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SMALL PERTURBATION OF DIFFUSIONS IN INHOMOGENEOUS MEDIA ☆

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ABSTRACT. – For the system of d-dim stochastic differential equations,

$$\begin{aligned} \mathrm{d}X^\varepsilon(t) &= b(X^\varepsilon(t))\,\mathrm{d}t + \varepsilon\sigma(X^\varepsilon(t))\,\mathrm{d}W(t), \quad t \in [0,1], \\ X^\varepsilon(0) &= x^0 \in R^d, \end{aligned}$$

where $b(x)$ and $\sigma(x)$ are smooth except possibly along the hyperplane $\{(x_1,\ldots,x_d); x_1 = 0\}$, we shall demonstrate that the natural setup of its large deviation principle is to consider the probability $\varepsilon^2 \log P(\|X^\varepsilon - \varphi\| < \delta,\ \|u^\varepsilon - \psi\| < \delta,\ \|\ell^\varepsilon - \eta\| < \delta) \sim -I(\varphi,\psi,\eta)$ of the triplet $(X^\varepsilon, u^\varepsilon, \ell^\varepsilon)$ simultaneously. Here, u^ε is the occupation time of $X_1^\varepsilon(\cdot)$ in the positive half line and $\ell^\varepsilon(\cdot)$ is the local time of $X_1^\varepsilon(\cdot)$ at 0. The explicit form of the rate function $I(\cdot,\cdot,\cdot)$ is obtained. The usual Wentzell–Friedlin theory concerns only probabilities of the form $\varepsilon^2 \log P(\|X^\varepsilon - \varphi\| < \delta)$ and its limit is a consequence of the contraction principle of our result. © 2002 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Pour l'équation differentielle stochastique sur R^d

$$\begin{aligned} \mathrm{d}X^\varepsilon(t) &= b(X^\varepsilon(t))\,\mathrm{d}t + \varepsilon\sigma(X^\varepsilon(t))\,\mathrm{d}W(t), \quad t \in [0,1], \\ X^\varepsilon(0) &= x^0 \in R^d, \end{aligned}$$

où $b(x)$ et $\sigma(x)$ sont lisses à l'exception de l'hyperplan $\{(x_1,\ldots,x_d); x_1 = 0\}$, nous démontrons que la forme natuelle du prinicipe des grande deviations s'eait

$$\varepsilon^2 \log P(\|X^\varepsilon - \varphi\| < \delta,\ \|u^\varepsilon - \psi\| < \delta,\ \|\ell^\varepsilon - \eta\| < \delta) \sim -I(\varphi,\psi,\eta).$$

Ici, u^ε est le temps possé par $X_1^\varepsilon(\cdot)$ sur la demidraite positive et $\ell^\varepsilon(\cdot)$ est le temps local de $X_1^\varepsilon(\cdot)$ en 0. Nous obtenons $I(\phi,\psi,\eta)$ explicitement. La théorie de Wentzell–Freidlin étudie seulement les probabilités de la forme

$$\varepsilon^2 \log P(\|X^\varepsilon - \varphi\| < \delta) \sim -I(\phi).$$

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La limité procède alors de notre resultat par le principé de contraction.

1. Introduction

For the stochastic differential equation in R^d,

$$\begin{aligned} &\mathrm{d}X^\varepsilon(t) = b\big(X^\varepsilon(t)\big)\,\mathrm{d}t + \varepsilon\sigma\big(X^\varepsilon(t)\big)\,\mathrm{d}W(t), \quad t \in [0,1], \\ &X^\varepsilon(0) = x^0, \end{aligned} \tag{1.1}$$

we shall be concerned with its large deviation principle (l.d.p) for $\varepsilon \downarrow 0$. Here, we assume the following conditions on (1.1) throughout the paper:

(1.2)

- (i) $b(\cdot)$ and $a(\cdot) = \sigma(\cdot)\sigma^*(\cdot)$ are smooth with possibly jump discontinuities along the hyperplane $H^0 = \{x = (x_1, \ldots, x_d), x_1 = 0\}$.
- (ii) There are positive constants c_1 and c_2 such that $c_1 I \leqslant a(x) \leqslant c_2 I$ for all $x \in R^d$ where I is the identity matrix.

More precisely, we assume that there are $b^+(\cdot), b^-(\cdot), a^+(\cdot)$ and $a^-(\cdot)$ with bounded derivatives such that

$$b(x) = \begin{cases} b^+(x) & \text{if } x_1 > 0, \\ b^-(x) & \text{if } x_1 \leqslant 0, \end{cases} \qquad a(x) = \begin{cases} a^+(x) & \text{if } x_1 > 0, \\ a^-(x) & \text{if } x_1 \leqslant 0. \end{cases}$$

For notational convenience, we shall write a vector x in R^d or a function f with value in R^d as $x = (x_1, \bar{x})$ or $f = (f_1, \bar{f})$ to distinguish the first components from the rest.

We shall briefly recall the definition of l.d.p and refer the details to [9,14] or [22]. A family of probability measures $\{P^\varepsilon\}_{\varepsilon>0}$ on a complete separable metric space $(\mathcal{X}, d)$ is said to satisfy the l.d.p with rate function $I(\cdot)$ if the following (i)–(iv) are satisfied:

(1.3)

- (i) $I : \mathcal{X} \to [0, \infty]$ is lower semicontinuous;
- (ii) $K_r := \{x \in \mathcal{X}, I(x) \leqslant r\}$ is precompact;
- (iii) For any M, there exists a compact set $K \subseteq \mathcal{X}$ such that for any $\delta > 0$ and ε small
$$P^\varepsilon\big(B_\delta(K)^c\big) \leqslant \exp\left(-\frac{M}{\varepsilon^2}\right);$$
- (iv) $$\lim_{\delta\to 0}\liminf_{\varepsilon\to 0} \varepsilon^2 \log P^\varepsilon\big(B_\delta(x)\big) = -I(x) = \lim_{\delta\to 0}\limsup_{\varepsilon\to 0} \varepsilon^2 \log P^\varepsilon\big(B_\delta(x)\big).$$

Here, $B_\delta(x)$ is the δ-neighborhood of x and K^c the complement of K in $\mathcal{X}$. We give a few words for the definition of l.d.p of this form. The definitions given in [14] and [22] are different but are equivalent. See Theorem 3.3, Chapter 3 in [14]. The result, Theorem 3.5, Chapter 3 [14], implies the properties in our definition. On the other hand, by using (iv), it is easy to prove that the following holds if A is a compact set,

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log P^\varepsilon(A) \leqslant -\inf\{I(x); x \in A\}.$$

Then, by using (iii), we can prove that this also holds if A is a closed set. Therefore, we can verify the properties in the definition of l.d.p in [22].

If $b(\cdot)$ and $\sigma(\cdot)$ are smooth and $a(\cdot)$ is strictly positive throughout R^d, the l.d.p of the trajectories for (1.1) was known [3]:

$$P\big(\|X^\varepsilon - \phi\|_{[0,1]} < \delta\big) \sim \exp\left(-\frac{I(\phi)}{\varepsilon^2}\right) \tag{1.4}$$

for $\phi \in C[0,1]$, where

$$I(\phi) = \frac{1}{2}\int_0^1 \big|\sigma^{-1}\big(\phi(t)\big)\big(\dot\phi(t) - b\big(\phi(t)\big)\big)\big|^2\,\mathrm{d}t$$

for an absolutely continuous ϕ and $I(\phi) = \infty$ otherwise. Here and hence after, $\|\cdot\|_{[a,b]}$ will denote the supnorm for a function defined on $[a,b]$. We sometimes omit the index $[a,b]$ when no ambiguity arises. The problem becomes more involved when discontinuities are allowed in (1.1). First, the strong solution of (1.1) may not exist and the usual Picard's approximation method thus fails. Secondly, the rate function $I(\cdot)$ in (1.4) cannot be interpreted literally (this can be most easily seen by taking $b(x) = \operatorname{sgn} x$, $\sigma(x) = 1$ and $\phi \equiv 0$ in one-dimensional case) and hence there is no candidate for performing the change of measure technique in large deviation theory. The first generalization of (1.1) with jump discontinuities (1.2) allowed was proved in [16,17] for the special case that $\sigma \equiv I$ and (in addition to some smoothness conditions on $b^+(\cdot), b^-(\cdot)$) a stability condition holds on $b(\cdot)$:

$$\inf_{\bar x\in R^{d-1}} \big(b_1^-(0,\bar x) - b_1^+(0,\bar x)\big) > 0. \tag{1.5}$$

Under these conditions, (1.4) was proved to hold with the rate function expressed in a complicated variational form. For the special case $d = 1$ but without the stability condition (1.5), the l.d.p was proved in [7]. We note that it is natural to do the analysis for such processes by using occupation time of X^ε in the positive half space $H^+ = \{(x_1, \bar x) \in R^d,\ x_1 > 0\}$. See some studies in [6,10,11]. However, for $d > 1$, the following setup for the l.d.p of (1.1) is first proposed in [8] and is then proved for the case of $\sigma \equiv I$:

$$P\big(\|X^\varepsilon - \phi\| < \delta,\ \|u^\varepsilon - \psi\| < \delta\big) \sim \exp\left(-\frac{I(\phi,\psi)}{\varepsilon^2}\right) \tag{1.6}$$

where $u^\varepsilon(t)(= \int_0^t \chi_{(0,\infty)}(X_1^\varepsilon(s))\,\mathrm{d}s)$ and ψ are occupation times of X^ε and ϕ in the positive half space $H^+ = \{(x_1,\bar x) \in R^d, x_1 > 0\}$ respectively. In general, for a continuous function $f:[0,1]\to R^d$, an absolutely continuous real function g is called an occupation time of f in H^+ if, for almost all t, $\dot g(t)\in[0,1]$ if $f_1(t)=0$ and

$$\dot g(t)=\begin{cases}1 & \text{if } f_1(t)>0,\\ 0 & \text{if } f_1(t)<0.\end{cases}$$

In general, occupation times are not unique. It is unique if and only if $m\{t\colon\ f_1(t)=0\}=0$ where $m(\cdot)$ is the Lebesgue measure. We shall write $g\in H^+(f)$ if g is an occupation time of f in H^+. Along this line we shall consider the l.d.p under (1.2) allowing general $\sigma(\cdot)$ and jump discontinuities in both $b(\cdot)$ and $\sigma(\cdot)$ in this paper. The novelty to have a nonconstant $\sigma(\cdot)$ is that the natural form of the l.d.p for such a system is a level even higer than (1.6):

$$P\left(\|X^\varepsilon-\phi\|<\delta,\ \|u^\varepsilon-\psi\|<\delta,\ \|\ell^\varepsilon-\eta\|<\delta\right)\sim\exp\left(-\frac{I(\phi,\psi,\eta)}{\varepsilon^2}\right) \tag{1.7}$$

where ℓ^ε and η are local times of X_1^ε and ϕ_1 at 0 respectively. Here the local time $\ell(t)$ at 0 for a continuous semimartingale $m(t)$ is defined as the increasing process such that

$$|m(t)|=|m(0)|+\int_0^t\operatorname{sgn}\big(m(s)\big)\,\mathrm{d}m(s)+\ell(t).$$

See [23]. For a real-valued function f on $[0,1]$, a local time of f at 0 is an absolutely continuous function $\eta(\cdot)$ satisfying $\eta(0)=0$ and

$$\begin{cases}\dot\eta(t)=0 & \text{if } f(t)\neq 0,\\ \dot\eta(t)\in[0,\infty) & \text{if } f(t)=0.\end{cases} \tag{1.8}$$

We shall denote by $L^0(f)$ the set of all local times of f at 0.

In [6], they directly proved the l.d.p for the trajectories of X^ε by using a weak convergence approach which is quite different from ours. The rate functional described below is suggested by their work. See also [8]. Their arguments also apply to the cases of nonconstant coefficient, but the existence of strong solution for the dynamics was required.

We now formalize our result as follows.

If $x_1>0,\ \rho=1$ and $\theta=0$ (or $x_1<0,\ \rho=0$ and $\theta=0$), then

$$L(x,p,\rho,\theta)=L(x;p)=\frac12\left|\sigma(x)^{-1}\big(p-b(x)\big)\right|^2.$$

For $x,p\in R^d$ such that $x_1=0$, $p_1=0$ and $0<\rho<1,\theta\geqslant 0$, let

$$\begin{aligned}&L(x,p,\rho,\theta)\\&\quad=\inf\Big\{\frac12\rho\left|\sigma^+(x)^{-1}\big(q^+-b^+(x)\big)\right|^2+\frac12(1-\rho)\left|\sigma^-(x)^{-1}\big(q^--b^-(x)\big)\right|^2;\\&\qquad q^+,q^-\in R^d \text{ satisfying } \rho q^++(1-\rho)q^-=p\\&\qquad \text{and } q_1^+=-\frac{\theta}{2\rho},\ q_1^-=\frac{\theta}{2(1-\rho)}\Big\};\end{aligned} \tag{1.9}$$

If $\rho = 0$ or 1, $\theta = 0$, then

$$L(x, p, \rho, \theta) = \begin{cases} \frac{1}{2}\left|\sigma^+(x)^{-1}\left(p - b^+(x)\right)\right|^2, & \rho = 1, \\ \frac{1}{2}\left|\sigma^-(x)^{-1}\left(p - b^-(x)\right)\right|^2, & \rho = 0. \end{cases}$$

For an absolutely continuous function $\phi \in C[0,1]$, $\psi \in H^+(\phi)$ and $\eta \in L^0(\phi)$, let

$$I(\phi, \psi, \eta) = \int_0^1 L\left(\phi(t), \dot\phi(t), \dot\psi(t), \dot\eta(t)\right) \mathrm{d}t. \tag{1.10}$$

For other triplets, $I(\phi, \psi, \eta) = \infty$.

Finally, let $AC^+[0,1]$ denote the set of all absolutely continuous functions on $[0,1]$ starting from 0 with derivative in $[0,1]$ and $C_0^+[0,1]$ be the set of real, continuous non-decreasing functions starting from 0.

The main result of the paper is the following:

THEOREM A. – *Let P^ε be the distribution of $(X^\varepsilon, u^\varepsilon, \ell^\varepsilon)$ induced by the solution of* (1.1) *on $\mathbb{C} = C[0,1] \times AC_0^+[0,1] \times C_0^+[0,1]$ where u^ε is the occupation time of $X^\varepsilon(\cdot)$ on H^+ and ℓ^ε is the local time of $X_1^\varepsilon(\cdot)$ at 0. Then $\{P^\varepsilon\}_{\varepsilon>0}$ satisfies the l.d.p with rate function* (1.10).

We shall denote $C[0,1] \times AC_0^+[0,1] \times C_0^+[0,1]$ by $\mathbb{C}$ in the sequel.

Remark. – Using the supnorm as the metric, the space $\mathbb{C}$ is a complete separable metric space. It is also easy to verify that $I(\cdot,\cdot,\cdot)$ has precompact level sets. We shall show the lower semi-continuity of $I(\cdot,\cdot,\cdot)$ in Appendix A. Our main work is to prove (1.3)(iv) which is given in Sections 2–4 under different conditions.

As a consequence of the contraction principle, we have the following result.

