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Computing the expectation of the Azéma-Yor stopping times

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ABSTRACT. – Given the maximum process $(S_t) = (\max_{0 \le r \le t} X_r)$ associated with a diffusion $((X_t), \mathbf{P}_x)$, and a continuous function g satisfying $g(s) < s$, we show how to compute the expectation of the Azéma-Yor stopping time

$$\tau_g = \inf\{t > 0 \mid X_t \le g(S_t)\}$$

as a function of x. The method of proof is based upon verifying that the expectation solves a differential equation with two boundary conditions. The third 'missing' condition is formulated in the form of a minimality principle which states that the expectation is the minimal non-negative solution to this system. It enables us to express this solution in a closed form. The result is applied in the case when (X_t) is a Bessel process and g is a linear function. © Elsevier, Paris

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RÉSUMÉ. – Étant donnés une diffusion $((X_t), \mathbf{P}_x)$, son maximum $(S_t) = (\max_{0 \le r \le t} X_r)$, et une fonction continue g vérifiant $g(s) < s$, nous montrons comment calculer explicitement l'espérance du temps d'arrêt d'Azéma-Yor

$$\tau_g = \inf\{t > 0 \mid X_t \le g(S_t)\}$$

en tant que fonction de x. La méthode de démonstration utilise le fait que celle-ci est solution d'une équation différentielle avec deux conditions « au bord ». Une troisième condition sous-jacente est formulée en terme d'un principe de minimalité, lequel énonce que cette espérance est la solution minimale non négative du système. Ceci permet d'expliciter cette solution comme une forme fermée. Nous appliquons ce résultat au cas où (X_t) est un processus de Bessel est g une fonction linéaire. © Elsevier, Paris

1. FORMULATION OF THE PROBLEM

Let $((X_t), \mathbf{P}_x)$ be a non-negative canonical diffusion with the infinitesimal operator on $(0, \infty)$ given by

$$\mathbf{L_X} = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2} + \mu(x)\frac{\partial}{\partial x}$$

where σ^2 and μ are continuous functions on $(0, \infty)$ and σ^2 is furthermore strictly positive (*see* [6]). Assume there exists a standard Wiener process (B_t) such that for every $x > 0$

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x \quad \mathbf{P}_x\text{–a.s.} \tag{1.1}$$

The main purpose of this paper is to compute the expectation of the Azéma-Yor stopping time (*see* [1]). More precisely, for any continuous function g on $[0, \infty)$ satisfying $0 < g(x) < x$ for $x > 0$, the Azéma-Yor stopping time is defined as follows

$$\tau_g = \inf\{t > 0 \mid X_t \le g(S_t)\}$$

where (S_t) is the maximum process associated with (X_t)

$$S_t = (\max_{0 \le r \le t} X_r) \vee s \tag{1.2}$$

started at $s > 0$. The main aim of this paper is to present a method for computing the function

$$m(x, s) = \mathbf{E}_{x,s}[\tau_g]$$

for $0 < x < s$. Here the expectation is taken with respect to the probability measure $\mathbf{P}_x := \mathbf{P}_{x,s}$ under which the process (X_t) starts at x and the process (S_t) starts at s.

The motivation to compute the expectation of such stopping times comes from some optimal stopping problems (*see* [2-4] and [7]). In these problems it is of interest to know the expected waiting time for the optimal stopping strategy which is of the form τ_g for some g. In view of this application we have assumed that the diffusion (X_t) and the function $s \mapsto g(s)$ are non-negative, but it will be clear from our considerations below that the results obtained are generally valid.

The method of proof relies upon showing that the expectation of the stopping time solves a differential equation with two boundary conditions. The third 'missing' condition is formulated in the form of a minimality principle which states that *the expectation is the minimal non-negative solution to this system* (*see* Fig. 1 below). It enables us to pick up the expectation among all possible candidates in a unique way. The minimality principle is the main novelty in this approach (compare with [2], [3] and [4]).

In Section 2 the minimality principle is formulated, and in Section 3 the existence and uniqueness of the minimal solution is proved. The main theorem is proved in Section 4, and in Section 5 an application of the theorem is given.

