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GLAUBER DYNAMICS OF SPIN GLASSES AT LOW AND HIGH TEMPERATURE

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ABSTRACT. – We consider an increasing sequence of finite boxes $\Lambda_L \subset \mathbf{Z}^2$ and a reversible stochastic frustrated Ising model having invariant measures satisfying free boundary conditions. We show that the spectral gap associated with the Edwards–Anderson model has a different asymptotic behavior in low and in high temperature.

In low temperature, associated with the spectral gap, there is a qualitatively slower relaxation to equilibrium than there is in high temperature. Some geometrical lemmas are employed in the paper to show that some regions are *almost* independent from their exterior. We use for this aim a Peierls' argument. © 2002 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Nous considérons, dans cet article, une suite strictement croissante de boites finies $\Lambda_L \subset \mathbf{Z}^2$ et un modèle d'Ising stochastique réversible non ferromagnétique ayant des mesures invariantes avec conditions de frontière libre. Nous montrons que le frou spectral associé au modèle d'Edwards–Anderson a des comportements asymptotiques différents à haute et basse température.

A basse température, la relaxation vers l'équilibre est qualitativement plus lente qu'à haute température. Nous employons dans cet article des lemmes géomêtriques pour montrer que certaines régions sont *presque* indépendantes de leur exterieur. À cette fin, nous utilisons un argument de type Peierls. © 2002 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

In the past two decades different dynamics have been analyzed for spin models. The most popular for physical reasons is the Glauber dynamics [15]. Other dynamics such as heat bath, Metropolis, etc. all share the characteristic of being single site dynamics, that is to say only one spin at the time is flipped. In this work we only deal with dynamics of this type and other local dynamics. Dynamics such as Swendsen–Wang, in which whole clusters are flipped at once are beyond the scope of the present paper (see [26,24] for the

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ferromagnetic Ising model). A paper is in preparation to prove a similar behavior for the Swendsen–Wang dynamics of a frustrated Ising model [4].

Some basic results for the Glauber dynamics in the ferromagnetic Ising model are in [27,21–23].

A fundamental paper for the ferromagnetic Ising model is [23]. In that paper, it is proved that for 2-dimensional spin systems with finite ferromagnetic interactions, the conditions of weak mixing imply strong mixing of Gibbs states. One of the most interesting consequences, from our point of view, is that for the ferromagnetic Ising model at all temperatures $T > T_c$ there is a rapid (i.e. exponential) convergence to equilibrium using a Glauber dynamics (some results in [21] and [22] are also needed).

Also in the context of Glauber dynamics, the paper [29,28,20] are relevant; in these works is proved that the Dobrushin–Shlosman mixing conditions are equivalent to the existence of a logarithmic Sobolev inequality for the associated Gibbs measure. In particular, in [20] it is proved that there is a positive spectral gap for the Kawasaki and for the Glauber dynamics if the mixing conditions are satisfied. These properties are proved in a general framework but they are applicable mainly for models at high temperature.

There are also some results concerning the zero-temperature case of Glauber dynamics, usually studied with a random initial spin configuration; for a review, see [25].

The case we study here is the dynamics of the Edwards–Anderson model in two dimensions. In this context some progress has been recently made. In the work of [2] there are upper and lower bounds on $q(J,t)$ for the Edwards–Anderson model. There the quantity $q(J,t)$ is the absolute value of the difference between the expectation at time t of the spin in the origin, starting with a fixed initial condition, and its value in the equilibrium (J is the strength of the coupling).

The bounds in [2] are almost optimal and show that the velocity of convergence to the stationary measure, in the Griffiths' region [16], for almost all initial conditions (with respect to the Gibbs measure), have the following bounds if the probability distribution of interactions decays more than exponentially and if there is a probability greater than zero to have interactions different in absolute value.

The lower bound is

$$q(J,t) \geqslant c_1(J)\exp\bigl[-t\exp\bigl[-k_1(\ln t)^{1-\frac{1}{d}}\bigr]\bigr]$$

and the upper bound

$$q(J,t) \leqslant c_2(J)\exp\bigl[-t\exp\bigl[-k_2(\ln t)^{1-\frac{1}{d}}(\ln\ln t)^{d-1}\bigr]\bigr].$$

Also in the context of Glauber dynamics, we mention the work [11], which discusses the spectral gap for the Random Energy Model. It is proved in there that the gap has the same dependence on the volume at any temperature, at least in the leading order. This shows that the situation for mean field models can be quite different from the short range situation that we discuss here.

Our work deals with Glauber dynamics for Edwards–Anderson spin-glass model in $\mathbf{Z}^2$. It differs in results and methods from [2], first they have results only for the diluted Ising models and at temperature regions different from ours. Moreover, in [2] the dynamics is defined on the infinite graph $\mathbf{Z}^d$, with $d \geqslant 2$, while we analyze a sequence of boxes Λ to study the spectral gap of the finite transition matrix.

Another difference from [2] consists in the choice of the interactions; in our case we can also take all interactions with equal absolute value. Some results for zero-temperature dynamics in the equal absolute value ($\pm J$) spin glass model have also been obtained in [13].

Also in our case it seems natural to think that a temperature T_0 exists, like in the ferromagnetic case, separating two regions of the temperature parameter range in which there is a qualitatively different relaxation speed to the equilibrium; i.e. we think that for every $T < T_0$ there is a slow relaxation to the equilibrium while for every $T > T_0$ a fast relaxation to the equilibrium takes place. Seeing also the work [23] we conjecture that T_0 is equal to T_c, *the critical temperature for the ferromagnetic Ising model*. It seems to be relevant also for spin-glasses because there are arbitrarily large ferromagnetic regions.

Our work is similar in construction and in techniques to [27], in which a bound is proved for the spectral gap at low temperature for the ferromagnetic stochastic Ising model; more precisely, it shows that in the low temperature regime, the spectral gap decreases exponentially fast in $|\Lambda|^{(d-1)/d}$ for all dimensions $d \geqslant 2$.

We will study the dynamics of a short-ranged Ising model (spin-glass), where the Hamiltonian is

$$H_{\Lambda_L,\mathbf{J}}(\sigma) = -\sum_{\langle i,k\rangle, i,k\in\Lambda_L} J_{i,k}\sigma_i\sigma_k. \tag{1}$$

Here $\Lambda_L = [-L, L]^2 \cap \mathbf{Z}^2$, and $\mathbf{J}$ denotes a specific realization of the interactions $\{J_{i,k}\}_{i,k\in\mathbf{Z}^2}$, the spins $\sigma_i = \pm 1$ and the sum is only over nearest-neighbor pairs, with i and k belonging to the finite region Λ_L. The $\{J_{i,k}\}_{i,k\in\mathbf{Z}^2}$ are independent identically distributed random variables on $\{-1, 1\}$, with a Bernoulli distribution of parameter $p \in (0, 1)$. Starting from (1) we define the Gibbs measure and we consider any *local irreducible Markov chain* having the Gibbs measure as stationary measure. The adjective *local* means that for any transition there are only finitely many different spins between the new configuration and the old one, and these different spins are all inside the translation of a fixed finite region independent of Λ. To fix ideas we can think of Glauber dynamics that in every transition can have only a single different spin.

We will also introduce a new kind of distance $D_\rho(\cdot,\cdot)$ between two measures that are defined on a regular lattice. We show that there are some spins for which the auto-correlation time decays slowly to zero (see Theorem 2). In Theorem 3 we will use the distance D_ρ with its induced topology to study the convergence of the measure at time t, $\mu^{(t)}_{\Lambda,\sigma,J,T}$, starting in the configuration σ (of the Markov chain) to the stationary measure $\mu_{\Lambda,J,T}$. We will show that for T large enough there exist $c > 0$ and $M > 0$ uniformly bounded in Λ and σ (see Theorem 3) such that

$$D_\rho\big(\mu^{(t)}_{\Lambda,\sigma,J,T}, \mu_{\Lambda,J,T}\big) < M\mathrm{e}^{-ct}. \tag{2}$$

On the other hand for T small enough we will show that there exists an $\varepsilon = \varepsilon(T) > 0$ and a box Λ depending only on the time t such that

$$\sup_\sigma D_\rho\big(\mu^{(t)}_{\Lambda,\sigma,J,T}, \mu_{\Lambda,J,T}\big) > \varepsilon \tag{3}$$

almost surely in the realization of the interactions **J** (see Theorem 3). So there are two regions $(0, T_B)$ – low temperature – and (T_A, ∞) – high temperature – in which the relaxation to the equilibrium measure of the Glauber process is qualitatively different.

In Section 5 we deal only with reversible Markov chains on boxes Λ_L at temperature $T = 1/\beta$. For the transition matrices associated to these Markov chains all the eigenvalues $\rho_{L,\beta,i}$ are real because the Markov chain is reversible. In high temperature (small β) it is known that the spectral gap $\lambda_{*,L,\beta}$ of the Markov chain is positive and uniformly larger than zero: $\lambda_{*,L,\beta} > C_\beta$; in low temperature we will prove that there is a constant $A_\beta > 1$ such that for all large L:

$$\lambda_{*,L,\beta} < A_\beta^{-\sqrt{\ln L}} \tag{4}$$

almost surely in the realization of the interactions **J**. The most important difference in results between this work on the stochastic Edwards–Anderson model and the article [27] on stochastic ferromagnetic Ising models is on the bounds of relaxation speed at low temperature. In fact, in the ferromagnetic model with free boundary conditions, there is an auto-correlation time that increases exponentially with the length L of the box $\Lambda_L = [-L, L]^2 \cap \mathbf{Z}^2$, while in the spin glass models here,we are able only to show that it increases with $A_\beta^{\sqrt{\ln L}}$ where $A_\beta > 1$ for large enough β (see also (4) and Proposition 1.2).

As mentioned, the study of the dynamical or equilibrium properties of spin glasses is more difficult than the corresponding problems for ferromagnetic systems because some simple and useful inequalities are lost; as examples we mention the FKG inequality and the attractive property for the Glauber process; for these reasons the results are weaker.

The idea at the origin of this work is extremely simple and it consists in observing that whenever there are some ferromagnetic zones in the spin system independent of the rest of the system and if these regions are arbitrarily large then, in these regions, we can use the results of ferromagnetic systems (see [7,27,21–23]) to find a slow convergence to the equilibrium measure. Although our proof only works for very small T, this argument suggest that the conclusions should remain valid for all $T < T_c$.

In a disordered model (with random interactions between the spins) the independence of a region from the rest of the system is obtainable by putting very small (or zero) interactions on its boundary. By taking a sequence of squares, with increasing side length, satisfying the previous property, we would obtain a slow convergence to equilibrium of Glauber dynamics. If, as in our case, all the interactions have equal absolute value, we have a more difficult problem; but also in this case we can get a region *almost* independent of the rest using a particular realization of the interactions that we present in Section 2.

The techniques of this work appear to be promising and generalizable to frustrated Potts models [3] and dimensions larger than two. The reader can see Section 1.1 for the main ideas and can find precise definitions in Section 2.

1.1. Main ideas

In this section we present the main ideas and results from Section 3 through Appendix A. In Appendix A we will prove Lemmas A.4 and A.5; in these two lemmas it is shown that if there is an interface (see definition in Section 2) in a particular region, denoted by $B_{O,l}$, then there will exist a dual circuit γ_3 in which the percentage of unsatisfied edges is larger than the percentage of satisfied edges. We denote with $(\gamma_3)_n$ (resp. $(\gamma_3)_r$) the unsatisfied (resp. satisfied) edges of the dual circuit γ_3 and with $|A|$ the cardinality of a set A. Lemmas A.4 and A.5 are combined in Proposition 1.1.