THEOREM B. – *The distribution generated by X^ε on $C[0,1]$ satisfies the l.d.p with rate function $I(\cdot)$ given by the following. For $\phi(t)$ absolute continuous, R^d valued function defined on $[0,1]$ with $\phi(0) = x^0$,*

$$I(\phi) = \int_0^1 L\left(\phi(t), \dot\phi(t)\right) \mathrm{d}t,$$

and $I(\phi) = \infty$ otherwise. Here if $x_1 \neq 0$,

$$L(x, p) = \frac{1}{2}\left|\sigma(x)^{-1}\left(p - b(x)\right)\right|^2;$$

if $x_1 = 0$ and $p_1 = 0$, then

$$L(x, p) = \inf\left\{\frac{1}{2}\rho\left|\sigma^+(x)^{-1}\left(q^+ - b^+(x)\right)\right|^2 + \frac{1}{2}(1-\rho)\left|\sigma^-(x)^{-1}\left(q^- - b^-(x)\right)\right|^2\right\}$$

the inf *is taken over all* $0 \leqslant \rho \leqslant 1$ *and* $q^+, q^- \in R^d$ *satisfying*

$$\rho q^+ + (1-\rho)q^- = p, \qquad q_1^+ < 0, \qquad q_1^- > 0.$$

2. Brownian motion: $b = 0, \sigma = I$

In this section, we shall establish the l.d.p for the 1-dim Brownian motion in the form (1.7). In this case, (1.9) can be calculated explicitly. We have

$$L(0,0,\rho,\theta) = \begin{cases} \dfrac{1}{8}\dfrac{\theta^2}{\rho(1-\rho)}, & 0 < \rho < 1 \text{ and } \theta \geqslant 0, \\ 0, & \rho = 0 \text{ or } 1 \text{ and } \theta = 0, \\ \infty, & \rho = 0 \text{ or } 1 \text{ and } \theta > 0. \end{cases}$$

For $x \neq 0$, $L(x,p,0,0) = \frac{1}{2}|p|^2$.

First, we define a notion of log-equivalence to simplify the notation.

DEFINITION 2.1. – *For two real positive functions* $f(\varepsilon,\delta)$ *and* $g(\varepsilon,\delta)$, *we say that* $f(\varepsilon,\delta) \stackrel{\log}{=} g(\varepsilon,\delta)$ *if*

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^2 \log f(\varepsilon,\delta) = \lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon^2 \log g(\varepsilon,\delta),$$
$$\lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2 \log f(\varepsilon,\delta) = \lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon^2 \log g(\varepsilon,\delta).$$

Obviously, if $\{P_1^\varepsilon\}$ and $\{P_2^\varepsilon\}$ are two family of probability measures such that $P_1^\varepsilon(B_\delta(x)) \stackrel{\log}{=} P_2^\varepsilon(B_\delta(x))$ for any x, then $\{P_1^\varepsilon\}$ satisfies (1.3)(iv) with rate $I(\cdot)$ if and only if $\{P_2^\varepsilon\}$ satisfies (1.3)(iv) with the same $I(\cdot)$.

We recall a lemma from [8] (see Lemma 3.3 in [8]).

LEMMA 2.2. – *Let* f *be a real-valued function on* R^d *with* $f(x) = f^+(x)$ *if* $x_1 > 0$ *and* $f(x) = f^-(x)$ *if* $x_1 \leqslant 0$ *where* f^+ *and* f^- *are bounded and continuous. Then the function* $(\phi,\psi) \to \int_0^1 f_{\phi,\psi}(t)\,\mathrm{d}t$ *is continuous. Here,* $f_{\phi,\psi}(t) := f^+(\phi(t))\dot\psi(t) + f^-(\phi(t))(1-\dot\psi(t))$.

The main result in this section is the following.

THEOREM 2.3. – *Let* P^ε *be the distribution of* $(\varepsilon W, u^\varepsilon, \ell^\varepsilon)$ *on* $\mathbb{C}$ *where* W *is the standard* 1-dim *Brownian motion,* u^ε *is the occupation time and* ℓ^ε *is the local time at* 0 *of* εW. *Then* P^ε *satisfies the l.d.p with rate function* $I(\phi,\psi,\eta)$, *where*

$$I(\phi,\psi,\eta) = \frac{1}{2}\int_0^1 \dot\phi^2(t)\,\mathrm{d}t + \frac{1}{8}\int_0^1 \frac{\dot\eta^2(t)}{\dot\psi(t)(1-\dot\psi(t))}\,\mathrm{d}t$$

if $\phi(t)$ *is absolutely continuous,* $\psi \in H^+(\phi)$ *and* $\eta \in L^0(\phi)$; *Otherwise,* $I(\phi,\psi,\eta) = \infty$.

Proof. – We shall only prove the case that

$$\phi \equiv 0, \qquad \psi(t) = \rho t, \qquad \eta(t) = \theta t$$

for some constants $0<\rho<1$ and $\theta>0$. We calculate the asymptotics of

$$P\big\{\|X^\varepsilon\|\leqslant\delta,\ \|u^\varepsilon-\psi\|\leqslant\delta,\ \|\ell^\varepsilon-\eta\|\leqslant\delta\big\}.$$

Here $X^\varepsilon(t)=\varepsilon W(t)$. The general cases can be treated through successive conditioning. Since this idea is used in several places of the paper, we shall give some detail of it in Appendix A.

We consider a diffusion $\tilde X^\varepsilon$ satisfying

$$\mathrm{d}\tilde X^\varepsilon(t)=v\big(\tilde X^\varepsilon(t)\big)\,\mathrm{d}t+\varepsilon\,\mathrm{d}\tilde W(t).$$

Let $\tilde u^\varepsilon$ be the occupation time of $\tilde X^\varepsilon$ in H^+ and $\tilde\ell^\varepsilon$ the local time of $\tilde X^\varepsilon$ at 0. Here $\tilde W$ is a standard 1-dim Brownian motion and $v(\cdot)$ is given by the following,

$$v(x)=-\frac{\theta}{2\rho},\quad \text{if } x_1>0\quad\text{and}\quad v(x)=\frac{\theta}{2(1-\rho)},\quad \text{if } x_1<0.$$

By Cameron–Martin–Girsanov Theorem,

$$\begin{aligned}&P\big\{\|X^\varepsilon\|\leqslant\delta,\ \|u^\varepsilon-\psi\|\leqslant\delta,\ \|\ell^\varepsilon-\eta\|\leqslant\delta\big\}\\&\quad=E\Bigg\{\exp\Bigg(-\frac{1}{\varepsilon}\int_0^1 v\big(\tilde X^\varepsilon(t)\big)\,\mathrm{d}\tilde W(t)-\frac{1}{2\varepsilon^2}\int_0^1\big|v\big(\tilde X^\varepsilon(t)\big)\big|^2\,\mathrm{d}t\Bigg);\\&\qquad\qquad\|\tilde X^\varepsilon\|\leqslant\delta,\ \|\tilde u^\varepsilon-\psi\|\leqslant\delta,\ \|\tilde\ell^\varepsilon-\eta\|\leqslant\delta\Bigg\}.\end{aligned}\tag{2.1}$$

For $(\tilde X^\varepsilon,\tilde u^\varepsilon,\tilde\ell^\varepsilon)$ in $B_\delta(\phi,\psi,\eta)$ we have the following estimates: from

$$\int_0^1\big|v\big(\tilde X^\varepsilon(t)\big)\big|^2\,\mathrm{d}t=\left(\frac{\theta}{2\rho}\right)^2\tilde u^\varepsilon(1)+\left(\frac{\theta}{2(1-\rho)}\right)^2\big(1-\tilde u^\varepsilon(1)\big),$$

this can be approximated by

$$\left(\frac{\theta}{2\rho}\right)^2\rho+\left(\frac{\theta}{2(1-\rho)}\right)^2(1-\rho)=\frac{\theta^2}{4\rho(1-\rho)}$$

with an error bounded by $c\delta$ for some $c>0$; Also, by Tanaka's formula,

$$\begin{aligned}f\big(\tilde X^\varepsilon(1)\big)&=f\big(\tilde X^\varepsilon(0)\big)+\int_0^1\big|v\big(\tilde X^\varepsilon(t)\big)\big|^2\,\mathrm{d}t+\varepsilon\int_0^1 v\big(\tilde X^\varepsilon(t)\big)\,\mathrm{d}\tilde W(t)\\&\quad+\frac12\int_0^1\left(-\frac{\theta}{2\rho}-\frac{\theta}{2(1-\rho)}\right)\mathrm{d}\tilde\ell^\varepsilon(t),\end{aligned}$$

where $f(x)=\int_0^x v(u)\,\mathrm{d}u$, then

$$\int_0^1 v\big(\tilde{X}^\varepsilon(t)\big)\,\mathrm{d}\tilde{W}(t) = \frac{1}{\varepsilon}\Bigg(f\big(\tilde{X}^\varepsilon(1)\big) - f\big(\tilde{X}^\varepsilon(0)\big) - \int_0^1 \big|v\big(\tilde{X}^\varepsilon(t)\big)\big|^2\,\mathrm{d}t + \frac{1}{2}\int_0^1 \left(\frac{\theta}{2\rho} + \frac{\theta}{2(1-\rho)}\right)\mathrm{d}\tilde{\ell}^\varepsilon(t)\Bigg),$$

which, after some calculation, is approximately equal to zero with an error bounded by $c\delta/\varepsilon$. Therefore,

$$P\big\{\|X^\varepsilon\| \leqslant \delta,\ \|u^\varepsilon - \psi\| \leqslant \delta,\ \|\ell^\varepsilon - \eta\| \leqslant \delta\big\} \stackrel{\log}{=} \exp\left(-\frac{\theta^2}{8\varepsilon^2\rho(1-\rho)}\right) P\big\{\|\tilde{X}^\varepsilon\| \leqslant \delta,\ \|\tilde{u}^\varepsilon - \psi\| \leqslant \delta,\ \|\tilde{\ell}^\varepsilon - \eta\| \leqslant \delta\big\}. \tag{2.2}$$

In the following, we estimate the probability on the right hand side by using ergodic theorem.

We define

$$X_0(t) = \frac{1}{\varepsilon^2}\tilde{X}^\varepsilon\big(\varepsilon^2 t\big), \qquad W_0(t) = \frac{1}{\varepsilon}\tilde{W}\big(\varepsilon^2 t\big).$$

Then $W_0(t)$ is a 1-dim Brownian motion and

$$\mathrm{d}X_0(t) = v\big(X_0(t)\big)\,\mathrm{d}t + \mathrm{d}W_0(t).$$

It is easy to see that $X_0(t)$ is ergodic and has the invariant density

$$p(x) = \begin{cases} \theta\exp\left(-\dfrac{\theta}{\rho}x\right), & x > 0,\\[2mm] \theta\exp\left(-\dfrac{\theta}{1-\rho}x\right), & x < 0. \end{cases}$$

By ergodic theorem, for each $t > 0$,

$$\tilde{u}^\varepsilon(t) = \varepsilon^2 \int_0^{t/\varepsilon^2} \chi_{(0,\infty)}\big(X_0(s)\big)\,\mathrm{d}s \to \rho t \tag{2.3}$$

in probability as $\varepsilon \to 0$, since $\int_0^\infty p(x)\,\mathrm{d}x = \rho$.

By Tanaka's formula,

$$\big|X_0(t)\big| = \big|X_0(0)\big| + \int_0^t \mathrm{sign}\big(X_0(s)\big)v\big(X_0(s)\big)\,\mathrm{d}s + \int_0^t \mathrm{sign}\big(X_0(s)\big)\,\mathrm{d}W_0(s) + \ell_0(t),$$

with $\ell_0(t) = \frac{1}{\varepsilon^2}\tilde{\ell}^\varepsilon(\varepsilon^2 t)$. Therefore,

$$\tilde{\ell}^\varepsilon(t) = \varepsilon^2\left(\left|X_0\left(\frac{t}{\varepsilon^2}\right)\right| - \big|X_0(0)\big|\right) - \varepsilon^2\bar{W}_0\left(\frac{t}{\varepsilon^2}\right) - \varepsilon^2\int_0^{t/\varepsilon^2} \mathrm{sign}\big(X_0(s)\big)v\big(X_0(s)\big)\,\mathrm{d}s,$$

where $\bar{W}_0(t) = \int_0^t \operatorname{sign}(X_0(s))\,\mathrm{d}W_0(s)$ is a Brownian motion. But, for each $t > 0$,

$$\varepsilon^2 \int_0^{t/\varepsilon^2} \operatorname{sign}\big(X_0(s)\big) v\big(X_0(s)\big)\,\mathrm{d}s \to t \int \operatorname{sign}(x) v(x) p(x)\,\mathrm{d}x = t\theta = \eta(t)$$

in probability as $\varepsilon \to 0$ by ergodic theorem. From this, it is not difficult to prove that

$$P\big\{\big\|\tilde{\ell}^\varepsilon - \eta\big\| \leqslant \delta\big\} \to 1 \tag{2.4}$$

as $\varepsilon \to 0$.

Also, by $\tilde{X}^\varepsilon(t) = \varepsilon^2 X_0(\frac{t}{\varepsilon^2})$ and the fact that $X_0(t)$ is ergodic, we can show that

$$P\big\{\big\|\tilde{X}^\varepsilon\big\| \leqslant \delta\big\} \to 1 \tag{2.5}$$

as $\varepsilon \to 0$. The proof of the theorem is complete by combining (2.1)–(2.5).

3. The case $a_{11}(x) = 1$

Let $a \in S_d^+$ be a symmetric positive definite $d \times d$ matrix on R^d such that $c_1 I \leqslant a \leqslant c_2 I$ for some $c_1, c_2 > 0$. We shall define a particular σ satisfying $\sigma\sigma^* = a$. Here are some notations.

Let "*" denote the transpose of a matrix. For a d-dim column vector v such that $v^* = (v_1, \ldots, v_d)$, we shall denote $\bar{v}$ the $(d-1)$-dim column vector and $\bar{v}^* = (v_2, \ldots, v_d)$. For a general $d \times d$ matrix $m = (m_{ij})_{i,j=1,\ldots,d}$, we shall adopt the following notation: $\bar{m}$ is the $(d-1) \times (d-1)$ submatrix of m omitting the first row and column from m i.e., $\bar{m} = (\bar{m}_{ij})_{i,j=1,\ldots,d-1}$ and

$$\bar{m}_{i,j} = m_{i+1,j+1}, \quad i, j = 1, \ldots, d-1;$$

m_1 is the first column of m,

$$m_1 = \begin{pmatrix} m_{11} \\ \vdots \\ m_{d1} \end{pmatrix} = \begin{pmatrix} m_{11} \\ \bar{m}_1 \end{pmatrix}.$$

The first row and column of σ are defined as follows.

$$\sigma_{11} = \sqrt{a_{11}}, \quad \sigma_{i1} = \frac{a_{i1}}{\sqrt{a_{11}}} \quad \text{and} \quad \sigma_{1i} = 0 \quad \text{for } i = 2, \ldots, d.$$

Denote

$$a = \begin{pmatrix} a_{11} & \bar{a}_1^* \\ \bar{a}_1 & \bar{a} \end{pmatrix}, \qquad \sigma = \begin{pmatrix} \sqrt{a_{11}} & 0 \\ \frac{\bar{a}_1}{\sqrt{a_{11}}} & \bar{\sigma} \end{pmatrix}.$$

Then

$$\sigma\sigma^* = \begin{pmatrix} \sqrt{a_{11}} & 0 \\ \frac{\bar a_1}{\sqrt{a_{11}}} & \bar\sigma \end{pmatrix} \begin{pmatrix} \sqrt{a_{11}} & \frac{\bar a_1^*}{\sqrt{a_{11}}} \\ 0 & \bar\sigma^* \end{pmatrix} = \begin{pmatrix} a_{11} & \bar a_1^* \\ \bar a_1 & \frac{\bar a_1 \bar a_1^*}{a_{11}} + \bar\sigma\bar\sigma^* \end{pmatrix}.$$

In order to define $\bar\sigma$ so that $\sigma\sigma^* = a$, we need the following lemma.

LEMMA 3.1. – $c_1 I \leqslant \bar a - \frac{\bar a_1 \bar a_1^*}{a_{11}} \leqslant c_2 I$.

