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# THE AVERAGE DENSITY OF SUPER-BROWNIAN MOTION

**Peter MÖRTERS** *

*Universität Kaiserslautern, Fachbereich Mathematik, 67663 Kaiserslautern, Germany*

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ABSTRACT. – In this paper we prove the existence of average densities for the support of a super-Brownian motion at a fixed time. Our result establishes a dimension-dependent fractal parameter for super-Brownian motion, which enables us to compare the local mass density of the super-Brownian motion at a fixed time with the local mass density of the occupation measure of a standard Brownian motion. © 2001 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Dans cet article est démontré l'existence des densités moyennes pour le support d'un super-mouvement brownien. Notre résultat établit un paramètre fractal dépendant de la dimension qui permet de comparer la densité locale de masse d'un super mouvement brownien à un instant fixe avec celle de la mesure d'occupation d'un mouvement brownien classique. © 2001 Éditions scientifiques et médicales Elsevier SAS

## 1. Introduction

The Hausdorff dimension and the exact Hausdorff dimension gauge are important fractal parameters, which describe the size of a fractal set. Of course, two fractal sets of the same Hausdorff dimension may have completely different topology or shape. Therefore it is important to study parameters which go beyond the measurement of size and characterize finer features of the set, like its local density or its geometric regularity. Not many such parameters are established in fractal geometry, the notion of *average density* introduced by Bedford and Fisher in [1] is one of the most popular concepts and it has given rise to a good deal of recent publications, see for example [6] and references therein.

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*E-mail address:* peter@mathematik.uni-kl.de (P. Mörters).

A striking example of two important random sets with the same exact Hausdorff dimension gauge are the path of a Brownian motion on the one hand and the support of a super-Brownian motion at a fixed positive time on the other hand. These two random sets look entirely different, the former is a curve and hence connected, the latter is totally disconnected (at least in higher dimensions), their Hausdorff dimension gauge, however, is the same,

$$\psi(r) = r^2 \log\log(1/r) \quad \text{in dimension } d \geqslant 3$$

and

$$\psi(r) = r^2 \log(1/r) \log\log\log(1/r) \quad \text{in dimension } d = 2.$$

It is therefore natural to try and compare them using a parameter describing their local density of mass like the average density of Bedford and Fisher. Whilst the average density of the Brownian path has been investigated in recent papers of Falconer and Xiao [7] and Mörters [17], it is the aim of this paper to do this for the support of a super-Brownian motion. We show that, for super-Brownian motion $\{Z_t\}$ in dimension $d \geqslant 3$ at a fixed time $t > 0$, the average density of order two exists at $Z_t$-almost every point $x$ and is equal to a constant (Theorem 1.1). This constant depends on the branching rate $\gamma$ of the super-Brownian motion and coincides with the average density of the Brownian occupation measure of the same dimension exactly if $\gamma = 4$. The constant can be interpreted in terms of the equilibrium measure of the super-Brownian motion (Theorem 5.1). In the planar case the situation is more subtle and a stronger averaging procedure is needed to get convergence of the average densities. We show that, for super-Brownian motion $\{Z_t\}$ in dimension $d = 2$ at a fixed positive time $t > 0$, the average density of order three exists and is constant at $Z_t$-almost every point $x$. If the branching rate is $\gamma = 4$, this constant agrees with the constant average density of the Brownian occupation measure in the plane (Theorem 1.2).

Beyond our motivation from fractal geometry our results constitute small scale ergodic theorems for super-Brownian motion, which are of independent interest. Our proofs are based on an interesting statement about the decay of correlation between the mass of concentric balls as the radii move apart (Lemma 3.2).

It should not remain unmentioned that other authors have used different fractal parameters to compare the support of super-Brownian motion at a fixed time and the Brownian path. Very interesting results were obtained by Le Gall, Perkins and Taylor [12] on the exact packing dimension gauge and by Perkins and Taylor [21] on the multifractal spectrum of super-Brownian motion.

In the remainder of this section we first introduce the notion of average densities and recall some basic facts about it and then describe our results about the average densities of super-Brownian motion and compare them with the known results about the Brownian path. In Section 2 we collect some facts and results about super-Brownian motion before embarking upon the finer details of the proofs of our results, which shall be given in Sections 3 and 4. Section 5 is devoted to the description of the average densities in terms of the equilibrium measure and we conclude the paper with some additional remarks and open questions.

### 1.1. Average densities

The heuristic idea of a density of a locally finite measure $\mu$ is based on the picture that the mass in a small closed ball $B(x,r)$ of radius $r$, which is centred in a point $x$ of the support, behaves like $\mu(B(x,r)) \sim D(x)r^\alpha$, in which case $\alpha$ describes the dimension of $\mu$ and the mass prefactor $D(x)$ the local density at $x$. In the case of a measure $\mu$ that is absolutely continuous with respect to Lebesgue measure this picture is correct. For singular measures, however, several difficulties occur.

The first problem consists in the fact that $D(x)$ cannot be defined as $\lim_{r\to 0}\mu(B(x, r))/r^\alpha$, as this limit fails to exist for all irregular measures and the function oscillates as $r \downarrow 0$ (see [22] or [15] for a precise statement of this fact). To handle this oscillation, Bedford and Fisher [1] suggested to use an averaging method based on classical summation techniques of Hardy and Riesz. For $n \geqslant 2$ they define the *average density of order* $n$ of $\mu$ at $x$ as

$$\lim_{k\to\infty} \frac{1}{k}\int_0^k \frac{\mu(B(x, 1/\exp^{(n-1)}(a))}{(1/\exp^{(n-1)}(a))^\alpha}\,da,$$

where $\exp^{(n)}$ is the $n$th iterate of the exponential function. The average densities of order two and three may also be written as

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\log(1/\varepsilon)}\int_\varepsilon^1 \frac{\mu(B(x,r))}{r^\alpha}\,\frac{dr}{r} \quad \text{and} \quad \lim_{\varepsilon\downarrow 0}\frac{1}{\log\log(1/\varepsilon)}\int_\varepsilon^{1/\mathrm{e}} \frac{\mu(B(x,r))}{r^\alpha}\,\frac{dr}{r\log(1/r)}.$$

For a large class of fractal measures $\mu$ possessing some self-similarity the average densities of order two were shown to exist and be equal to a constant at $\mu$-almost every $x$. Examples include the natural measures on random and deterministic self-similar sets, see, e.g., [19,20,8], mixing repellers, see [5], the zero set and path of Brownian motion, see [1,7], and intersections of Brownian paths in 3-space, see [18]. It was also shown that average densities can distinguish between different $m$-part Cantor sets of equal dimension, see [13] or [6].

In many cases, particularly in the context of stochastic processes, a further phenomenon occurs: the upper hull behaviour and the lower hull behaviour of $\mu(B(x,r))$ are governed by different functions $\psi$ and $\theta$ with the property

$$0 < \liminf_{r\downarrow 0}\frac{\mu(B(x,r))}{\theta(r)} < \infty \quad \text{and} \quad 0 < \limsup_{r\downarrow 0}\frac{\mu(B(x,r))}{\psi(r)} < \infty,$$

sometimes even such functions fail to exist and the lower or upper hull have to be determined by means of an integral test. Typically, in these cases a third gauge function $\varphi$ enters, which governs the typical behaviour between these hulls and which allows the definition of the average density for $\mu(B(x,r))/\varphi(r)$. Examples of fractal sets, for which an average density of order three can be defined using a density gauge function different from the exponential type $\varphi(r) = r^\alpha$ are the path of a Brownian motion in the plane, [17], and intersections of independent planar Brownian paths, [18].

The family of average densities of order $n$ is *consistent* in the sense that existence of average densities of order $n$ implies existence of average densities of all higher orders with the same value. The minimal number $n$ with the property that the average density of order $n$ exists at $\mu$-almost every point is sometimes called the *order of regularity* of $\mu$. It is intuitively plausible that this parameter describes regularity properties of $\mu$, although this point of view seems to have so far very little rigorous justification. There are however interesting recent results relating the geometric regularity of measures to the relation of the average densities and the lower and upper densities, see [6,14,16].

Let us now recall the known results about the average densities of the Brownian path $\{B(t)\colon\ 0 \leqslant t \leqslant 1\}$. The path is equipped with a natural measure $\mu$, the occupation measure defined by

$$\mu(A) = \int_0^1 \mathbb{1}_A(s)\,ds \quad \text{for } A \subset \mathbb{R}^d \text{ Borel.}$$

By classical results of Ciesielski, Taylor and Ray the occupation measure is almost surely a constant multiple of the generalized Hausdorff measure on the Brownian path with respect to the gauge functions $\psi$ mentioned at the beginning of this introduction. Here is what we know about the average densities of $\mu$.

- In dimension $d \geqslant 3$ Falconer and Xiao [7] found that, almost surely, average densities of order *two* exist at $\mu$-almost every point for the occupation measure $\mu$ using the density gauge $\varphi(r) = r^2$. The actual value of the average density is deterministic and independent of the point and equal to the expectation of the total occupation time of the Brownian path in the unit ball, which is easily seen to be equal to $D(d) = 2/(d-2)$.
- In dimension $d = 2$ Mörters [17] found that, almost surely, average densities of order *three* exist at $\mu$-almost every point. The appropriate density gauge is $\varphi(r) = r^2 \log(1/r)$ and the actual value of the average density is 2. The average density of order two fails to exist, so that the order of regularity is *three*.

### 1.2. Statement of the main theorems

Super-Brownian motion is a continuous Markov process with values in the space $\mathcal{M}_F(\mathbb{R}^d)$ of finite measures on $\mathbb{R}^d$. It was originally defined as a high density limit of a system of critically branching particle systems, but the enormous interest super-Brownian motion has found in the last fifteen years is also due to its many connections to the theory of certain semi-linear partial differential equations and to its rich and interesting geometric phenomenology, see for example [2] for some of these aspects.

Throughout this paper we suppose that the measure valued process $\{Z_t\}$ is a super-Brownian motion with arbitrary finite starting mass $\mu$ and constant branching rate $\gamma > 0$. Its precise definition and some basic properties are recalled in Section 2. We now give the statement of our principal results.

THEOREM 1.1. – *Suppose that $d \geqslant 3$ and $\{Z_t\}$ is a super-Brownian motion in $\mathbb{R}^d$ with branching rate $\gamma > 0$. Then, for every $t > 0$ with probability one, at $Z_t$-almost* *every* $x$,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{\varepsilon}^{1} \frac{Z_t(B(x,r))}{r^2} \frac{dr}{r} = D(d,\gamma),$$

*where the constant average density is given by* $D(d,\gamma) = \gamma/(2d-4)$.

*Remark.* – A description of the average density $D(d,\gamma)$ in terms of the equilibrium random measure $Z_\infty$ of the super-Brownian motion will be given in Section 5. This description connects the value of $D(d,\gamma)$ to the long term behaviour of the super-Brownian motion.

In the critical dimension $d = 2$ we encounter a completely different situation. It can be shown that the average density of order *two* fails to exist for the support of a super-Brownian motion in dimension 2. However, similarly as in the case of the planar Brownian path, averaging of higher order helps.