PROPOSITION 1.1. – *If there is an interface in $B_{O,l}$ then there exists a dual circuit γ_3 in B_l such that*: $|(\gamma_3)_n| - |(\gamma_3)_r| \geqslant |\gamma_3|/10$.

This proposition has a pure geometrical flavor, but it becomes relevant in conjunction with a result of Section 3. In Section 3, it is shown that the probability of a configuration σ in which there is a circuit γ_3 with these characteristics is exponentially decreasing with $|\gamma_3|$. Now using the strong Markov property and other arguments (see Lemma A.6) it is quite simple to show that the auto-correlation time for a spin inside $B_{O,l}$ remains large for times that are increasing with the side length of Λ (see Theorem 2). We introduce a new distance between measures defined on a regular lattice that will be denoted by D_ρ. In Theorem 3 we will use this distance D_ρ with its induced topology to study the convergence of the measure at time t of the Markov chain, called $\mu^{(t)}_{\Lambda,\sigma,J\beta}$, to its stationary measure; we show that at high temperature there is an exponential convergence of $\mu^{(t)}_{\Lambda,\sigma,J\beta}$ to the stationary measure independent of the size of Λ (ergodicity). But at low temperature, the time to converge to the stationary measure grows to infinity with increasing $|\Lambda|$.

In Section 5 we present a result on the spectral gap of the associated reversible Markov chains that follows from the slow convergence results of the earlier section. We give some basic definitions now to be able to present that result. In all of this paper we deal with irreducible and aperiodic Markov chains. We assume in the last section that the transition matrix $Q_{L,\beta}$ associated with our Markov chain is reversible relative to the Gibbs measure $\mu_{\Lambda_L,\mathbf{J}\beta}$ of the Ising model on the box $\Lambda_L = [-L, L]^2 \cap \mathbf{Z}^2$; that means:

$$\mu_{\Lambda_L,\mathbf{J}\beta}(\sigma) Q_{L,\beta}(\sigma,\omega) = \mu_{\Lambda_L,\mathbf{J}\beta}(\omega) Q_{L,\beta}(\omega,\sigma) \tag{5}$$

for all σ, ω in the space $\Omega_{\Lambda_L} = \{-1, 1\}^{\Lambda_L}$. This implies that the Gibbs measure is the stationary distribution for $Q_{L,\beta}$ and also that $\mu_{\Lambda_L,\mathbf{J}\beta}(\sigma) > 0$ for all $\sigma \in \Omega_\Lambda$. The transition matrix $Q_{L,\beta}$ can be thought as the multiplication of single spin flip transition matrices. Let us define $T^x_{L,\beta}$ as a reversible transition matrix that leaves unchanged the configuration out of the vertex $x \in V$ and that satisfies the detailed balanced with respect to the Gibbs measure conditioned to the configuration $\sigma_{\setminus x}$ out of x. Let us define

$$Q_{L,\beta} = \prod_{x\in\Lambda} T^x_{L,\beta}; \tag{6}$$

trivially $Q_{L,\beta}$ is dependent on the order of the multiplication of the matrices $T^x_{L,\beta}$ but the result will be independent of this order. The operator $Q_{L,\beta}$ is a self-adjoint contraction on $\mathbf{L}^2$ of the Gibbs measure; so all its eigenvalues are in $(-1, 1]$ and the eigenvectors have all the components in $\mathbf{R}$. We order the eigenvalues $\rho_{i,L,\beta}$ of $Q_{L,\beta}$

$$1 = \rho_{0,L,\beta} \geqslant \rho_{1,L,\beta} \geqslant \cdots \geqslant \rho_{n-1,L,\beta} > -1. \tag{7}$$

Let us define $\rho_{*,L,\beta} = \max\{\rho_{1,L,\beta}, |\rho_{n-1,L,\beta}|\}$ and let $\|\mu - \nu\|_{\mathrm{Var}}$ be the distance in variation between the measures μ and ν; then Dobrushin's bound says

$$\rho_{*,L,\beta} \leqslant \max_{\sigma,\omega} \big\| Q_{L,\beta}(\sigma,\cdot) - Q_{L,\beta}(\omega,\cdot)\big\|_{\mathrm{Var}}. \tag{8}$$

As a consequence of our hypotheses it is easy to show the equality

$$\ln \rho_{*,L,\beta} = \lim_{n\to\infty} \frac{1}{n} \ln\Big[\max_{\sigma,\omega} \big\| Q^n_{L,\beta}(\sigma,\cdot) - Q^n_{L,\beta}(\omega,\cdot)\big\|_{\mathrm{Var}}\Big]. \tag{9}$$

The equality (9) follows by Dobrushin and Diaconis–Stroock inequalities (see [7, pp. 42–43]). In an interval $(0, T_B)$ the process has a slow relaxation to equilibrium; but in a high temperature interval (T_A, ∞) we obtain a mixing property for the process (also for the process on $\mathbf{Z}^d$ with $d \geqslant 2$) and so a rapid convergence of the process to equilibrium. We consider a sequence of boxes $\Lambda_L = [-L, L]^2 \cap \mathbf{Z}^2$ and we have a bound on the spectral gap $\lambda_{*,L,\beta} := 1 - \rho_{*,L,\beta}$ of the transition matrix $Q_{L,\beta}$.

We prove in Section 5 the following proposition for appropriate β_A and β_B with $0 < \beta_A < \beta_B < \infty$.

PROPOSITION 1.2. – (i) *For all $\beta < \beta_A$ there is $C_\beta > 0$ such that for all L we have*:

$$\lambda_{*,L,\beta} > C_\beta > 0. \tag{10}$$

(ii) *For all $\beta > \beta_B$ there exist a constant $A_\beta > 1$ such that almost surely in the realization of the interactions* $\mathbf{J}$ *for all large L we have*:

$$\lambda_{*,L,\beta} < A_\beta^{-\sqrt{\ln L}}. \tag{11}$$

The conclusion (i) is an old result of Aizenman and Holley (see [1]), and the conclusion (ii) will be proved in Section 5 as a consequence of Theorem 3 and Proposition 4.1 via the variational characterization of $\lambda_{1,L,\beta}$ (see [7]). We believe that the spectral gap $\lambda_{*,L,\beta}$ converges to zero faster than we are able to show.

2. The model and some definitions

The graph. We consider a graph $G = (V, E)$ in which the vertices are the points of $\mathbf{Z}^2 \cap \Lambda$ where Λ is a finite subset of $\mathbf{R}^2$. The edges E are pairs of vertices $\{v, w\}$ where $v = (v_1, v_2)$, $w = (w_1, w_2)$ and $|v_1 - w_1| + |v_2 - w_2| = 1$; the vertices belonging to the same edge are called *neighbors*. We, sometimes, denote the set of edges by $E(\Lambda)$ to make clear the dependence on Λ; V or $V(\Lambda)$ is the set of vertices. With abuse of notation, in the following, we will denote a graph only by writing the set of its vertices.

An edge e is *incident* with a vertex v if $v \in e$; two distinct edges e_1, e_2 are incident if there exists a vertex v with $v \in e_1$ and $v \in e_2$.

Paths and circuits. A *path* π is a sequence of vertices and edges, say $x_0, b_1, x_1, b_2, \ldots, b_l, x_l$ where $b_i = \{x_{i-1}, x_i\}$, $1 \leqslant i \leqslant l$, and all its edges are distinct; π is a *path* *between* x_0 and x_l and its *length* is l. A path whose end-vertices coincide (a close path) is called a *circuit*. A path is called *self-avoiding* if all its vertices are distinct, i.e. given a path α with $v_i, v_j \in \alpha$ then $v_i \neq v_j$ if $i \neq j$. A circuit $\gamma = x_0, b_1, x_1, b_2, \ldots, b_l, x_0$ is called *self-avoiding* or a *cycle* if $\gamma_1 = x_0, b_1, x_1, b_2, \ldots, b_{l-1}, x_{l-1}$ is a self-avoiding path. It is easy to see that one can identify a path or a circuit with the set of its edges. The cardinality of a set A is indicated by $|A|$ and for the length of a path π we also write $|\pi|$.

Some sets. Given a set $B \subset E$ we define the *vertex set of* B, $V(B)$, as all the vertices $v \in V$ such that there exists $w \in V$ with $\{v, w\} \in B$; for a vertex set A we put $V(A) = A$. Given a set of vertices A we define $Edges(A)$ as the set of edges $e \in E$ such that $V(\{e\}) \in A$; we remark that given a set of vertices $V_1 \subset V$ we have $V(Edges(V_1)) \subset V_1$ and given a set of edges $E_1 \subset E$ we have $Edges(V(E_1)) = E_1$. In the following if we are doing a set operation using a vertex set A and an edge set B then the two sets are thought as vertex sets; for example $A \cap B$ corresponds to $A \cap V(B)$.

Boundary of a set. Given a set C of vertices we define the boundary ∂C to be the set of edges $b \in E$ with $b \cap C \neq \emptyset$ and $b \cap C^c \neq \emptyset$; given a set of edges A we abbreviate the notation $\partial V(A)$ with ∂A.

Distance in the graph. The distance between two vertices $x, y \in \mathbf{Z}^2$ is given by $d(x, y) = \sup_{i=1,2} |x_i - y_i|$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$; the distance between two sets of vertices A and B is indicated $d(A, B) = \inf_{x \in A, y \in B} d(x, y)$; if we have two sets A, B of vertices or edges we put $d(A, B) = d(V(A), V(B))$, so the distance is well defined also in the case that we have a set of vertices and a set of edges.

Translation. We denote with $A + i$ the translation of the set A with the vector $i = (i_1, i_2)$.

Dual graph. The graph of vertices $\mathbf{Z}^2$ and edges between the neighbor vertices has a *dual graph* having vertex set $(\frac{1}{2}, \frac{1}{2}) + \mathbf{Z}^2$ and the edges between all the pairs of vertices verifying $|v_1^* - w_1^*| + |v_2^* - w_2^*| = 1$ with $v^* = (v_1^*, v_2^*)$ and $w^* = (w_1^*, w_2^*)$. We call *dual path, dual circuit, etc.* a path, a circuit, etc. in the dual graph. This notion can be made more general for all the planar graph, but for our aim it is not necessary.

Each edge e^* of the dual graph – seen as a line connecting two vertices – crosses an edge e of the original graph; so there is a bijection between the edges in the dual graph and the edges in the original graph, therefore we can also denote a dual path with the set of its intersected edges. So we consider a dual path or a dual circuit ω in two different ways; sometimes it is regarded as the set of the intersected edges and we will write ω, some other times it is relevant the order of the dual vertices and of the dual edges and in this case we will write $\Gamma(\omega)$ to stress that is an ordered set. We notice that Γ is not a function because given a set of edges related to a dual path or a dual circuit could be several manners to order them to form a dual path or a dual circuit. So $\Gamma(\omega)$ means a particular choice of the allowed orders and if the order will be relevant we will explicitly write it. Given an edge b_i the associated dual edge is denoted with b_i^*.