Proof. – Since $\bar a_1 \bar a_1^* \geqslant 0$ and $\bar a \leqslant c_2 I$, it is obvious that the upper estimate holds. For the lower bound, let $\bar y = (y_2, \ldots, y_d) \in R^{d-1}$. Let $y_1 = -\sum_{j=2}^d a_{1j} \frac{y_j}{a_{11}}$. Then

$$\begin{aligned} c_1 \sum_{i=1}^d y_i^2 &\leqslant \sum_{i,j=1}^d a_{ij} y_i y_j \\ &= \sum_{i,j=2}^d a_{ij} y_i y_j + a_{11} y_1^2 + 2 \sum_{i=2}^d a_{1i} y_i y_1 \\ &= \sum_{i,j=2}^d a_{ij} y_i y_j + \left(\sum_{j=2}^d a_{1j} y_j \right)^2 \Big/ a_{11} - 2 \left(\sum_{i=2}^d a_{1i} y_i \right)^2 \Big/ a_{11} \\ &= \sum_{i,j=2}^d a_{ij} y_i y_j - \left(\sum_{j=2}^d a_{1j} y_j \right)^2 \Big/ a_{11}. \end{aligned}$$

This implies $c_1 I \leqslant \bar a - \frac{\bar a_1 \bar a_1^*}{a_{11}}$. The proof is complete. □

Now, let $\bar\sigma$ be the square root of $\bar a - \frac{\bar a_1 \bar a_1^*}{a_{11}}$. Then σ satisfies $\sigma\sigma^* = a$.

Let $\sigma^+(x), \sigma^-(x)$ be chosen as above for $a^+(x), a^-(x)$ respectively and $\sigma(x) = \sigma^+(x)$ if $x_1 > 0$ and $\sigma(x) = \sigma^-(x)$ if $x_1 \leqslant 0$. Then $\sigma(x)$ has bounded derivatives except along hyperplane $\{(x_1, \ldots, x_d); x_1 = 0\}$. Now SDE (1.1) has the following form:

$$\begin{aligned} \mathrm{d}X_1^\varepsilon(t) &= b_1\big(X^\varepsilon(t)\big)\,\mathrm{d}t + \varepsilon\sigma_{11}\big(X^\varepsilon(t)\big)\,\mathrm{d}W_1(t), \\ \mathrm{d}X_i^\varepsilon(t) &= b_i\big(X^\varepsilon(t)\big)\,\mathrm{d}t + \varepsilon\sigma_{i1}\big(X^\varepsilon(t)\big)\,\mathrm{d}W_1(t) + \varepsilon \sum_{j=2}^d \sigma_{ij}\big(X^\varepsilon(t)\big)\,\mathrm{d}W_j(t), \\ i &= 2, \ldots, d. \end{aligned}$$

or

$$\begin{aligned} \mathrm{d}X_1^\varepsilon(t) &= b_1\big(X^\varepsilon(t)\big)\,\mathrm{d}t + \varepsilon\sigma_{11}\big(X^\varepsilon(t)\big)\,\mathrm{d}W_1(t), \\ \mathrm{d}\bar X^\varepsilon(t) &= \bar b\big(X^\varepsilon(t)\big)\,\mathrm{d}t + \varepsilon \frac{\bar a_1(X^\varepsilon(t))}{\sqrt{a_{11}(X^\varepsilon(t))}}\,\mathrm{d}W_1(t) + \varepsilon\bar\sigma\big(X^\varepsilon(t)\big)\,\mathrm{d}\bar W(t), \end{aligned} \tag{3.1}$$

where $W = \binom{W_1}{\bar W}$. For technical reasons, we shall assume $\sigma_{11}(x) = 1$ for each $x \in R^d$ in the rest of this section. This assumption will be romoved later.

Let $X^\varepsilon(t)$ be the solution to (3.1) and $u^\varepsilon(t)$ and $\ell^\varepsilon(t)$ be its occupation time in H^+ and local time at 0 respectively. In other words, $u^\varepsilon(t) = \int_0^t \chi_{(0,\infty)}(X_1^\varepsilon(s))\,\mathrm{d}s$ and $\ell^\varepsilon(t)$ is the local time of $X_1^\varepsilon(\cdot)$ at 0 up to time t. Let $\mathbb{C}$ be defined as in Section 1. The main result in this section is the following theorem.

THEOREM 3.2. – *Assume that* $\sigma_{11}(x) = 1$ *and* P^ε *is the distribution of* $(X^\varepsilon, u^\varepsilon, \ell^\varepsilon)$ *defined above. Then* $\{P^\varepsilon\}$ *satisfies the l.d.p with the rate function given in* (1.10).

For a triplet $(\phi, \psi, \eta) \in \mathbb{C}$, we shall consider $P^\varepsilon(B_\delta(\phi, \psi, \eta))$ where

$$B_\delta(\phi, \psi, \eta) = \{(X, u, \ell): \|X - \phi\| < \delta,\ \|u - \psi\| < \delta,\ \|\ell - \eta\| < \delta\}.$$

We first use Cameron–Martin–Girsanov theorem to change $X_1^\varepsilon(t)$ to a Brownian motion

$$P^\varepsilon\big(B_\delta(\phi, \psi, \eta)\big) = \int\limits_{B_\delta(\phi,\psi,\eta)} \exp\Bigg(\int_0^1 c\big(X(t)\big) \cdot \mathrm{d}X(t) - \int_0^1 c\big(X(t)\big) \cdot \hat{b}\big(X(t)\big)\,\mathrm{d}t \\ - \frac{\varepsilon^2}{2}\int_0^1 c\big(X(t)\big) \cdot a\big(X(t)\big)c\big(X(t)\big)\,\mathrm{d}t\Bigg)\,\mathrm{d}Q^\varepsilon, \tag{3.2}$$

where $\mathrm{d}X(t) = \hat{b}(X(t))\,\mathrm{d}t + \varepsilon\sigma(X(t))\,\mathrm{d}W(t)$ under Q^ε and $b = \hat{b} + \varepsilon^2 ac$. Here we choose $c = (\frac{b_1}{\varepsilon^2}, 0, \ldots, 0)^*$. Then

$$\hat{b} = b - \varepsilon^2 ac = b - b_1\, a_1 = \begin{pmatrix} 0 \\ \bar{b} - b_1 \bar{a}_1 \end{pmatrix}.$$

Hence, under Q^ε,

$$\begin{cases} \mathrm{d}X_1(t) = \varepsilon\,\mathrm{d}W_1(t), \\ \mathrm{d}\bar{X}(t) = \bar{\hat{b}}(X(t))\,\mathrm{d}t + \varepsilon\bar{\sigma}_1(X(t))\,\mathrm{d}W_1(t) + \varepsilon\bar{\sigma}(X(t))\,\mathrm{d}\bar{W}(t). \end{cases}$$

Let $u(t), \ell(t)$ be the occupation time in H^+ and the local time at 0. They are uniquely determined by $X(\cdot)$. In the following, we shall write $X \in B_\delta(\phi, \psi, \eta)$ instead of $(X, u, \ell) \in B_\delta(\phi, \psi, \eta)$ if no confusion occurs.

Hence

$$P^\varepsilon\big(X \in B_\delta(\phi, \psi, \eta)\big) \\ = \int\limits_{B_\delta(\phi,\psi,\eta)} \exp\Bigg(\frac{1}{\varepsilon^2}\int_0^1 b_1\big(X(t)\big)\,\mathrm{d}X_1(t) - \frac{1}{2\varepsilon^2}\int_0^1 b_1^2\big(X(t)\big)\,\mathrm{d}t\Bigg)\,\mathrm{d}Q^\varepsilon. \tag{3.3}$$

By using Lemma 2.2, it is not difficult to deduce the following:

$$P^\varepsilon\big(B_\delta(\phi, \psi, \eta)\big) \\ \overset{\log}{=} \exp\Bigg(-\frac{1}{2\varepsilon^2}\int_0^1 b_{1,\phi,\psi}^2(t)\,\mathrm{d}t\Bigg) \int\limits_{B_\delta(\phi,\psi,\eta)} \exp\Bigg(\frac{1}{\varepsilon^2}\int_0^1 b_1\big(X(t)\big)\,\mathrm{d}X_1(t)\Bigg)\,\mathrm{d}Q^\varepsilon. \tag{3.4}$$

We next apply an Ito's formula to estimate the stochastic integral on the right hand side of (3.4). We want to remark that the main difficulty of a direct estimate is caused by the discontinuity of $b_1(\cdot)$. We will overcome this difficulty by expressing $\int_0^1 b_1(X(t))\,\mathrm{d}X_1(t)$ as a combination of a stochastic integral with respect to $X_1(t)$ and the local time of $X_1(t)$ through Ito's formula, but now the new stochastic integral has a smooth integrand.

Let $F(x) = F(x_1, \bar{x}) = \int_0^{x_1} b_1(u, \bar{x})\,\mathrm{d}u$. Then by Ito's formula,

$$\begin{aligned} F\big(X(t)\big) &= F\big(X(0)\big) + \int_0^t \nabla F\big(X(s)\big)\cdot \mathrm{d}X(s) + \frac{\varepsilon^2}{2}\int_0^t \sum_{i,j} a_{ij}\big(X(s)\big)\frac{\partial^2 F}{\partial x_i \partial x_j}\big(X(s)\big)\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_0^t \big(b_1^+\big(X(s)\big) - b_1^-\big(X(s)\big)\big)\,\mathrm{d}\ell(s) \\ &= F\big(X(0)\big) + \int_0^t b_1\big(X(s)\big)\,\mathrm{d}X_1(s) + \int_0^t \bar{\nabla} F\big(X(s)\big)\cdot \mathrm{d}\bar{X}(s) \\ &\quad + \frac{\varepsilon^2}{2}\int_0^t \sum_{i,j} a_{ij}\big(X(s)\big)\frac{\partial^2 F}{\partial x_i \partial x_j}\big(X(s)\big)\,\mathrm{d}s + \frac{1}{2}\int_0^t (b_1^+ - b_1^-)\big(X(s)\big)\,\mathrm{d}\ell(s), \end{aligned}$$

where $\bar{\nabla} F = (\frac{\partial F}{\partial x_2}, \ldots, \frac{\partial F}{\partial x_n}) \in R^{d-1}$. Hence

$$\begin{aligned} &\frac{1}{\varepsilon^2}\int_0^1 b_1\big(X(s)\big)\,\mathrm{d}X_1(s) \\ &\quad = \frac{1}{\varepsilon^2}\left(F\big(X(1)\big) - F\big(X(0)\big) - \int_0^1 \bar{\nabla} F\big(X(s)\big)\cdot \mathrm{d}\bar{X}(s)\right) \\ &\qquad - \frac{1}{2}\int_0^1 \sum_{i,j} a_{ij}\big(X(s)\big)\frac{\partial^2 F}{\partial x_i \partial x_j}\big(X(s)\big)\,\mathrm{d}s - \frac{1}{2\varepsilon^2}\int_0^1 (b_1^+ - b_1^-)\big(X(s)\big)\,\mathrm{d}\ell(s). \end{aligned} \tag{3.5}$$

Since F is a bounded Lipschitz function, we therefore have, for any $\gamma > 0$, the existence of an δ_0 such that

$$\big|\big(F\big(X(1)\big) - F\big(X(0)\big)\big) - \big(F\big(\phi(1)\big) - F\big(\phi(0)\big)\big)\big| < \gamma \tag{3.6}$$

for any $X(\cdot) \in B_\delta(\phi)$ with $\delta < \delta_0$. Also, by the boundedness of $a_{ij}(x)$ and $\frac{\partial^2 F(x)}{\partial x_i \partial x_j}$ (for $x_1 \neq 0$), for any $\gamma > 0$ we have

$$\frac{\varepsilon^2}{2}\left|\int_0^1 \sum_{i,j} a_{ij}\big(X(t)\big)\frac{\partial^2 F}{\partial x_i \partial x_j}\big(X(t)\big)\,\mathrm{d}t\right| < \gamma \tag{3.7}$$

if ε is small.

The other two integrals on the right hand side of (3.5) are estimated in the following.

LEMMA 3.3. – *For any $\gamma > 0$, there exists an $\delta_0 > 0$ such that in $B_\delta(\phi, \psi, \eta)$,*

$$\left|\int_0^1 \big(b_1^+\big(X(t)\big) - b_1^-\big(X(t)\big)\big)\,\mathrm{d}\ell(t) - \int_0^1 \big(b_1^+\big(\phi(t)\big) - b_1^-\big(\phi(t)\big)\big)\dot{\eta}(t)\,\mathrm{d}t\right| < \gamma$$

for any $\delta < \delta_0$.

Proof. –

$$
\begin{aligned}
&\left|\int_0^1 \big(b_1^+\big(X(t)\big) - b_1^-\big(X(t)\big)\big)\,\mathrm{d}\ell(t) - \int_0^1 \big(b_1^+\big(\phi(t)\big) - b_1^-\big(\phi(t)\big)\big)\dot\eta(t)\,\mathrm{d}t\right| \\
&\quad\leqslant \int_0^1 \big|(b_1^+ - b_1^-)\big(X(t)\big) - (b_1^+ - b_1^-)\big(\phi(t)\big)\big|\,\mathrm{d}\ell(t) \\
&\qquad + \left|\int_0^1 (b_1^+ - b_1^-)\big(\phi(t)\big)\big(\mathrm{d}\ell(t) - \dot\eta(t)\,\mathrm{d}t\big)\right| \\
&\quad\leqslant 2K\|X-\phi\|\ell(1) + \left|\int_0^1 (b_1^+ - b_1^-)\big(\phi(t)\big)\,\mathrm{d}\ell(t)\right. \\
&\qquad \left.- \sum_{i=1}^N (b_1^+ - b_1^-)\big(\phi(t_i)\big)\big(\ell(t_{i+1}) - \ell(t_i)\big)\right| \\
&\qquad + \left|\int_0^1 (b_1^+ - b_1^-)\big(\phi(t)\big)\dot\eta(t)\,\mathrm{d}t - \sum_{i=1}^N (b_1^+ - b_1^-)\big(\phi(t_i)\big)\big(\eta(t_{i+1}) - \eta(t_i)\big)\right| \\
&\qquad + \sum_{i=1}^N \big|(b_1^+ - b_1^-)\big(\phi(t_i)\big)\big|\big|\big(\ell(t_{i+1}) - \ell(t_i)\big) - \big(\eta(t_{i+1}) - \eta(t_i)\big)\big|,
\end{aligned}
$$

where K is the Lipschitz constant of b^+ and b^- and $\{t_i\}_{i=1}^{N+1}$ is a partition of [0, 1]. Since ϕ is uniformly continuous on [0, 1],

$$
\left|\int_0^1 (b_1^+ - b_1^-)\big(\phi(t)\big)\,\mathrm{d}\ell(t) - \sum_{i=1}^N (b_1^+ - b_1^-)\big(\phi(t_i)\big)\big(\ell(t_{i+1}) - \ell(t_i)\big)\right| \leqslant \frac{\gamma}{4}
$$

if the partition $\{t_i\}_{i=1}^{N+1}$ is fine enough. Similarly, if $\{t_i\}_{i=1}^{N+1}$ is fine enough,

$$
\left|\int_0^1 (b_1^+ - b_1^-)\big(\phi(t)\big)\dot\eta(t)\,\mathrm{d}t - \sum_{i=1}^N (b_1^+ - b_1^-)\big(\phi(t_i)\big)\big(\eta(t_{i+1}) - \eta(t_i)\big)\right| \leqslant \frac{\gamma}{4}.
$$

For $X(\cdot) \in B_\delta(\phi, \psi, \eta)$, $|(\ell - \eta)(t)| < \delta$ for every t. Hence

$$
\sum_{i=1}^N \big|(b_1^+ - b_1^-)\big(\phi(t_i)\big)\big|\big|\ell(t_{i+1}) - \eta(t_{i+1}) - \ell(t_i) + \eta(t_i)\big| \leqslant 2MN\delta < \frac{\gamma}{4}
$$

if $\delta < \gamma/8MN$, where $M = \sup|b_1^+(x) - b_1^-(x)|$. The proof is complete by combining the above estimates.