THEOREM 1.2. – *Suppose that* $d = 2$ *and* $\{Z_t\}$ *is a super-Brownian motion in* $\mathbb{R}^2$. *Then, for every* $t > 0$, *with probability one, at* $Z_t$*-almost every* $x$,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_{\varepsilon}^{1/\mathrm{e}} \frac{Z_t(B(x,r))}{r^2\log(1/r)} \frac{dr}{r\log(1/r)} = \gamma/2.$$

*Remark.* – Both our theorems are based on an ergodic phenomenon: the scale average over $Z_t(B(x,r))/\varphi(r)$ converges to the average over all random paths, because of the decay in the correlation of $Z_t(B(x,r))$ and $Z_t(B(x,s))$ as $r$ moves away from $s$. The different ways of averaging reflect qualitatively different types of decay: a decay proportional to a power of $r/s$ in the case $d \geqslant 3$ and a decay proportional to a power of $\log(s)/\log(r)$ in the case $d = 2$. All this will be made precise in Section 3.

Let us now compare the average densities of the Brownian path and the super-Brownian motion in different dimensions. In dimension $d \geqslant 3$ the order of regularity of super-Brownian motion is *two*, whereas in dimension $d = 2$, it is *three*, which heuristically means that the measure is less regular in the plane. We have encountered the same behaviour already in the case of the occupation measure of the Brownian path. For both measures the density gauge is $\varphi(r) = r^2$ in dimensions $d \geqslant 3$ and $\varphi(r) = r^2\log(1/r)$ in dimension 2 and so it makes sense to compare the actual values of the average density. These values coincide in each dimension exactly for the super-Brownian motion with branching rate $\gamma = 4$. This is also the natural choice in view of Le Gall's path valued process, see [10] and the next section. Hence, for the critical branching rate of $\gamma = 4$, the concept of average density is unable to distinguish between a Brownian path and the support of a super-Brownian motion equipped with their natural measures.

Heuristically, the smallness of the average densities of super-Brownian motion in higher dimension can be explained by the fact that in every scale there are large massless areas between separate clumps of mass, so that typical balls centred in the support cover a large portion of massless area. Such an observation was made rigorous by Tribe [23] to prove a disconnectedness property of super-Brownian motion, but his statement is to weak to have a direct influence on our result. The phenomenon of separation of mass in a fractal measure by large holes has been termed “fractal lacunarity” by Mandelbrot.

## 2. Preliminaries on super-Brownian motion

Denote by $\mathcal{M}(\mathbb{R}^d)$ the space of locally finite measures on the Borel $\sigma$-algebra on $\mathbb{R}^d$ equipped with the vague topology and by $\mathcal{M}_F(\mathbb{R}^d)$ its subspace consisting of the finite measures. Let $(\Omega_0, \mathcal{A}_0)$ be the canonical space of continuous $\mathcal{M}_F(\mathbb{R}^d)$-valued paths on $[0,\infty)$ with the Borel $\sigma$-algebra and denote the coordinate process by $\{Z_t\}$. With respect to a probability measure $\mathbb{Q}_\mu^\gamma$ on $(\Omega_0, \mathcal{A}_0)$, the Markov process $\{Z_t\}$ is a *super-Brownian motion in* $\mathbb{R}^d$ with starting mass $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and branching rate $\gamma > 0$ if, for every $\phi : \mathbb{R}^d \to [0,\infty)$ bounded, measurable,

$$\mathbb{Q}_\mu^\gamma\left(\exp\left(-\int \phi\, dZ_t\right)\right) = \exp\left(-\int U_t^\gamma \phi\, d\mu\right), \tag{1}$$

where $U_t = U_t^\gamma \phi : \mathbb{R}^d \to \mathbb{R}$ is the unique solution of the (integrated form of)

$$\frac{\partial U_t}{\partial t}(x) = \frac{\Delta}{2} U_t(x) - \frac{\gamma}{2} U_t(x)^2, \quad U_0(x) = \phi(x). \tag{2}$$

The scaling properties of this equation show that

$$\mathbb{Q}_\mu^\gamma(Z \in M) = \mathbb{Q}_{\mu/\gamma}^1(\gamma Z \in M) \quad \text{for } M \in \mathcal{A}_0. \tag{3}$$

Hence, in our proofs, it suffices to study the average density of $\{Z_t\}$ for a single branching rate $\gamma$, which we choose to be $\gamma = 4$ in the sequel. We let $\mathbb{Q}_\mu = \mathbb{Q}_\mu^4$ and even write $\mathbb{Q}_x = \mathbb{Q}_\mu$ if $\mu = \delta_x$ is the Dirac measure in $x$.

We point out two important properties of super-Brownian motion. The *superprocess property* states that the intensity measures $\mathbb{E}Z_t$ evolve like a heat flow, i.e. denoting the Brownian transition kernel by $p$ we have

$$\mathbb{E}\left\{\int \phi\, dZ_t\right\} = \int p(x-z,t)\phi(z)\, d\mu(x).$$

The second important feature, which can be seen from (1), is the so-called *branching property* of super-Brownian motion: If we start with a finite mass $\mu = \mu_1 + \mu_2$ the contributions of $\mu_1$ and $\mu_2$ evolve independently of each other. Consequently, for any fixed time $t > 0$, the random measure $Z_t$ is infinitely divisible and we can characterize it via its *canonical measure*. Very useful (and beautiful) descriptions of $Z_t$ and its canonical measure were given by Le Gall in [9] and [10], we briefly sketch the formulae relevant for our purpose.

Consider the space of stopped, continuous paths in $\mathbb{R}^d$ defined as

$$\mathcal{W} = \left\{(W,\zeta) \in \mathcal{C}\left([0,\infty), \mathbb{R}^d\right) \times [0,\infty):\ W(s) = W(\zeta) \text{ for } s \geqslant \zeta\right\},$$

equipped with the metric

$$d\big((W_1,\zeta_1),(W_2,\zeta_2)\big)=\|W_1-W_2\|_\infty+|\zeta_1-\zeta_2|.$$

Most of the time we write $W$ for $(W,\zeta)$, as $\zeta$ is clear from the context. Denote by $\Omega_1=\mathcal{C}([0,\infty),\mathcal{W})$ the space of continuous mappings from $[0,\infty)$ to $\mathcal{W}$ equipped with the Borel $\sigma$-algebra $\mathcal{A}_1$ coming from the compact-open topology. Denote by $W=\{W_s\}_{s\geqslant 0}$ the coordinate process on $\Omega_1$, by $\zeta_s$ the lifetime of $W_s$ and by $\hat{W}_s=W(\zeta_s)$ the endpoint of $W_s$. By $P_w$ we denote the law on $(\Omega_1,\mathcal{A}_1)$ of the path-valued process associated with $d$-dimensional Brownian motion starting at $w$. This law was introduced in [10]. Under $P_w$ the process $\{W_s\}_{s\geqslant 0}$ is a $\mathcal{W}$-valued continuous Markov process and $\{\zeta_s\}_{s\geqslant 0}$ is a one-dimensional reflecting Brownian motion. The intuitive picture is that $\{W_s\}$ grows like a Brownian motion in $\mathbb{R}^d$ when $\{\zeta_s\}$ increases and is erased, when $\{\zeta_s\}$ decreases (though, of course, $\{\zeta_s\}$ has strictly speaking neither points of increase, nor of decrease). Let $\{L_s^t\colon s\geqslant 0\}$ be the continuous local time of $\{\zeta_s\}$ at $t$ normalized to be a density of the occupation measure of $\{\zeta_s\}$.

We identify a point $y\in\mathbb{R}^d$ with the constant path $y$ of zero lifetime and write $P_y$ for the law of the path-valued process started in the constant path $y$. Every $y$ is a regular point for $\{W_s\}$, so that we may introduce the *Itô excursion measure* $N_y$ of excursions of $\{W_s\}$ from $y$. It is convenient to define an excursion on the whole time interval by letting it remain in $y$, once the excursion from $y$ is finished. Then $N_y$ is a $\sigma$-finite measure on $(\Omega_1,\mathcal{A}_1)$, which we normalize so that it is the intensity measure of the Poisson process $\Pi^y$ of excursions of $W$ from $y$, completed up to time $\tau[W]=\inf\{s\colon L_s^0=1\}$. The distribution of $\{\zeta_s\}$ under $N_y$ is the Itô excursion measure for reflecting Brownian motion normalized so that

$$N_y\Big(\sup_s\zeta_s>h\Big)=\frac{1}{2h}\quad\text{for } h>0.$$

Define a continuous $\mathcal{M}_F(\mathbb{R}^d)$-valued process $\{X_t\}$ on the $\sigma$-finite measure space $(\Omega_1,\mathcal{A}_1,N_y)$ by

$$X_t[W](A)=\int_0^{\sigma[W]}\mathsf{1}_A(\hat{W}_s)L_\cdot^t(ds)\quad\text{for } A\subseteq\mathbb{R}^d \text{ Borel and } t>0,$$

where $\sigma[W]=\inf\{s>0\colon\ \zeta_s=0\}$, and an $\mathcal{M}_F(\mathbb{R}^d)$-valued process on the probability space $(\Omega_1,\mathcal{A}_1,P_y)$ by

$$Z_t[W](A)=\int_0^{\tau[W]}\mathsf{1}_A(\hat{W}_s)L_\cdot^t(ds)\quad\text{for } A\subseteq\mathbb{R}^d \text{ Borel.}$$

Then, by [10, 2.1], $\{Z_t\}$ is a super-Brownian motion with branching rate $\gamma=4$ started with a unit mass in $y$. Furthermore, $\{Z_t\}$ has a Poisson representation in terms of $\{X_t\}$, i.e. for all $t > 0$,

$$Z_t[W] = \int_{\Omega_1} X_t[W] \Pi^y(dW), \qquad P_y\text{-almost surely}, \tag{4}$$

and we infer from the formula for Laplace transforms of general Poisson processes, that

$$\mathbb{Q}_y\Big(\exp\Big(-\int \phi dZ_t\Big)\Big) = \exp\Big(-\int \Big(1-\exp\Big(-\int \phi\, dX_t[W]\Big)\Big) N_y(dW)\Big). \tag{5}$$

This means that the distribution $R_t$ of $X_t$ under $N_y$ is the canonical measure associated with $Z_t$ under $\mathbb{Q}_y$.