The interior and exterior part of a circuit. We recall that $(\mathbf{Z}^2, E(\mathbf{Z}^2))$ is a planar graph and that there is a standard unique way to associate a regular curve in $\mathbf{R}^2$ with a

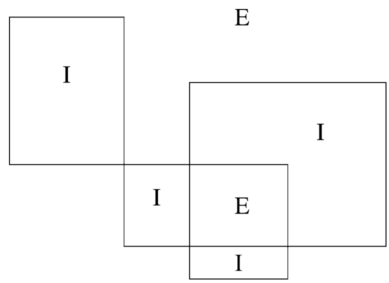

Fig. 1. An example of the interior and the exterior part of a circuit.

(dual) circuit γ for a planar graph. The regular curve associated to a circuit γ divides the plane into regions that we can color with two colors in such a way that all the bordering regions have different colors (see [6]). Exactly one of these regions will be infinite; any vertex inside a region that has the same color as this infinite region is in the *exterior part* of γ and it is denoted by $Ext(\gamma)$. *The interior part* $Int(\gamma)$ is the set of all the vertices that are inside the other regions (see Fig. 1). This construction is possible because the regions and the borders can be put in a 1–1 relation with a bipartite graph (see [6]).

The block. Let us define a *block* $B_l(v)$ as the set of all the edges in a square of side length $4l$ and center v; $B_{O,l}(v) \subset B_l(v)$ is the set of all the edges in a square of side length $2l$ and center v. We call *frame of a block* the set:

$$\Delta B_{O,l}(v) = \big\{b = \{u_1, u_2\} \in E\colon\ d(b, v) = l + 1 \text{ and } d(u_1, v) = d(u_2, v)\big\}. \qquad (12)$$

The spin space. The space $\Omega_\Lambda = \{-1, 1\}^\Lambda$ is endowed with the discrete topology, a spin configuration is an element $\sigma \in \Omega_\Lambda$; for a bounded Λ the σ-algebra $\mathcal{F}_\sigma$ consists of all the subsets of Ω_Λ. We denote with σ_i or $\sigma(i)$ the *spin* on the vertex $i \in V$; we indicate with $\sigma(i, t)$ the spin on the vertex i at a time t and with $\sigma(\cdot, t)$ the spin configuration at a time t.

Clusters. We consider a subset E_0 of the edge set E as an *active edge set*; given a specification of the active edges E_0 we say that two vertices v_1, v_2 are *connected* if there exists a path π between v_1 and v_2 made only of active edges. Given the set E_0 the vertex set V is divided into maximal connected components called *clusters*. In the following the active edges set, unless otherwise stated, is $E_0 = \{b = \{x, y\} \in E \mid \sigma_x = \sigma_y\}$. We will always deal with the infinite graph $\mathbf{Z}^2$ or with a finite subset of it in which we preserve the original structure of $\mathbf{Z}^2$. We define the external boundary of a vertex set V_1 as

$$\partial^E V_1 = \big\{b \in \partial V_1\colon \text{ there exists an infinite path } \gamma \text{ such that } Edges(\gamma) \cap \partial V_1 = b\big\}. \quad (13)$$

We say that a cluster A *surrounds* a vertex v if there is a circuit ω with all the vertices inside A and $v \in Int(\omega)$.

In the following proposition we consider the *clusters of positive spins* or positive vertex cluster where an edge $e = \{x, y\} \in E$ is active if $\sigma_x = \sigma_y = 1$; we consider also clusters of a single vertex x if $\sigma_x = 1$ and the incident edges are not active; analogously we could define the *clusters of negative spins*. For a cluster of positive spins C we have that if $b = \{x, y\} \in \partial C$ then $\sigma_x \sigma_y = -1$. We recall that a graph is called *Eulerian* if in each vertex there is an even number of incident edges. The following result is known

PROPOSITION 2.1. – *Let $\{A_i^+\}_{i=1,\dots,n}$ be a set of distinct finite clusters of positive spins. Then*
(1) $\partial A_i^+ \cap \partial A_j^+ = \emptyset$ *if* $i \neq j$.
(2) $\bigcup_i \partial A_i^+$ *and* $\bigcup_i \partial^E A_i^+$ *are sets of dual circuits.*
(3) *Every maximal dual connected set in* $\bigcup_i \partial A_i^+$ *or in* $\bigcup_i \partial^E A_i^+$ *is a dual circuit.*

Proof. – (1) See [17, p. 387].
(2) Because all the considered graphs are Eulerian having, in every vertex, 0, 2 or 4 incident edges. This is equivalent to the group structure exposed in [6].
(3) This follows from (2) and the connection of an Eulerian set. □

The Hamiltonian. We now define the Gibbs measure on the measurable space $(\Omega_\Lambda, \mathcal{F}_\sigma)$ (in a standard way); the Hamiltonian for a finite Λ is

$$H_{\Lambda,\mathbf{J}}(\sigma) = -\sum_{b=\{i,k\}\in E(\Lambda)} J_b \sigma_i \sigma_k \tag{14}$$

where $J_{\{i,k\}}$ is the interaction between the spins in i and k; $\mathbf{J}$ is the interaction configuration. Let us define the Gibbs measure, with free boundary conditions, on the single configuration σ on a finite volume as

$$\mu_{\Lambda,\mathbf{J}\beta}(\sigma) = \frac{\exp(-\beta H_{\Lambda,\mathbf{J}}(\sigma))}{Z} \tag{15}$$

where $Z = Z_{\Lambda,\mathbf{J}\beta}$ is the normalizing factor (*partition function*) and β is called *inverse temperature* and it is defined on $(0, \infty)$. Following [8] and [18] it can also be defined a Gibbs measure $\mu_{\mathbf{J}\beta}$ on the infinite graph $\mathbf{Z}^2$.

Space of interactions. We introduce the space $\Omega_{\Lambda,\mathbf{J}} = \{-1, 1\}^{E(\Lambda)}$; an interaction configuration is an element $\mathbf{J} \in \Omega_\mathbf{J}$ where we omit, as in the following, the index Λ; the σ-algebra $\mathcal{F}_\mathbf{J}$ consists of all the subset of $\Omega_\mathbf{J}$. The probability measure on $\Omega_\mathbf{J}$ is a Bernoulli distribution with parameter $p \in (0, 1)$ and $P_\mathbf{J}(J_b = 1) = p$; we do not study the case $p = 1, 0$ that is well known in literature and corresponds to the ferromagnetic and to the anti-ferromagnetic Ising model. For the symmetric distribution $P_\mathbf{J}(J_b = 1) = 1/2$ we say that (15) define the Edwards–Anderson model in the finite volume Λ. In a standard way it is possible to define the model on the infinite graph $\mathbf{Z}^2$. An edge b is called *negative* or *anti-ferromagnetic* if its interaction is $J_b = -1$ and it is called *positive* or *ferromagnetic* if its interaction is $J_b = 1$.

Satisfied edges. We define

$$\eta_b = \eta_b(\sigma) = \operatorname{sign}(J_b \sigma_x \sigma_y) \quad \text{with } b = \{x, y\} \tag{16}$$

and we say that η_b is *satisfied* if $\eta_b = 1$; $\eta \in \{-1, 1\}^{E(\Lambda)}$ is called the *configuration on the edges*. It is simple to see that given $\mathbf{J}$ and η it is not always possible to find a configuration

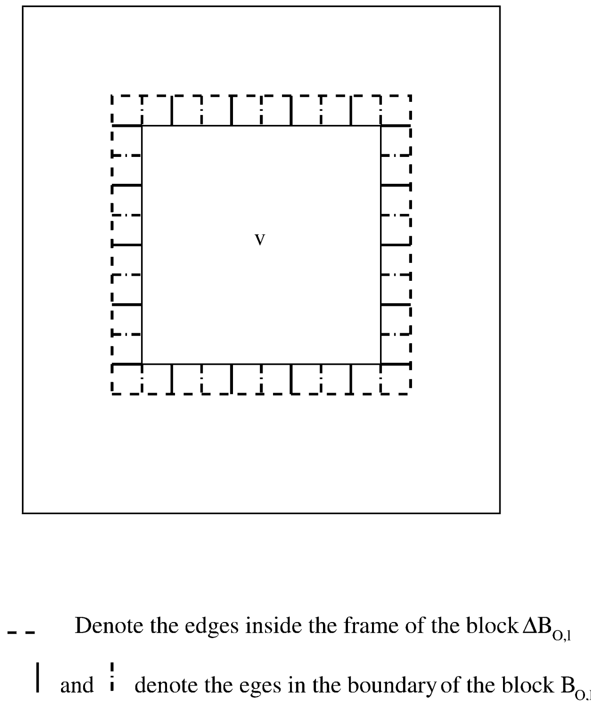

Fig. 2. An example of a block.

σ that is compatible with the prescriptions given by **J** and η, but that if there exists such a configuration σ then also $-\sigma$ is compatible and no other configurations have this property. Given **J**, σ and an edge set A, we denote by $A_r(\sigma)$ the set of satisfied edges in A and we set $A_n(\sigma) = A \setminus A_r(\sigma)$ the unsatisfied edges; we do not write the dependence on **J** because it is a quenched random variable, so **J** is assumed to be fixed once for all; **J** is the random environment on which we will construct random measures and stochastic processes. Sometimes we omit also the dependence on σ and we write simply A_r and A_n. We will also use γ_n and γ_r to denote the satisfied edges of a circuit γ.

The event $C_{l,\Lambda}$. We say that $C_{l,\Lambda}$ is the event consisting in the existence in $E(\Lambda)$ of a block $B_l(v)$ in which if $b \in B_l(v) \setminus \partial B_{O,l}(v)$ then $J_b = 1$ and if $b \in \partial B_{O,l}(v)$ then alternatively J_b is equal to 1 and -1. Sometime in the following we omit the index v because in every box Λ we will fix a unique square $B_l(v)$, if there is one, initially chosen with the specified properties (see Fig. 2).

Interface. We say that an interface α is a dual path with all the edges inside $B_{O,l}(v)$ and the ends vertices on $\partial B_{O,l}$ such that all the edges $e \in \alpha$ are unsatisfied and $|\alpha| \geqslant 2l$ (see Fig. 3).

Distance between measures. We write σ_A or $\sigma|_A$ for the restriction of the configuration σ to a set of vertices A or for the corresponding cylinder event on Ω_Λ.

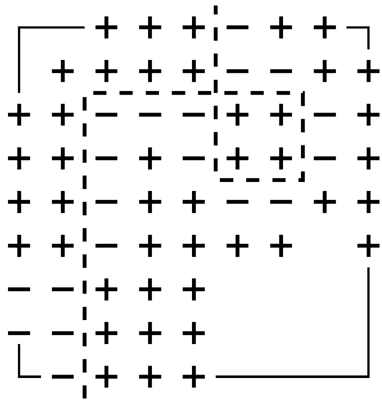

Fig. 3. An example of an interface.

We introduce the following distance between measures

$$D_\rho(\mu_1,\mu_2)=\sup_{i\in\Lambda}\sum_{A,i\in A\subset\Lambda}\sum_{\sigma_A\in\{-1,1\}^A}\big|\mu_1(\sigma_A)-\mu_2(\sigma_A)\big|\rho^{|A|} \tag{17}$$

where A is a connected vertex set containing i; in this case all the edges of *Edges*(A) are active and $\rho\in(0,1)$ is a parameter. It is easy to check that if ρ is small enough then the distance $D(\cdot,\cdot)$ is uniformly bounded for every couple of probability measures on $(\{-1,1\}^{\mathbf{Z}^2},\mathcal{F})$ (where $\mathcal{F}$ is the σ-algebra generated by cylinders).