2. THE MINIMALITY PRINCIPLE

In the first part of this section we shall observe that the function $x \mapsto m(x, s)$ solves a differential equation with two boundary conditions. In the remaining part of the section we will present the minimality principle as the 'missing' condition, which will enable us to select the expectation of the stopping time in a unique way.

In the sequel we need the following definitions and results. The scale function is for $x > 0$ given by

$$S(x) = \int_1^x \phi(t)dt$$

where

$$\phi(x) = \exp\Big(-2 \int_1^x \mu(t)/\sigma^2(t)dt \Big).$$

We define as usual the first exit time from an interval by

$$\tau_{a,b} = \inf\{t > 0 \mid X_t \notin (a,b)\}$$

for $0 < a < b$, and the following formulas for $0 < a < x < b$ are well-known

$$\mathbf{P}_x(X_{\tau_{a,b}} = a) = 1 - \mathbf{P}_x(X_{\tau_{a,b}} = b) = \frac{S(b) - S(x)}{S(b) - S(a)} \tag{2.1}$$

$$\mathbf{E}_x[\tau_{a,b}] = 2\int_a^x \frac{S(b) - S(x)}{S(b) - S(a)} \frac{S(t) - S(a)}{\sigma^2(t)\phi(t)} dt + 2\int_x^b \frac{S(x) - S(a)}{S(b) - S(a)} \frac{S(b) - S(t)}{\sigma^2(t)\phi(t)} dt. \tag{2.2}$$

Let g be a continuous function satisfying $0 < g(x) < x$ for $x > 0$ such that $m(x,s) = \mathbf{E}_{x,s}[\tau_g]$ is finite for all $0 < x \leq s$. We will now state the first result. Whenever $s > 0$ is given and fixed, the function $x \mapsto m(x,s)$ solves the differential equation

$$\mathbf{L_X} m(x,s) = -1 \text{ for } g(s) < x < s \tag{2.3}$$

with the following two boundary conditions

$$m(x,s)\Big|_{x=g(s)+} = 0 \quad \text{(instantaneous stopping)} \tag{2.4}$$

$$\frac{\partial m}{\partial s}(x,s)\Big|_{x=s-} = 0 \quad \text{(normal reflection)}. \tag{2.5}$$

A first step in the direction of verifying that $(x,s) \mapsto m(x,s)$ satisfies the system above is contained in the following result.

LEMMA 2.1. – *The function $s \mapsto m(s,s) := M(s)$ is C^1 and satisfies the equation*

$$M'(s) = \frac{\phi(s)}{S(s) - S(g(s))} \Big(M(s) - 2\int_{g(s)}^s \frac{S(t) - S(g(s))}{\sigma^2(t)\phi(t)} dt \Big). \tag{2.6}$$

Proof. – The proof is essentially contained in Lemma 1 and Lemma 2 in [3]. Alternatively, to obtain a better feeling why (2.6) holds, as well to derive it in another way, one could use (2.7) below with (2.1)+(2.2) above to verify that $(\partial m/\partial x)(s-,s)$ equals the right-hand side in (2.6), thus showing that (2.5) is equivalent to (2.6), and then follow the second part of the proof of Theorem 4.1 below. ∎

Let $g(s) < x < s$ be given and fixed. It is immediately seen that

$$\tau_g = \tau_{g(s),s} + \tau_g \circ \theta_{\tau_{g(s),s}} \mathbf{1}_{\{X_{\tau_{g(s),s}} = s\}} \ \mathbf{P}_x\text{-a.s.}$$

and by applying strong Markov property we get

$$m(x,s) = \mathbf{E}_x[\tau_{g(s),s}] + m(s,s)\mathbf{P}_x(X_{\tau_{g(s),s}} = s). \tag{2.7}$$

From (2.1) and (2.2) we see that $G(x) = \mathbf{P}_x(X_{\tau_{g(s),s}} = s)$ and $H(x) = \mathbf{E}_x[\tau_{g(s),s}]$ solve the following well-known systems respectively

$$\begin{aligned} &\mathbf{L_X}G(x) = 0 \text{ for } g(s) < x < s \\ &G(g(s)) = 1 - G(s) = 0 \end{aligned} \tag{2.8}$$

$$\begin{aligned} &\mathbf{L_X}H(x) = -1 \text{ for } g(s) < x < s \\ &H(g(s)) = H(s) = 0. \end{aligned} \tag{2.9}$$

Consequently, by (2.6)-(2.9) we easily verify that $x \mapsto m(x,s)$ solves the system (2.3)-(2.5).