Let us now fix a time, say $t = 1$, and a starting mass, say $Z_0 = \delta_y$. Later in this paper we work mainly with the Campbell measure associated with the canonical measure of $Z_1$, that is the measure on $\mathcal{M}_F(\mathbb{R}^d) \times \mathbb{R}^d$ defined by $P(d\mu, dx) = \mu(dx) R_1(d\mu)$. A Poisson representation of $P$ can be found in [11, (5.4)]: Denote by $\mathbb{P}_y$ the law of a Brownian motion in $\mathbb{R}^d$ started in $y$. For each $w \in \mathcal{C}([0,1], \mathbb{R}^d)$ let $\mathcal{M}_2(dt, dW)$ be a Poisson random measure on $[0,1] \times \Omega_1$ with intensity $4dt\, N_{w(t)}(dW)$ and assume that $\mathcal{M}_2$ is defined on a canonical probability space $(\Omega_2, \mathcal{A}_2, P^{(w)})$ with $w \mapsto P^{(w)}(A)$ measurable for each $A \in \mathcal{A}_2$. Then, for every measurable function $\phi : \mathbb{R}^d \times \mathcal{M}_F(\mathbb{R}^d) \to [0, \infty)$,

$$\begin{aligned} &\iint \phi\big(x, X_1[W]\big) X_1[W](dx) N_y(dW) \\ &\quad = \int P^{(w)}\Big(\phi\Big(w(1), \iint X_{1-t}[W] \mathcal{M}_2(dt, dW)\Big)\Big) \mathbb{P}_y(dw). \end{aligned} \tag{6}$$

Finally, we recall the following useful formula for the second moments of the super-Brownian motion, which may be inferred directly from (1). Denote the transition density of Brownian motion by

$$p(x-y, t) = \frac{1}{\sqrt{2\pi t}^{\,d}} \mathrm{e}^{\frac{-\|x-y\|^2}{2t}},$$

and write $dx$ for integration with respect to Lebesgue measure $\ell^d$. Then

$$\begin{aligned} &\mathbb{Q}_\mu\Big(\int \varphi dZ_t \int \psi\, dZ_t\Big) - \mathbb{Q}_\mu\Big(\int \varphi\, dZ_t\Big) \mathbb{Q}_\mu\Big(\int \psi\, dZ_t\Big) \\ &\quad = 4\int_0^t ds \int dz \int \mu(dv) \Big\{ p(v-z, t-s) \\ &\qquad \times \iint \varphi(x)\psi(y) p(z-x, s) p(z-y, s)\, dx\, dy \Big\}. \end{aligned} \tag{7}$$

## 3. Main lemma: The decay of correlation

We consider the starting mass $\mu = \delta_0$ and the corresponding super-Brownian motion $\{Z_t\}$ at time $t = 1$. The canonical measure $R_1$ associated with the random measure $Z_1$

has been described in the previous section. We work on the space $\Omega = \mathcal{M}_F(\mathbb{R}^d) \times \mathbb{R}^d$ equipped with the Borel $\sigma$-algebra $\mathcal{A}$. We define the associated Campbell measure $P$ on $\Omega$ by

$$P(M \times B) = \int_M X(B)\, R_1(dX)$$

and observe that $P$ is a probability measure on $\Omega$. On the space $(\Omega, \mathcal{A}, P)$ we define the stochastic process $\{X(r)\colon r > 0\}$ by

$$X(r)[X, x] = X(B(x,r)).$$

This process describes the mass in a ball around a random point in the support of our super-Brownian motion at time 1. We first study the expectations of $X(r)$.

LEMMA 3.1. – *For the random variable $X(r)$ on the probability space $(\Omega, \mathcal{A}, P)$ we have*

$$\begin{aligned}\mathbb{E}X(r) &= 4\int_0^1 dt \int\int_{\|v-x\|\leqslant r} p(x,t)p(v,t)\,dv\,dx\\ &= 2\int_0^2 dt \int_{\|y\|\leqslant r} p(y,t)\,dy.\end{aligned}$$

*Furthermore, in dimension $d \geqslant 3$ we have*

$$\lim_{r\downarrow 0} \mathbb{E}\left\{\frac{X(r)}{r^2}\right\} = \frac{2}{d-2} \quad \text{and} \quad \mathbb{E}\left\{\frac{X(\eta r) - X(r)}{r^2}\right\} \leqslant \frac{2}{d-2}\big(\eta^2 - 1\big),$$

*for all $\eta > 1$ and $r > 0$. In dimension $d = 2$ we have*

$$\lim_{r\downarrow 0} \mathbb{E}\left\{\frac{X(r)}{r^2 \log(1/r)}\right\} = 2 \quad \text{and} \quad \mathbb{E}\left\{\frac{X(\eta r) - X(r)}{r^2\log(1/r)}\right\} \leqslant 3\big(\eta^2 - 1\big),$$

*for all sufficiently small $r > 0$ and all $1 < \eta < 2$.*

*Proof.* – Recall the notation from the previous section. We use the Poisson representation (6) of the Campbell measure to obtain

$$\begin{aligned}\mathbb{E}X(r) &= \int\int X_1[W]\big(B(x,r)\big) X_1[W](dx)\, N_0(dW)\\ &= \int\left[\int_0^1\int X_{1-t}[W]\big(B\big(w(1),r\big)\big) N_{w(t)}(dW) 4\,dt\right]\mathbb{P}_0(dw).\end{aligned}$$

For the innermost integral we note that, by (5) and the superprocess property,

$$\int X_{1-t}[W]\big(B(y,r)\big) N_z(dW) = \mathbb{Q}_z Z_{1-t}\big(B(y,r)\big) = \int_{B(y,r)} p(x-z, 1-t)\,dx. \quad (8)$$

Hence we obtain

$$\int \left[ \int_0^1 \int X_{1-t}[W]\big(B\big(w(1),r\big)\big) N_{w(t)}(dW) 4\,dt \right] \mathbb{P}_0(dw)$$
$$= \int \left[ \int_0^1 \int_{B(w(1),r)} p(x-w(t),1-t)\,dx\,4\,dt \right] \mathbb{P}_0(dw)$$
$$= \int_0^1 \left[ \iint \int_{\|v-x\|\leqslant r} p(x-z,1-t)p(z,t)p(v-z,1-t)\,dv\,dz\,dx \right] 4\,dt$$
$$= 4\int_0^1 dt \left[ \int \int_{\|v-x\|\leqslant r} p(x,t)p(v,t)\,dv\,dx \right]$$
$$= 4\int_0^1 dt \left[ \int \int_{\|y\|\leqslant r} p(x,t)p(y-x,t)\,dx\,dy \right]$$
$$= 4\int_0^1 dt \left[ \int_{\|y\|\leqslant r} p(y,2t)\,dy \right] = 2\int_0^2 dt \left[ \int_{\|y\|\leqslant r} p(y,t)\,dy \right],$$

where we have denoted $v = w(1)$ and $z = w(t)$ and used the Chapman–Kolmogorov equation in the penultimate step. This proves the general formula. We now substitute the space variables and change the order of integration, so that

$$2\int_0^2 dt \left[ \int_{\|y\|\leqslant r} p(y,t)\,dy \right] = 2r^2 \int_{\|y\|\leqslant 1} \left[ \int_0^{2/r^2} p(y,t)\,dt \right] dy.$$

Hence, in dimension $d \geqslant 3$, we obtain

$$\lim_{r\downarrow 0} \mathbb{E}\left\{ \frac{X(r)}{r^2} \right\} = 2\int_{\|y\|\leqslant 1} \left[ \int_0^\infty p(y,t)\,dt \right] dy = \frac{\Gamma(d/2-1)}{\sqrt{\pi}^d} \int_{\|y\|\leqslant 1} \frac{1}{\|y\|^{d-2}}\,dy = \frac{2}{d-2},$$

and similarly, for all $\eta > 1$, $r > 0$,

$$\mathbb{E}\left\{ \frac{X(\eta r) - X(r)}{r^2} \right\} \leqslant \big(\eta^2 - 1\big)\, 2\int_{\|y\|\leqslant 1} \left[ \int_0^\infty p(y,t)\,dt \right] dy = \big(\eta^2-1\big)\frac{2}{d-2}.$$

In dimension $d = 2$ we obtain for the inner integral an explicit solution in terms of the integral exponential function Ei,

$$\int_0^{2/r^2} p(y,t)\,dt = \frac{1}{2\pi} \int_0^{2/r^2} \mathrm{e}^{-\|y\|^2/2t} \frac{dt}{t} = \frac{1}{2\pi} \int_{\|y\|^2 r^2/4}^\infty \mathrm{e}^{-t}\,\frac{dt}{t} = \frac{\mathrm{Ei}(-\|y\|^2 r^2/4)}{2\pi}.$$

This yields, as $\mathrm{Ei}(-r) = \pi \log(1/r) + \mathrm{O}(r)$ for $r \downarrow 0$,

$$\lim_{r\downarrow 0} \mathbb{E}\left\{\frac{X(r)}{r^2 \log(1/r)}\right\} = \frac{1}{\pi} \lim_{r\downarrow 0} \int_{\|y\|\leqslant 1} \frac{\mathrm{Ei}(-\|y\|^2 r^2/4)}{\log(1/r)}\, dy = 2,$$

and, similarly,

$$\mathbb{E}\left\{\frac{X(\eta r) - X(r)}{r^2}\right\} \leqslant (\eta^2 - 1) \int_{\|y\|\leqslant 1} \frac{\mathrm{Ei}(-\|y\|^2 r^2/4)}{\pi}\, dy \leqslant 3(\eta^2 - 1)\log(1/r),$$

for all sufficiently small $r > 0$ and $1 < \eta < 2$. □

It is natural to conjecture a decay of the correlation of $X(r)$ and $X(\varrho)$ when $r$ moves away from $\varrho$. The crucial tool in the proofs of our main theorems makes this conjecture precise, recall that the covariances refer to the Campbell measure.

LEMMA 3.2 (Main Lemma). – *Define the process* $\{X(r)\}$ *on the probability space* $(\Omega, \mathcal{A}, P)$ *as above. Then in dimension* $d \geqslant 3$, *for every* $0 < \alpha < (d-2)/(2d-3)$, *there is a* $r_0 > 0$ *and a constant* $C > 0$ *such that, for all* $0 < r \leqslant \varrho < r_0$,

$$\mathrm{Cov}\left\{\frac{X(r)}{r^2}, \frac{X(\varrho)}{\varrho^2}\right\} \leqslant C \cdot \left(\frac{r}{\varrho}\right)^{\alpha}.$$

*In dimension* $d = 2$ *there is* $r_0 > 0$ *and a constant* $C > 0$ *such that, for all* $0 < r \leqslant \varrho < r_0$,

$$\mathrm{Cov}\left\{\frac{X(r)}{r^2 \log(1/r)}, \frac{X(\varrho)}{\varrho^2 \log(1/\varrho)}\right\} \leqslant C \cdot \sqrt{\frac{\log(1/\varrho)}{\log(1/r)}}.$$

*Remark.* – I have not tried to optimize the powers appearing in the theorem, as they are unimportant for our purpose.

The remainder of this section will be devoted to the proof of this lemma. We start by deriving an explicit formula for the covariance of $\{X(r)\}$ using the Poisson representation provided in the previous section.

LEMMA 3.3. – *For all* $d \geqslant 2$ *and* $0 < r \leqslant \varrho$ *we have*

$$\mathrm{Cov}\{X(r), X(\varrho)\} = 16 \int_0^1 dt \int_0^t ds \iiint \int_{\substack{\|w-x\|\leqslant r\\ \|w-y\|\leqslant \varrho}} p(w,t)\, p(z, t-s) \times p(z-x, s)\, p(z-y, s)\, dw\, dx\, dy\, dz \tag{9}$$

$$+ 16 \int_0^1 dt \int_0^t ds \iiint \int_{\substack{\|w-x\|\leqslant r\\ \|w-y\|\leqslant \varrho}} p(v, t-s)\, p(w-v, s) \times p(x,t)\, p(v-y, s)\, dv\, dw\, dx\, dy \tag{10}$$

$$+16\int_0^1 dt\int_0^t ds\iiint\int_{\substack{\|w-x\|\leqslant r\\ \|w-y\|\leqslant \varrho}} p(v,t-s)p(w-v,s) \times p(v-x,s)p(y,t)\,dv\,dw\,dx\,dy \tag{11}$$

$$-16\int_0^1 dt\int_0^1 ds\iiint\int_{\substack{\|v-x\|\leqslant r\\ \|w-y\|\leqslant \varrho}} p(v,t)p(x,t) \times p(w,s)p(y,s)\,dv\,dw\,dx\,dy. \tag{12}$$