Dynamics. We consider a Glauber process acting on the spins variables; a Glauber process changes one spin at a time. We call Glauber (process) different kinds of processes with discrete or continuous time that update each single spin in a fixed or in a random order.

Now we give the notions that allow to define a Markov process with a continuous time for a spin system on a finite or infinite graph. We follow the exposition in [2] and [19]. Let us define the *gradient* of a function f as

$$(\nabla_x f)(\sigma)=f\big(\sigma^x\big)-f(\sigma)$$

where $(\sigma^x)_x=-\sigma_x$ and $(\sigma^x)_y=\sigma_y$ if $y\neq x$; by $\|f\|_\infty$ we mean the supremum norm of f. We also define the norm

$$|||f|||=\sum_{x\in\mathbf{Z}^d}\|\nabla_x f\|_\infty. \tag{18}$$

The dynamics of the Markov process on $\{-1,1\}^\Lambda$ for a finite Λ is defined by the generator

$$(L_{\Lambda,\mathbf{J}}f)(\sigma)=\sum_{x\in\Lambda}c_{\mathbf{J}}(x,\sigma)(\nabla_x f)(\sigma),\quad \sigma\in\{-1,1\}^\Lambda, \tag{19}$$

where the non-negative quantities $c_{\mathbf{J}}(x,\sigma)$ are the *transition rates* for the process. We assume that for the $c_{\mathbf{J}}(x,\sigma)$'s the following holds:

(1) *Nearest neighbor interactions*, if $\sigma(y) = \sigma'(y)$ for all y adjacent to x then $c_{\mathbf{J}}(x, \sigma) = c_{\mathbf{J}}(x, \sigma')$.
(2) *Reversible process* or *detailed balance*.

$$\exp\left[-H_{\Lambda,\mathbf{J}}(\sigma)\right] c_{\mathbf{J}}(x, \sigma) = \exp\left[-H_{\Lambda,\mathbf{J}}\left(\sigma^x\right)\right] c_{\mathbf{J}}\left(x, \sigma^x\right). \tag{20}$$

(3) *Positivity and boundedness*. There exist positive real numbers c_1 and c_2 such that

$$c_1 \leqslant \inf_{x,\sigma} c_{\mathbf{J}}(x, \sigma) \leqslant \sup_{x,\sigma} c_{\mathbf{J}}(x, \sigma) \leqslant c_2. \tag{21}$$

The most frequently used transition rates are those of the *Metropolis dynamic* and of the *heat-bath* dynamic, both have the Gibbs measure as stationary measure. The first has the following transition rate:

$$c_{\mathbf{J}}(x, \sigma) = \min\left\{\mathrm{e}^{-\nabla_x H_{\Lambda,\mathbf{J}}(\sigma)}, 1\right\}; \tag{22}$$

the second is defined by the following transition rate:

$$c_{\mathbf{J}}(x, \sigma) = \mu_{\Lambda,\mathbf{J}\beta}\left(\sigma_x^x \mid \sigma_{\Lambda\setminus x}\right) = \left[1 + \mathrm{e}^{\nabla_x H_{\Lambda,\mathbf{J}}(\sigma)}\right]^{-1}. \tag{23}$$

At this point we can define the generator for finite or infinite spaces as for example $\{-1, 1\}^{\mathbf{Z}^d}$. If the nearest neighbor interactions condition holds and $c_2 < \infty$ we obtain that the generator is well defined on the space of all functions having finite $|||f|||$. For the construction of the generator, weaker hypotheses are needed as, for example, finite range interaction (see [2] and [19]). It has been proved that the closure in $C(\Omega_{\mathbf{Z}^2})$ of the generator (19) is also a Markov generator (see [19]). If we consider also the detailed balance (20) then it is natural to consider the closure of the generator in $L^2(\Omega_{\mathbf{Z}^2}, d\mu_{\mathbf{J}\beta})$ (where $\mu_{\mathbf{J}\beta}$ is a reversible measure for the process); also in this case it has been proved that the closure of the generator is a Markov generator (see [19]). Moreover, the generator on all the infinite graph can be obtained as limit of the generators on finite Λ's.

In the following Λ is a finite box or is equal to $\mathbf{Z}^2$. We call $S_{\Lambda,\mathbf{J}\beta}$ the semigroup of our Markov process and by construction we have

$$\lim_{t\to 0^+} \frac{S^t_{\Lambda,\mathbf{J}\beta} - I}{t} = L_{\Lambda,\mathbf{J}\beta}. \tag{24}$$

We set $\mu^{(t)}_{\Lambda,\mathbf{J}\beta} = \mu_{\Lambda,0} S^t_{\mathbf{J}\beta}$ where $\mu^{(t)}_{\Lambda,\mathbf{J}\beta}$ is the induced measure in time t with initial measure $\mu_{\Lambda,0}$. In the case in which $\mu_{\Lambda,0} = \delta(\sigma)$ (the initial measure is concentrated on the configuration $\sigma = \sigma_\Lambda$) we write $\mu^{(t)}_{\Lambda,\sigma,\mathbf{J}\beta,} = \delta(\sigma) S^t_{\Lambda,\mathbf{J}\beta}$.

3. The exponential decay of the probability of large interfaces at low temperature

Let us consider a change of sign, called *spin flip*, of the configuration σ on a vertex set A; it is easy to see that this corresponds to change the configuration on the edges with the following rules:

$$(\partial A)_n \to (\partial A)_r \quad \text{and} \quad (\partial A)_r \to (\partial A)_n. \tag{25}$$

The spin flip in the region A does not change the configuration on the edges which have two vertices in A in fact $J_{x,y}\sigma_x\sigma_y = J_b(-\sigma_x)(-\sigma_y)$; similarly for A^c. The variable $\eta_{x,y}$ change only if a vertex is in A and the other in A^c. The vice versa holds, the transformation (25) on a dual circuit corresponds to a spin flip in the region $\mathrm{Int}(\gamma)$ or $\mathrm{Ext}(\gamma)$. Note that the dual circuit γ can be either self-avoiding or not self-avoiding. Using this property it is easy to prove the following lemma (see also [5] where a similar technique is applied for the percolation in frustrated systems).

LEMMA 3.1. – *Given* $\varepsilon > 0$ *and the probability measure* $\mu_{\Lambda,\mathbf{J}\beta}$ *of the Ising model in dimension* 2 *we have that for every dual circuit* γ *the probability to have* $|\gamma_n| - |\gamma_r| > \varepsilon|\gamma|$ *is exponentially decreasing with the length* $|\gamma|$ *and the parameter* β*, uniformly in* Λ.

Proof. – Let us fix a dual circuit γ. We have:

$$\mu_{\Lambda,\mathbf{J}\beta}\big(\{|\gamma_n| - |\gamma_r| > \varepsilon|\gamma|\}\big) \leqslant \max_{\eta_{\setminus\gamma}} \mu_{\Lambda,\mathbf{J}\beta}\big(\{|\gamma_n| - |\gamma_r| > \varepsilon|\gamma|\}|\eta_{\setminus\gamma}\big) \tag{26}$$

where $\eta_{\setminus\gamma}$ is a configuration η in $E\setminus\gamma$ that has to be compatible with a spin configuration σ, i.e. there exists σ such that $\eta_b J_b\sigma_x\sigma_y \geqslant 0$ with $b = \{x, y\} \in E\setminus\gamma$. For every $\eta_{\setminus\gamma}$ there are four spin configurations that are compatible. By (25) for every spin configuration compatible with $\eta_{\setminus\gamma}$ having $|\gamma_r| - |\gamma_n| > \varepsilon|\gamma|$ there is a spin configuration compatible with $\eta_{\setminus\gamma}$, obtained flipping the region A, with $|\gamma_n| - |\gamma_r| > \varepsilon|\gamma|$. We can write the Hamiltonian of the Ising model as a function of η. We write the Hamiltonian:

$$H = -\sum_{b\in E(\Lambda)} \eta_b; \tag{27}$$

so we have

$$H = \big|\big(E(\Lambda)\big)_n\big| - \big|\big(E(\Lambda)\big)_r\big| \tag{28}$$

where $E(\Lambda)$ are the edge in the box Λ. On a single dual circuit γ, fixing $\eta_{\setminus\gamma}$, there is an energy difference between the compatible spin configuration that is

$$\Delta H = 2\big(|\gamma_n| - |\gamma_r|\big); \tag{29}$$

so adding on the compatible configurations we have:

$$\begin{aligned}\max_{\eta\setminus\gamma} \mu_{\Lambda,\mathbf{J}\beta}\big(\{|\gamma_n| - |\gamma_r| > \varepsilon|\gamma|\}|\eta_{\setminus\gamma}\big) &\leqslant \frac{\mathrm{e}^{-\beta(|\gamma_n|-|\gamma_r|)}}{\mathrm{e}^{-\beta(|\gamma_n|-|\gamma_r|)} + \mathrm{e}^{-\beta(|\gamma_r|-|\gamma_n|)}} \\ &\leqslant \frac{1}{1+\mathrm{e}^{2\beta|\gamma|\varepsilon}} < \mathrm{e}^{-2\beta|\gamma|\varepsilon}\end{aligned} \tag{30}$$

which is the announced exponential bound. □

4. Auto-correlation time for the Edwards–Anderson spin-glass model for finite volume at low temperature

In this section we give the results on the dynamics. From the symmetry of the Hamiltonian, at zero magnetic field with free boundary condition, we have $\mu_{\Lambda,\mathbf{J}\beta}(\sigma_A) = \mu_{\Lambda,\mathbf{J}\beta}(-\sigma_A)$ for every vertex set A. For a single vertex, given the initial configuration, we will find that the probability to have $\sigma(v,t) \neq \sigma(v,0)$ will be for long time, with Λ invading $\mathbf{Z}^2$, separated by the equilibrium value $1/2$. We will prove these results using the strong Markov property and the FKG inequality which in a ferromagnetic Ising model, with measure $\mu^f_{\Lambda,\beta}$, give the inequality

$$\mu^f_{\Lambda,\beta}\big(\sigma(v) = 1 \mid \sigma(\omega) = 1\big) \geqslant \mu^{f,+}_{\Lambda,\beta}\big(\sigma(v) = 1\big) \tag{31}$$

where $\mu^{f,+}_{\Lambda,\beta}$ is the ferromagnetic Ising measure with positive boundary conditions, ω is a circuit that surround v with all positive spins; obviously, by symmetry, we can consider the same relation putting all the spins in formula (31) equal to -1 including the boundary conditions.

We define Pr_β as the measure at temperature $1/\beta$ on the trajectories for the process in discrete time that update the single spin variable in Λ with a lexicographic order and we choose the Gibbs measure as initial measure; at every time $t \in \mathbf{N}$ only one spin is updated. $\mathrm{Pr}_{\Lambda,\beta}$ is the same measure on the trajectories with re-scaled time; in a unit time we update all the spins in Λ. P_β is the analogous measure on the trajectories in a continuous time.

THEOREM 1. – *Let us take as initial distribution of the process the Gibbs measure. If* $\beta > 10\ln 3$ *then the probability that in the time interval* $[0, t_0]$ *there is an interface in the block* $B_l(v)$ *can be bounded as follows*:

$$\mathrm{Pr}_\beta(\exists \textit{ an interface in } B_l \textit{ for some } t \leqslant t_0) \leqslant 8l(t_0+1)\sum_{n=2l}^{\infty} 3^n \mathrm{e}^{-cn} < \infty \tag{32}$$

where $c = c_\beta = \beta/10 > \ln 3$.