Note since τ_g may be viewed as the exit time by diffusion (X_t, S_t) from an open set, the equation (2.3) is well-known and the condition (2.4) is evident. The condition (2.5) is less evident but is known to be satisfied in a similar context (*see* [6] p. 118-119).

Unfortunately $(x,s) \mapsto m(x,s)$ is not uniquely determined by (2.3) and the two boundary conditions (2.4) and (2.5). Thus we need another condition to determine $(x,s) \mapsto m(x,s)$ uniquely. We formulate the third 'missing' condition in the form of a *minimality principle* (*see* Fig. 1 below): The expectation $m(x,s)$ is the minimal non-negative solution to the system (2.3)-(2.5).

3. EXISTENCE AND UNIQUENESS OF THE MINIMAL SOLUTION

Since $m(x,s) = 0$ for $0 < x \le g(s)$ we only need to consider m on $g(s) < x \le s$. Throughout we shall consider the system

$$\begin{aligned} &\mathbf{L_X}m(x,s) = -1 \text{ for } g(s) < x < s \\ &m(x,s)\Big|_{x=g(s)+} = 0 \\ &\frac{\partial m}{\partial s}(x,s)\Big|_{x=s-} = 0. \end{aligned} \tag{3.1}$$

Motivated by the minimality principle, in this section we shall prove the existence (and uniqueness) of a minimal non-negative solution to (3.1). Let us introduce the following notation:

$$\mathcal{M} = \{m : D \to \mathbf{R} \mid m \in \mathcal{C}^{2,1}(D^\circ) \cap C(D), m \text{ solves } (3.1) \text{ and } m \geq 0\} \tag{3.2}$$

where $D = \{(x,s) \mid g(s) \leq x \leq s, s > 0\}$ and D° is the interior of D. The function m belongs to $C^{2,1}(D^\circ)$ if $x \mapsto m(x,s)$ is C^2 and $s \mapsto m(x,s)$ is C^1 on D°.

The main result of this section may be now formulated as follows. If $\mathcal{M}$ is non-empty then it contains a minimal element, *i.e.*

$$m_* = \inf\,\{m \mid m \in \mathcal{M}\} \in \mathcal{M}$$

where the infimum is taken pointwise. Combined with the results in Section 2 this will be deduced in the proof of Theorem 4.1 below by using Itô calculus. The proof we present here is based upon the uniqueness theorem for the first and second-order differential equations.

For this note that the uniqueness theorem implies that if m_1 and m_2 belong to $\mathcal{M}$ then either $m_1 > m_2$ or $m_1 < m_2$ on D°. Let $(x_0, s_0) \in D$ be given and let $\{m_n\}_{n\geq 1}$ be a sequence of functions from $\mathcal{M}$ such that $m_n(x_0,s_0) \downarrow m_*(x_0,s_0)$. Due to the remark just mentioned, the sequence $\{m_n\}_{n\geq 1}$ is decreasing, and therefore the limit exists everywhere, *i.e.*

$$m_n(x,s) \downarrow \tilde{m}(x,s)$$

for all $(x,s) \in D$. If we can show that

$$\tilde{m} \in \mathcal{M} \tag{3.3}$$

then using the uniqueness theorem it follows that $\tilde{m} = m_*$.