LEMMA 3.4. – *For any $M>0$ and $\gamma>0$, there exists an δ_0 such that*

$$Q^\varepsilon\left(B_\delta(\phi,\psi,\eta),\ \sup_{0\leqslant t\leqslant 1}\left|\int_0^t \bar\nabla F\big(X(s)\big)\cdot \mathrm{d}\bar X(s)-\int_0^t \bar\nabla F\big(\phi(s)\big)\cdot \dot{\bar\phi}(s)\,\mathrm{d}s\right|>\gamma\right)$$
$$\leqslant \exp\left(-\frac{M}{\varepsilon^2}\right)$$

if $\delta<\delta_0$.

Proof. – Let $\pi=\{t_0=0,t_1,\dots,t_N=1\}$ be a partition of [0,1] and let define $\phi_\pi(t)=\phi(t_i)$ if $t_i\leqslant t<t_{i+1}$ for $i=0,1,\dots,N-1$. Then

$$Q^\varepsilon\left(B_\delta(\phi,\psi,\eta),\ \sup_{0\leqslant t\leqslant 1}\left|\int_0^t \bar\nabla F\big(X(s)\big)\cdot \mathrm{d}\bar X(s)-\int_0^t \bar\nabla F\big(\phi(s)\big)\cdot \dot{\bar\phi}(s)\,\mathrm{d}s\right|>\gamma\right)$$
$$\leqslant Q^\varepsilon\left(B_\delta(\phi,\psi,\eta),\ \sup_{0\leqslant t\leqslant 1}\left|\int_0^t \big(\bar\nabla F\big(X(s)\big)-\bar\nabla F\big(\phi(s)\big)\big)\cdot \mathrm{d}\bar X(s)\right|>\frac{\gamma}{4}\right)$$
$$+Q^\varepsilon\left(B_\delta(\phi,\psi,\eta),\ \sup_{0\leqslant t\leqslant 1}\left|\int_0^t \big(\bar\nabla F\big(\phi(s)\big)-\bar\nabla F\big(\phi_\pi(s)\big)\big)\cdot \mathrm{d}\bar X(s)\right|>\frac{\gamma}{4}\right)$$
$$+Q^\varepsilon\left(B_\delta(\phi,\psi,\eta),\ \sup_{0\leqslant t\leqslant 1}\left|\int_0^t \bar\nabla F\big(\phi_\pi(s)\big)\cdot\big(\mathrm{d}\bar X(s)-\dot{\bar\phi}(s)\,\mathrm{d}s\big)\right|>\frac{\gamma}{4}\right)$$
$$+Q^\varepsilon\left(B_\delta(\phi,\psi,\eta),\ \sup_{0\leqslant t\leqslant 1}\left|\int_0^t \big(\bar\nabla F\big(\phi_\pi(s)\big)-\bar\nabla F\big(\phi(s)\big)\big)\cdot \dot{\bar\phi}(s)\,\mathrm{d}s\right|>\frac{\gamma}{4}\right)$$
$$=I_1+I_2+I_3+I_4.$$

We have $I_1\leqslant I_{11}+I_{12}$ where

$$I_{11}=Q^\varepsilon\left(B_\delta(\phi,\psi,\eta),\ \sup_{0\leqslant t\leqslant 1}\left(\int_0^t \big(\bar\nabla F\big(X(s)\big)-\bar\nabla F\big(\phi(s)\big)\big)\cdot \mathrm{d}\bar X(s)\right)>\frac{\gamma}{4}\right)$$

and

$$I_{12}=Q^\varepsilon\left(B_\delta(\phi,\psi,\eta),\ \sup_{0\leqslant t\leqslant 1}\left(-\int_0^t \big(\bar\nabla F\big(X(s)\big)-\bar\nabla F\big(\phi(s)\big)\big)\cdot \mathrm{d}\bar X(s)\right)>\frac{\gamma}{4}\right).$$

Let $G(t)=(0,\bar G(t))$, $\bar G(t)=\bar\nabla F(X(t))-\bar\nabla F(\phi(t))$. Since $\bar\nabla F$ is Lipschitz, we have $|G(t)|\leqslant K\delta$ for some constant K independent of δ if $X(\cdot)\in B_\delta(\phi,\psi,\eta)$. Hence for $\theta>0$,

$$
\begin{aligned}
I_{11} &= Q^{\varepsilon}\left(B_{\delta}(\phi,\psi,\eta),\ \sup_{0\leqslant t\leqslant 1}\int_0^t \bar{G}(s)\cdot \mathrm{d}\bar{X}(s) > \frac{\gamma}{4}\right)\\
&= Q^{\varepsilon}\left(B_{\delta}(\phi,\psi,\eta),\ \sup_{0\leqslant t\leqslant 1}\left(\int_0^t \theta\bar{G}(s)\cdot\big(\mathrm{d}\bar{X}(s)-\bar{\hat{b}}\big(X(s)\big)\,\mathrm{d}s\big)\right.\right.\\
&\quad - \frac{\theta^2\varepsilon^2}{2}\int_0^t G(s)\cdot a\big(X(s)\big)G(s)\,\mathrm{d}s + \theta\int_0^t \bar{G}(s)\cdot\bar{\hat{b}}\big(X(s)\big)\,\mathrm{d}s\\
&\quad \left.\left. + \frac{\theta^2\varepsilon^2}{2}\int_0^t G(s)\cdot a\big(X(s)\big)G(s)\,\mathrm{d}s\right) > \frac{\theta\gamma}{4}\right)\\
&\leqslant Q^{\varepsilon}\left(B_{\delta}(\phi,\psi,\eta), \exp\left(\sup_{0\leqslant t\leqslant 1}\left(\int_0^t \theta\bar{G}(s)\cdot\big(\mathrm{d}\bar{X}(s)-\bar{\hat{b}}\big(X(s)\big)\,\mathrm{d}s\big)\right.\right.\right.\\
&\quad \left.\left.\left. - \frac{\theta^2\varepsilon^2}{2}\int_0^t G(s)\cdot a\big(X(s)\big)G(s)\,\mathrm{d}s\right)\right) > \exp\left(\frac{\theta\gamma}{4} - \theta K'\delta - \frac{\theta^2\varepsilon^2}{2}K'\delta\right)\right)
\end{aligned}
$$

where K' is a constant satisfying

$$
\int_0^1 |G(s)|\big|\bar{\hat{b}}\big(X(s)\big)\big|\,\mathrm{d}s < K'\delta, \quad \text{and} \quad \int_0^1 G(s)\cdot a\big(X(s)\big)G(s)\,\mathrm{d}s < K'\delta
$$

for $X(\cdot)\in B_{\delta}(\phi,\psi,\eta)$. Thus

$$
\begin{aligned}
I_{11} &\leqslant Q^{\varepsilon}\left(\sup_{0\leqslant t\leqslant 1}\exp\left(\int_0^t \theta\bar{G}(s)\cdot\big(\mathrm{d}\bar{X}(s)-\bar{\hat{b}}\big(X(s)\big)\,\mathrm{d}s\big)\right.\right.\\
&\quad \left.\left. - \frac{\theta^2\varepsilon^2}{2}\int_0^t G(s)\cdot a\big(X(s)\big)G(s)\,\mathrm{d}s\right) > \exp\left(\frac{\theta\gamma}{4} - \theta K'\delta - \frac{\theta^2\varepsilon^2}{2}K'\delta\right)\right)\\
&\leqslant \exp\left(-\frac{\theta\gamma}{4} + \theta K'\delta + \frac{\theta^2\varepsilon^2}{2}K'\delta\right) E\left(\exp\left(\int_0^1 \theta\bar{G}(s)\cdot\big(\mathrm{d}\bar{X}(s)-\bar{\hat{b}}\big(X(s)\big)\,\mathrm{d}s\big)\right.\right.\\
&\quad \left.\left. - \frac{\theta^2\varepsilon^2}{2}\int_0^1 G(s)\cdot a\big(X(s)\big)G(s)\,\mathrm{d}s\right)\right)\\
&= \exp\left(-\frac{\theta\gamma}{4} + \theta K'\delta + \frac{\theta^2\varepsilon^2}{2}K'\delta\right) = \exp\left(-\frac{(\frac{\gamma}{4}-\delta K')^2}{2\varepsilon^2 K'\delta}\right)
\end{aligned}
$$

if $\theta = \frac{\frac{\gamma}{4}-\delta K'}{\varepsilon^2 K'\delta}$ and $\frac{\gamma}{4}-\delta K' > 0$. From the first relation to the second relation we use the martingale inequality. See [15, Theorem 6.2, Chapter 1]. Therefore, $I_{11}\leqslant \exp(-M/\varepsilon^2)$

if δ is small. We can treat I_{12}, I_2 similarly; I_3, I_4 can be proved to be 0 if δ is small enough by an argument as in the proof of Lemma 3.3. The proof is complete. □

In particular, Lemma 3.4 implies that for any $\gamma > 0$

$$Q^\varepsilon\big(B_\delta(\phi,\psi,\eta)\big) \stackrel{\log}{=} Q^\varepsilon\left(B_\delta(\phi,\psi,\eta), \left|\int_0^1 \bar\nabla F\big(X(s)\big)\cdot \mathrm{d}\bar X(s) - \int_0^t \bar\nabla F\big(\phi(s)\big)\cdot \dot{\bar\phi}(s)\,\mathrm{d}s\right| < \gamma\right).$$

Now, by (3.5), (3.6) and (3.7) and Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned}
&\int_{B_\delta(\phi,\psi,\eta)} \exp\left(\int_0^1 b_1\big(X(t)\big)\,\mathrm{d}X_1(t)\right)\mathrm{d}Q^\varepsilon \\
&= \int_{B_\delta(\phi,\psi,\eta)} \exp\left(\frac{1}{\varepsilon^2}\left(F\big(X(1)\big) - F\big(X(0)\big) - \int_0^1 \bar\nabla F\big(X(t)\big)\cdot \mathrm{d}\bar X(t)\right.\right. \\
&\quad \left.\left. - \frac{\varepsilon^2}{2}\int_0^1 \sum_{i,j} a_{ij}\big(X(t)\big)\frac{\partial^2 F}{\partial x_i\partial x_j}\big(X(t)\big)\,\mathrm{d}t - \frac{1}{2}\int_0^1 (b_1^+ - b_1^-)\big(X(t)\big)\,\mathrm{d}\ell(t)\right)\right)\mathrm{d}Q^\varepsilon \\
&\stackrel{\log}{=} \exp\left(\frac{1}{\varepsilon^2}\left(F\big(\phi(1)\big) - F\big(\phi(0)\big) - \frac{1}{2}\int_0^1 (b_1^+ - b_1^-)\big(\phi(t)\big)\dot\eta(t)\,\mathrm{d}t\right.\right. \\
&\quad \left.\left. - \int_0^1 \bar\nabla F\big(\phi(t)\big)\cdot \dot{\bar\phi}(t)\,\mathrm{d}t\right)\right) \int_{B_\delta(\phi,\psi,\eta)} \exp\left(\frac{1}{\varepsilon^2}\left(-\int_0^1 \bar\nabla F\big(X(t)\big)\cdot \mathrm{d}\bar X(t)\right.\right. \\
&\quad \left.\left. + \int_0^1 \bar\nabla F\big(\phi(t)\big)\cdot \dot{\bar\phi}(t)\,\mathrm{d}t\right)\right)\mathrm{d}Q^\varepsilon.
\end{aligned}$$

Let $\Delta = \int_0^1 \bar\nabla F(\phi(t))\cdot \dot{\bar\phi}(t)\,\mathrm{d}t - \int_0^1 \bar\nabla F(X(t))\cdot \mathrm{d}\bar X(t)$. Then

$$\begin{aligned}
&\int_{B_\delta(\phi,\psi,\eta)} \exp\left(\frac{1}{\varepsilon^2}\left(-\int_0^1 \bar\nabla F\big(X(t)\big)\cdot \mathrm{d}\bar X(t) + \int_0^1 \bar\nabla F\big(\phi(t)\big)\cdot \dot{\bar\phi}(t)\,\mathrm{d}t\right)\right)\mathrm{d}Q^\varepsilon \\
&\quad \leqslant \mathrm{e}^{\frac{\gamma}{\varepsilon^2}} Q^\varepsilon\big(B_\delta(\phi,\psi,\eta), |\Delta| < \gamma\big) + \int_{B_\delta(\phi,\psi,\eta), |\Delta|\geqslant\gamma} \exp\left(\frac{1}{\varepsilon^2}\Delta\right)\mathrm{d}Q^\varepsilon \\
&\quad \leqslant \mathrm{e}^{\frac{\gamma}{\varepsilon^2}} Q^\varepsilon\big(B_\delta(\phi,\psi,\eta), |\Delta| < \gamma\big) + \left(\int \exp\left(\frac{2\Delta}{\varepsilon^2}\right)\mathrm{d}Q^\varepsilon\right)^{\frac{1}{2}} \\
&\qquad \times \big(Q^\varepsilon\big(B_\delta(\phi,\psi,\eta), |\Delta| \geqslant \gamma\big)\big)^{\frac{1}{2}} \\
&\quad \leqslant \mathrm{e}^{\frac{\gamma}{\varepsilon^2}} Q^\varepsilon\big(B_\delta(\phi,\psi,\eta), |\Delta| < \gamma\big) + \mathrm{e}^{\frac{K}{\varepsilon^2}}\mathrm{e}^{-\frac{M}{2\varepsilon^2}},
\end{aligned}$$

where K is a constant depending on a, b, ϕ and M is as in Lemma 3.4. From these calculations, we have

$$\int_{B_\delta(\phi,\psi,\eta)} \exp\left(\frac{1}{\varepsilon^2}\left(-\int_0^1 \bar\nabla F\big(X(t)\big)\cdot \mathrm{d}\bar X(t) + \int_0^1 \bar\nabla F\big(\phi(t)\big)\cdot \dot{\bar\phi}(t)\,\mathrm{d}t\right)\right)\mathrm{d}Q^\varepsilon$$
$$\stackrel{\log}{=} Q^\varepsilon\big(B_\delta(\phi,\psi,\eta), |\Delta|<\gamma\big) \stackrel{\log}{=} Q^\varepsilon\big(B_\delta(\phi,\psi,\eta)\big).$$

These can be summarized in the following lemma.

LEMMA 3.5. – *Let* $\mathrm{d}X(t) = \hat b(X(t))\,\mathrm{d}t + \varepsilon\sigma(X(t))\,\mathrm{d}W(t)$ *under* Q^ε *and* $\hat b(x) = b(x) - b_1(x)a_1(x)$, *where* $b(x) = \binom{b_1(x)}{\bar b(x)}$ *and* $a_1(x)$ *is the first column of* $a(x)$. *Then*

$$P^\varepsilon\big(B_\delta(\phi,\psi,\eta)\big) \stackrel{\log}{=} \exp\left(-\frac{1}{2\varepsilon^2}\int_0^1 b^2_{1\phi,\psi}(t)\,\mathrm{d}t\right)\exp\left(\frac{1}{\varepsilon^2}\left(F\big(\phi(1)\big) - F\big(\phi(0)\big)\right.\right.$$
$$\left.\left. - \frac{1}{2}\int_0^1 (b_1^+ - b_1^-)\big(\phi(t)\big)\dot\eta(t)\,\mathrm{d}t - \int_0^1 \bar\nabla F\big(\phi(t)\big)\cdot\dot{\bar\phi}(t)\,\mathrm{d}t\right)\right)$$
$$\times\, Q^\varepsilon\big(B_\delta(\phi,\psi,\eta)\big).$$

It is clear from Lemma 3.5 that we now only need to obtain the l.d.p for the measures $\{Q^\varepsilon\}$. The advantage of $\{Q^\varepsilon\}$ over $\{P^\varepsilon\}$ is that under Q^ε, $\mathrm{d}X_1(t) = \varepsilon\,\mathrm{d}W_1(t)$ and can be solved independently of $\bar X(t)$. It is important to solve $X_1(t)$ first because the discontinuities of $b(\cdot)$ and $\sigma(\cdot)$ only occur at $X_1 = 0$.