Proof. – We remark that the probability to have an interface is time-invariant, in fact we suppose that the process has as initial distribution the Gibbs measure which is also the stationary measure of the process. We have

$$\begin{aligned}
&\Pr_\beta(\exists \text{ an interface in } B_l \text{ with } t \leqslant t_0)\\
&\quad\leqslant \sum_{t=0}^{t_0} \mathrm{Pr}_\beta(\exists \text{ an interface in } B_l \text{ at time } t) = (t_0+1)\mu_{\Lambda,\mathbf{J}\beta}(\exists \text{ an interface in } B_l).
\end{aligned}$$

By Lemmas A.4 and A.5 we know that an interface in B_l implies that there is a dual circuit γ with $|\gamma| \geqslant 2l$ in which is verified: $|\gamma_r| - |\gamma_n| < -|\gamma|/10$. Then, by Lemma 3.1, (setting $\varepsilon = \beta/10$) we deduce that the probability that there exists such a circuit γ is less than $\mathrm{e}^{-\beta|\gamma|/10}$. Now we need only to bound the number of circuits of fixed length n and to sum over n. From a standard calculation we have that the circuits starting in a fixed vertex and with length n are less than 3^n on $\mathbf{Z}^2$. Taking into account that at least one of the vertex of the considered circuits is on ∂B_O and remembering that $|\partial B_O| = 8l$ we obtain the desired bound (32). $\square$

The following theorem shows that there is a positive time auto-correlation for the spin v in the center of the square $B_l(v)$, i.e. the auto-correlation remains far away from zero for times that are exponentially increasing with l. This means that if there is a block $B_l(v)$ in Λ that respects $C_{l,\Lambda}$, then we can have a bound for the correlation for long times. We recall that $\Pr_\beta$, $\Pr_{\Lambda,\beta}$ and P_β are trajectory measures with Gibbs measure as initial distribution on two discrete time processes and a continuous time process.

THEOREM 2. – *Given $B_l(v) \subset \Lambda$, for every $\delta \in (0,1)$ there are $\beta_0 = \beta_0(\delta) > 0$ and $a = a(\delta) > 1$ (independent of Λ) such that for all $\beta > \beta_0$ we have*

$$\Pr_\beta\bigl(\sigma(v,0)\sigma(v,t) = -1\bigr) \leqslant \delta \quad \textit{for all } t < a^l,\ t \in \mathbf{N}, \tag{33}$$

$$\Pr_{\Lambda,\beta}\bigl(\sigma(v,0)\sigma(v,t) = -1\bigr) \leqslant \delta \quad \textit{for all } t < a^l,\ t = \frac{n}{|\Lambda|} \textit{ and } n \in \mathbf{N}, \tag{34}$$

$$P_\beta\bigl(\sigma(v,0)\sigma(v,t) = -1\bigr) \leqslant \delta \quad \textit{for all } t < a^l,\ t \in \mathbf{R}. \tag{35}$$

Proof. – Define the events $C_{+,t_0}(v)$ and $C_{-,t_0}(v)$ as

$$C_{\pm,t_0}(v) = \{\forall t \leqslant t_0\ \exists \text{ a circuit} \pm \text{surrounding } v\}. \tag{36}$$

By Lemma A.6 in appendix we have the following inclusion

$$C^c_{\pm,t_0}(v) \subset \bigl\{\exists t \leqslant t_0 \text{ in which there is an interface in } B_l(v)\bigr\}. \tag{37}$$

We can write (we omit the parameter β):

$$\begin{aligned}
\Pr\bigl(\sigma(v,0)\sigma(v,t) = -1\bigr) &= \Pr\bigl(\sigma(v,0)\sigma(v,t) = -1 \mid \bigl(C_{\pm,t_0}(v)\bigr)^c\bigr)\Pr\bigl(\bigl(C_{\pm,t_0}(v)\bigr)^c\bigr) \\
&\quad + \Pr\bigl(\sigma(v,0)\sigma(v,t) = -1 \mid C_{\pm,t_0}(v)\bigr)\Pr\bigl(C_{\pm,t_0}(v)\bigr) \\
&\leqslant \Pr(\exists \text{ an interface in } B_l(v) \text{ for } t \leqslant t_0) \\
&\quad + \Pr\bigl(\sigma(v,0)\sigma(v,t) = -1 \mid C_{\pm,t_0}(v)\bigr)\Pr\bigl(C_{\pm,t_0}(v)\bigr)
\end{aligned} \tag{38}$$

where we used the inclusion (37). By Lemma A.6 we know that, until there is not an interface, at every time t we can find a circuit that surrounds v having the same sign of the initial one. We can equivalently choose the event $C_+(v)$ or $C_-(v)$ and the proof follows the same idea. Using also Theorem 1, we find the following upper bound for (38):

$$\begin{aligned}
&\leqslant \bigl[\Pr\bigl(\sigma(v,0) = -1 \mid C_{+,t_0}(v)\bigr) + \Pr\bigl(\sigma(v,t) = -1 \mid C_{+,t_0}(v)\bigr)\bigr]\Pr\bigl(C_{+,t_0}(v)\bigr) \\
&\quad + 8l(t_0+1)\sum_{n=2l}^{\infty} 3^n \mathrm{e}^{-\beta n/10}.
\end{aligned} \tag{39}$$

Formula (39) becomes:

$$8l(t_0+1)\sum_{n=2l}^{\infty}3^n\mathrm{e}^{-\beta n/10}+\Pr\bigl(\{\sigma(v,0)=-1\}\cap C_{+,t_0}(v)\bigr)$$
$$+\Pr\bigl(\{\sigma(v,t)=-1\}\cap C_{+,t_0}(v)\bigr)\leqslant \tag{40}$$

and for all $t\leqslant t_0$:

$$\leqslant 8l(t_0+1)\sum_{n=2l}^{\infty}3^n\mathrm{e}^{-\beta n/10}$$
$$+\Pr\bigl(\{\sigma(v,0)=-1\}\cap\{\text{for } t=0 \text{ there is a positive circuit surrounding } v\}\bigr)$$
$$+\Pr\bigl(\{\sigma(v,t)=-1\}\cap\{\text{for } t \text{ there is a positive circuit surrounding } v\}\bigr). \tag{41}$$

Using the FKG inequality (31), the property that the Gibbs measure is stationary with respect to the Glauber process and the strong Markov property (i.e. independence from the configuration external to τ^+ which is the random circuit defined in Lemma A.6 that has all spin equal to $+1$ and is inside the ferromagnetic region $B_{O,l}$) we find the following bound for (41)

$$\leqslant 8l(t_0+1)\sum_{n=2l}^{\infty}3^n\mathrm{e}^{-\beta n/10}+2\mu_\beta^{f,+}\bigl(\sigma(v)=-1\bigr) \tag{42}$$

and we have $2\mu_\beta^{f,+}(\sigma(v)=-1)=1-m_{\beta,+}$ where $m_{\beta,+}>0$ is the magnetization of the Ising model with positive boundary condition and temperature $T=1/\beta$. Increasing β, we can make $1-m_{\beta,+}$ smaller than every positive constant and series in (42) is exponentially small with l for large enough β. This is enough to complete the proof of the first assertion of the theorem. To prove the second assertion it is enough to observe that at a time t_0 there are $|\Lambda|t_0$ updated spins but the interface can appear only updating a spin in $V(B_{O,l})$ and we remember that $|V(B_{O,l})|=4l^2$. The spins are updated in a lexicographic order, so we obtain the inequality (42) multiplying by $4l^2$ and taking the integer part of t_0+2. To prove formula (35) we only need to find an upper bound of the number of updated spins; in fact the lexicographic order is used only to estimate this number. From the *boundedness* of the process we have:

$$\sup_{x,\sigma}c_{\mathbf{J}}(x,\sigma)\leqslant c_2.$$

We define $C(t)$ as the random variable of the number of updated spins in the time interval $[0,t]$ in the block B_l. We write

$$P_\beta\bigl(\sigma(v,0)\sigma(v,t)=-1\bigr)$$
$$=P_\beta\bigl(\sigma(v,0)\sigma(v,t)=-1\mid C(t)<40c_2tl^2\bigr)P_\beta\bigl(C(t)<40c_2tl^2\bigr)$$
$$+P_\beta\bigl(\sigma(v,0)\sigma(v,t)=-1\mid C(t)\geqslant 40c_2tl^2\bigr)P_\beta\bigl(C(t)\geqslant 40c_2tl^2\bigr), \tag{43}$$

where c_2 is the constant in (21). For the first addendum of (43) we can use the same bound of formula (34) and for the second addendum we can use a large deviation estimate to write

$$P_\beta\bigl(C(t)\geqslant 40c_2tl^2\bigr)<\mathrm{e}^{-\Delta tl^2}$$

with $\Delta > 0$. So, collecting all the results, we have a correlation time that increases exponentially with the parameter l of the block B_l and (35) is verified. This completes the proof. $\square$

We take a sequence of boxes $\Lambda_L = [-L, L]^2$; we will give an upper bound for the time in which the system remains auto-correlated as a function of the box side L (the result is $P_{\mathbf{J}}$-a.e. in the limit of $L \to \infty$).

We say that $\{J_b^*\}_{b\in E}$ is a *gauge transformation of the interactions* $\{J_b\}_{b\in E}$ if there exists a spin configuration $\sigma \in \Omega$ such that for every edge $b = \{x, y\}$ we have $J_b^* = J_b \sigma_x \sigma_y$.

Given the event $C_{l,\Lambda}$ we say that $C_{l,\Lambda}^*$ is the event that there exists a gauge transformation of the interactions $\{J_b\}_{b\in E}$ verifying $C_{l,\Lambda}$. The events $C_{l,\Lambda}^*$ and $C_{l,\Lambda}$ are isomorphic with respect to the Glauber process. We remember that p denote the probability that an edge is ferromagnetic. We define the following function

$$f(L) = \left[\frac{1}{4}\sqrt{\frac{\ln L}{\ln 4}}\right]$$

where $[\cdot]$ is the integer part.

LEMMA 4.1. – *Given* $p \in (0, 1)$ *and a sequence* $\Lambda_L = [-L, L]^2$ *then*

$$P_{p,J}\left(\limsup_{L\to\infty}(C^*_{[f(L)],\Lambda_L})^c\right) = 0.$$

Proof. – We give the proof for all $p \in [1/2, 1)$ therefore, observing that $\mathbf{Z}^2$ is a bipartite graph, we can deduce the proof for $p \in (0, 1/2)$; since $\mathbf{Z}^2$ is a bipartite graph the transformation $\mathbf{J} \to -\mathbf{J}$ is a gauge transformation and it is equivalent to change the parameter of the Bernoulli distribution with the rule $p \to 1 - p$, so we will study only the interval $[1/2, 1)$. We will use the Borell–Cantelli lemma to show that only a finite number of times the event $C_{f(L),\Lambda_L}$ fails. Now we will study separately the case $p \in (1/2, 1)$ and $p = 1/2$.