*Proof.* – Denote $\phi(x,\mu)=\mu(B(x,r))\mu(B(x,\varrho))$. We use the Poisson representation (6) and the formula for the variance of the Poisson process to get

$$\begin{aligned}
&\mathbb{E}\{X_1(B(x,r))X_1(B(x,\varrho))\}\\
&\quad=\iint \phi(x,X_1[W])X_1[W](dx)\,N_0(dW)\\
&\quad=\int P^{(w)}\Big\{\phi\Big(w(1),\iint X_{1-t}[W]\,\mathcal{M}_2(dt,dW)\Big)\Big\}\,\mathbb{P}_0(dw)\\
&\quad=\int P^{(w)}\Big\{\iint X_{1-t}(B(w(1),r))\,\mathcal{M}_2(dt,dW)\\
&\qquad\times\iint X_{1-t}(B(w(1),\varrho))\mathcal{M}_2(dt,dW)\Big\}\,\mathbb{P}_0(dw)\\
&\quad=\int \mathbb{P}_0(dw)\Big\{4\int_0^1\int X_{1-t}(B(w(1),r))X_{1-t}(B(w(1),\varrho))\,N_{w(t)}(dW)\,dt\\
&\qquad+16\Big(\int_0^1\int X_{1-t}(B(w(1),r))N_{w(t)}(dW)\,dt\Big)\\
&\qquad\times\Big(\int_0^1\int X_{1-t}(B(w(1),\varrho))N_{w(t)}(dW)\,dt\Big)\Big\}.
\end{aligned}$$

We have already seen in (8) that

$$\int_0^1\int X_{1-t}[W](B(w(1),r))N_{w(t)}(dW)\,dt=\int_0^1\int_{\|w(1)-x\|\leqslant r} p(w(t)-x,1-t)\,dx\,dt.$$

From (5) and the moment formula (7) we infer that

$$\begin{aligned}
&\int X_{1-t}[W](B(w(1),r))X_{1-t}[W](B(w(1),\varrho))N_{w(t)}(dW)\\
&\quad=\mathbb{Q}_{w(t)}(Z_{1-t}(B(w(1),r))Z_{1-t}(B(w(1),\varrho)))\\
&\qquad-\mathbb{Q}_{w(t)}(Z_{1-t}(B(w(1),r)))\mathbb{Q}_{w(t)}(Z_{1-t}(B(w(1),\varrho)))
\end{aligned}$$

$$= 4 \int_0^{1-t} ds \int dz \Big\{ p\big(w(t) - z, 1 - t - s\big) \times \int \int_{\substack{\|w(1)-x\| \leqslant r \\ \|w(1)-y\| \leqslant \varrho}} p(z - x, s) p(z - y, s)\, dx\, dy \Big\}.$$

We can plug these expressions in our formula, simplify, and get

$$\begin{aligned}
&\mathbb{E}\big\{X_1\big(B(x,r)\big)X_1\big(B(x,\varrho)\big)\big\} \\
&\quad = 16 \int \mathbb{P}_0(dw) \Big\{ \int_0^1 dt \int_0^{1-t} ds \int dz \Big\{ p\big(w(t) - z, 1 - t - s\big) \\
&\qquad \times \int \int_{\substack{\|w(1)-x\| \leqslant r \\ \|w(1)-y\| \leqslant \varrho}} p(z - x, s) p(z - y, s)\, dx\, dy \Big\}\Big\} \\
&\quad + 16 \int \mathbb{P}_0(dw) \Big\{ \Big\{ \int_0^1 \int_{\|w(1)-x\| \leqslant r} p\big(w(t) - x, 1 - t\big)\, dx\, dt \Big\} \\
&\qquad \times \Big\{ \int_0^1 \int_{\|w(1)-y\| \leqslant \varrho} p\big(w(s) - y, 1 - s\big)\, dy\, ds \Big\}\Big\} \\
&\quad = 16 \int_0^1 dt \int_0^{1-t} ds \iint dv\, dw\, p(v, t) p(w - v, 1 - t) \\
&\qquad \times \Big\{ \iint \int_{\substack{\|w-x\| \leqslant r \\ \|w-y\| \leqslant \varrho}} dx\, dy\, dz\, p(v - z, 1 - t - s) p(z - x, s) p(z - y, s) \Big\} \\
&\quad + 16 \int_0^1 dt \int_t^1 ds \iiint du\, dv\, dw\, p(u, t) p(v - u, s - t) p(w - v, 1 - s) \\
&\qquad \times \int \int_{\substack{\|w-x\| \leqslant r \\ \|w-y\| \leqslant \varrho}} dx\, dy\, p(u - x, 1 - t) p(v - y, 1 - s) \\
&\quad + 16 \int_0^1 dt \int_0^t ds \iiint du\, dv\, dw\, p(u, s) p(v - u, t - s) p(w - v, 1 - t) \\
&\qquad \times \int \int_{\substack{\|w-x\| \leqslant r \\ \|w-y\| \leqslant \varrho}} dx\, dy\, p(v - x, 1 - t) p(u - y, 1 - s)
\end{aligned}$$

$$
\begin{aligned}
&= 16\int_0^1 dt\int_0^t ds \iiint\int_{\substack{\|w-x\|\leqslant r\\ \|w-y\|\leqslant \varrho}} p(w,t)p(z,t-s)\\
&\quad\times p(z-x,s)p(z-y,s)\,dw\,dx\,dy\,dz\\
&\quad+16\int_0^1 dt\int_0^t ds \iiint\int_{\substack{\|w-x\|\leqslant r\\ \|w-y\|\leqslant \varrho}} p(v,t-s)p(w-v,s)\\
&\quad\times p(x,t)p(v-y,s)\,dv\,dw\,dx\,dy\\
&\quad+16\int_0^1 dt\int_0^t ds \iiint\int_{\substack{\|w-x\|\leqslant r\\ \|w-y\|\leqslant \varrho}} p(v,t-s)p(w-v,s)\\
&\quad\times p(v-x,s)p(y,t)\,dv\,dw\,dx\,dy,
\end{aligned}
$$

where, in the penultimate step, we have split the second summand in two parts according as $s \leqslant t$ or $s > t$. We then get the final form of our formula by subtracting the expression for $\mathbb{E}X(r)\mathbb{E}X(\varrho)$, which was established in Lemma 3.1. □

The expression in Lemma 3.3 could be simplified a little more, but instead of doing this, we express all integrals as integrals over the same integrand, by means of a change of variable.

LEMMA 3.4. – *For $0 < r \leqslant \varrho$ denote $q = r/\varrho$. For $w, x, y, z \in \mathbb{R}^d$ we denote*

$$
\Phi_\varrho[w,x,y,z] = \varrho^{4d}\int_0^1 t^{1-2d} p\left(\frac{w\varrho}{\sqrt{t}},1\right)p\left(\frac{x\varrho}{\sqrt{t}},1\right)p\left(\frac{y\varrho}{\sqrt{t}},1\right)p\left(\frac{z\varrho}{\sqrt{t}},1\right)dt.
$$

*Then we can write*

$$
\begin{aligned}
\operatorname{Cov}\left\{\frac{X(r)}{r^2},\frac{X(\varrho)}{\varrho^2}\right\} = \frac{1}{q^2}\int_0^1 ds\Bigg\{ & 16\iiint\int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant q\\ \|w-\sqrt{s}y-\sqrt{1-s}z\|\leqslant 1}} + 16\iiint\int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant q\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant 1}}\\
&+16\iiint\int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant 1\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q}} - 16\iiint\int_{\substack{\|w-z\|\leqslant q\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant 1}}\\
&-16\iiint\int_{\substack{\|w-z\|\leqslant 1\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q}}\Bigg\}\frac{\Phi_\varrho[w,x,y,z]}{\varrho^4}\,dw\,dx\,dy\,dz. \qquad (13)
\end{aligned}
$$

*Furthermore,we have, for all $w, x, y, z \in \mathbb{R}^d$,*

$$
\lim_{\varrho\downarrow 0}\frac{\Phi_\varrho[w,x,y,z]}{\varrho^4} = \frac{\Gamma(2d-2)}{4\pi^{2d}}\left[\|w\|^2+\|x\|^2+\|y\|^2+\|z\|^2\right]^{2-2d}
$$

*and this limit is monotonically increasing. If $d = 2$ we have, for all $w, x, y, z \in \mathbb{R}^2$,*

$$\frac{\Phi_\varrho[w,x,y,z]}{\varrho^4} = \frac{1}{4\pi^4}\left\{\frac{\varrho^2}{2}\,\frac{\exp[-\varrho^2(\|w\|^2+\|x\|^2+\|y\|^2+\|z\|^2)/2]}{\|w\|^2+\|x\|^2+\|y\|^2+\|z\|^2}\right.$$
$$\left. + \frac{\exp[-\varrho^2(\|w\|^2+\|x\|^2+\|y\|^2+\|z\|^2)/2]}{[\|w\|^2+\|x\|^2+\|y\|^2+\|z\|^2]^2}\right\}.$$

*Proof.* – We carry out the change of variables for (9). Substitute $w$ for $w/\varrho$, $z$ for $z/\varrho$, $x$ for $(x-z)/\varrho$ and $y$ for $(y-z)/\varrho$ and obtain

$$16\varrho^{4d}\int_0^1 dt \int_0^t ds \iiint \int_{\substack{\|w-x-z\|\leqslant q\\ \|w-y-z\|\leqslant 1}} p(w\varrho,t)\,p(z\varrho,t-s)\,p(x\varrho,s)\,p(y\varrho,s)\,dw\,dx\,dy\,dz.$$

Recall that $p(x,s) = s^{-d/2}p(x/\sqrt{s},1)$ and change the order of integration to obtain

$$16\iiint \int_{\substack{\|w-x-z\|\leqslant q\\ \|w-y-z\|\leqslant 1}} dw\,dx\,dy\,dz\left\{\varrho^{4d}\int_0^1 dt\int_0^t ds\left(\frac{1}{t(t-s)s^2}\right)^{d/2}\right.$$
$$\left.\times\, p\left(\frac{w\varrho}{\sqrt{t}},1\right)p\left(\frac{z\varrho}{\sqrt{t-s}},1\right)p\left(\frac{x\varrho}{\sqrt{s}},1\right)p\left(\frac{y\varrho}{\sqrt{s}},1\right)\right\}.$$

Now substitute $s$ for $s/t$, $x$ for $x/\sqrt{s}$, $y$ for $y/\sqrt{s}$ and $z$ for $z/\sqrt{1-s}$. This finally yields

$$16\int_0^1 ds \iiint \int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant q\\ \|w-\sqrt{s}y-\sqrt{1-s}z\|\leqslant 1}} dw\,dx\,dy\,dz$$
$$\times\left\{\varrho^{4d}\int_0^1 t^{1-2d}\,p\left(\frac{w\varrho}{\sqrt{t}},1\right)p\left(\frac{z\varrho}{\sqrt{t}},1\right)p\left(\frac{x\varrho}{\sqrt{t}},1\right)p\left(\frac{y\varrho}{\sqrt{t}},1\right)dt\right\}.$$

This corresponds to the form given in the lemma and analogous substitutions may be performed for the other terms. Additionally, we have split the last term in two parts according as $s < t$ or $s \geqslant t$. This proves (13). To obtain the asymptotics of $\Phi_\varrho[w,x,y,z]$ we abbreviate $a = \|w\|^2+\|x\|^2+\|y\|^2+\|z\|^2$ and get

$$\lim_{\varrho\downarrow 0}\frac{\Phi_\varrho[w,x,y,z]}{\varrho^4} = \lim_{\varrho\downarrow 0}\frac{\varrho^{4d-4}}{(2\pi)^{2d}}\int_0^1 t^{1-2d}\exp\bigl[-a\varrho^2/2t\bigr]\,dt$$
$$= \lim_{\varrho\downarrow 0}\frac{1}{(2\pi)^{2d}}(a/2)^{2-2d}\int_{a\varrho^2/2}^{\infty} t^{2d-3}\mathrm{e}^{-t}\,dt = \frac{\Gamma(2d-2)}{4\pi^{2d}a^{2d-2}}.$$

In the case $d = 2$ the limit above is also valid, but we have to evaluate the integral explicitly,

$$\frac{\Phi_\varrho[w,x,y,z]}{\varrho^4} = \frac{1}{4\pi^4a^2}\int_{a\varrho^2/2}^{\infty} t\mathrm{e}^{-t}\,dt = \frac{1}{4\pi^4}\left(\frac{\varrho^2}{2a}\mathrm{e}^{-a\varrho^2/2} + \frac{1}{a^2}\mathrm{e}^{-a\varrho^2/2}\right). \qquad \square$$

It is now necessary to distinguish the cases $d=2$ and $d\geqslant 3$. We start the estimates necessary for the *proof of the main lemma in the case* $d=2$. In our estimates $C$ denotes the value of a constant that may change from line to line, $\varrho$ is assumed to be sufficiently small.