Hence we concentrate on

$$Q^\varepsilon\big(B_\delta(\phi,\psi,\eta)\big) = Q^\varepsilon\big(\|X-\phi\|<\delta,\ \|u-\psi\|<\delta,\ \|\ell-\eta\|<\delta\big) \tag{3.8}$$

where under Q^ε, $X(t)$ satisfies the stochastic differential equation of the special form:

$$\begin{aligned}
&\mathrm{d}X_1(t) = \varepsilon\,\mathrm{d}W_1(t),\\
&\mathrm{d}\bar X(t) = \bar{\hat b}\big(X(t)\big)\,\mathrm{d}t + \varepsilon\bar\sigma_1\big(X(t)\big)\,\mathrm{d}W_1(t) + \varepsilon\bar\sigma\big(X(t)\big)\,\mathrm{d}\bar W(t),\\
&X(0) = x.
\end{aligned}\tag{3.9}$$

We assume that $x_1 = 0$ in the rest. Our strategy to study the l.d.p for Q^ε is the following. Define the following auxiliary processes,

$$V_1^+(t) = \int_0^t \chi_{(0,\infty)}\big(X_1(s)\big)\,\mathrm{d}W_1(s), \qquad V_1^-(t) = \int_0^t \chi_{(-\infty,0)}\big(X_1(s)\big)\,\mathrm{d}W_1(s),$$
$$\bar V^+(t) = \int_0^t \chi_{(0,\infty)}\big(X_1(s)\big)\,\mathrm{d}\bar W(s), \qquad \bar V^-(t) = \int_0^t \chi_{(-\infty,0)}\big(X_1(s)\big)\,\mathrm{d}\bar W(s),$$

and

$$V^+(t) = \big(V_1^+(t), \bar V^+(t)\big)^*, \qquad V^-(t) = \big(V_1^-(t), \bar V^-(t)\big)^*.$$

Then (3.9) can be rewritten as

$$
\begin{aligned}
\mathrm{d}X_1(t) &= \varepsilon\,\mathrm{d}W_1(t),\\
\mathrm{d}\bar{X}(t) &= \bar{\hat{b}}^{+}\big(X(t)\big)\,\mathrm{d}u(t) + \bar{\hat{b}}^{-}\big(X(t)\big)\big(\mathrm{d}t - \mathrm{d}u(t)\big) + \varepsilon\big(\bar{\sigma}_1^{+}\big(X_1(t), \bar{X}(t)\big)\,\mathrm{d}V_1^{+}(t)\\
&\quad + \bar{\sigma}_1^{-}\big(X_1(t), \bar{X}(t)\big)\,\mathrm{d}V_1^{-}(t) + \bar{\sigma}^{+}\big(X_1(t), \bar{X}(t)\big)\,\mathrm{d}\bar{V}^{+}(t)\\
&\quad + \bar{\sigma}^{-}\big(X_1(t), \bar{X}(t)\big)\,\mathrm{d}\bar{V}^{-}(t),\\
X(0) &= x.
\end{aligned}
\tag{3.10}
$$

Now the coefficients in (3.10) are smooth. Following the argument in [3, Chapter III], if we can show that the process $(X_1(t), u(t), \ell(t), \varepsilon V_1^+(t), \varepsilon V_1^-(t), \varepsilon\bar{V}^+(t), \varepsilon\bar{V}^-(t))$ under Q^ε satisfies l.d.p, then the process $(X(t), u(t), \ell(t))$ under Q^ε satisfies the l.d.p.

For $V_1^+(t)$ (and similarly for $V_1^-(t)$), we consider a function G^+ such that $G^+(y) = y$ if $y > 0$ and $G^+(y) = 0$ otherwise. Then by Tanaka's formula, we have

$$
\mathrm{d}G^+\big(X_1(t)\big) = \chi_{(0,\infty)}\big(X_1(t)\big)\,\mathrm{d}X_1(t) + \frac{1}{2}\,\mathrm{d}\ell(t),
$$

that is,

$$
\varepsilon V_1^+(t) = G^+\big(X_1(t)\big) - \frac{1}{2}\ell(t). \tag{3.11}
$$

Similarly,

$$
\varepsilon V_1^-(t) = G^-\big(X_1(t)\big) + \frac{1}{2}\ell(t), \tag{3.12}
$$

where $G^-(y) = y - G^+(y)$. Therefore, $V_1^+(\cdot)$ and $V_1^-(\cdot)$ can be considered as the image of $(X_1(\cdot), u(\cdot), \ell(\cdot))$ under the continuous mappings given in (3.11) and (3.12). We denote these mappings by $\mathcal{H}^+$ and $\mathcal{H}^-$. That is,

$$
\varepsilon V_1^+ = \mathcal{H}^+(X_1, u, \ell), \qquad \varepsilon V_1^- = \mathcal{H}^-(X_1, u, \ell). \tag{3.13}
$$

From this, it is enough to show that the process $(X_1(t), u(t), \ell(t), \varepsilon\bar{V}^+(t), \varepsilon\bar{V}^-(t))$ satisfies l.d.p. This is given in the following lemma. Before state it, we need some notations. Let denote

$$
I^{(0)}(\phi_1, \psi, \eta) = \frac{1}{2}\int_0^1 \big|\dot{\phi}_1(t)\big|^2\,\mathrm{d}t + \frac{1}{8}\int_0^1 \frac{\dot{\eta}(t)^2}{\dot{\psi}(t)(1 - \dot{\psi}(t))}\,\mathrm{d}t \tag{3.14}
$$

if $\psi \in H^+(\phi_1)$, $\eta \in L^{(0)}(\phi_1)$, and $I^{(0)}(\phi_1, \psi, \eta) = \infty$ otherwise. By Theorem 2.3,

$$
Q^\varepsilon\big((X_1, u, \ell) \in B_\delta(\phi_1, \psi, \eta)\big) \overset{\log}{=} \exp\left(-\frac{1}{\varepsilon^2} I^{(0)}(\phi_1, \psi, \eta)\right).
$$

Let $\bar{v}^+, \bar{v}^-$ be absolutely continuous, R^{d-1} valued functions on [0.1]. Define

$$
I_\psi(\bar{v}^+, \bar{v}^-) = \frac{1}{2}\int_0^1 \left(\frac{|\dot{\bar{v}}^+(t)|^2}{\dot{\psi}(t)} + \frac{|\dot{\bar{v}}^-(t)|^2}{1 - \dot{\psi}(t)}\right)\mathrm{d}t.
$$

LEMMA 3.6. – *Let $\bar v^+(\cdot), \bar v^-(\cdot)$ be R^{d-1} valued and absolutely continuous. Then*

$$Q^\varepsilon\big((X_1,u,\ell)\in B_\delta(\phi_1,\psi,\eta),\, (\varepsilon\bar V^+,\varepsilon\bar V^-)\in B_\delta(\bar v^+,\bar v^-)\big)$$
$$\overset{\log}{=}\exp\bigg(-\frac{1}{\varepsilon^2}\big(I^{(0)}(\phi_1,\psi,\eta)+I_\psi(\bar v^+,\bar v^-)\big)\bigg).$$

Proof. – Let $\bar W^+(t)$, $\bar W^-(t)$ be two independent $(d-1)$-dim Brownian motions which are independent with $W(t)$. Then $(X_1,u,\ell,\varepsilon\bar V^+,\varepsilon\bar V^-)$ and $(X_1,u,\ell,\varepsilon\hat V^+,\varepsilon\hat V^-)$ have the same distribution, where

$$\hat V^+(t)=\bar W^+\big(u(t)\big),\qquad \hat V^-(t)=\bar W^-\big(t-u(t)\big).$$

Let θ be a small positive number and define

$$\hat V^+_\theta(t)=\bar W^+\big(u(t)+\theta t\big),\qquad \hat V^-_\theta(t)=\bar W^-\big((1+\theta)t-u(t)\big).$$

Note that $(X_1,u,\ell)\in B_\delta(\phi_1,\psi,\eta)$ and

$$\big|\varepsilon\hat V^+_\theta(t)-\bar v^+(t)\big|<\delta\quad\text{for all } t\in[0,1]$$

imply

$$\big|\varepsilon\bar W^+(t)-\bar v^+\big(\alpha(t)\big)\big|<\delta\quad\text{for all } t\in\big[0,\psi(1)+\theta-\delta\big],$$

where $\alpha(\cdot)$ is the inverse of the mapping

$$t\to\psi(t)+\theta t.$$

Then by a routine argument and using Theorem 2.3, it is not difficult to show that

$$Q^\varepsilon\big((X_1,u,\ell)\in B_\delta(\phi_1,\psi,\eta),\, (\varepsilon\hat V^+_\theta,\varepsilon\hat V^-_\theta)\in B_\delta(\bar v^+,\bar v^-)\big)$$
$$\overset{\log}{=}\exp\bigg(-\frac{1}{\varepsilon^2}\big(I^{(0)}(\phi_1,\psi,\eta)+I^{(\theta)}_\psi(\bar v^+,\bar v^-)\big)\bigg),$$

where

$$I^{(\theta)}_\psi(\bar v^+,\bar v^-)=\frac12\int_0^1\bigg(\frac{|\dot{\bar v}^+(t)|^2}{\theta+\dot\psi(t)}+\frac{|\dot{\bar v}^-(t)|^2}{\theta+1-\dot\psi(t)}\bigg)\,\mathrm{d}t.$$

On the other hand, we can also show that for any $\gamma>0$, $M>0$, there are $\theta_0>0$, $\delta_0>0$ such that for $\theta<\theta_0$, $\delta<\delta_0$, we have

$$Q^\varepsilon\big((X_1,u,\ell)\in B_\delta(\phi_1,\psi,\eta),\, (\varepsilon\hat V^+,\varepsilon\hat V^-)\in B_\delta(\bar v^+,\bar v^-),\, (\varepsilon\hat V^+_\theta,\varepsilon\hat V^-_\theta)\notin B_\gamma(\bar v^+,\bar v^-)\big)$$
$$+\,Q^\varepsilon\big((X_1,u,\ell)\in B_\delta(\phi_1,\psi,\eta),\, (\varepsilon\hat V^+_\theta,\varepsilon\hat V^-_\theta)\in B_\delta(\bar v^+,\bar v^-),$$
$$(\varepsilon\hat V^+,\varepsilon\hat V^-)\notin B_\gamma(\bar v^+,\bar v^-)\big)\leqslant\exp\bigg(-\frac{M}{\varepsilon^2}\bigg).$$

The lemma follows from these two relations. This completes the proof. □

COROLLARY 3.7. – *Let $v^+(\cdot), v^-(\cdot)$ be R^d valued and absolutely continuous. Assume that*

$$v_1^+(t) = \mathcal{H}^+(\phi_1, \psi, \eta)(t), \qquad v_1^-(t) = \mathcal{H}^-(\phi_1, \psi, \eta)(t), \tag{3.15}$$

with the mappings $\mathcal{H}^\pm$ given in (3.13). *Then*

$$Q^\varepsilon\big((X_1, u, \ell) \in B_\delta(\phi_1, \psi, \eta), \big(\varepsilon V^+, \varepsilon V^-\big) \in B_\delta(v^+, v^-)\big) \stackrel{\log}{=} \exp\bigg(-\frac{1}{\varepsilon^2}\big(I^{(0)}(\phi_1, \psi, \eta) + I_\psi(\bar{v}^+, \bar{v}^-)\big)\bigg).$$

Given (ϕ, ψ, η), we shall consider $\bar{v}^+(\cdot), \bar{v}^-(\cdot)$ which are absolutely continuous and satisfy

$$\bar{\phi}(t) = \bar{\phi}(0) + \int_0^t \bar{B}(s)\,\mathrm{d}s + \int_0^t \bar{\sigma}^+\big(\phi(s)\big)\dot{\bar{v}}^+(s)\,\mathrm{d}s + \int_0^t \bar{\sigma}^-\big(\phi(s)\big)\dot{\bar{v}}^-(s)\,\mathrm{d}s. \tag{3.16}$$

Here

$$\bar{B}(t) = \bar{\hat{b}}^+\big(\phi(t)\big)\dot{\psi}(t) + \bar{\hat{b}}^-\big(\phi(t)\big)\big(1 - \dot{\psi}(t)\big) + \dot{\phi}_1(t)\big(\bar{\sigma}_1^+\big(\phi(t)\big)\chi_{(0,\infty)}\big(\phi_1(t)\big) + \bar{\sigma}_1^-\big(\phi(t)\big)\chi_{(-\infty,0]}\big(\phi_1(t)\big)\big) - \frac{1}{2}\dot{\eta}(t)\big(\bar{\sigma}_1^+\big(\phi(t)\big) - \bar{\sigma}_1^-\big(\phi(t)\big)\big).$$

Here we note that (3.16) is just the relation (3.10) if $(X_1, u, \ell) = (\phi_1, \psi, \eta)$, $(\varepsilon\bar{V}^+, \varepsilon\bar{V}^-) = (\bar{v}^+, \bar{v}^-)$ and $(\varepsilon V_1^+, \varepsilon V_1^-) = (v_1^+, v_1^-)$ given by (3.15). Define

$$J(\phi, \psi, \eta) = \inf_{(\bar{v}^+, \bar{v}^-)}\big\{I_\psi(\bar{v}^+, \bar{v}^-);\ (\bar{v}^+, \bar{v}^-) \text{ satisfies (3.16)}\big\}. \tag{3.17}$$

LEMMA 3.8. – *Let $I^{(0)}(\phi_1, \psi, \eta)$, $J(\phi, \psi, \eta)$ be given in* (3.14) *and* (3.17). *Then*

$$Q^\varepsilon\big((X, u, \ell) \in B_\delta(\phi, \psi, \eta)\big) \stackrel{\log}{=} \exp\bigg(-\frac{1}{\varepsilon^2}\big(I^{(0)}(\phi_1, \psi, \eta) + J(\phi, \psi, \eta)\big)\bigg).$$

Proof. – This follows from the argument in [3, p. 91, Theorem 2.13] by using Corollary 3.7. □

Proof of Theorem 3.2. – By Lemma 3.8, it remains to simplify

$$I(\phi, \psi, \eta) = I^{(0)}(\phi_1, \psi, \eta) + J(\phi, \psi, \eta).$$

Similar to (3.16), we consider (f^+, f^-) which satisfies

$$\bar{\phi}(t) = \bar{\phi}(0) + \int_0^t \bar{B}(s)\,\mathrm{d}s + \int_0^t \bar{\sigma}^+\big(\phi(s)\big)\sqrt{\dot{\psi}}\,\dot{f}^+(s)\,\mathrm{d}s + \int_0^t \bar{\sigma}^-\big(\phi(s)\big)\sqrt{1 - \dot{\psi}}\,\dot{f}^-(s)\,\mathrm{d}s.$$

Then

$$J(\phi,\psi,\eta)=\inf_{f^+,f^-}\left\{\frac{1}{2}\int_0^1\left(\left|\dot f^+(s)\right|^2+\left|\dot f^-(s)\right|^2\right)\mathrm{d}s\right\}$$
$$=\int_0^1\tilde L\big(\phi(s),\dot\phi(s),\dot\psi(s),\dot\eta(s)\big)\,\mathrm{d}s,$$

where the inf is taken over such pairs,

$$\tilde L(x,p,\rho,\theta)=\frac{1}{2}\left|\bar\sigma(x)^{-1}\big(\bar p-\big(\bar b(x)+\big(p_1-b_1(x)\big)\bar\sigma_1(x)\big)\big)\right|^2\quad\text{if } x_1\neq 0,$$

and

$$\tilde L(x,p,\rho,\theta)=\inf\left\{\frac{1}{2}\big(|\bar p^+|^2+|\bar p^-|^2\big)\right\}\tag{3.18}$$

if $x_1=0$, $p_1=0$, $\theta\geqslant 0$, $0\leqslant\rho\leqslant 1$, where the inf is taken over all pairs (p^+,p^-) satisfying

$$\begin{aligned}&\bar\sigma^+(x)\sqrt{\rho}\,\bar p^+ +\bar\sigma^-(x)\sqrt{1-\rho}\,\bar p^-\\&\quad=\bar p-\rho\big(\bar b^+(x)-b_1^+(x)\bar\sigma_1^+(x)\big)-(1-\rho)\big(\bar b^-(x)-b_1^-(x)\bar\sigma_1^-(x)\big)\\&\qquad+\frac{1}{2}\theta\big(\bar\sigma_1^+(x)-\bar\sigma_1^-(x)\big).\end{aligned}\tag{3.19}$$