We divide the box $\Lambda_L = [-L, L]^2$ into squares of side $l_0 = 4[\frac{1}{4}\sqrt{\frac{\ln L}{\ln 4}}]$ (so for the blocks B_l we have $l_0 = 4l$); the number of these squares in Λ_L is approximately $L^2 \ln 4/\ln L$; for $p \in (1/2, 1)$, for every $\varepsilon \in (0, p)$ and for large enough L (so that also l_0 large) the probability that a block B_l verifies the event $C_{l,\Lambda}$ is larger than

$$(p-\varepsilon)^{2{l_0}^2}, \tag{44}$$

where $2{l_0}^2$ is the number of edges in a square of side l_0. In fact we are omitting a factor of order $(1-p)^{l_0} < 1$ which is a surface factor but we are also missing $[(p-\varepsilon)/p]^{-2{l_0}^2} > 1$ which is a volume factor. Therefore using $(p - \varepsilon)$ in place of p we can bound the probability that a block B_l verifies the event $C_{l,\Lambda}$ with formula (44). So we can write:

$$\sum_{L=1}^{\infty} P_p\big((C^*_{[f(L)],\Lambda_L})^c\big) \leqslant \sum_{L=1}^{\infty} P_p\big((C_{[f(L)],\Lambda_L})^c\big) \leqslant M\sum_{L=1}^{\infty}\big(1 - p_1^{\frac{2\ln L}{\ln 4}}\big)^{\frac{L^2\ln 4}{\ln L}} < \infty \tag{45}$$

where $p_1 \in (1/2, p)$, $\varepsilon = p - p_1$ and M a positive constant. For $p = 1/2$ we have that all the interaction configurations on a square of side l_0 have probability equal to $(1/2)^{2l_0^2}$; there are $2^{l_0^2}$ configurations (gauge transformation) that verify $C^*_{f(L),\Lambda_L}$ so the probability of the event $C^*_{f(L),\Lambda_L}$ is equal to $(1/2)^{l_0^2}$; adding up as in (45) the convergence of the sum is easily proved also in this case.

By the first Borell–Cantelli lemma we have that only a finite number of times $C^*_{f(L),\Lambda_L}$ is false $P_{\mathbf{J}}$-a.e. □

So far all the assertions are true independently of the boundary conditions; instead for the following results we have to use free boundary conditions because we want the symmetry $\mu_{\Lambda,\mathbf{J}\beta}(\sigma_A) = \mu_{\Lambda,\mathbf{J}\beta}(-\sigma_A)$. Theorem 3 below has a deep connection with the ferromagnetic systems in which it is possible to find the average time to have an interface. It is known that, at low temperatures for ferromagnetic systems, the average time to find an interface is exponentially increasing with L (side of the box Λ_L). Collecting the results of Lemma A.8 and of Theorem 2 we have the following proposition.

PROPOSITION 4.1. – *For all β large enough there exists $\delta \in (0, 1/2)$ and $A_\beta > 1$ such that definitively in L ($P_{\mathbf{J}}$-a.e.) and for all $t < A_\beta^{\sqrt{\ln L}}$ we have*

$$\Pr_{\beta,\Lambda}\big(\sigma(v, 0)\sigma(v, t) = -1\big) < \delta < 1/2.$$

In the following theorem we use definition (17) for the distance between measures D_ρ. The parameter $\rho > 0$ can get any positive value but if we want T_A, in the following theorem, to be finite we need to put $\rho < (1/2d)^2$. Let us define

$$\Lambda(a, t) = \big[-\mathrm{e}^{a(\ln t)^2}, \mathrm{e}^{a(\ln t)^2}\big]^2 \cap \mathbf{Z}^2. \tag{46}$$

In the following theorem we index the measure with the parameter $T = 1/\beta$; so $\mu^{(t)}_{\sigma_1,\Lambda,T}$ is the measure at a time t induced by the process $\Pr_{\Lambda,\beta}$ or the measure induced by the process P_β; in fact the proof is the same using both the definitions.

THEOREM 3. – *For the Glauber process at discrete time of the Edwards–Anderson model in $\mathbf{Z}^2$ we have*:

(I) (High temperature) *There is $T_A > 0$ for which it is true the following. For all $T > T_A$ there exists $M > 1$ and $c > 0$ such that for all Λ, for all $\sigma_1, \sigma_2 \in \{-1, 1\}^\Lambda$ and for all $t > 0$*

$$D_\rho\big(\mu^{(t)}_{\sigma_1,\Lambda,T}, \mu^{(t)}_{\sigma_2,\Lambda,T}\big) < M\mathrm{e}^{-ct}. \tag{47}$$

(II) (Low temperature) *There is $T_B > 0$ for which it is true the following. For all $T \in [0, T_B]$ there is $\varepsilon > 0$ and $a > 0$ such that for all t large enough we have*: *$\forall t_1 < t$ that there exists $\sigma_1 \in \{-1, 1\}^{\Lambda(a,t)}$ such that*

$$D_\rho\big(\mu^{(t_1)}_{\sigma_1,\Lambda(a,t),T}, \mu^{(t_1)}_{-\sigma_1,\Lambda(a,t),T}\big) > \varepsilon \ P_{\mathbf{J}}\text{-}a.e. \tag{48}$$

Part (I) is known in literature (see [23]). For this first point we can actually find an estimate of T_A in $\mathbf{Z}^d$ and not only in $\mathbf{Z}^2$. It can be shown that $T_A \leqslant 4d^2 \max_b |J_b|$ for all

ρ small enough. In our analyzed case all the interactions have absolute value equal to 1, so we obtain $T_A \leqslant 4d^2$. We will not give the proof of this point that can be found in [14] and we concentrate our attention on the proof of the second point.

Proof of (II). – We know by the Lemma A.8 that definitively in t and $P_{\mathbf{J}}$-a.e. there exists in $\Lambda(a,t)$ a block $B_l(v)$ that verify the event $C^*_{f(L),\Lambda_L}$ with $l = \ln t/\ln(a_1)$, and $a_1 > 1$ a constant which is independent of $\Lambda(a,t)$. By Theorem 2 we have

$$\mathrm{Pr}_{\Lambda,\beta}\big(\sigma(v,0)\sigma(v,t) = -1\big) \leqslant \delta \quad \text{for all } t \leqslant a^l \tag{49}$$

with $a > 1$; we can partition the probability (49) conditioning on the initial configurations $\sigma(\cdot,0)$ and writing:

$$\begin{aligned}&\mathrm{Pr}_{\Lambda,\beta}\big(\sigma(v,0)\sigma(v,t) = -1\big)\\&\quad = \sum_{\sigma(\cdot,0)\in\Omega_\Lambda} \mathrm{Pr}_{\Lambda,\beta}\big(\sigma(v,0)\sigma(v,t) = -1 \mid \sigma(\cdot,0)\big)\mu_{\Lambda,\mathbf{J}\beta}\big(\sigma(\cdot,0)\big);\end{aligned}$$

then there must be at least one initial configuration $\sigma(\cdot,0) \in \Omega_\Lambda$ for which at a time $t_1 < t$ we have

$$\mathrm{Pr}_{\Lambda,\beta}\big(\sigma(v,0)\sigma(v,t_1) = -1 \mid \sigma(\cdot,0)\big) \leqslant \delta$$

but, by definition, we have

$$\mathrm{Pr}_{\Lambda,\beta}\big(\sigma(v,0)\sigma(v,t_1) = -1 \mid \sigma(\cdot,0)\big) = \mu^{(t)}_{\sigma(\cdot,0),\Lambda,T}\big(\sigma(v,0)\sigma(v,t) = -1\big)$$

with $T = 1/\beta$.

Using Lemma A.8 and the previous result, we notice that definitively in t for all $t_1 < t$ there exists a σ_1 such that:

$$\big|\mu^{(t_1)}_{\sigma_1,\Lambda(a,t),T}\big(\sigma_1(v)\sigma(v,t) = 1\big) - \mu^{(t_1)}_{\sigma_1,\Lambda(a,t),T}\big(\sigma_1(v)\sigma(v,t) = -1\big)\big| > 1 - 2\delta > 0 \tag{50}$$

and using the symmetry of the Gibbs measure with respect to the change $\sigma \to -\sigma$ we obtain

$$\begin{aligned}&D_\rho\big(\mu^{(t_1)}_{\sigma_1,\Lambda(a,t),T}, \mu^{(t_1)}_{-\sigma_1,\Lambda(a,t),T}\big)\\&\quad > \rho\big|\mu^{(t_1)}_{\sigma_1,\Lambda(a,t),T}\big(\sigma(v,t_1) = 1\big) - \mu^{(t_1)}_{-\sigma_1,\Lambda(a,t),T}\big(\sigma(v,t_1) = 1\big)\big|\\&\quad = \rho\big|\mu^{(t_1)}_{\sigma_1,\Lambda(a,t),T}\big(\sigma_1(v,0)\sigma(v,t_1) = 1\big) - \mu^{(t_1)}_{-\sigma_1,\Lambda(a,t),T}\big(\sigma_1(v,0)\sigma(v,t_1) = 1\big)\big|\\&\quad > \rho(1 - 2\delta) > 0\end{aligned} \tag{51}$$

with δ given by Theorem 2. This complete the proof. □

We remark that in the second point of Theorem 3 the configurations σ_1 can depend on Λ_L; to be precise the configurations $\sigma_{1,L} \in \{-1,1\}^{\Lambda_L}$ and $\sigma_{1,M} \in \{-1,1\}^{\Lambda_M}$, which verify point (II) of Theorem 3, can not coincide on $\Lambda_L \cap \Lambda_M$, but we can choose a sub-sequence $\{\sigma_{1,L_n}\}_n$ of $\{\sigma_{1,L}\}_L$, since the space is finite-dimensional, in which the configurations σ_{1,L_m} and σ_{1,L_n} coincide on $\Lambda_{L_m} \cap \Lambda_{L_n}$.

5. Bounds on the spectral gap for the transition matrix

In this paragraph we study a reversible Markov chain associated to the Gibbs measure on the finite graph.

We only analyze the Markov chain which follows by the re-scaling of the time but the same considerations can be made, with the same aim, also for the process at continuous time. As just remarked the process is uniquely identified with the transition matrix which is a stochastic matrix.

In general the maximal eigenvalue of the Markov process is equal to 1; for all the irreducible chain the maximal eigenvalue is unique and the minimal eigenvalue is larger than -1 if, as in our case, the transition matrix is aperiodic. The eigenvalues of the irreducible transition matrix are ordered with respect to their size (magnitude), in fact they take only real values; usually the larger eigenvalue is denoted with $\rho_0 = 1$; we write the eigenvalues

$$1 = \rho_0 = 1 > \rho_1 \geqslant \cdots \geqslant \rho_n$$

with ρ_1 that denote the second eigenvalue. A very important question is the study of $\rho_* = \max\{\rho_1, |\rho_n|\}$ that drives the speed of relaxation to the stationary measure (see [7]).

There are some advantages to consider the Laplacian $(\mathbf{1} - P)$ and its eigenvalues $\lambda_i = 1 - \rho_i$ also called the *gap of the transition matrix*. In our case we investigate a sequence of transition matrices and we look for some asymptotic bounds as function of the matrix dimension (the matrix dimension is the number of configurations in Ω_Λ). We denote with $Q_{L,\beta}$ the transition matrix associated to the process $\mathrm{Pr}_{\Lambda_L,\beta}$ with $\Lambda_L = [-L, L]^2$ that is taken reversible as respect to the Gibbs measure. We indicate the eigenvalues of $(\mathbf{1} - Q_{L,\beta})$ with $\lambda_{i,L,\beta}$ and the eigenvalues of $Q_{L,\beta}$ with $\rho_{i,L,\beta}$; the eigenvalues are parameterized by the side L and the inverse temperature β; we will show two different asymptotic bounds in low and high temperature.