We proceed in two steps. We first prove that

$$\frac{1}{q^2}\int_0^1 ds \iiint \int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant q\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant 2}} \frac{\Phi_\varrho[w,x,y,z]}{\varrho^4}\,dw\,dx\,dy\,dz \leqslant C\cdot\big(\log(1/\varrho)\big)^2. \tag{14}$$

Since $q\leqslant 1$ this gives favourable estimates for the first two summands in (13). Observe first that $\Phi_\varrho[w,x,y,z]\leqslant\Phi_\varrho[0,x,y,z]$. Hence we can integrate, first with respect to $w$ and then with respect to $s$ and obtain

$$\begin{aligned}
&\frac{1}{q^2}\int_0^1 ds \iiint \int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant q\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant 2}} \frac{\Phi_\varrho[w,x,y,z]}{\varrho^4}\,dw\,dx\,dy\,dz\\
&\quad\leqslant C\cdot\int_0^1 ds\bigg\{\iint\int_{\|\sqrt{s}x-\sqrt{s}y\|\leqslant 2}\frac{\Phi_\varrho[0,x,y,z]}{\varrho^4}\,dx\,dy\,dz\bigg\}\\
&\quad\leqslant C\cdot\iiint\min\bigg(1,\frac{4}{\|x-y\|^2}\bigg)\frac{\Phi_\varrho[0,x,y,z]}{\varrho^4}\,dx\,dy\,dz\\
&\quad\leqslant C\cdot\iint\int_{\|x-y\|>2}\frac{\Phi_\varrho[0,x,y,z]}{(\|x-y\|^2)\varrho^4}\,dx\,dy\,dz\\
&\qquad+C\cdot\iint\int_{\|x-y\|\leqslant 2}\frac{\Phi_\varrho[0,x,y,z]}{\varrho^4}\,dx\,dy\,dz.
\end{aligned}\tag{15}$$

We now use the expression for $\Phi_\varrho$ from Lemma 3.4. Using the symmetry in $x$ and $y$ we can restrict integration to $\|x\|\geqslant\|y\|$. The second summand can be bounded easily by

$$\begin{aligned}
&\iint\int_{\substack{\|x-y\|\leqslant 2\\ \|x\|\geqslant\|y\|}}\frac{\Phi_\varrho[0,x,y,z]}{\varrho^4}\,dx\,dy\,dz\\
&\quad\leqslant C\cdot\bigg\{\varrho^2\iint\frac{\exp[-\varrho^2(\|x\|^2+\|z\|^2)/2]}{[\|x\|^2+\|z\|^2]}\,dx\,dz\\
&\qquad+\iint\int_{\|x\|^2+\|y\|^2+\|z\|^2\leqslant 2}\frac{\exp[-\varrho^2(\|x\|^2+\|y\|^2+\|z\|^2)/2]}{[\|x\|^2+\|y\|^2+\|z\|^2]^2}\,dx\,dy\,dz\\
&\qquad+\iint\int_{\|x\|^2+\|z\|^2\geqslant 1}\frac{\exp[-\varrho^2(\|x\|^2+\|z\|^2)/2]}{[\|x\|^2+\|z\|^2]^2}\,dx\,dz\bigg\}
\end{aligned}$$

$$\leqslant C \cdot \left\{ \varrho^2 \int_0^\infty s \exp\left[-\varrho^2 s^2/2\right] ds + \int_0^{\sqrt{2}} s\, ds + \int_1^\infty \frac{\exp[-\varrho^2 s^2/2]}{s}\, ds \right\}$$
$$\leqslant C \cdot \log(1/\varrho).$$

For the first summand we can find an upper bound of

$$\iint \int_{\substack{\|x-y\|>2\\ \|x\|\geqslant\|y\|}} \frac{\Phi_\varrho[0,x,y,z]}{(\|x-y\|^2)\varrho^4}\, dx\, dy\, dz$$
$$\leqslant C \cdot \left\{ \varrho^2 \iint \int_{2\|x\|\geqslant\|x-y\|>2} \frac{\exp[-\varrho^2(\|x\|^2+\|y\|^2+\|z\|^2)/2]}{(\|x-y\|^2)[\|x\|^2+\|y\|^2+\|z\|^2]}\, dx\, dy\, dz \right.$$
$$\left. + \iint \int_{2\|x\|\geqslant\|x-y\|>2} \frac{\exp[-\varrho^2(\|x\|^2+\|y\|^2+\|z\|^2)/2]}{(\|x-y\|^2)[\|x\|^2+\|y\|^2+\|z\|^2]^2}\, dx\, dy\, dz \right\}$$
$$\leqslant C \cdot \left\{ \varrho^2 \iint \frac{\exp[-\varrho^2(\|x\|^2+\|z\|^2)/2]}{\|x\|^2+\|z\|^2} \int_{2\|x\|\geqslant\|x-y\|>2} \frac{dy}{\|x-y\|^2}\, dx\, dz \right.$$
$$+ \iint \int_{\|x\|^2+\|y\|^2+\|z\|^2\leqslant 2} \frac{dx\, dy\, dz}{[\|x\|^2+\|y\|^2+\|z\|^2]^2}$$
$$\left. + \int \int_{\|x\|^2+\|z\|^2\geqslant 1} \frac{\exp[-\varrho^2(\|x\|^2+\|z\|^2)/2]}{[\|x\|^2+\|z\|^2]^2} \int_{2\|x\|\geqslant\|x-y\|>2} \frac{dy}{\|x-y\|^2}\, dx\, dz \right\}$$
$$\leqslant C \cdot \left\{ \varrho^2 \int_0^\infty \exp\left(-\varrho^2 s^2/2\right) s \log(s)\, ds \right.$$
$$\left. + \int_0^{\sqrt{2}} s\, ds + \int_1^\infty \frac{\exp(-\varrho^2 s^2/2)}{s} \cdot \log(s)\, ds \right\}$$
$$\leqslant C \cdot \left(\log(1/\varrho)\right)^2.$$

This establishes an upper bound of $C(\log(1/\varrho))^2$ for (14) and we are done.

In the second step we show that

$$\frac{1}{q^2} \int_0^1 ds \left\{ \iiint \int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant 1\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q}} - \iiint \int_{\substack{\|w-z\|\leqslant 1\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q}} \right\} \frac{\Phi_\varrho[w,x,y,z]}{\varrho^4}\, dw\, dx\, dy\, dz$$
$$\leqslant C \cdot \log(1/\varrho)^{3/2} \log(1/r)^{1/2}. \tag{16}$$

This gives a favourable estimate of the last three summands of (13). In this term the cancellation occurs. Fix $a = a(r,\varrho) \leqslant 1$. We can bound (16) from above by

$$\frac{C}{q^2}\int_0^1 ds\Bigg\{\iiint\int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant 1\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q,\,\|\sqrt{s}x\|\geqslant a}} + \iiint\int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant 1\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q,\,(1-\sqrt{1-s})\|z\|\geqslant a}}$$
$$+\iiint\int_{\substack{1\leqslant\|w-z\|\leqslant 1+2a\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q}}\Bigg\}\frac{\Phi_\varrho[w,x,y,z]}{\varrho^4}\,dw\,dx\,dy\,dz. \tag{17}$$

Observe that in the last integral we have decoupled the variables $w, z$ from the variables $x, y$. Hence this term is the product of two integrals as they were considered in Lemma 3.1 and we have

$$\frac{C}{q^2}\int_0^1 ds\iiint\int_{\substack{1\leqslant\|w-z\|\leqslant 1+2a\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q}}\frac{\Phi_\varrho[w,x,y,z]}{\varrho^4}\,dw\,dx\,dy\,dz$$
$$\leqslant C\cdot\frac{\mathbb{E}X_1B(x,r)}{r^2}\frac{\mathbb{E}X_1(B(x,\varrho(1+2a))\setminus B(x,\varrho))}{\varrho^2}$$
$$\leqslant C\cdot\log(1/r)\log(1/\varrho)\big((1+2a)^2-1\big).$$

The last inequality used the estimates for the expected values of small annuli obtained in Lemma 3.1. Choosing $a=\log(1/\varrho)/\log(1/r)$ gives an upper bound of $C(\log(1/\varrho))^2$. It remains to estimate the first two integrals in (17) for this choice of $a$. Let us begin with the first integral. Integrating with respect to $y$ and $w$ and using Lemma 3.4 yields

$$\frac{1}{q^2}\int_0^1 ds\iiint\int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant 1\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q,\,\|\sqrt{s}x\|\geqslant a}}\frac{\Phi_\varrho[w,x,y,z]}{\varrho^4}\,dw\,dx\,dy\,dz$$
$$\leqslant C\cdot\int_0^1\frac{ds}{s}\int\int_{\|\sqrt{s}x\|\geqslant a}\frac{\Phi_\varrho[0,x,0,z]}{\varrho^4}\,dx\,dz$$
$$\leqslant C\cdot\Bigg\{\varrho^2\int_0^1\frac{ds}{s}\int_{\|\sqrt{s}x\|\geqslant a}dx\int\frac{\exp[-\varrho^2\|x\|^2(1+\|z\|^2)/2]}{1+\|z\|^2}\,dz \tag{18}$$
$$+\int_0^1\frac{ds}{s}\int_{\|\sqrt{s}x\|\geqslant a}\frac{dx}{\|x\|^2}\int\frac{\exp[-\varrho^2\|x\|^2(1+\|z\|^2)/2]}{[1+\|z\|^2]^2}\,dz\Bigg\}. \tag{19}$$

For (18) we obtain, writing $\log^+$ for the positive part of the logarithm,

$$\varrho^2\int_0^1\frac{ds}{s}\int_{\|\sqrt{s}x\|\geqslant a}dx\left\{\mathrm{e}^{-\varrho^2\|x\|^2/2}\int\frac{\exp[-\varrho^2\|x\|^2\|z\|^2/2]}{1+\|z\|^2}\,dz\right\}$$
$$\leqslant C\cdot\varrho^2\cdot\int_0^1\frac{ds}{s}\int_{\|x\|\geqslant a/\sqrt{s}}dx\big\{\mathrm{e}^{-\varrho^2\|x\|^2/2}\big(\log^+\big(1/\|x\|\varrho\big)+1\big)\big\}$$

$$
\begin{aligned}
&\leqslant C \cdot \varrho^2 \cdot \int dx \left\{ \mathrm{e}^{-\varrho^2 \|x\|^2/2} \big( \log^+ \big(1/\|x\|\varrho\big) + 1\big) \log^+ \big(\|x\|/a\big) \right\} \\
&\leqslant C \cdot \int_0^\infty \left\{ \mathrm{e}^{-p^2/2} \big( \log^+ (1/p) + 1\big) \log^+ (p/a\varrho) \right\} p\,dp \\
&\leqslant C \cdot \log(1/a\varrho).
\end{aligned}
$$