Now we simplify (3.18). Given (x,p,ρ,θ), consider $\frac{1}{2}(|\bar p^+|^2+|\bar p^-|^2)$ with $(\bar p^+,\bar p^-)$ satisfying (3.19). Define (q^+,q^-) by the relations

$$\sqrt{\rho}\,\bar\sigma^+(x)\bar p^+=\rho\left(\bar q^+-\bar b^+(x)+b_1^+(x)\bar\sigma_1^+(x)+\frac{\theta}{2\rho}\bar\sigma_1^+(x)\right),$$
$$\sqrt{1-\rho}\,\bar\sigma^-(x)\bar p^-=(1-\rho)\left(\bar q^- -\bar b^-(x)+b_1^-(x)\bar\sigma_1^-(x)+\frac{\theta}{2(1-\rho)}\bar\sigma_1^-(x)\right),$$

and

$$q_1^+=-\frac{\theta}{2\rho},\qquad q_1^-=\frac{\theta}{2(1-\rho)}.$$

Then

$$\begin{aligned}&\rho q^+ +(1-\rho)q^-=p,\\&q_1^+=-\frac{\theta}{2\rho},\qquad q_1^-=\frac{\theta}{2(1-\rho)}.\end{aligned}\tag{3.20}$$

Moreover,

$$\sigma^+(x)^{-1}\big(q^+-b^+(x)\big)=\frac{1}{\sqrt{\rho}}p^+,$$
$$\sigma^-(x)^{-1}\big(q^- -b^-(x)\big)=\frac{1}{\sqrt{1-\rho}}p^-$$

with

$$p_1^+ = \sqrt{\rho}\big(q_1^+ - b_1^+(x)\big), \qquad p_1^- = \sqrt{1-\rho}\big(q_1^- - b_1^-(x)\big).$$

Therefore,

$$\frac{1}{2}\big(|\bar{p}^+|^2 + |\bar{p}^-|^2\big) + \frac{1}{2}\big(|p_1^+|^2 + |p_1^-|^2\big) = \frac{1}{2}\Big(\rho\big|\sigma^+(x)^{-1}\big(q^+ - b^+(x)\big)\big|^2 + (1-\rho)\big|\sigma^-(x)^{-1}\big(q^- - b^-(x)\big)\big|^2\Big).$$

We then have for $p_1 = 0$,

$$\tilde{L}(x, p, \rho, \theta) = \inf\Big\{\frac{1}{2}\Big(\rho\big|\sigma^+(x)^{-1}\big(q^+ - b^+(x)\big)\big|^2 + (1-\rho)\big|\sigma^-(x)^{-1}\big(q^- - b^-(x)\big)\big|^2\Big);\ (q^+, q^-) \text{ satisfies (3.20)}\Big\} - \frac{1}{2}\left(\rho\left|\frac{\theta}{2\rho} + b_1^+(x)\right|^2 + (1-\rho)\left|\frac{\theta}{2(1-\rho)} - b_1^-(x)\right|^2\right).$$

Since,

$$\frac{1}{2}\left(\rho\left|\frac{\theta}{2\rho} + b_1^+(x)\right|^2 + (1-\rho)\left|\frac{\theta}{2(1-\rho)} - b_1^-(x)\right|^2\right) = \frac{\theta^2}{4\rho(1-\rho)} + \big(\rho b_1^+(x)^2 + (1-\rho)b_1^-(x)^2\big) + \theta\big(b_1^+(x) - b_1^-(x)\big),$$

then using also the expression for $I^{(0)}(\phi_1, \psi, \eta)$ in (3.14), we have

$$I(\phi, \psi, \eta) = \int_0^1 L\big(\phi(s), \dot{\phi}(s), \dot{\psi}(s), \dot{\eta}(s)\big)\,\mathrm{d}s,$$

where

$$L(x, p, \rho, \theta) = \frac{1}{2}\big|\sigma(x)^{-1}\big(p - b(x)\big)\big|^2$$

if $x_1 \neq 0$, and

$$L(x, p, \rho, \theta) = \inf\Big\{\frac{1}{2}\Big(\rho\big|\sigma^+(x)^{-1}\big(q^+ - b^+(x)\big)\big|^2 + (1-\rho)\big|\sigma^-(x)^{-1}\big(q^- - b^-(x)\big)\big|^2\Big);\ (q^+, q^-) \text{ satisfies (3.20)}\Big\}$$

if $x_1 = 0$, $p_1 = 0$. This completes the proof. □

LEMMA 3.9. – *$I(\phi, \psi, \eta)$ is a lower semi-continuous function.*

This will be proved in Appendix A.

4. General cases

The system we are concerned with is the following,

$$\begin{aligned} dX^\varepsilon(t) &= b\big(X^\varepsilon(t)\big)\,dt + \varepsilon\sigma\big(X^\varepsilon(t)\big)\,dW(t), \\ X^\varepsilon(0) &= x(0), \end{aligned} \tag{4.1}$$

where $\sigma_{11} = \sqrt{a_{11}}$ and $\sigma_{1j} = 0$ for $j = 2, \ldots, d$. The purpose of this section is to show that the l.d.p of $X^\varepsilon(\cdot)$ in (4.1) holds if and only if it holds for $Y^\varepsilon(\cdot)$ in (4.3) (Theorem 4.5), where the leading coefficient $\sigma_{11}(x) \equiv 1$. The new process $Y^\varepsilon(\cdot)$ is obtained from $X^\varepsilon(\cdot)$ through scaling (4.2).

Let

$$\beta^\varepsilon(t) = \int_0^t \sigma_{11}\big(X^\varepsilon(s)\big)^2\,ds$$

and

$$\beta(t) = \int_0^t \big(\sigma_{11}^+\big(\phi(s)\big)^2\dot\psi(s) + \sigma_{11}^-\big(\phi(s)\big)^2\big(1-\dot\psi(s)\big)\big)\,ds.$$

Also, $\gamma^\varepsilon(t)$ and $\gamma(t)$ are the inverses of β^ε and β respectively. Let $Y^\varepsilon(t) = X^\varepsilon(\gamma^\varepsilon(t))$,

$$\begin{aligned} &\phi^Y(t) = \phi\big(\gamma(t)\big), \qquad \eta^Y(t) = \eta\big(\gamma(t)\big), \\ &\psi^Y(t) = \int_0^{\gamma(t)} \dot\psi(s)\sigma_{11}^+\big(\phi(s)\big)^2\,ds = \int_0^t \dot\psi\big(\gamma(s)\big)\sigma_{11}^+\big(\phi\big(\gamma(s)\big)\big)^2\dot\gamma(s)\,ds. \end{aligned} \tag{4.2}$$

Let $u^Y(\cdot)$ and $\ell^Y(\cdot)$ be the occupation time of Y_1^ε in H^+ and the local time of Y_1^ε at 0 respectively. In the following, we shall omit ε in X^ε, Y^ε when no ambiguity arises.

From (4.1) and (4.2), it follows that $Y(t)$ satisfies the following SDE:

$$\begin{aligned} dY(t) &= \frac{b(Y(t))}{\sigma_{11}(Y(t))^2}\,dt + \varepsilon\frac{\sigma(Y(t))}{\sigma_{11}(Y(t))}\,dW(t), \\ Y(0) &= x(0). \end{aligned} \tag{4.3}$$

LEMMA 4.1. – *For any $\gamma > 0$, there exists a δ_0 such that $|\beta^\varepsilon(t) - \beta(t)| < \gamma$ and $|\gamma^\varepsilon(t) - \gamma(t)| < \gamma$ for all $t \in [0,1]$ and $\varepsilon > 0$ if $\|X^\varepsilon - \phi\| < \delta_0$, $\|u^\varepsilon - \psi\| < \delta_0$.*

Proof. – This is easy, we omit the detail. □

We shall next demonstrate that

$$P^\varepsilon\big(B_\delta(\phi,\psi,\eta)\big) \overset{\log}{=} P^\varepsilon\big(B_\delta(\phi^Y,\psi^Y,\eta^Y)\big), \tag{4.4}$$

where

$$\begin{aligned} &P^\varepsilon\big(B_\delta(\phi^Y,\psi^Y,\eta^Y)\big) \\ &\quad = P^\varepsilon\big(\|Y-\phi^Y\|_{[0,\beta(1)]} < \delta,\ \|u^Y-\psi^Y\|_{[0,\beta(1)]} < \delta,\ \|\ell-\eta^Y\|_{[0,\beta(1)]} < \delta\big). \end{aligned}$$

Supposing this, since (4.3) has the leading coefficient $a_{11} \equiv 1$, $P^{\varepsilon}(B_{\delta}(\phi^{Y}, \psi^{Y}, \eta^{Y}))$ satisfies the l.d.p with rate function

$$I^{Y}(\phi^{Y}, \psi^{Y}, \eta^{Y}) = \int_{0}^{\beta(1)} L^{Y}(\phi^{Y}(t), \dot{\phi}^{Y}(t), \dot{\psi}^{Y}(t), \dot{\eta}^{Y}(t))\, \mathrm{d}t.$$

Here

$$L^{Y}(x, p, \rho, \theta) = \frac{1}{2}\left|\left(\frac{\sigma(x)}{\sigma_{11}(x)}\right)^{-1}\left(p - \frac{b(x)}{\sigma_{11}(x)^{2}}\right)\right|^{2}$$

if $x_1 \neq 0$, and

$$L^{Y}(x, p, \rho, \theta) = \inf\left\{\frac{1}{2}\rho\left|\left(\frac{\sigma^{+}(x)}{\sigma_{11}^{+}(x)}\right)^{-1}\left(q^{+} - \frac{b^{+}(x)}{\sigma_{11}^{+}(x)^{2}}\right)\right|^{2} + \frac{1}{2}(1-\rho)\left|\left(\frac{\sigma^{-}(x)}{\sigma_{11}^{-}(x)}\right)^{-1}\left(q^{-} - \frac{b^{-}(x)}{\sigma_{11}^{-}(x)^{2}}\right)\right|^{2}\right\}$$

if $x_1 = 0$, $p_1 = 0$, where the inf is taken over all (q^{+}, q^{-}) satisfying

$$\rho q^{+} + (1-\rho)q^{-} = p, \qquad q_1^{+} = -\frac{1}{2}\frac{\theta}{\rho}, \qquad q_1^{-} = \frac{1}{2}\frac{\theta}{1-\rho}.$$

From (4.4) and the following result, $P^{\varepsilon}(B_{\delta}(\phi, \psi, \eta))$ satisfies l.d.p with rate function $I(\phi, \psi, \eta)$. This proves our main theorem for the general cases.

LEMMA 4.2. – *We have*

$$L^{Y}(\phi^{Y}(t), \dot{\phi}^{Y}(t), \dot{\psi}^{Y}(t), \dot{\eta}^{Y}(t)) = L(\phi(\gamma(t)), \dot{\phi}(\gamma(t)), \dot{\psi}(\gamma(t)), \dot{\eta}(\gamma(t)))\dot{\gamma}(t)$$

and

$$I^{Y}(\phi^{Y}, \psi^{Y}, \eta^{Y}) = I(\phi, \psi, \eta).$$

Proof. – First, we have

$$\begin{aligned} &L^{Y}(\phi^{Y}(t), \dot{\phi}^{Y}(t), \dot{\psi}^{Y}(t), \dot{\eta}^{Y}(t)) \\ &\quad = L^{Y}\big(\phi(\gamma(t)), \dot{\phi}(\gamma(t))\dot{\gamma}(t), \dot{\psi}(\gamma(t))\sigma_{11}^{+}(\phi(\gamma(t)))^{2}\dot{\gamma}(t), \dot{\eta}(\gamma(t))\dot{\gamma}(t)\big). \end{aligned}$$

Let $x = \phi(\gamma(t))$, $p = \dot{\phi}(\gamma(t))$, $\rho = \dot{\psi}(\gamma(t))$ and $\theta = \dot{\eta}(\gamma(t))$. For $x_1 = 0$, $p_1 = 0$,

$$\begin{aligned} &L^{Y}(x, p\dot{\gamma}(t), \rho\sigma_{11}^{+}(x)^{2}\dot{\gamma}(t), \theta\dot{\gamma}(t)) \\ &\quad = \inf\left\{\frac{1}{2}\rho\dot{\gamma}(t)\left|\sigma^{+}(x)^{-1}(q^{+}\sigma_{11}^{+}(x)^{2} - b^{+}(x))\right|^{2} \right. \\ &\qquad \left. + \frac{1}{2}(1-\rho)\dot{\gamma}(t)\left|\sigma^{-}(x)^{-1}(q^{-}\sigma_{11}^{-}(x)^{2} - b^{-}(x))\right|^{2}\right\}, \end{aligned}$$

where

$$\dot{\gamma}(t)\rho\sigma_{11}^{+}(x)^{2}q^{+} + \dot{\gamma}(t)(1-\rho)\sigma_{11}^{-}(x)^{2}q^{-} = p\dot{\gamma}(t) \tag{4.5}$$

and

$$q_1^+ = -\frac{\theta\dot\gamma(t)}{2\rho\sigma_{11}^+(x)^2\dot\gamma(t)} = -\frac{\theta}{2\rho\sigma_{11}^+(x)^2}, \qquad q_1^- = \frac{\theta}{2(1-\rho)\sigma_{11}^+(x)^2}. \tag{4.6}$$

In the above relation we use

$$1-\rho\sigma_{11}^+(x)^2\dot\gamma(t) = (1-\rho)\sigma_{11}^-(x)^2\dot\gamma(t)$$

by the definition of $\gamma(t)$. Let $q^+\sigma_{11}^+(x)^2 = \tilde q^+$ and $q^-\sigma_{11}^-(x)^2 = \tilde q^-$. Then from (4.5) and (4.6),

$$\rho\tilde q^+ + (1-\rho)\tilde q^- = p, \qquad \tilde q_1^+ = -\frac{\theta}{2\rho}, \qquad \tilde q_1^- = -\frac{\theta}{2(1-\rho)}.$$

Thus

$$\begin{aligned}
&L^Y\big(x, p\dot\gamma(t), \rho\sigma_{11}^+(x)^2\dot\gamma(t), \theta\dot\gamma(t)\big)\\
&\quad = \dot\gamma(t)\inf_{\tilde q^+,\tilde q^-}\left(\frac{1}{2}\rho\big|\sigma^+(x)^{-1}\big(\tilde q^+ - b^+(x)\big)\big|^2 + \frac{1}{2}(1-\rho)\big|\sigma^-(x)^{-1}\big(\tilde q^- - b^-(x)\big)\big|^2\right)\\
&\quad = \dot\gamma(t)L(x,p,\rho,\theta).
\end{aligned}$$

The proof is complete.

