g^{(\rho)}(v)dv = \rho.$$

Replacing X by $-X$ gives $\limsup_{t\to 0+} P\{X_t \geq 0\} \leq \rho$, and the result follows.

4. REMARKS

(i) It is easy to adapt the argument given in Section 3.2 to large times (for a Lévy process or a random walk).

(ii) Although we don't actually need it in this paper, we would like to point out that there is a local limit theorem for large t for Lévy processes, and that the following result can be established in exactly the same way that the corresponding random walk result was in Stone [10].

PROPOSITION 2. – *Let X by a Lévy process which is not a compound Poisson process whose jump measure is lattice. Suppose there exists $a(t)$ and $b(t)$ such that $(X(t) - b(t))/a(t)$ converges in distribution to a stable law as $t \to \infty$. Then, uniformly for h in compact subsets of $\mathbb{R}^+$ and $x \in \mathbb{R}$,*

$$a(t)P\left\{X_t \in \left[x-\frac{1}{2}h, x+\frac{1}{2}h\right]\right\} = hp^{(\alpha)}((x-b(t)/a(t))+o(1) \text{ as } t \to \infty,$$

where $p^{(\alpha)}$ is the continuous density function of the limit law.

(iii) We would like to thank Harry Kesten for allowing his argument to be given in Section 2.1, and would also like to point out that his argument can be adapted to deal with the 'small t' Lévy process case. On the other hand, the argument given in Section 3.1 for this case can also be adapted to deal with the random walk case.

(iv) We would also like to thank Cindy Greenwood, who suggested to one of us the usefulness of Lemma 4.

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