With our choice of $a$ the last expression is easily seen to be bounded by $C \cdot \log(1/\varrho)^{3/2} \log(1/r)^{1/2}$. This gives the necessary bound for (18). For (19) we can split our domain in two parts depending whether $\|x\| \leqslant \varrho$ or $\|x\| > \varrho$ and obtain an estimate

$$
\begin{aligned}
&\int_0^1 \frac{ds}{s} \int_{\|\sqrt{s}x\| \geqslant a} dx \left\{ \frac{1}{\|x\|^2} \int \frac{\exp[-\varrho^2 \|x\|^2 (1 + \|z\|^2)/2]}{[1 + \|z\|^2]^2} \, dz \right\} \\
&\quad \leqslant C \cdot \int_0^1 \frac{ds}{s} \int_{\|\sqrt{s}x\| \geqslant a} \frac{dx}{\|x\|^2} \mathrm{e}^{-\varrho^2 \|x\|^2/2} \int \frac{1}{[1 + \|z\|^2]^2} \, dz \\
&\quad \leqslant C \cdot \left\{ \int_{a^2/\varrho^2}^1 \frac{ds}{s} \int_{\varrho \geqslant \|x\| \geqslant a/\sqrt{s}} \frac{dx}{\|x\|^2} \mathrm{e}^{-\varrho^2 \|x\|^2/2} + \int_{\|x\| \geqslant \varrho} \frac{dx}{\|x\|^2} \mathrm{e}^{-\varrho^2 \|x\|^2/2} \int_{a^2/\|x\|^2}^1 \frac{ds}{s} \right\} \\
&\quad \leqslant C \cdot \left\{ \int_{a^2/\varrho^2}^1 \frac{ds}{s} \log(\sqrt{s}/a) + \int_\varrho^\infty \frac{1}{p} \mathrm{e}^{-\varrho^2 p^2/2} \log(p/a) \, dp \right\} \\
&\quad \leqslant C \cdot \big\{ \log(1/a) \log(1/\varrho) + \log(1/\varrho)^2 \big\}.
\end{aligned}
$$

Both expressions are bounded by a constant multiple of $\log(1/\varrho)^{3/2} \log(1/r)^{1/2}$ and hence we have established the necessary bound for the first integral in (17). Upon observing that $1 - \sqrt{1-s} \leqslant \sqrt{s}$ for all $0 < s < 1$, the second integral may be bounded in the same manner. Thus we have established the necessary bounds for all expressions in (17) and (16) is proved. Altogether, (14) and (16) yield the main lemma in the case $d = 2$.

Let us now work out the *proof of the main lemma in the case* $d \geqslant 3$. Given $0 < \alpha < (d-2)/(2d-3)$ we choose $\alpha < \eta < (d-2)/(2d-3)$ such that $\delta = (1-2\eta)(d-2) > \alpha$.

To make our proof as compact as possible we start with a general estimate. Let $0 \leqslant a, b, c, \theta \leqslant 1$, then there is a constant $C > 0$ independent of them, such that

$$
\begin{aligned}
&\int_0^1 ds \iiint \int_{\substack{\|w - \sqrt{s}x - \sqrt{1-s}z\| \leqslant a \\ \|x - y\| \leqslant b/\sqrt{s}, \|y\| \geqslant c/s^\theta}} \frac{dw\,dx\,dy\,dz}{[\|w\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2]^{2d-2}} \\
&\quad \leqslant C \cdot a^d \int_0^1 ds \iint \int_{\substack{\|x - y\| \leqslant b/\sqrt{s} \\ \|y\| \geqslant c/s^\theta}} \frac{dx\,dy\,dz}{[\|x\|^2 + \|y\|^2 + \|z\|^2]^{2d-2}}
\end{aligned}
$$

$$\leqslant C\cdot a^d\int_0^1 ds\int_{\|y\|\geqslant c/s^\theta} dy\left\{\|y\|^{4-2d}\int\int_{\|x\|\leqslant b/(\sqrt{s}\|y\|)}\frac{dx\,dz}{[1+\|x\|^2+\|z\|^2]^{2d-2}}\right\}$$

$$\leqslant C\cdot a^d\int_0^1 ds\left\{\left(\int_{c/s^\theta}^{b/\sqrt{s}} p^{3-d}\,dp\right)^+ + s^{-d/2}b^d\left[\max\left(\frac{b}{\sqrt{s}},\frac{c}{s^\theta}\right)\right]^{4-2d}\right\}. \tag{20}$$

Having provided this general estimate, we now start with the estimates leading to the statement of the main lemma. Again we proceed in two steps. In the first step we show that

$$\frac{1}{q^2}\int_0^1 ds\int\int\int\int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant q\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant 2}}\frac{dw\,dx\,dy\,dz}{[\|w\|^2+\|x\|^2+\|y\|^2+\|z\|^2]^{2d-2}}\leqslant C\cdot q, \tag{21}$$

thus providing a favourable estimate for the first two summands in (13).

If $d=3$ we can use (20) straight away with $a=q$, $b=2$ and $c=0$ and obtain an upper bound for (21) of

$$C\cdot q\int_0^1 ds\left\{\int_0^{2/\sqrt{s}} dp + s^{-3/2}\left(\frac{2}{\sqrt{s}}\right)^{-2}\right\}\leqslant C\cdot q\int_0^1\frac{ds}{\sqrt{s}}\leqslant Cq.$$

In dimensions $d\geqslant 4$ we first look at a restriction of our domain of integration by assuming $\|y\|\geqslant q$. From (20) with $a=c=q$, $b=2$ and $\theta=0$ we obtain an upper bound of

$$Cq^{d-2}\int_0^1 ds\left\{\int_q^{2/\sqrt{s}} p^{3-d}\,dp + s^{-d/2}\left(\frac{2}{\sqrt{s}}\right)^{4-2d}\right\}$$
$$\leqslant Cq^{d-2}\max\left(q^{4-d},\,\log(1/q)\right)\leqslant Cq.$$

It remains to integrate over the part of the domain satisfying $\|y\|\leqslant q$.

$$\frac{1}{q^2}\int_0^1 ds\int\int\int\int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant q\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant 2,\|y\|\leqslant q}}\frac{dw\,dx\,dy\,dz}{[\|w\|^2+\|x\|^2+\|y\|^2+\|z\|^2]^{2d-2}}$$

$$\leqslant\frac{C}{q^2}\int_0^1 ds\int_0^q dp\left\{p^3\int\int\int_{\substack{\|x\|\leqslant 2/(\sqrt{s}p)\\ \|w\|\leqslant 1/p}}\frac{dw\,dx\,dz}{(1+\|w\|^2+\|x\|^2+\|z\|^2)^{2d-2}}\right\}.$$

Recall that

$$\int_0^\infty a^{d-1}\frac{da}{(b+a^2)^{2d-2}}\leqslant\frac{C}{b^{3d/2-2}}\quad\text{for } b\geqslant 1.$$

This allows us to estimate

$$\frac{1}{q^2}\int_0^1 ds\int_0^q dp\left\{p^3\iint_{\substack{\|x\|\leqslant 2/(\sqrt{s}p)\\ \|w\|\leqslant 1/p}}\int \frac{dw\,dx\,dz}{(1+\|w\|^2+\|x\|^2+\|z\|^2)^{2d-2}}\right\}$$
$$\leqslant \frac{C}{q^2}\int_0^1 ds\int_0^q dp\left\{p^3\int_{\substack{\|x\|\leqslant 2/(\sqrt{s}p)\\ \|w\|\leqslant 1/p}}\int dw\,dx\right.$$
$$\left.\times\int_0^\infty a^{d-1}\frac{da}{(1+\|w\|^2+\|x\|^2+a^2)^{2d-2}}\right\}$$
$$\leqslant \frac{C}{q^2}\int_0^1 ds\int_0^q dp\left\{p^3\int_{\substack{\|x\|\leqslant 2/(\sqrt{s}p)\\ \|w\|\leqslant 1/p}}\int \frac{dw\,dx}{(1+\|w\|^2+\|x\|^2)^{3d/2-2}}\right\}$$
$$\leqslant \frac{C}{q^2}\int_0^1 ds\int_0^q dp\left\{p^3\left(1+\log\left(\frac{1}{\sqrt{s}p}\right)\right)\right\}\leqslant C\cdot q.$$

This finishes the proof of (21) in all dimensions $d\geqslant 3$.

In the second step we have to establish that,

$$\frac{1}{q^2}\int_0^1 ds\left\{\iiint_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant 1\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q}}\int - \iiint_{\substack{\|w-z\|\leqslant 1\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q}}\int\right\}$$
$$\times\frac{\Phi_\varrho[w,x,y,z]}{\varrho^4}\,dw\,dx\,dy\,dz\leqslant C\cdot q^\alpha. \tag{22}$$

We proceed as in the case of $d=2$. One can bound the left hand side from above by

$$\frac{C}{q^2}\int_0^1 ds\left\{\iiint_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant 1\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q,\,\|\sqrt{s}x\|\geqslant q^\eta}}\int + \iiint_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant 1\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q,\,(1-\sqrt{1-s})\|z\|\geqslant q^\eta}}\int\right.$$
$$\left.+\iiint_{\substack{1\leqslant\|w-z\|\leqslant 1+2q^\eta\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q}}\int\right\}\frac{\Phi_\varrho[w,x,y,z]}{\varrho^4}\,dw\,dx\,dy\,dz. \tag{23}$$

As before we have decoupled the variables $w,z$ from the variables $x,y$ in the last term. Hence this term is the product of two integrals as in Lemma 3.1 and we may estimate

$$\frac{C}{q^2}\int_0^1 ds\iiint_{\substack{1\leqslant\|w-z\|\leqslant 1+2q^\eta\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q}}\int\frac{\Phi_\varrho[w,x,y,z]}{\varrho^4}\,dw\,dx\,dy\,dz$$
$$\leqslant C\cdot\frac{\mathbb{E}X_1B(x,r)}{r^2}\,\frac{\mathbb{E}X_1(B(x,\varrho(1+2q^\eta))\setminus B(x,\varrho))}{\varrho^2}$$
$$\leqslant C\cdot\left(\left(1+2q^\eta\right)^2-1\right)\leqslant C\cdot q^\eta,$$

where we have used the estimate of Lemma 3.1 for the small annuli. As $\eta \geqslant \alpha$ this is sufficient. The first error term may be estimated by means of (20), with $a = 1$, $b = q$, $c = q^\eta$ and $\theta = 1/2$,

$$\frac{1}{q^2}\int_0^1 ds \iiint \int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant 1\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q,\ \|\sqrt{s}x\|\geqslant q^\eta}} \frac{\Phi_\varrho[w,x,y,z]}{\varrho^4}\,dw\,dx\,dy\,dz$$

$$\leqslant \frac{1}{q^2}\int_0^1 ds \iiint \int_{\substack{\|w-\sqrt{s}x-\sqrt{1-s}z\|\leqslant 1\\ \|\sqrt{s}x-\sqrt{s}y\|\leqslant q,\ \|\sqrt{s}x\|\geqslant q^\eta}} \frac{dw\,dx\,dy\,dz}{[\|w\|^2+\|x\|^2+\|y\|^2+\|z\|^2]^{2d-2}}$$

$$\leqslant \frac{C}{q^2}\int_0^1 ds\left\{s^{-d/2}q^d\left(\frac{q^\eta}{\sqrt{s}}\right)^{4-2d}\right\} \leqslant C\cdot q^\delta,$$

and this estimate is good, as $\delta \geqslant \alpha$. The second term may be estimated completely analogously. Hence (22) is established and this finishes the proof of our main lemma. □

## 4. Proof of the main theorems

By a result of Evans and Perkins [4], for all nonzero measures $\nu, \mu \in \mathcal{M}_F(\mathbb{R}^d)$ and $0 < s < t$ the laws of $Z_s$ under $\mathbb{Q}^1_\nu$ and $Z_t$ under $\mathbb{Q}^1_\mu$ are mutually equivalent, so that it suffices to consider the case $\mu = \delta_0$ and $t = 1$.