LEMMA 4.3. – *For any $\gamma > 0$, there exists an δ_0 such that $\|u^Y - \psi^Y\|_{[0,\beta(1)-\gamma]} \leqslant \gamma$ and $\|\ell^Y - \eta^Y\|_{[0,\beta(1)-\gamma]} \leqslant \gamma$ for all ε if $X \in B_\delta(\phi,\psi,\eta)$, $\delta < \delta_0$.*

Proof. – By definition,

$$\begin{aligned}
u^Y(t) &= \int_0^t \chi_{(0,\infty)}\big(Y^\varepsilon(s)\big)\,\mathrm{d}s = \int_0^t \chi_{(0,\infty)}\big(X^\varepsilon\big(\gamma^\varepsilon(s)\big)\big)\,\mathrm{d}s\\
&= \int_0^{\gamma^\varepsilon(t)} \chi_{(0,\infty)}\big(X^\varepsilon(s)\big)\dot\beta^\varepsilon(s)\,\mathrm{d}s = \int_0^{\gamma^\varepsilon(t)} \chi_{(0,\infty)}\big(X^\varepsilon(s)\big)\sigma_{11}^+\big(X^\varepsilon(s)\big)^2\,\mathrm{d}s.
\end{aligned}$$

Since $|\int_0^t \chi_{(0,\infty)}(X^\varepsilon(s))\,\mathrm{d}s - \psi(t)| \to 0$ uniformly over t as $\delta \to 0$ if $X^\varepsilon \in B_\delta(\phi,\psi,\eta)$, we have

$$\int_0^{\gamma^\varepsilon(t)} \sigma_{11}^+\big(\phi(s)\big)^2\dot\psi(s)\,\mathrm{d}s - \int_0^{\gamma^\varepsilon(t)} \sigma_{11}^+\big(X^\varepsilon(s)\big)^2\chi_{(0,\infty)}\big(X^\varepsilon(s)\big)\,\mathrm{d}s \to 0$$

as $\delta \downarrow 0$ because $\gamma^\varepsilon(t) \to \gamma(t)$ uniformly in t by Lemma 4.1 and σ_{11}^+ is a continuous function. Hence $\|u^Y - \psi^Y\|_{[0,\beta(1)-\gamma]} \to 0$ as $\delta \to 0$. Similarly, $\|\ell^Y - \eta^Y\|_{[0,\beta(1)-\gamma]} \leqslant \gamma$ for all ε because $\ell^Y(t) = \ell^X(\gamma^\varepsilon(t))$ by Tanaka's formula and Lemma 4.1. This completes the proof. □

On the other hand, the following converse of Lemma 4.3 can be proven using the same arguments.

LEMMA 4.4. – *For any $\delta > 0$, there exists an θ_0 such that for $\theta < \theta_0$,*

$$\|X - \phi\|_{[0,1-\delta]} \leqslant \delta, \qquad \|u - \psi\|_{[0,1-\delta]} \leqslant \delta, \qquad \|\ell - \eta\|_{[0,1-\delta]} \leqslant \delta$$

if

$$\left\|Y - \phi^Y\right\|_{[0,\beta(1)-\theta]} \leqslant \theta, \qquad \left\|u^Y - \psi^Y\right\|_{[0,\beta(1)-\theta]} \leqslant \theta, \qquad \left\|\ell^Y - \eta^Y\right\|_{[0,\beta(1)-\theta]} < \theta.$$

Finally, we now prove (4.4). Let denote $Y \in B_{\gamma,[0,\beta(1)-\gamma]}(\phi^Y, \psi^Y, \eta^Y)$ if

$$\left\|Y - \phi^Y\right\|_{[0,\beta(1)-\gamma]} \leqslant \gamma, \qquad \left\|u^Y - \psi^Y\right\|_{[0,\beta(1)-\gamma]} \leqslant \gamma, \qquad \left\|\ell^Y - \eta^Y\right\|_{[0,\beta(1)-\gamma]} < \gamma.$$

Then

$$P^\varepsilon\big(B_\delta(\phi, \psi, \eta)\big) \leqslant P^\varepsilon\big(B_{\gamma,[0,\beta(1)-\gamma]}(\phi^Y, \psi^Y, \eta^Y)\big)$$

for δ small, thus

$$\begin{aligned}\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\varepsilon^2 \log P^\varepsilon\big(B_\delta(\phi, \psi, \eta)\big) &\leqslant \lim_{\gamma\to 0}\overline{\lim_{\varepsilon\to 0}}\varepsilon^2 \log P^\varepsilon\big(B_{\gamma,[0,\beta(1)-\gamma]}(\phi^Y, \psi^Y, \eta^Y)\big)\\ &= I^Y_{[0,\beta(1)]}(\phi^Y, \psi^Y, \eta^Y).\end{aligned} \tag{4.7}$$

Similarly, for any $M > 0$, choose δ so small that for all small θ,

$$\begin{aligned}P^\varepsilon\big(B_\delta(\phi, \psi, \eta)\big) &\geqslant P^\varepsilon\big(B_{\delta,[0,1-\delta]}(\phi, \psi, \eta)\big) - \exp\left(-\frac{M}{\varepsilon^2}\right)\\ &\geqslant P^\varepsilon\big(B_{\theta,[0,\beta(1)-\theta]}(\phi^Y, \psi^Y, \eta^Y)\big) - \exp\left(-\frac{M}{\varepsilon^2}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\underline{\lim}_{\varepsilon\to 0}\, \varepsilon^2 \log P^\varepsilon\big(B_\delta(\phi, \psi, \eta)\big) &\geqslant \inf\big(I^Y_{[0,\beta(1)]}(\phi^Y, \psi^Y, \eta^Y), M\big)\\ &= I^Y_{[0,\beta(1)]}(\phi^Y, \psi^Y, \eta^Y)\end{aligned}$$

if M is large enough. From this and (4.7), then (4.4) follows.

Appendix A

In the first part of this appendix we shall give a proof of

$$P\big\{\left\|X^\varepsilon - \phi\right\| \leqslant \delta,\ \left\|u^\varepsilon - \psi\right\| \leqslant \delta,\ \left\|\ell^\varepsilon - \eta\right\| \leqslant \delta\big\} \stackrel{\log}{=} \exp\left(-\frac{I(\phi, \psi, \eta)}{\varepsilon^2}\right) \tag{A.1}$$

for general ϕ, ψ, η with $I(\phi, \psi, \eta) < \infty$, when $X^\varepsilon(t) = \varepsilon W(t)$. Here

$$I(\phi, \psi, \eta) = \frac{1}{2}\int_0^1 \left|\dot\phi(t)\right|^2 \mathrm{d}t + \frac{1}{8}\int_0^1 \frac{\dot\eta^2(t)}{\dot\psi(t)(1 - \dot\psi(t))}\,\mathrm{d}t.$$

In the second part, we shall prove Lemma 3.9.

To prove (A.1), we need to construct ϕ_n, ψ_n, η_n satisfying the following properties:

$$\phi_n(t) \to \phi(t), \qquad \psi_n(t) \to \psi(t), \qquad \eta_n(t) \to \eta(t)$$

uniformly for $t \in [0,1]$ as $n \to \infty$; $\psi_n \in H^+(\phi_n)$ and $\eta_n \in L^0(\phi_n)$; For each n, there are $0 = a_0^{(n)} < a_1^{(n)} < a_2^{(n)} < \cdots < a_{M_n}^{(n)} < a_{M_n+1}^{(n)} = 1$ such that $\sup_i (a_{i+1}^{(n)} - a_i^{(n)}) \to 0$ as $n \to \infty$ and for each i, ϕ_n, ψ_n, η_n are linear on $[a_i^{(n)}, a_{i+1}^{(n)}]$ and

$$\phi_n\big(a_i^{(n)}\big) = \phi\big(a_i^{(n)}\big), \qquad \psi_n\big(a_i^{(n)}\big) = \psi\big(a_i^{(n)}\big), \qquad \eta_n\big(a_i^{(n)}\big) = \eta\big(a_i^{(n)}\big).$$

Moreover,

$$\big\{t \in [0,1]; \phi(t) = 0\big\} \subseteq \cup \big\{\big[a_i^{(n)}, a_{i+1}^{(n)}\big]; \phi\big(a_i^{(n)}\big) = \phi\big(a_{i+1}^{(n)}\big) = 0\big\};$$

We have

$$\lim_{n\to\infty} I(\phi_n, \psi_n, \eta_n) = I(\phi, \psi, \eta).$$

Before proving this, we remark that $I(\cdot,\cdot,\cdot)$ is lower semicontinuous. To see this, first, we have the relation,

$$\int_0^1 \big|\dot\phi(t)\big|^2 \,\mathrm{d}t = \sup_{0=t_0<t_1<\cdots<t_m=1} \frac{|\phi(t_{i+1}) - \phi(t_i)|^2}{t_{i+1} - t_i}.$$

See [14, Lemma 2.1, Chapter 3]. From this, it is easy to see that the mapping

$$\phi \to \int_0^1 \big|\dot\phi(t)\big|^2 \,\mathrm{d}t$$

is lower semicontinuous. On the other hand, we have the relations,

$$\int_0^1 \frac{\dot\eta^2(t)}{\dot\psi(t)(1-\dot\psi(t))} \,\mathrm{d}t = \int_0^1 \frac{\dot\eta^2(t)}{\dot\psi(t)} \,\mathrm{d}t + \int_0^1 \frac{\dot\eta^2(t)}{1-\dot\psi(t)} \,\mathrm{d}t,$$

and

$$\int_0^1 \frac{\dot\eta^2(t)}{\dot\psi(t)} \,\mathrm{d}t = \sup_{\theta>0} \int_0^1 \frac{\dot\eta^2(t)}{\theta + \dot\psi(t)} \,\mathrm{d}t,$$

$$\int_0^1 \frac{\dot\eta^2(t)}{1-\dot\psi(t)} \,\mathrm{d}t = \sup_{\theta>0} \int_0^1 \frac{\dot\eta^2(t)}{\theta + 1 - \dot\psi(t)} \,\mathrm{d}t.$$

For each $\theta > 0$, we define $\hat\eta(t)$ by

$$\hat\eta\big(\theta t + \psi(t)\big) = \eta(t).$$

Then,

$$\int_0^1 \frac{\dot\eta^2(t)}{\theta+\dot\psi(t)}\,\mathrm{d}t = \int_0^{\theta+\psi(1)} \left|\dot{\hat\eta}(t)\right|^2 \mathrm{d}t.$$

From this and above reasoning, we see

$$(\psi,\eta) \to \int_0^1 \frac{\dot\eta^2(t)}{\theta+\dot\psi(t)}\,\mathrm{d}t$$

is lower semicontinuous. Hence,

$$(\psi,\eta) \to \int_0^1 \frac{\dot\eta^2(t)}{\dot\psi(t)}\,\mathrm{d}t$$

is lower semicontinuous. Similarly,

$$(\psi,\eta) \to \int_0^1 \frac{\dot\eta^2(t)}{1-\dot\psi(t)}\,\mathrm{d}t$$

is lower semicontinuous. We conclude that $I(\cdot,\cdot,\cdot)$ is lower semicontinuous.

We now show the existence of ϕ_n, ψ_n, η_n. For this, let fix n. We first choose a subset U of [0, 1] such that it is a disjoint union of finitely many closed intervals,

$$U = \bigcup_{i=1}^{N_0} [a_i, b_i],$$

such that

$$U \supseteq \{t; \phi(t) = 0\}$$

and

$$\left|\phi(t)\right| \leqslant \frac{1}{n} \quad \text{for } t \in U, \qquad \int_U \left|\dot\phi(t)\right|^2 \mathrm{d}t < \frac{1}{n}.$$

We may choose a_i, b_i in such a way that $\phi(a_i) = \phi(b_i) = 0$. Otherwise, we can replace a_i, b_i by

$$\hat a_i = \inf\{t \geqslant a_i; \phi(t) = 0\}, \qquad \hat b_i = \sup\{t < b_i; \phi(t) = 0\}.$$

Denote

$$[0,1]\backslash U = \bigcup_{i=1}^{N_1} (c_i, d_i).$$

We may divide each (c_i, d_i) into subintervals with lengths smaller than $1/n$ and define ϕ_n such that ϕ_n is linear on each subinterval, the values of ϕ_n and ϕ are equal at endpoints of each subinterval and the following properties hold:

$$\phi_n(t) = 0, \quad t \in U,$$

$$\|\phi_n - \phi\| \leqslant \frac{1}{n}, \qquad \int_0^1 \left|\dot\phi_n(t) - \dot\phi(t)\right|^2 < \frac{1}{n}.$$

We now define ψ_n and η_n. For each i,

$$\psi_n(c_i) = \psi_n(d_i) = \psi(c_i), \qquad \eta_n(c_i) = \eta_n(d_i) = \eta(c_i).$$

We remark,

$$\int_0^1 \frac{\dot\eta^2(t)}{\dot\psi(t)}\,\mathrm{d}t = \sup_{0=t_0<t_1<t_2<\cdots<t_m=1} \frac{|\eta(t_{i+1}) - \eta(t_i)|^2}{\psi(t_{i+1}) - \psi(t_i)}$$

and

$$\int_0^1 \frac{\dot\eta^2(t)}{1-\dot\psi(t)}\,\mathrm{d}t = \sup_{0=t_0<t_1<t_2<\cdots<t_m=1} \frac{|\eta(t_{i+1}) - \eta(t_i)|^2}{t_{i+1} - t_i - \psi(t_{i+1}) + \psi(t_i)}$$

by a argument similar to that in [14, Lemma 2.1, Chapter 3]. Therefore, we can divide each (a_i, b_i) into subintervals finer enough such that the length of each subinterval is smaller than $1/n$ and define $\psi_n,\ \eta_n$ such that ψ_n, η_n are linear on each subinterval and the values of ψ_n, ψ (η_n, η) at the endpoints of these subintervals are equal and the properties hold:

$$\|\psi_n - \psi\| < \frac{1}{n}, \qquad \|\eta_n - \eta\| < \frac{1}{n},$$

$$\left| \int_0^1 \frac{\dot\eta_n^2(t)}{\dot\psi_n(t)(1-\dot\psi_n(t))}\,\mathrm{d}t - \int_0^1 \frac{\dot\eta^2(t)}{\dot\psi(t)(1-\dot\psi(t))}\,\mathrm{d}t \right| < \frac{1}{n}.$$

This completes the construction of ϕ_n, ψ_n, η_n.

We now show (A.1). We shall show the following statements separately,

$$\lim_{\delta\to 0} \liminf_{\varepsilon\to 0} \varepsilon^2 \log P\left\{\left\|X^\varepsilon - \phi\right\| \leqslant \delta,\ \left\|u^\varepsilon - \psi\right\| \leqslant \delta,\ \left\|\ell^\varepsilon - \eta\right\| \leqslant \delta\right\}$$
$$\geqslant -I(\phi, \psi, \eta), \tag{A.2}$$

and

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \varepsilon^2 \log P\left\{\left\|X^\varepsilon - \phi\right\| \leqslant \delta,\ \left\|u^\varepsilon - \psi\right\| \leqslant \delta,\ \left\|\ell^\varepsilon - \eta\right\| \leqslant \delta\right\}$$
$$\leqslant -I(\phi, \psi, \eta). \tag{A.3}$$

We first treat (A.2).