In order to prove (3.3) we first show that $x \mapsto \tilde{m}(x,s)$ solves the differential equation in (3.1). By the instantaneous stopping condition, and the uniqueness theorem, m_n can be written as

$$m_n(x,s) = \mathbf{E}_x[\tau_{g(s),s}] + A_n(s)\mathbf{P}_x(X_{\tau_{g(s),s}} = s)$$

where $s \mapsto A_n(s)$ is a C^1-function. Since $\{m_n\}_{n\geq 1}$ is a decreasing sequence of functions, the sequence $\{A_n\}_{1\geq m}$ is also decreasing, and therefore it converges pointwise to a function $\tilde{A}$, *i.e.*

$$A_n(s) \downarrow \tilde{A}(s)$$

for $n \to \infty$. Hence $x \mapsto \tilde{m}(x,s)$ solves the differential equation in (3.1).

Obviously $x \mapsto \tilde{m}(x,s)$ satisfies the first boundary condition (instantaneous stopping)

$$\tilde{m}(x,s)\Big|_{x=g(s)+} = 0$$

since each $x \mapsto \tilde{m}_n(x,s)$ satisfies this condition.

Finally, to verify the second boundary condition (normal reflection), note that straightforward computations based on the normal reflection condition and (2.1)+(2.2) show that $s \mapsto A_n(s)$ solves the following differential equation

$$A_n'(s) = \frac{\phi(s)}{S(s) - S(g(s))}\left(A_n(s) - 2\int_{g(s)}^{s} \frac{S(t) - S(g(s))}{\sigma^2(t)\phi(t)}dt\right) \qquad (3.4)$$

or equivalently

$$\begin{aligned} A_n(s) - A_n(s_0) = & \int_{s_0}^{s} \frac{\phi(t)}{S(t) - S(g(t))} A_n(t)dt \\ & - 2\int_{s_0}^{s} \frac{\phi(t)}{S(t) - S(g(t))}\left(\int_{g(t)}^{t} \frac{S(r) - S(g(t))}{\sigma^2(r)\phi(r)}dr\right)dt. \end{aligned}$$

Applying the monotone convergence theorem, we find that $s \mapsto \tilde{A}(s)$ also solves the differential equation (3.4), and we can conclude that $\tilde{m} \in \mathcal{M}$.

4. THE EXPECTATION OF THE AZÉMA-YOR STOPPING TIMES

The main result of the paper is contained in the following theorem.

THEOREM 4.1. – *Let $((X_t), \mathbf{P}_x)$ be the non-negative diffusion defined in (1.1) and let (S_t) be the maximum process associated with (X_t) defined in (1.2). Let g be a continuous function on $[0, \infty)$ satisfying $0 < g(x) < x$ for $x > 0$, and let us define the stopping time*

$$\tau_g = \inf\{t > 0 \mid X_t \leq g(S_t)\}.$$

If $\mathcal{M}$ from (3.2) is non-empty, then $\mathbf{E}_{x,s}[\tau_g]$ is finite and is given by

$$\mathbf{E}_{x,s}[\tau_g] = \begin{cases} m_*(x,s) & for\ g(s) < x \leq s \\ 0 & for\ 0 < x \leq g(s) \end{cases}$$

where $(x,s) \mapsto m_*(x,s)$ is the minimal element in $\mathcal{M}$. The converse is also true, and we have the following explicit formula and a criterion for verifying that $\mathcal{M}$ is non-empty

$$\begin{aligned}\mathbf{E}_{x,s}[\tau_g] = 2\frac{S(s)-S(x)}{S(s)-S(g(s))}\int_{g(s)}^{x}\frac{S(t)-S(g(s))}{\sigma^2(t)\phi(t)}dt \\ + 2\frac{S(x)-S(g(s))}{S(s)-S(g(s))}\bigg\{\int_x^s \frac{S(s)-S(t)}{\sigma^2(t)\phi(t)}dt \\ + \int_s^\infty \frac{\phi(t)}{S(t)-S(g(t))}\bigg(\int_{g(t)}^t \frac{S(r)-S(g(t))}{\sigma^2(r)\phi(r)}dr\bigg) \\ \times \exp\bigg(-\int_s^t \frac{\phi(r)}{S(r)-S(g(r))}dr\bigg)dt\bigg\}\end{aligned}$$

for $g(s) < x \le s$ with $\mathbf{E}_{x,s}[\tau_g] = 0$ for $0 < x \le g(s)$, which is valid in the usual sense (if the right-hand side in (4.1) is finite, then so is the left-hand side, and vice versa).

