ANNALES DE L'I. H. P., SECTION B

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Annales de l'I. H. P., section B, tome 33, n° 4 (1997), p. 491-495

<http://www.numdam.org/item?id=AIHPB_1997__33_4_491_0>

Ann. Inst. Henri Poincaré,

Vol. 33, n° 4, 1997, p. 491-495. *Probabilités et Statistiques*

The return time theorem fails on infinite measure-preserving systems

by

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ABSTRACT. – The Return Time Theorem of Bourgain [BFKO] cannot be extended to the infinite measure-preserving case. Specifically, there exist a sigma-finite measure-preserving system $(X, \mathcal{A}, \mu, T)$ and a set $A \subset X$ of positive finite measure so that for almost every $x \in X$ the following undesirable behavior occurs. For every aperiodic measure-preserving system $(Y, \mathcal{B}, \nu, S)$, with $\nu(S) = 1$, there is a square-integrable g on Y so that the averages $\tau_n^{-1} \sum_{m=1}^{n} 1_A(T^m x)\, g(S^m y)$ diverge a.e. (y), where $\tau_n = \tau_n(x) = \sum_{m=1}^{n} 1_A(T^n x)$.

RÉSUMÉ. – Le théorème du temps de retour de Bourgain ne peut pas être étendu au cas des transformations préservant une mesure infinie. Précisément il existe un système préservant une mesure σ-finite, $(X, \mathcal{A}, \mu, T)$ et un ensemble $A \subset X$ de mesure positive et finie, tels que pour presque tout $x \in X$, on a le comportement indésirable suivant. Pour chaque système préservant la mesure apériodique $(Y, \mathcal{B}, \nu, S)$, avec $\nu(S) = 1$, il existe une fonction g de carré intégrable sur Y telle que les moyennes $\tau_n^{-1} \sum_{m=1}^{n} 1_A(T^m x)\, g(S^m y)$ divergent pour presque tout (y), où $\tau_n = \tau_n(x) = \sum_{m=1}^{n} 1_A(T^n x)$.

Consider an ergodic measure-preserving system $(X, \mathcal{A}, \mu, T)$. Initially assume $\mu(X) < +\infty$; for a set $A \subset X$ of positive measure and $x \in X$ fixed consider the return times $\mathcal{N}_x = \{n \geq 0 : T^n x \in A\}$ of the orbit x,

Tx, $T^2 x$, ... to the set A. These sets of integers have a rich arithmetical structure in both a mean and pointwise sense. We are interested in the latter, and in particular in using the return times as sets of integers along which to form new ergodic theorems. The ergodic theorem itself implies that $\mathcal{N}_x$ has density $\mu(A)$ for a.e. x. the Wiener-Wintner Theorem [WW] goes further, asserting that an L^2 ergodic theorem holds along $\mathcal{N}_x$: for a.e. x, for all second systems $(Y, \mathcal{B}, \nu, S)$ and $g \in L^2(Y)$

$$\frac{1}{N} \sum_{n=1}^{N} 1_A (T^n x)\, g(S^n y)$$

converges in $L^2(Y)$. The much more recent Return Time Theorem of Bourgain [BFKO] strengthens converge to pointwise in Y. This is a very delicate result.

The form of these last two theorems is to consider sets of integers arising from a "timing system" X and, pointwise in X, establish a result valid for all "test systems" Y. We pose the question of replacing the timing system X by a sigma-finite measure-preserving system, while still requiring the test systems to be finite. The effect of this change on the Wiener-Wintner and Return Time theorems is dramatic. It appears that an interesting form of the first theorem could be true, but it can no longer be seen from a simple weak-convergence principle, as is illustrated in [BL, section 2]. But the Return Time Theorem in this new formulation is false. This note gives a proof of the result stated in the abstract. It will follow from

PROPOSITION. – *Let X_m be non-negative i.i.d. integer-valued random variables such that $P(X_1 > \lambda) \sim \lambda^{-\alpha}$ as $\lambda \to +\infty$. Here $0 < \alpha < 1$, so that $EX_1 = +\infty$. Then with probability 1, for every aperiodic finite measure-preserving system $(Y, \mathcal{B}, \nu, S)$ there is a square-integrable function g on Y for which*

$$A_n\, g(y) = \frac{1}{n} \sum_{m=1}^{n} g(S^{\tau_m} y) \text{ diverges a.e. } (y),$$

where the power of S above is $\tau_n = \sum_{m=1}^{n} X_m$.

It is well known that τ_m can be realized as the return times to the base of a tower of total measure $EX_1 = +\infty$, and so the Proposition proves the Theorem of the abstract. Also we derive as a corollary an interesting example for the simple random walk. Note that the interarrival times of a simple random walk to the origin satisfy the hypotheses of the Proposition with $\alpha = 1/2$. Therefore the returns of the walk to the origin are almost surely a bad sequence along which to try to form a pointwise ergodic theorem.

The key point of the proof, much as in Theorem 4 of [LPRW], is to isolate an unfortunate, but almost sure, random fluctuation with the aid of a functional law of the iterated logarithm (LIL). An important difference is that in the current setting there is no lim sup LIL for $\tau_n = \sum_{m=1}^{n} X_m$ because of the infinite mean. Instead there is a lim inf LIL, and in particular there is the following Corollary to the main result of Wichura [W]. We need some notation. Define random functions on [0, 1] by

$$F_n(t, \omega) = F_n(t) = \frac{\sum_{m=1}^{[nt]} X_m}{K_\alpha\, n^{1/\alpha}\, (LLn)^{1-1/\alpha}}, \qquad 0 \le t \le 1.$$

Here, $[\,\cdot\,]$ is the greatest integer function, $Ln = e \vee \log n$, $LLn = \log(Ln)$, and K_α is a normalizing constant.