The usual minimax characterization of eigenvalues gives

$$\lambda_1 = \inf\left\{\frac{E(\phi,\phi)}{\mathrm{Var}(\phi)}\colon \phi \text{ is nonconstant}\right\} \tag{52}$$

where $\mathrm{Var}(\phi)$ is the variance of ϕ relative to the stationary measure (or invariant measure of the process) which is, by construction, the Gibbs measure and $E(\phi,\phi)$ is the *Dirichlet form*

$$E(\phi,\phi) = \frac{1}{2}\sum_{\sigma,\tilde{\sigma}\in\Omega}\big(\phi(\sigma) - \phi(\tilde{\sigma})\big)^2 \widetilde{Q}(\sigma,\tilde{\sigma}) \tag{53}$$

where

$$\widetilde{Q}(\sigma,\tilde{\sigma}) = \mu(\sigma) Q(\sigma,\tilde{\sigma}) \tag{54}$$

and $Q(\cdot,\cdot)$ denote the transition matrix of the Markov process. With this characterization we get an interesting estimate of the spectral gap using as function in (52) $\phi = \phi_v(\sigma) = \sigma_v$ with v given by the definition of the block $B_l(v)$ verifying the event C^*_{l,Λ_L}. We use, in place of Q, the transition matrix Q^t which is the tth power of Q and we mark with the index t the quantity related to Q^t. We have

$$1-\rho^t_{1,L,\beta}\leqslant \frac{E_t(\phi_v,\phi_v)}{\mathrm{Var}_t(\phi_v)}=2\mathrm{Pr}_{\Lambda,\beta}\big(\sigma(v,0)\sigma(v,t)=-1\big)\leqslant 2\delta<1 \tag{55}$$

for β large enough and $t<A_\beta^{\sqrt{\ln L}}$ where A_β is an opportune constant larger then 1. In fact $\mathrm{Var}_t(\phi_v)=1$ because we are using free boundary conditions, and

$$E_t(\phi_v,\phi_v)=\frac{1}{2}\sum_{\sigma,\sigma'\in\Omega}(\sigma_v-\sigma'_v)^2\mu_{\Lambda,\mathbf{J}\beta}(\sigma)P^t(\sigma,\sigma')=2\mathrm{Pr}_{\Lambda,\beta}\big(\sigma(v,0)\sigma(v,t)=-1\big). \tag{56}$$

Now using Proposition 4.1 we find that the (56) can be bounded with the constant $2\delta<1$ where δ is the same constant of Proposition 4.1. Hence $P_{\mathbf{J}}$-a.e. and definitively in L for every β large enough we have

$$\rho_{1,L,\beta}\geqslant \exp\left(-\frac{k_\beta}{A_\beta^{\sqrt{\ln L}}}\right)\quad \text{with } k_\beta>0 \text{ and } A_\beta>1.$$

So we have the following bound for the spectral gap

$$\lambda_{*,L,\beta}=1-\rho_{*,L,\beta}\leqslant\left(-\frac{k_\beta}{A_\beta^{\sqrt{\ln L}}}\right).$$

At large enough temperature $T=\beta^{-1}$ we quote the result of [14] where we have:

$$\lambda_{*,L,\beta}=1-\rho_{*,L,\beta}\geqslant C_\beta\quad \text{with } C_\beta>0; \tag{57}$$

hence the spectral gap is uniformly larger than C_β. One has two different behaviors; for β_1 large enough and β_2 small enough we find, for example:

$$\lim_{L\to\infty}\frac{\ln\lambda_{*,L,\beta_2}}{\ln\lambda_{*,L,\beta_1}}=0. \tag{58}$$

This different behavior of the spectral gap in high and low temperature is the cause of the different convergence to the stationary measure. In fact the relation

$$\ln\rho_*=\lim_{t\to\infty}\frac{1}{t}\ln\big(\max_{\sigma,\omega}\big\|P^t(\sigma,\cdot)-P^t(\omega,\cdot)\big\|_{\mathrm{var}}\big) \tag{59}$$

is asymptotically satisfied for an aperiodic irreducible Markov chain. We remark that also in this last paragraph we could obtain similar results for a no-reversible dynamics using [10].

Appendix A. Geometrical results

In this appendix we present the main geometrical results. For the sake of simplicity we deal only with Glauber processes but it is quite simple to generalize the results to local non-reversible processes. In this appendix we will prove some geometrical results that are needed to use Peierls' arguments, as in [27], in Section 3.

We recall that given a dual path (or a dual circuit) γ we denote with γ directly the set of intersected edges and we denote with $\Gamma(\gamma)$ the ordered set of dual edges and dual vertices. We recall the important result that a graph is a circuit if and only if it is Eulerian (see [6] for general properties of graphs). We will write $\triangle$ for the symmetric difference.

LEMMA A.1. – *Given the* (*dual*) *circuits* (*or union of circuits without edges in common*) γ_a *and* γ_b *then* $\gamma_a \triangle \gamma_b$ *is a family of* (*dual*) *circuits and each connected component is a* (*dual*) *circuit.*

Proof. – If two graphs are Eulerian they remain Eulerian also after the symmetric difference that can change the number of incident edges on a vertex only by an even number; so the symmetric difference of circuits is a set of circuits (see [6]). □

For the sake of brevity we will indicate with $B_l(v)$ the square that verifies the event $C_{l,\Lambda}$ (see definition in Section 2). Given the region $V(B_l(v)) \setminus V(B_{O,l}(v))$ we call C_i the maximal connected regions in $V(B_l(v)) \setminus V(B_{O,l}(v))$ with all spin $+1$ or -1 that have nonempty intersection with $\partial B_{O,l}(v)$ and with $\partial B_l(v)$; the sets C_i are clusters in this restricted region. The set of all the C_i's is indicated with $\mathcal{C}$; $\mathcal{C}^+$ is the family of all the clusters $\{C_i\}_i$ made of positive spins and analogously we define $\mathcal{C}^-$. We call A_l the number of clusters C_i in the fixed block $B_l(v)$, i.e. $A_l = |\mathcal{C}|$.

Remark A.1. – The clusters $\{C_i\}_{i=1,\dots,n}$ can be ordered by their intersection with the dual circuit $\partial B_{O,l}$.

For a formal proof of the remark see [17].

Given a dual circuit $\Gamma(\partial^E C_i^+) = (x_0^*, b_0^*, \dots, x_0^*)$, with x_0^* that is a dual vertex external to B_l, we define $b_p(i)$ (resp. $b_u(i)$) as the first (resp. the last) edge in $\partial B_{O,l} \cap \partial^E C_i^+$, where the elements of $\partial B_{O,l} \cap \partial^E C_i^+$ are ordered as in $\Gamma(\partial^E C_i^+)$. We remark that if there is a unique cluster C_1 then can happen that there is not any edge $b_p(1)$ and $b_u(1)$ but it is not a problem and we only have to ignore the following construction setting $\bigcup_i \tilde{\gamma}_i = \emptyset$ in (A.1). We also observe that $b_u(i)$ and $b_p(i)$ can be the same edge and we set

$$F_0 = \bigcup_i \{b_u(i) \cup b_p(i)\},$$

so that $|F_0| \leqslant 2|\mathcal{C}|$. We write the dual circuit $\Gamma(\partial^E C_i^+)$ as

$$\left(x_0^*, \dots, x_p^*(i), b_p^*(i), \dots, b_u^*(i), x_u^*(i), \dots, x_0^*\right)$$

and this implicitly defines the dual vertices $x_p^*(i)$ and $x_u^*(i)$. We call $\Gamma(\theta_i)$ the dual path in $\partial B_{O,l}$ that has as initial vertex $x_p^*(i)$, edge $b_p^*(i)$ and is connected to $x_u^*(i)$; θ_i is the set of the intersected edges. Call $\Gamma(r_i)$ the path that results from the circuit $\Gamma(\partial^E C_i)$ eliminating all the dual edges and the dual vertices between $b_p^*(i)$ and $b_u^*(i)$ (taking $x_u^*(i)$ as the first vertex of the path r_i). Let us define $\tilde{\gamma}_i$ as the dual circuit union of the paths r_i and θ_i (see Fig. 4). It is easy to check the following properties of dual paths r_i and θ_i, seen as edge sets.

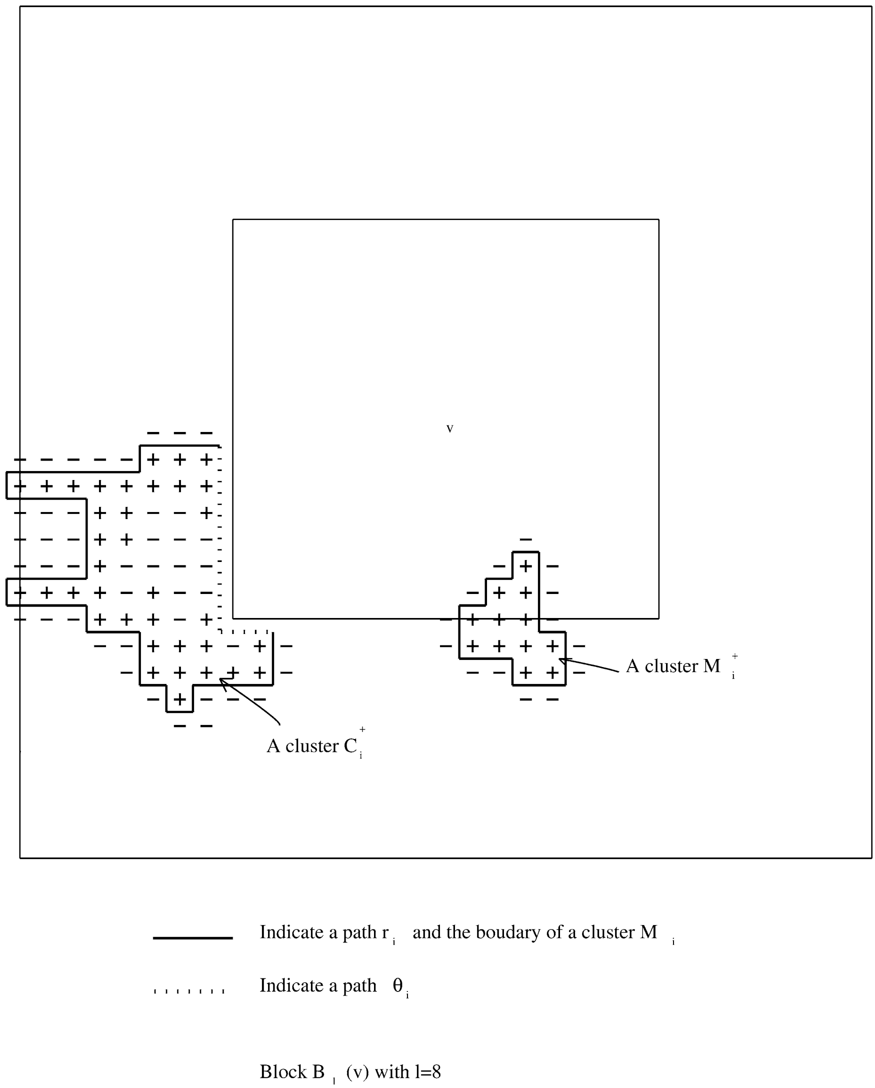

Fig. 4. A block $B_l(v)$ with a cluster C_i.