By the Poisson representation (4) we can write $Z_1[W]$ as a sum of clusters $X_1[W^i]$, where $W^i$ are those excursions of $W$ from the constant path 0 of lifetime 0, which are completed at time $\tau[W]$ and whose lifetime $\zeta^i$ reaches level 1,

$$Z_1[W] = \sum_{i=1}^M X_1\left[W^i\right].$$

Here $M$ is a Poisson random variable with mean $1/2$ (by our choice of the normalization of $N_0$) and, given $M = m$, the $X_1[W^i]$ are independent with law $N_0(X_1 \in \cdot \mid X_1 \neq 0)$. As the support $S^i$ of $X_1[W^i]$ is almost surely a Lebesgue nullset, we have $\mathbb{E}X_1[W^j](S^i) = 0$ for $i \neq j$. Hence, $X_1[W^i]$ and $X_1[W^j]$ are mutually singular measures and, e.g., by [15, 2.13], almost surely, at $X_1[W^i]$-almost every $x$ the density $dX_1[W^j]/dX_1[W^i]$, $i \neq j$, vanishes. It therefore suffices to prove our theorems for the random measure $X_1$ whose law is $N_0(X_1 \in \cdot \mid X_1 \neq 0)$. We look again at the Campbell measure $P$ associated with the canonical measure of $Z_1$. From our main lemma we infer the following.

LEMMA 4.1. – *For the process $\{X(r)\}$ on the probability space $(\Omega, \mathcal{A}, P)$ we have, for some constant $C > 0$ and all sufficiently small $\varepsilon > 0$, in the case $d \geqslant 3$,*

$$\mathrm{Var}\left\{\frac{1}{\log(1/\varepsilon)}\int_\varepsilon^1 \frac{X(r)}{r^2}\,\frac{dr}{r}\right\} \leqslant \frac{C}{\log(1/\varepsilon)},$$

*and, in the case $d = 2$,*

$$\mathrm{Var}\left\{\frac{1}{\log\log(1/\varepsilon)}\int_{\varepsilon}^{1/\mathrm{e}}\frac{X(r)}{r^2\log(1/r)}\,\frac{dr}{r\log(1/r)}\right\}\leqslant\frac{C}{\log\log(1/\varepsilon)}.$$

*Proof.* – Recall that $C$ denotes a constant whose value may be different in each occurence. In the case $d \geqslant 3$ we can bound the variance by

$$\frac{2C}{(\log(1/\varepsilon))^2}\int_{\varepsilon}^{1}\int_{\varepsilon}^{\varrho}\left(\frac{r}{\varrho}\right)^{\alpha}\frac{dr}{r}\,\frac{d\varrho}{\varrho}=\frac{C}{(\log(1/\varepsilon))^2}\int_{\varepsilon}^{1}\frac{1}{\varrho^{\alpha}}\,\frac{\varrho^{\alpha}}{\alpha}\,\frac{d\varrho}{\varrho}\leqslant\frac{C}{\log(1/\varepsilon)}.$$

In the case $d = 2$ we substitute $s = (\log(1/r))^{-1}$, $t = (\log(1/\varrho))^{-1}$ and let $\delta = (\log(1/\varepsilon))^{-1}$. Then we can bound the variance by

$$\begin{aligned}&\frac{C}{(\log\log(1/\varepsilon))^2}\int_{\varepsilon}^{1/\mathrm{e}}\int_{\varepsilon}^{\varrho}\sqrt{\frac{\log(1/\varrho)}{\log(1/r)}}\,\frac{dr}{r\log(1/r)}\,\frac{d\varrho}{\varrho\log(1/\varrho)}\\&\quad=\frac{C}{(\log(1/\delta))^2}\int_{\delta}^{1}\int_{\delta}^{t}\sqrt{\frac{s}{t}}\,\frac{ds}{s}\,\frac{dt}{t},\end{aligned}$$

which again is bounded by $C/\log(1/\delta)$, as in the first part. □

By a straightforward Borel–Cantelli argument we infer from the previous lemma that, if $d \geqslant 3$ and choosing $\varepsilon_n = \exp(-n^2)$,

$$\lim_{n\to\infty}\frac{1}{\log(1/\varepsilon_n)}\int_{\varepsilon_n}^{1}\frac{X_1(B(x,r))}{r^2}\,\frac{dr}{r}=\lim_{r\to 0}\mathbb{E}\left\{\frac{X(r)}{r^2}\right\}=D(d,4)\quad P\text{-almost surely.}$$

Using the monotonicity of the integrals and the fact that $\log(1/\varepsilon_n)/\log(1/\varepsilon_{n+1}) \to 1$ we infer that the sequence $\{\varepsilon_n\}$ is sufficiently rich to ensure the convergence along every sequence. In the case $d = 2$ we define $\varepsilon_n = \exp(-\exp(n^2))$ to obtain

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{\log\log(1/\varepsilon_n)}\int_{\varepsilon_n}^{1/\mathrm{e}}\frac{X_1(B(x,r))}{r^2\log(1/r)}\,\frac{dr}{r\log(1/r)}\\&\quad=\lim_{r\to 0}\mathbb{E}\left\{\frac{X(r)}{r^2\log(1/r)}\right\}=2\quad P\text{-almost surely,}\end{aligned}$$

and again the sequence $\varepsilon_n$ is rich enough to ensure full convergence. This proves Theorems 1.1 and 1.2 in the case $\gamma = 4$ and the general case follows by recalling (3).

Let us briefly sketch an argument leading to nonexistence of the average densities of order two of $\{Z_t\}$ in dimension 2. Suppose they exist on a set of positive measure with positive probability. Then, arguing with a zero-one law as Le Gall and Perkins in [11, Section 7], they exist $Z_t$-almost everywhere, almost surely. By the consistency of the averaging procedure, the average densities of order two must be equal to the average densities of order three, and in particular they must be constant. One can check that the family

$$\left\{\left(\frac{1}{\log(1/\varepsilon)}\int_{\varepsilon}^{1}\frac{X(r)}{r^2\log(1/r)}\frac{dr}{r}-2\right)^2\right\}$$

is uniformly integrable and hence the expectation of this family has to tend to zero. It may be shown, using calculations analogous to those in Section 3 above, that this is not the case, and one arrives at a contradiction.

## 5. Average densities and long time behaviour

In this section we restrict attention to the case $d \geqslant 3$. We recall some well-known facts about the long-term behaviour of super-Brownian motion (see Dawson and Perkins [3]) and point out the connection to the average densities. In our current setting, if the super-Brownian motion is started with a finite mass, almost surely, the process $\{Z_t\}$ suffers extinction in finite time, i.e. there is a finite random time $T$ such that, almost surely, $Z_t = 0$ for all $t \geqslant T$. It is however possible to obtain a nontrivial longtime behaviour if we extend our process to a process on the space

$$\mathcal{M}_p(\mathbb{R}^d) = \left\{\mu \in \mathcal{M}(\mathbb{R}^d)\colon \int \varphi_p \, d\mu < \infty\right\},$$

for $\varphi_p(x) = (1 + \|x\|^2)^{-p}$, equipped with the $p$-vague topology, generated by the functionals $\mu \mapsto \int \varphi \, d\mu$ for all $\varphi : \mathbb{R}^d \to [0, \infty)$ satisfying $\sup |\varphi(x)/\varphi_p(x)| < \infty$. Such an extension is possible and allows the definition of the process $\{Z_t\}$ started in $Z_0 = \ell^d$, the Lebesgue measure, if $p \in (d/2, d/2 + 1)$. Then it is easy to see that

$$\lim_{t\to\infty} Z_t = Z_\infty \quad \text{weakly in } \mathcal{M}_p(\mathbb{R}^d),$$

for some random variable $Z_\infty$ on the space $\mathcal{M}_p(\mathbb{R}^d)$, see [3, 6.1]. $Z_\infty$ is called the *equilibrium random measure* of the super-Brownian motion. By the superprocess property we have $\mathbb{E}Z_\infty = \ell^d$.

It is not hard to see (using the branching property) that $Z_\infty$ is an infinitely divisible random measure and hence we can associate a canonical measure $R_\infty$ with $Z_\infty$. $R_\infty$ is a $\sigma$-finite and translation invariant measure on $\mathcal{M}(\mathbb{R}^d)$. The Palm distributions $R_\infty^x$ associated with $R_\infty$ are given by the formula

$$\int \exp\left[-\int \varphi \, d\mu\right] R_\infty^x(d\mu) = \mathbb{P}_x\left\{\mathrm{e}^{-\gamma \int_0^\infty U_s^\gamma \varphi(w_s)\, ds}\right\}, \tag{24}$$

where $\mathbb{P}_x$ is the distribution of Brownian motion $W$ started in $x$, see [3, 6.1] for a proof. Note that, due to the translation invariance of $R_\infty$ the Palm distributions $R_\infty^x$ at $x$ are given as translates of the Palm distribution $R_\infty^0$ at 0.

THEOREM 5.1. – *Suppose that $X^0$ is distributed according to the Palm distribution $R^0_\infty$ at 0, which is associated with the canonical measure of the equilibrium random measure $Z_\infty$.*

(i) *The finite dimensional distributions of the process $\{X(rs)/(rs)^2\}_{s>0}$ converge, as $r\downarrow 0$, to the finite-dimensional distributions of the process $\{X^0(B(0,s))/s^2\}_{s>0}$.*

(ii) *The Palm distribution $R^0_\infty$ is invariant under the scaling flow $\{S_\lambda\}_{\lambda\in\mathbb{R}}$, which is defined by $S_\lambda\mu(A)=\mu(\mathrm{e}^{-\lambda}A)/\mathrm{e}^{-2\lambda}$. This implies that there exists a random variable $D^0$ such that*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\log(1/\varepsilon)}\int_\varepsilon^1\frac{X^0(B(0,r))}{r^2}\,\frac{dr}{r}=D^0,\qquad R^0_\infty\text{-almost surely}.$$

(iii) *The random variable $D^0$ is almost surely constant and we can describe the average density of Theorem 1.1 as $D(d,\gamma)=D^0=\mathbb{E}\{X^0(B(0,1))\}$.*