Let $\delta > 0$ be fixed. We have

$$\begin{aligned} &P\{\|X^\varepsilon - \phi\| \leqslant \delta,\ \|u^\varepsilon - \psi\| \leqslant \delta,\ \|\ell^\varepsilon - \eta\| \leqslant \delta\} \\ &\quad \geqslant P\left\{\|X^\varepsilon - \phi_n\| \leqslant \frac{\delta}{2},\ \|u^\varepsilon - \psi_n\| \leqslant \frac{\delta}{2},\ \|\ell^\varepsilon - \eta_n\| \leqslant \frac{\delta}{2}\right\} \end{aligned}$$

if n is large enough. We may assume that $\dot{\psi}_n(t) > 0$ if $\phi_n(t) = 0$. Otherwise, we replace $\psi_n(t)$ by

$$\psi_n(t) + \theta \int_0^t \chi_{\phi_n(s)=0}\, \mathrm{d}s$$

for a small θ. We fix such n. To simplify the notation, we write a_i for $a_i^{(n)}$ and M for M_n. Then we have the following relation,

$$\begin{aligned} &P\left\{\|X^\varepsilon - \phi_n\| \leqslant \frac{\delta}{2},\ \|u^\varepsilon - \psi_n\| \leqslant \frac{\delta}{2},\ \|\ell^\varepsilon - \eta_n\| \leqslant \frac{\delta}{2}\right\} \\ &\quad \geqslant \prod_{i=0}^{M} \inf_{|x-\phi(a_i)|\leqslant\delta_i} P_{a_i,x}\{\|X^\varepsilon - \phi_n\|_{[a_i,a_{i+1}]} \leqslant \delta_{i+1},\ \|u^\varepsilon - \hat{\psi}_n^{(i)}\|_{[a_i,a_{i+1}]} \leqslant \delta_{i+1}, \\ &\qquad \|\ell^\varepsilon - \hat{\eta}_n^{(i)}\|_{[a_i,a_{i+1}]} \leqslant \delta_{i+1}\}, \end{aligned} \tag{A.4}$$

where $\delta_{i+1} = K\delta_i$, $K\delta_M = \delta/4$, and $K = 2 + \max\{8\rho_i/\theta_i, \theta_i \neq 0\}$, ρ_i, θ_i are the derivatives of ψ_n, η_n on (a_i, a_{i+1}). Here we use the notations,

$$\hat{\psi}_n^{(i)}(t) = \psi_n(t) - \psi_n(a_i), \qquad \hat{\eta}_n^{(i)}(t) = \eta_n(t) - \eta_n(a_i)$$

for $t \in [a_i, a_{i+1}]$, and $P_{s,x}[\cdot]$ is the probability measure generated by the process starting from x at time s. To see (A.4), applying the conditioning given $X^\varepsilon(t), 0 \leqslant t \leqslant a_M$, and the Markovian property of the process, we get

$$\begin{aligned} &P\left\{\|X^\varepsilon - \phi_n\| \leqslant \frac{\delta}{2},\ \|u^\varepsilon - \psi_n\| \leqslant \frac{\delta}{2},\ \|\ell^\varepsilon - \eta_n\| \leqslant \frac{\delta}{2}\right\} \\ &\quad \geqslant P\left\{\|X^\varepsilon - \phi_n\|_{[0,a_M]} \leqslant \frac{\delta}{2},\ \|u^\varepsilon - \psi_n\|_{[0,a_M]} \leqslant \frac{\delta}{4},\ \|\ell^\varepsilon - \eta_n\|_{[0,a_M]} \leqslant \frac{\delta}{4}\right\} \\ &\qquad \times \inf_{|x-\phi_n(a_M)|\leqslant\delta_M} P_{a_M,x}\left\{\|X^\varepsilon - \phi_n\|_{[a_M,1]} \leqslant \frac{\delta}{2},\ \|u^\varepsilon - \hat{\psi}_n^{(M)}\|_{[a_M,1]} \leqslant \frac{\delta}{4}, \right. \\ &\qquad\qquad \left. \|\ell^\varepsilon - \hat{\eta}_n^{(M)}\|_{[a_M,1]} \leqslant \frac{\delta}{4}\right\}. \end{aligned}$$

Then (A.4) follows by repeating this procedure.

Next, we use the following relation,

$$\begin{aligned} &\lim_{\delta\to 0} \liminf_{\varepsilon\to 0} \varepsilon^2 \log \inf_{|x-\phi_n(a_i)|\leqslant\delta_i} P_{a_i,x}\left\{\|X^\varepsilon - \phi_n\|_{[a_i,a_{i+1}]} \leqslant \delta_{i+1},\ \|u^\varepsilon - \hat{\psi}_n^{(i)}\|_{[a_i,a_{i+1}]} \leqslant \frac{\delta}{2}, \right. \\ &\qquad \left. \|\ell^\varepsilon - \hat{\eta}_n^{(i)}\|_{[a_i,a_{i+1}]} \leqslant \frac{\delta}{2}\right\} \end{aligned} \tag{A.5}$$

is equal to $I_{[a_i,a_{i+1}]}(\phi_n,\psi_n,\eta_n)$. Here we use the notation,

$$I_{[a,b]}(\phi,\psi,\eta)=\frac{1}{2}\int_a^b\left|\dot\phi(t)\right|^2\mathrm{d}t+\frac{1}{8}\int_a^b\frac{\dot\eta^2(t)}{\dot\psi(t)(1-\dot\psi(t))}\,\mathrm{d}t.$$

Then (A.2) can be proved by letting $\varepsilon\to 0,\delta\to 0$ and $n\to\infty$. To see an argument for (A.5), we consider only the case that $a_i=0,\ a_{i+1}=1,\ \phi_n(t)=0,\ \psi_n(t)=\psi(t)=\rho t,\ \eta_n(t)=\eta(t)=\theta t$ for $t\in[0,1]$. We note that this is the case considered in the proof of Theorem 2.3. Then we have the relation (2.2). That is, for $|x|\leqslant\delta$,

$$\begin{aligned}&P_x\left\{\left\|X^\varepsilon\right\|\leqslant K\delta,\ \left\|u^\varepsilon-\psi\right\|\leqslant\delta,\ \left\|\ell^\varepsilon-\eta\right\|\leqslant\delta\right\}\\&\quad\stackrel{\log}{=}\exp\left(-\frac{\theta^2}{8\varepsilon^2\rho(1-\rho)}\right)P_x\left\{\left\|\tilde X^\varepsilon\right\|\leqslant K\delta,\ \left\|\tilde u^\varepsilon-\psi\right\|\leqslant\delta,\ \left\|\tilde\ell^\varepsilon-\eta\right\|\leqslant\delta\right\}.\end{aligned}$$

We refer to the proof of Theorem 2.3 for the notations. Since $\tilde X^\varepsilon(t)$ satisfies

$$\begin{aligned}&\mathrm{d}\tilde X^\varepsilon(t)=v\big(\tilde X^\varepsilon(t)\big)\,\mathrm{d}t+\varepsilon\,\mathrm{d}\tilde W(t),\\&X^\varepsilon(0)=x,\end{aligned}$$

it is easy to see that

$$P_x\{\tau>K_0\delta\}\leqslant\exp\left(-c\frac{\delta}{K_0\varepsilon^2}\right)\tag{A.6}$$

for some $c>0$ if $|x|<\delta$, where $K_0=4\max\{\rho,1-\rho\}/\theta$. Here

$$\tau=\inf\left\{t>0;\ \tilde X^\varepsilon(t)=0\right\}.$$

Then

$$\begin{aligned}&P_x\left\{\left\|\tilde X^\varepsilon\right\|\leqslant K\delta,\ \left\|\tilde u^\varepsilon-\psi\right\|\leqslant\delta,\ \left\|\tilde\ell^\varepsilon-\eta\right\|\leqslant\delta\right\}\\&\quad\geqslant P_x\{\tau\leqslant K_0\delta\}P_0\left\{\left\|\tilde X^\varepsilon\right\|\leqslant K\delta,\ \left\|\tilde u^\varepsilon-\psi\right\|\leqslant\delta,\ \left\|\tilde\ell^\varepsilon-\eta\right\|\leqslant\delta\right\}.\end{aligned}$$

By the proof of Theorem 2.3 and (A.6), the probabilities on the right hand side converges to 1 uniformly for $|x|\leqslant\delta$ as $\varepsilon\to 0$. We conclude that the probability on the left hand side converges to 1 uniformly for $|x|\leqslant\delta$ as $\varepsilon\to 0$. This completes the proof of (A.5), hence (A.2).

Now we prove (A.3). Let fix $\rho>0$ and take n such that

$$I(\phi_n,\psi_n,\eta_n)>I(\phi,\psi,\eta)-\frac{\rho}{2}.$$

Again, we write a_i for $a_i^{(n)}$ and M for M_n. Similar to (A.4), we have the relation,

$$\begin{aligned}&P\left\{\left\|X^\varepsilon-\phi\right\|\leqslant\delta,\ \left\|u^\varepsilon-\psi\right\|\leqslant\delta,\ \left\|\ell^\varepsilon-\eta\right\|\leqslant\delta\right\}\\&\quad\leqslant P\left\{\left|X^\varepsilon(a_i)-\phi(a_i)\right|\leqslant\delta,\ \left|u^\varepsilon(a_i)-\psi(a_i)\right|\leqslant\delta,\ \left|\ell^\varepsilon(a_i)-\eta(a_i)\right|\leqslant\delta,\right.\\&\qquad\left.i=0,1,\ldots,M\right\}\end{aligned}$$

$$\leqslant \prod_{i=0}^{M} \sup_{|x-\phi(a_i)|\leqslant\delta} P_{a_i,x}\big\{\big|X^\varepsilon(a_{i+1})-\phi(a_{i+1})\big|\leqslant\delta,\\ \big|u^\varepsilon(a_{i+1})-\big(\psi(a_{i+1})-\psi(a_i)\big)\big|\leqslant 2\delta,\\ \big|\ell^\varepsilon(a_{i+1})-\big(\eta(a_{i+1})-\eta(a_i)\big)\big|\leqslant 2\delta\big\}.$$

Using a argument similar to that for (2.2), we can deduce

$$P_{a_i,x}\big\{\big|X^\varepsilon(a_{i+1})-\phi(a_{i+1})\big|\leqslant\delta,\ \big|u^\varepsilon(a_{i+1})-\big(\psi(a_{i+1})-\psi(a_i)\big)\big|\leqslant 2\delta,\\ \big|\ell^\varepsilon(a_{i+1})-\big(\eta(a_{i+1})-\eta(a_i)\big)\big|\leqslant 2\delta\big\}\\ \leqslant \exp\left(-\frac{|\eta(a_{i+1})-\eta(a_i)|^2}{8\varepsilon^2(\psi(a_{i+1})-\psi(a_i))((a_{i+1}-a_i)-(\psi(a_{i+1})-\psi(a_i)))}\right)\exp\left(\frac{cK\delta}{\varepsilon^2}\right)$$

if $\phi(a_i)=\phi(a_{i+1})=0,\ |x|\leqslant\delta$, where c is a constant and

$$K=\sup\left\{\frac{\eta(a_{i+1})-\eta(a_i)}{\psi(a_{i+1})-\psi(a_i)},\ \frac{\eta(a_{i+1})-\eta(a_i)}{(a_{i+1}-a_i)-(\psi(a_{i+1})-\psi(a_i))};\\ \psi(a_{i+1})-\psi(a_i)\neq 0, (a_{i+1}-a_i)-\big(\psi(a_{i+1})-\psi(a_i)\big)\neq 0\right\}.$$

Otherwise,

$$P_{a_i,x}\big\{\big|X^\varepsilon(a_{i+1})-\phi(a_{i+1})\big|\leqslant\delta,\ \big|u^\varepsilon(a_{i+1})-\big(\psi(a_{i+1})-\psi(a_i)\big)\big|\leqslant 2\delta,\\ \big|\ell^\varepsilon(a_{i+1})-\big(\eta(a_{i+1})-\eta(a_i)\big)\big|\leqslant 2\delta\big\}\\ \leqslant \exp\left(-\frac{|\phi(a_{i+1})-\phi(a_i)|^2}{2\varepsilon^2(a_{i+1}-a_i)}\right)\exp\left(\frac{cK'\delta}{\varepsilon^2}\right)$$

if $|x-\phi(a_i)|\leqslant\delta$, where

$$K'=\sup\left\{\frac{|\phi(a_{i+1})-\phi(a_i)|}{a_{i+1}-a_i};\ i=0,1,\ldots,M\right\}.$$

Then

$$P\big\{\|X^\varepsilon-\phi\|\leqslant\delta,\ \|u^\varepsilon-\psi\|\leqslant\delta,\ \|\ell^\varepsilon-\eta\|\leqslant\delta\big\}\\ \leqslant \exp\left(-\frac{I(\phi_n,\psi_n,\eta_n)}{\varepsilon^2}+\frac{c(K+K')M\delta}{\varepsilon^2}\right)\leqslant \exp\left(-\frac{I(\phi,\psi,\eta)}{\varepsilon^2}+\frac{\rho}{\varepsilon^2}\right)$$

if δ is small enough. Since $\rho>0$ can be arbitrarily small, this proves (A.3).

Proof of Lemma 3.9. – First we prove that the mapping

$$(\phi,\psi,\eta,v^+,v^-)\to I^{(0)}(\phi_1,\psi,\eta)+I_\psi^{(\theta)}(v^+,v^-) \tag{A.7}$$

is lower semicontinuous for each $\theta>0$, where

$$I_\psi^{(\theta)}(v^+,v^-)=\frac{1}{2}\int_0^1\left(\frac{1}{\theta+\dot\psi(t)}\big|\dot v^+(t)\big|^2+\frac{1}{1+\theta-\dot\psi(t)}\big|\dot v^-(t)\big|^2\right)\mathrm{d}t.$$

Define the functions $\hat{v}^+$, $\hat{v}^-$ by the following relation,

$$\hat{v}^+\big(\theta t + \psi(t)\big) = v^+(t), \qquad \hat{v}^-\big(t + \theta t - \psi(t)\big) = v^-(t).$$

Then

$$I_\psi^{(\theta)}(v^+, v^-) = \frac{1}{2}\left(\int_0^{\theta+\psi(1)} \big|\dot{\hat{v}}^+(s)\big|^2 + \int_0^{1+\theta-\psi(1)} \big|\dot{\hat{v}}^-(s)\big|^2 \,\mathrm{d}s\right).$$

From this expression, it follows that the mapping in (A.7) is lower semicontinuous.

Define

$$I^{(\theta)}(\phi, \psi, \eta) = I^{(0)}(\phi_1, \psi, \eta) + J^{(\theta)}(\phi, \psi, \eta),$$

where

$$J^{(\theta)}(\phi, \psi, \eta) = \inf\big\{I_\psi^{(\theta)}(v^+, v^-); (v^+, v^-) \text{ satisfies (3.16)}\big\}.$$

Then $I^{(\theta)}(\cdot,\cdot,\cdot)$ is lower semicontinuous by the following standard argument. Let

$$\big(\phi^{(n)}, \psi^{(n)}, \eta^{(n)}\big) \to (\phi, \psi, \eta)$$

as $n \to \infty$. For each n, there is $(v^{+,(n)}, v^{-,(n)})$ such that

$$J^{(\theta)}\big(\phi^{(n)}, \psi^{(n)}, \eta^{(n)}\big) = I_{\psi^{(n)}}^{(\theta)}\big(v^{+,(n)}, v^{-,(n)}\big)$$

and $(v^{+,(n)}, v^{-,(n)})$ satisfies (3.16) with $(\phi, \psi, \eta) = (\phi^{(n)}, \psi^{(n)}, \eta^{(n)})$. We may assume that

$$\big(v^{+,(n)}, v^{-,(n)}\big) \to (v^+, v^-)$$

as $n \to \infty$ for some (v^+, v^-). Then

$$\begin{aligned}\liminf_{n\to\infty} I^{(\theta)}\big(\phi^{(n)}, \psi^{(n)}, \eta^{(n)}\big) &= \liminf_{n\to\infty}\big(I^{(0)}\big(\phi_1^{(n)}, \psi^{(n)}, \eta^{(n)}\big) + I_{\psi^{(n)}}^{(\theta)}\big(v^{+,(n)}, v^{-,(n)}\big)\big)\\ &\geqslant I^{(0)}(\phi_1, \psi, \eta) + I_\psi^{(\theta)}(v^+, v^-)\\ &\geqslant I^{(\theta)}(\phi, \psi, \eta).\end{aligned}$$

The last step is by the definition of $I^{(\theta)}$ and the fact that (v^+, v^-) satisfies (3.16). This proves the lower semicontinuity of $I^{(\theta)}$.

It is easy to see that $I^{(\theta)}(\cdot) \nearrow I(\cdot)$ as $\theta \searrow 0$. Together with the lower semicontinuity of $I^{(\theta)}(\cdot)$, we can easily deduce that $I(\cdot)$ is lower semicontinuous. This completes the proof of Lemma 3.9. □

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