For large integers M, set

$$f(t) = t + M^{M-k} \qquad \text{for} \quad M^{-k-1} \le t < M^{-k} \quad \text{and} \quad k = 0, 1, 2, \dots .$$

The important features of f are that for all $k \ge 0$

$$M^k \int_0^{M^{-k}} \exp(2\pi i\,\theta\, f(t))\, dt \simeq \exp(2\pi i\, M^{M-k}\,\theta)\, \frac{e^{2\pi i\theta} - 1}{2\pi i\,\theta}, \qquad (1)$$

and for $0 < k \le M$ the functions of θ on the right side of (1) are essentially orthogonal on $0 \le \theta \le 1$.

LEMMA. – *With probability* 1, *for every M there is a (random) sequence of integers $n_k(\omega) = n_k$ such that*

$$F_{n_k}(\,\cdot\,) \xrightarrow{w^*} f(\,\cdot\,) \quad \textit{as} \;\; k \to +\infty. \qquad (2)$$

The convergence is to be understood in the sense of convergence in distribution, viewing $F_n(\cdot, \omega)$ and $f(\,\cdot\,)$ as random variables defined on the probability space [0, 1].

The reader who refers to [W] will have to patiently unwind definitions to find this Lemma. Aaronson and Denker [AD] have generalized Whichura's results, and most readers will find their paper easier to refer to.

Proof of Proposition. – We will invoke Bourgain's Entropy Criterion [B]. To this end, it is enough to verify that for almost all ω the following holds: For every integer M there are a subset of the integers $\mathcal{M}$ of cardinality M and g of $\ell^2(\mathbb{Z})$ norm 1 so that for all $n, n' \in \mathcal{M}$

$$\|A_n\, g - A_{n'}\, g\|_{\ell^2(\mathbb{Z})} > 1/2. \qquad (3)$$

By transference, the same assertion holds with $\mathbb{Z}$ and the shift replaced by any aperiodic dynamical system. This will violate the L^2-entropy criterion and so prove the proposition.

Fix a large M. The good ω's are those which satisfy the conclusion of the Lemma. To define $\mathcal{M}$, first select n for which the approximation in (2) is very good. Specifically, choose n so that

$$\left| \int \phi(F_n(t)) - \phi(f(t))\, dt \right| \leq M^{-10M} (\|\phi\|_\infty + \|\phi'\|_\infty) \qquad (4)$$

for all bounded, Lipschitz $\phi : \mathbb{R} \to \mathbb{R}$ (*see* [D, p. 310]). Take

$$\mathcal{M} = \{[n\, M^{-M}],\ [n\, M^{-M+1}],\ \ldots,\ [n\, M^{-1}]\},$$

and define g by

$$|\hat{g}(\theta)|^2 = \beta_n\, 1_{[-1/2\beta_n,\, 1/2\beta_n]}(\theta)$$

where $\beta_n = K_\alpha\, n^{1/\alpha}\, (LLn)^{1-1/\alpha}$ is the normalizing constant in the definition of $F_n(t)$. This allows us to exploit the functional LIL.

We can now verify (3) as follows. For $m \simeq nM^{-k}$ and $m \simeq nM^{-k'}$ in $\mathcal{M}$,

$$\begin{aligned} &\|A_m\, g - A_{m'}\, g\|^2_{\ell^2(\mathbb{Z})} \\ &= \int_{-1/2}^{1/2} \left| \frac{1}{m} \sum_{j=1}^{m} e^{2\pi i \tau_j \theta} - \frac{1}{m'} \sum_{j=1}^{m'} e^{2\pi i \tau_j \theta} \right|^2 |\hat{g}(\theta)|^2 d\theta \\ &= \beta_n \int_{-1/2\,\beta_n}^{1/2\,\beta_n} \left| \frac{1}{m} \sum_{j=1}^{m} \exp(2\pi\, i\, \beta_n\, \theta\, (\tau_j/\beta_n)) \right. \\ &\qquad \left. - \frac{1}{m'} \sum_{j=1}^{m'} \exp(2\pi\, i\, \beta_n\, \theta\, (\tau_j/\beta_n)) \right|^2 d\theta. \end{aligned}$$

These last two sums can be expressed as integrals of $F_n(t)$. In (4) we use for $\phi(x)$ a differentiable extension $\phi_{m,\theta}(x)$ of (the real or imaginary part of) $(2\pi\, i\, \beta_n\, \theta\, t)\ 1_{[1/n,\, m/n]}(t)$ which is bounded by 1 and has derivative bounded by $\beta_n\, \theta$. Then, using (4), the preceding expression is estimated by

$$\begin{aligned} &\int_{-1/2}^{1/2} \left| M^k \int_0^{M^{-k}} \exp(2\pi\, i\, \theta\, f(t))\, dt \right. \\ &\qquad \left. - M^{k'} \int_0^{M^{-k'}} \exp(2\pi\, i\, \theta\, f(t))\, dt \right|^2 d\theta \\ &\qquad + M^{-20M}\, M^{2(k+k')}\, \beta_n \int_{-1/2\,\beta_n}^{1/2\,\beta_n} (1 + 2\,\beta_n\, \theta)^2\, d\theta. \end{aligned}$$

Recalling (1), we see that this is essentially equal to 2, finishing the proof of Proposition. □

I am grateful to Maté Wierdl, and Karl Petersen. They provided the inspiration for this paper.

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(Manuscript received December 21, 1995; revised November 20, 1996.)