*Proof.* – Plugging $\gamma U_t^\gamma(\phi)=U_t^1(\gamma\phi)$ into (24) yields that all the involved quantities have the same scaling behaviour with respect to $\gamma$ and hence we do not lose generality by assuming $\gamma=4$. The distribution of $\{X(r)/r^2\}$ on the space $(\Omega,\mathcal{A},P)$ equipped with the Campbell measure $P$ associated with the canonical cluster is equal, for every $y\in\mathbb{R}^d$, to the distribution of $\{X_1[W](B(x,r))/r^2\}$ under $X_1[W](dx)\,N_y(dW)$. We pick positive numbers $\lambda_1,\dots,\lambda_k$ and $r_1,\dots,r_k$ and argue with the Laplace transform of the $k$-dimensional marginals. We obtain, using (6) and the moment formula for Poisson processes,

$$\begin{aligned}
&\int\exp\left[-\sum_{i=1}^k\lambda_i\frac{\mu(B(x,r_i))}{r_i^2}\right]P(d\mu\,dx)\\
&\quad=\int\exp\left[-\sum_{i=1}^k\lambda_i\frac{X_1(B(x,r_i))}{r_i^2}\right]X_1[W](dx)\,N_y(dW)\\
&\quad=\int P^{(w)}\left\{\exp\left[-\sum_{i=1}^k\frac{\lambda_i}{r_i^2}\iint X_{1-t}[W]\big(B\big(w(1),r_i\big)\big)\mathcal{M}_2(dt,dW)\right]\right\}\mathbb{P}_y(dw)\\
&\quad=\int\exp\left[4\int_0^1dt\int N_{w(t)}(dW)\left(\exp\left[-\sum_{i=1}^k\frac{\lambda_i}{r_i^2}\right.\right.\right.\\
&\qquad\left.\left.\left.\times X_{1-t}[W]\big(B\big(w(1),r_i\big)\big)\right]-1\right)\right]\mathbb{P}_y(dw).
\end{aligned}$$

Using (5) and (1) we can infer that

$$\begin{aligned}
&\int N_{w(t)}(dW)\left(\exp\left[-\sum_{i=1}^k\frac{\lambda_i}{r_i^2}X_{1-t}[W]\big(B\big(w(1),r_i\big)\big)\right]-1\right)\\
&\quad=\log\mathbb{Q}_{w(t)}\left\{\exp\left[-\sum_{i=1}^k\frac{\lambda_i}{r_i^2}Z_{1-t}\big(B\big(w(1),r_i\big)\big)\right]\right\}\\
&\quad=-U_{1-t}\left(\sum_{i=1}^k\frac{\lambda_i}{r_i^2}\mathbb{1}_{B(w(1),r_i)}\right)\big(w(t)\big).
\end{aligned}$$

Altogether, we obtain

$$\int \exp\left[-\sum_{i=1}^{k}\lambda_i\frac{\mu(B(x,r_i))}{r_i^2}\right]P(dx,d\mu) = \int \exp\left[-4\int_0^1 dt\, U_{1-t}\left(\sum_{i=1}^{k}\frac{\lambda_i}{r_i^2}1_{B(w(1),r_i)}\right)\big(w(t)\big)\right]\mathbb{P}_y(dw).$$

Choose $g:\mathbb{R}^d\to[0,\infty)$ such that $\int g(x)\,dx=1$. As the above expression is independent of $y$, we may take an average over $y$ and then use the reversibility of Brownian motion. This yields

$$\begin{aligned}
&\int \exp\left[-\sum_{i=1}^{k}\lambda_i\frac{\mu(B(x,r_i))}{r_i^2}\right]P(dx,d\mu)\\
&= \iint g(y)\exp\left[-4\int_0^1 U_{1-t}\left(\sum_{i=1}^{k}\frac{\lambda_i}{r_i^2}1_{B(w(1),r_i)}\right)\big(w(t)\big)\,dt\right]\mathbb{P}_y(dw)\,dy\\
&= \iint g\big(w(1)\big)\exp\left[-4\int_0^1 U_{1-t}\left(\sum_{i=1}^{k}\frac{\lambda_i}{r_i^2}1_{B(z,r_i)}\right)\big(w(1-t)\big)dt\right]\mathbb{P}_z(dw)\,dz\\
&= \int \exp\left[-4\int_0^1 U_{1-t}\left(\sum_{i=1}^{k}\frac{\lambda_i}{r_i^2}1_{B(0,r_i)}\right)\big(w(1-t)\big)\,dt\right]\mathbb{P}_0(dw).
\end{aligned}$$

We now write $r_i=rs_i$ and use the scaling property of Eq. (2) in the form (see [2, Lemma 4.5.1])

$$U_t\left(\sum_{i=1}^{k}\frac{\lambda_i}{(rs_i)^2}1_{B(0,rs_i)}\right)(y)=\frac{1}{r^2}U_{t/r^2}\left(\sum_{i=1}^{k}\lambda_i 1_{B(0,s_i)}\right)(y/r), \tag{25}$$

and afterwards Brownian scaling to obtain

$$\begin{aligned}
&\int \exp\left[-4\int_0^1 U_t\left(\sum_{i=1}^{k}\frac{\lambda_i}{r_i^2}1_{B(0,r_i)}\right)\big(w(t)\big)\,dt\right]\mathbb{P}_0(dw)\\
&= \int \exp\left[-4\int_0^1 \frac{1}{r^2}U_{t/r^2}\left(\sum_{i=1}^{k}\frac{\lambda_i}{s_i^2}1_{B(0,s_i)}\right)\big(w(t)/r\big)\,dt\right]\mathbb{P}_0(dw)\\
&= \int \exp\left[-4\int_0^{1/r^2} U_s\left(\sum_{i=1}^{k}\frac{\lambda_i}{s_i^2}1_{B(0,s_i)}\right)\big(w(s)\big)\,ds\right]\mathbb{P}_0(dw)\\
&\xrightarrow{r\downarrow 0} \int \exp\left[-4\int_0^{\infty} U_s\left(\sum_{i=1}^{k}\frac{\lambda_i}{s_i^2}1_{B(0,s_i)}\right)\big(w(s)\big)\,ds\right]\mathbb{P}_0(dw)\\
&= \mathbb{E}\left\{\exp\left[-\sum_{i=1}^{k}\lambda_i\frac{X^0(B(0,s_i))}{s_i^2}\right]\right\},
\end{aligned}$$

where we have used (24) in the last step. This proves the first statement of Theorem 5.1. From (24), the scaling property of Brownian motion and the scaling property (25) we get, for every $\varphi:\mathbb{R}^d \to [0,\infty)$ and $r>0$,

$$\begin{aligned}\int \exp\Big[-\int \varphi\,d\mu\Big] R^0_\infty(d\mu) &= \mathbb{P}_0\Big\{\exp\Big[-4\int_0^\infty U_s(\varphi)(w_s)\,ds\Big]\Big\}\\ &= \mathbb{P}_0\Big\{\exp\Big[-4\int_0^\infty U_s(\varphi)(w_{r^2 s}/r)\,ds\Big]\Big\}\\ &= \mathbb{P}_0\Big\{\exp\Big[-4\int_0^\infty \frac{1}{r^2}U_{t/r^2}(\varphi)(w_t/r)\,dt\Big]\Big\}\\ &= \mathbb{P}_0\Big\{\exp\Big[-4\int_0^\infty U_t\Big(\frac{\varphi(\cdot/r)}{r^2}\Big)(w_t)\,dt\Big]\Big\}\\ &= \int \exp\Big[\frac{-\int\varphi(x/r)\,\mu(dx)}{r^2}\Big] R^0_\infty(d\mu).\end{aligned}$$

In other words, the Palm distribution $R^0_\infty$ on the space $\mathcal{M}(\mathbb{R}^d)$ is invariant under the scaling flow. This allows the use of Birkhoff's Ergodic Theorem, which yields the existence of a random variable $D^0$ with $\mathbb{E}D^0 = \mathbb{E}\{X^0(B(0,1))\}$ such that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{X^0(B(0,\mathrm{e}^{-t}))}{\mathrm{e}^{-2t}}\,dt = D^0, \qquad R^0_\infty\text{-almost surely and in } L^1,$$

and (ii) follows by a change of variable. Finally, to establish (iii), we first show that the family $\{X(r)/r^2\colon 0<r<1\}$ of random variables is exponentially bounded. Recall from Lemma 3.1 that, for all $t>0$,

$$G(r,t) := \int_0^t \int_{\|x\|\leqslant r} p(x,s)\,dx\,ds \leqslant \frac{r^2}{d-2}.$$

Le Gall and Perkins [11, Lemma 3.1] given an easy argument that, for all $0<\theta<\frac{r^2}{2G(r,t)}$,

$$\int N_x(dW)\Big(\exp\Big(\frac{\theta X_t[W](B(y,r))}{r^2}\Big)-1\Big) \leqslant \frac{\theta\int_{\|z-y\|\leqslant r} p(z-x,t)\,dz}{r^2-2\theta G(r,t)}.$$

Hence we infer, for all $0<\theta<(d-2)/2$,

$$\begin{aligned}\mathbb{E}\Big\{\exp\Big(\theta\frac{X(r)}{r^2}\Big)\Big\} &= \int \exp\Big(4\int_0^1 dt\int N_{w(t)}(dW)\\ &\quad\times\Big(\exp\Big(\theta\frac{X_{1-t}[W](B(w(1),r)}{r^2}\Big)-1\Big)\Big)\mathbb{P}_y(dw)\\ &\leqslant \exp\Big(\frac{4\theta}{d-2-2\theta}\Big).\end{aligned}$$

This implies that $\{X(r)/r^2\colon 0<r<1\}$ and, by the first part, also $\{X^0(B(0,r))/r^2\colon 0<r<1\}$ are $L^q$-bounded for all $1\leqslant q<\infty$. The argument of (ii) with a discrete time version of Birkhoff's ergodic theorem yields that there exists a random variable $\overline{D}^{\,0}$ with

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{X^0(B(0,\mathrm{e}^{-k}))}{\mathrm{e}^{-2k}}=\overline{D}^{\,0},\qquad R^0_\infty\text{-almost surely and in } L^1.$$

Part (i), uniform intergability and, in the penultimate step, the main lemma now give, for a suitable constant $C>0$,

$$\begin{aligned}\mathrm{Var}\{\overline{D}^{\,0}\}&=\lim_{n\to\infty}\mathrm{Var}\left\{\frac{1}{n}\sum_{k=1}^{n}\frac{X^0(B(0,\mathrm{e}^{-k}))}{\mathrm{e}^{-2k}}\right\}\\&=\lim_{n\to\infty}\lim_{r\downarrow 0}\mathrm{Var}\left\{\frac{1}{n}\sum_{k=1}^{n}\frac{X(r\mathrm{e}^{-k})}{r^2\mathrm{e}^{-2k}}\right\}\\&\leqslant C\limsup_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sum_{l=k}^{n}\mathrm{e}^{k-l}=0.\end{aligned}$$

Hence, $\overline{D}^{\,0}$ is constant $R^0_\infty$-almost surely, and equals

$$\overline{D}^{\,0}=D^0=\mathbb{E}\{X^0(B(0,1))\}=\lim_{r\downarrow 0}\frac{\mathbb{E}\{X(r)\}}{r^2}=D(d,4),$$

as required to finish the proof. □

*Remarks*. – Note that the proof of the first two parts of Theorem 5.1 do not make use the calculation of the variances in the main section. It looks as if Theorem 1.1 can be obtained from these or similar arguments, but I have not been able to achieve this.

## 6. Further comments and open questions

- An interesting line of generalization one might want to follow is the replacement of the Brownian motion as underlying particle movement by a general diffusion. In the case of a scalar diffusion coefficient $\sigma:\mathbb{R}^d\to(0,\infty)$ it would be interesting to see whether the values of the average densities of the critically branching measure-valued diffusion $\{Z_t\}$ with underlying particle motion $dX_t=\sigma(X_t)\,dB_t$ at a fixed time $t$ allow a reconstruction of the scalar field $\sigma$ at $X_t$-almost every point. In a similar vein one could make the branching rate space-dependent and ask whether it can be recovered by means of the average densities.
- It would also be interesting to give finer descriptions of the fluctuations of the function $r\mapsto Z_1(B(x,r))$, for example the lacunarity distributions studied for the case of planar Brownian motions in [17].

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