Proof. – For $(x_0, s_0) \in D$ given and fixed, consider the set

$$G = \{(x,s) \mid g(s) < x < s+1\}.$$

Choose bounded open sets $G_1 \subseteq G_2 \subseteq \dots$ such that

$$(x_0, s_0) \in G_1 \text{ and } \bigcup_{n=1}^{\infty} G_n = G.$$

Define the exit time of the two-dimensional diffusion (X_t, S_t) from G_n by

$$\sigma_n = \inf\{t > 0 \mid (X_t, S_t) \notin G_n\}.$$

Note that $\mathbf{E}_{x_0,s_0}[\sigma_n] < \infty$ and $\sigma_n \uparrow \tau_g$ $\mathbf{P}_{x_0,s_0}$-a.s. as $n \to \infty$.

Let m be any function in $\mathcal{M}$. Note that $m \in C^{2,1}$ and (S_t) is of bounded variation so that Itô's formula can be applied (*see* Remark 1 in [5] p. 139). In this way we get $\mathbf{P}_{x_0,s_0}$-a.s.

$$\begin{aligned}m(X_{t\wedge\sigma_n}, S_{t\wedge\sigma_n}) =& m(x_0,s_0) + \int_0^{t\wedge\sigma_n}\frac{\partial m}{\partial x}(X_u,S_u)\sigma(X_u)dB_u \\ &+ \int_0^{t\wedge\sigma_n}\mathbf{L_X} m(X_u,S_u)du + \int_0^{t\wedge\sigma_n}\frac{\partial m}{\partial s}(X_u,S_u)dS_u.\end{aligned}$$

Due to the normal reflection condition the last integral is identically zero. Since the set of those $u > 0$ for which $X_u = S_u$ is of Lebesgue measure zero, and $\mathbf{L_X} m(x,s) = -1$ for $g(s) < x < s$, we can conclude that $\mathbf{P}_{x_0,s_0}$-a.s.

$$\int_0^{\sigma_n} \mathbf{L_X} m(X_u, S_u) du = -\sigma_n.$$

Let $\{T_k\}_{k\geq 1}$ be a localization for the local martingale

$$\int_0^{t\wedge\sigma_n} \frac{\partial m}{\partial x}(X_u, S_u)\sigma(X_u) dB_u.$$

Then by Fatou's lemma and the optional sampling theorem we get

$$\begin{aligned}
&\mathbf{E}_{x_0,s_0}[m(X_{\sigma_n}, S_{\sigma_n})] \leq \liminf_{k\to\infty} \mathbf{E}_{x_0,s_0}[m(X_{\sigma_n\wedge T_k}, S_{\sigma_n\wedge T_k})] = m(x_0, s_0) \\
&\quad + \liminf_{k\to\infty} \mathbf{E}_{x_0,s_0}\left[\int_0^{\sigma_n\wedge T_k} \frac{\partial m}{\partial x}(X_u, S_u)\sigma(X_u) dB_u - (\sigma_n \wedge T_k)\right] \\
&\leq m(x_0, s_0) - \mathbf{E}_{x_0,s_0}[\sigma_n].
\end{aligned}$$

Thus we have the inequality

$$\mathbf{E}_{x_0,s_0}[\sigma_n] \leq m(x_0, s_0)$$

for all $n \geq 1$, and by monotone convergence it follows

$$\mathbf{E}_{x_0,s_0}[\tau_g] \leq m(x_0, s_0).$$

From this we see that $\mathbf{E}_{x_0,s_0}[\tau_g]$ is finite. By the results in Section 2 we know that the function $(x,s) \mapsto \mathbf{E}_{x,s}[\tau_g]$ satisfies the system (3.1), and since m is arbitrary, hence we obtain

$$\mathbf{E}_{x_0,s_0}[\tau_g] = m_*(x_0, s_0).$$

This completes the first part of the proof.

To derive (4.1) note that (2.6) is a first-order linear differential equation whose general solution is easily found to be given by the formula

$$\begin{aligned}
M(s) &= C \exp\left(\int_{.}^{s} \frac{\phi(t)}{S(t) - S(g(t))} dt\right) - 2\exp\left(\int_{.}^{s} \frac{\phi(t)}{S(t) - S(g(t))} dt\right) \\
&\quad \times \int_{.}^{s} \left\{ \frac{\phi(t)}{S(t) - S(g(t))} \left(\int_{g(t)}^{t} \frac{S(r) - S(g(t))}{\sigma^2(r)\phi(r)} dr\right) \right. \\
&\quad \times \left. \exp\left(-\int_{.}^{t} \frac{\phi(r)}{S(r) - S(g(r))} dr\right)\right\} dt \\
&= C \exp\left(\int_{.}^{s} \frac{\phi(t)}{S(t) - S(g(t))} dt\right) \\
&\quad + \int_{s}^{\infty} \left\{ \frac{\phi(t)}{S(t) - S(g(t))} \left(\int_{g(t)}^{t} \frac{S(r) - S(g(t))}{\sigma^2(r)\phi(r)} dr\right) \right. \\
&\quad \times \left. \exp\left(-\int_{s}^{t} \frac{\phi(r)}{S(r) - S(g(r))} dr\right)\right\} dt
\end{aligned}$$

whenever the last integral is finite, where C is a real constant. Letting $s \to \infty$ we find that $C = 0$ corresponds to the minimal non-negative solution. Combining this with (2.7) and (2.1)+(2.2), we obtain the explicit formula (4.1). The proof of the theorem is complete. ■

5. AN EXAMPLE

Let $((X_t), \mathbf{P}_x)$ denote the Bessel process of dimension α, where for simplicity we assume that $\alpha > 1$ but $\alpha \neq 2$. (The other cases of α could be treated similarly.) Thus (X_t) is a non-negative diffusion with the infinitesimal operator on $(0, \infty)$ given by

$$\mathbf{L_X} = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\alpha - 1}{2x}\frac{\partial}{\partial x}.$$

(For more information about Bessel processes see [5].) Let g be a linear function given by

$$g(s) = \lambda s$$

where $0 < \lambda < 1$. Denote the stopping time τ_g by

$$\tau_\lambda = \inf\{t > 0 \mid X_t \leq \lambda S_t\}.$$

It is our aim in this section to present a closed formula for the expectation of the stopping time τ_λ. More precisely, denote the function

$$m_\lambda(x, s) = \mathbf{E}_{x,s}[\tau_\lambda]$$

for $0 < x \leq s$. Then our main task is to compute explicitly the function m_λ. Instead of using (4.1) directly, we shall rather make use of the minimality principle within the system (3.1).

According to Theorem 4.1 we shall consider the system

$$\mathbf{L_X} m_\lambda(x, s) = -1 \text{ for } \lambda s < x < s \tag{5.2}$$

$$m_\lambda(x, s)\Big|_{x=\lambda s+} = 0 \tag{5.3}$$

$$\frac{\partial m_\lambda}{\partial s}(x, s)\Big|_{x=s-} = 0. \tag{5.4}$$

Let $\lambda s < x \le s$ be given and fixed. The general solution to (5.2) is given by

$$m_\lambda(x,s) = A(s) + B(s)x^{2-\alpha} - \frac{1}{\alpha}x^2 \tag{5.5}$$

where $s \mapsto A(s)$ and $s \mapsto B(s)$ are unknown functions. By (5.3) and (5.4) we find

$$A(s) = \frac{\lambda^\alpha}{\alpha\lambda^{\alpha-2}-2}s^2 - C\lambda^{2-\alpha}s^{2-\alpha-\Delta} \tag{5.6}$$

$$B(s) = \Big(\frac{\lambda^\alpha}{\alpha} - \frac{\lambda^{2\alpha-2}}{\alpha\lambda^{\alpha-2}-2}\Big)s^\alpha + Cs^{-\Delta} \tag{5.7}$$

whenever $(2/\alpha)^{1/(\alpha-2)} < \lambda < 1$, where

$$\Delta = (2-\alpha)\frac{\lambda^{2-\alpha}}{\lambda^{2-\alpha}-1}$$

and C is an unknown constant.

In order to determine the constant C we shall use the minimality principle. It is easily verified that the minimal non-negative solution corresponds to $C = 0$. Thus by (5.5)-(5.7) with $C = 0$ we have the following candidate for $\mathbf{E}_{x,s}[\tau_\lambda]$:

$$m_\lambda(x,s) = -\frac{1}{\alpha}x^2 + \left(\frac{\lambda^\alpha}{\alpha} - \frac{\lambda^{2\alpha-2}}{\alpha\lambda^{\alpha-2}-2}\right)\frac{s^\alpha}{x^{\alpha-2}} + \frac{\lambda^\alpha}{\alpha\lambda^{\alpha-2}-2}s^2$$

when $\lambda s < x \le s$. Hence by applying Theorem 4.1 we obtain the following result. Observe that this example is also studied in [2] and [3].

PROPOSITION 5.1. – *Let $((X_t), \mathbf{P}_x)$ be a Bessel process of dimension α started at $x > 0$ under $\mathbf{P}_x$, where $\alpha > 1$ but $\alpha \neq 2$. Then for the stopping time τ_λ defined in* (5.1) *we have*

$$\mathbf{E}_{x,s}[\tau_\lambda] = \begin{cases} -\frac{1}{\alpha}x^2 + \left(\frac{\lambda^\alpha}{\alpha} - \frac{\lambda^{2\alpha-2}}{\alpha\lambda^{\alpha-2}-2}\right)\left(\frac{s}{x}\right)^\alpha x^2 & \\ \quad + \frac{\lambda^\alpha}{\alpha\lambda^{\alpha-2}-2}s^2 & \text{if } (2/\alpha)^{1/(\alpha-2)} < \lambda < 1 \\ \infty & \text{if } 0 < \lambda \le (2/\alpha)^{1/(\alpha-2)}. \end{cases}$$

(*Note that* $\mathbf{E}_{x,s}[\tau_\lambda] = 0$ *for* $0 < x \le \lambda s$.)

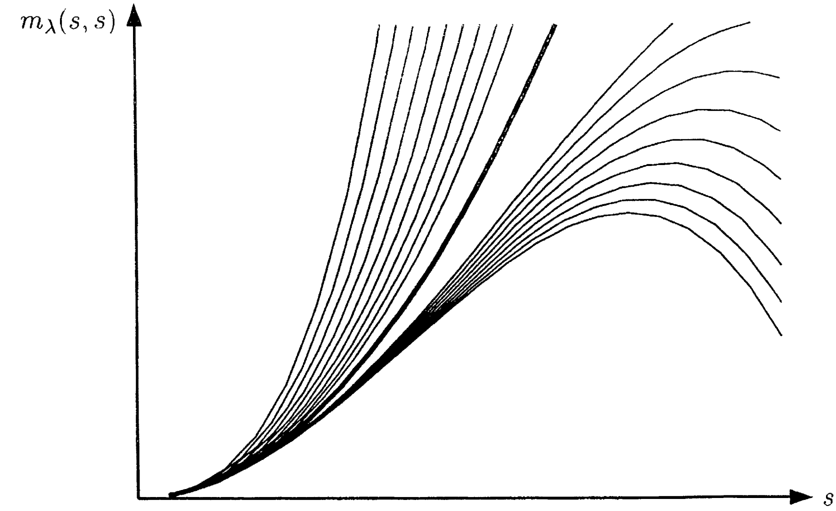

Fig. 1. – A computer drawing of solutions of the differential equation (2.6) in the case when $\alpha = 4$ and $\lambda = 4/5$. The bold line is the minimal non-negative solution (which never hits zero). By the minimality principle proved above, this solution equals $m_\lambda(s, s) = \mathbf{E}_{s,s}[\tau_\lambda]$ for all $s > 0$.

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