Remark A.2. –
(1) No r_i has edges in $\partial B_{O,l}$, by construction.
(2) Every r_i has only two edges in $\Delta B_{O,l}$ and they must be incident to $b_p(i)$ or to $b_u(i)$; otherwise r_i should use also edges in $\partial B_{O,l}$ and this is impossible by Remark A.2(1).
(3) There are only two dual vertices that belong to $\Gamma(r_i)$ and $\Gamma(\theta_i)$, namely $x_p^*(i)$ and $x_u^*(i)$. This follows directly by construction and from Remark A.2(1).
(4) Given $\tilde{\gamma}_i := r_i \cup \theta_i$ then each $\Gamma(\tilde{\gamma}_i)$ is a dual self-avoiding circuit. In fact $\Gamma(r_i)$ is a dual self-avoiding path having only $x_p^*(i)$ and $x_u^*(i)$ as dual vertices in $\Gamma(\partial B_{O,l})$.

(5) We have: $\tilde{\gamma}_i \cap \tilde{\gamma}_j = \emptyset$ $\forall i \neq j$. From the Remark A.1 on the order of clusters in $\mathcal{C}$ we deduce that $\theta_i \cap \theta_j = \emptyset$ if $i \neq j$ and, moreover, $r_i \cap r_j = \emptyset$ by Proposition 2.1.
(6) $\tilde{\gamma}_i \cap B_{O,l} = \emptyset$, by construction.

We define $\{M_j^+\}_{j=1,\dots,n}$ as the sets of vertices clusters restricted to the region $V(B_l)$ having each vertex v in $V(B_l)$, $\sigma_v = 1$ and $M_j^+ \cap V(B_{O,l} \cup \partial B_{O,l}) \neq \emptyset$. Set

$$\gamma_2 := \left(\bigcup_j \partial M_j^+\right) \triangle \left(\bigcup_i \tilde{\gamma}_i\right). \tag{A.1}$$

By Proposition 2.1 the intersection of the boundaries ∂M_j^+ is empty, so $\bigcup_j \partial M_j^+$ is equivalent to the symmetric difference of these sets of edges, the same remark is true also for the union of the dual circuits $\{\tilde{\gamma}_i\}$. By Lemma A.1 we have that γ_2 is a family of dual circuits. If there is an interface α the whole interface is in γ_2 because $\alpha \subset \bigcup_j M_j^+$ and $\bigcup_i \tilde{\gamma}_i$ are edges out of $B_{O,l}$. Moreover α, being a dual path, is connected and so it is inside one maximal connected component of γ_2 that is a circuit by the second part of Lemma A.1. If in $B_{O,l}$ there exists an interface we call α one of these interfaces, we define the dual circuit γ_3 as the maximal connected subset of γ_2 containing α. Let us note that γ_3 is a dual circuit but it is not necessarily self-avoiding.

LEMMA A.2. – *The dual circuit γ_3 has all the edges in B_l and all its satisfied edges are in $\gamma_3 \cap \partial B_{O,l}$.*

Proof. – We first show that all the edges in $\gamma_3 \cap B_l$ can be satisfied only if they are in $\partial B_{O,l}$. We know that the edges in $\gamma_3 \cap B_l \setminus \partial B_{O,l}$ are in some ∂M_i^+ or in some $r_i \subset \partial^E C_i^+$. All these sets are boundaries of clusters of positive spins and they have the property to have as satisfied edges only the negative edges ($J_b = -1$) because on each such a edge $b = \{x, y\}$ is $\sigma_x = -\sigma_y$, but we know that the negative edges of B_l are only in $\partial B_{O,l}$.

Now we show that all the edges belonging to γ_3 are in B_l. The only edges in γ_3 that are not in B_l can only be the edges in ∂B_l, in fact all the considered clusters are in B_l. Suppose $\gamma_3 \cap \partial B_l \neq \emptyset$; this can be true only in the following two cases:
(1) $\partial B_l \cap (\bigcup_j \partial M_j^+) \not\subset (\bigcup_i \tilde{\gamma}_i)$,
(2) $\partial B_l \cap (\bigcup_i \tilde{\gamma}_i) \not\subset (\bigcup_j \partial M_j^+)$.
We will show that (2) cannot hold; in fact we have $\partial B_l \cap (\bigcup_i \tilde{\gamma}_i) \subset \partial B_l \cap (\bigcup_i \partial^E C_i^+)$ that is a subset of $\partial B_l \cap (\bigcup_i \partial C_i^+)$. We know by hypothesis that C_i^+ belong to some M_k^+ so if a vertex x is in C_i^+ it belong also to M_k^+ and if $x \in b$ with $b \in \partial B_l$ then $b \in \partial C_i^+$ being the edge b connected to a vertex outside B_l that do not belong to M_k^+ or to C_i^+.

Now we show that the point (1) cannot be true; we suppose that an edge b is in $\partial B_l \cap (\bigcup_j \partial M_j^+)$ with $b = \{x, y\}$ and $x \in M_j$, we show that x is in some C_k^+, in fact the vertex x is connected (by definition of M_j^+) to the circuit $\partial B_{O,l}$ so it belong to a cluster C_k^+ that is a subset of M_j^+, being a cluster intersecting $\partial B_{O,l}$ and ∂B_l, and, as in the previous point, $b \in \partial C_k^+$. But also it is $b \in \partial^E C_k$ being possible construct an infinite path intersecting ∂B_l only in b and having as initial vertex x. □

In the following lemma we see that in the circuit γ_3 there cannot be two consecutive satisfied edges if they are not in F_0 or on the corners of $\partial B_{O,l}$.

LEMMA A.3. – *Let b_1 and b_2 be two edges that are consecutive in the dual circuit $\partial B_{O,l}$ and on the same one of its four sides, then it is impossible for the following three conditions to be simultaneously valid*:

(a) *the edges b_1 and b_2 are both satisfied*;
(b) *neither b_1 nor b_2 is in F_0*;
(c) *the edges b_1 and b_2 are both in γ_2*.

Proof. – For notation refer to Fig. 5. Let us suppose that $b_4 \in \Delta B_{O,l}$ and $b_3 \in B_{O,l}$ and that, moreover b_1, b_2 are in $\partial B_{O,l}$ with b_1 a positive edge and b_2 a negative edge. Let us remark that all the vertices x_1, x_2, x_3, x_4 are in $V(\partial B_{O,l})$ and that therefore each positive cluster containing one of these vertices is an M_k^+. There are only 4 spin configurations with the edges b_1 and b_2 satisfied (see Fig. 5). For the configuration (1) $b_3 \in B_{O,l}$ belongs to some ∂M_i^+ because it is in the boundary of a positive cluster with a vertex in $V(\partial B_{O,l})$. One need only notice that the edge b_3 has two different spins so it belongs to a boundary of a positive cluster. Moreover it cannot belong to any $\tilde{\gamma}_i$ because the $\tilde{\gamma}_i$'s have not edges in $B_{O,l}$, by construction as stressed in Remark A.2(6); so b_3 is in γ_2. The edge b_4 is not in any ∂M_j^+ or ∂C_i^+ therefore it is not in γ_2. But γ_2 is, by definition,

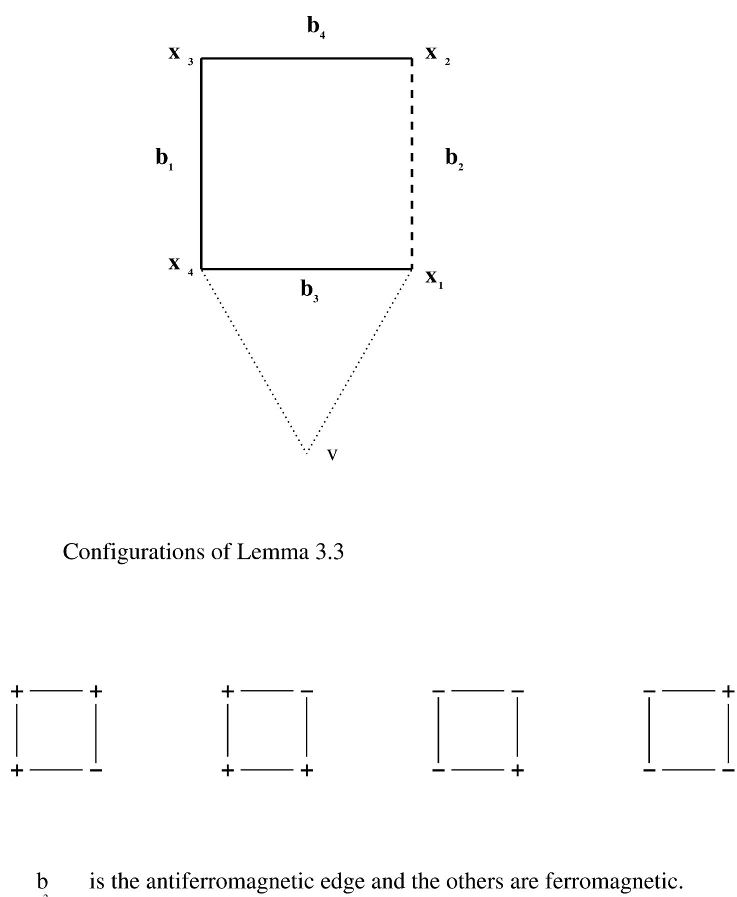

Fig. 5. The four configurations of Lemma A.3.

Eulerian because it is union of distinct dual circuits, so in $\Gamma(\gamma_2)$ there cannot be three dual edges b_1^*, b_2^* and b_3^* incident in the dual vertex x_1^*; this implies that either the edge b_1 or the edge b_2 is not in γ_2 because $b_3 \in \gamma_2$ and $b_4 \notin \gamma_2$. The same argument solve the configuration (3). So for the configurations (1) and (3) it is impossible to verify simultaneously points (a) and (c) without any regard for point (b).

For the configuration (2) we argue in the same way to observe that $b_3 \notin \gamma_2$. We can hold $b_4 \notin \gamma_2$ only if b_4 is in some $\tilde{\gamma}_i$ but this can be true only if b_1 or b_2 are in F_0, see Remark A.2(2), which contradicts the hypotheses; therefore we should have an odd number of incident dual edges in the dual vertex x_1^* which is impossible because γ_2 is Eulerian. The same argument solve the configuration (4). We notice that we have used also the point (b) for the configurations (2) and (4). □

Obviously Lemma A.3 is true also if in point (c) we change γ_2 with its subset γ_3. Given an interface $\Gamma(\alpha) = x_0, e_1, x_1, e_2, \ldots, e_n, x_n$ we call *interior part of the interface* the path

$$\Gamma(\alpha_0) = x_1, e_2, x_2, e_3, \ldots, e_{n-1}, x_{n-1}.$$

Remark A.3. – Given the dual circuit $\Gamma(\gamma_3) = y_0, b_1, y_1 b_2, \ldots, b_n, y_0$ if $b_i \in \gamma_3 \cap \partial B_{O,l}$ then $b_{i+1} \notin \alpha_0$. In fact in α_0 there are no edges that are in ΔB or in $\partial B_{O,l}$.

The following lemma shows that if $A_l = |\mathcal{C}| > 80$ then there is a dual circuit with a fraction of not satisfied edges bigger than the fraction of satisfied edges. With a different construction we can find a circuit with the same characteristic also when there is an interface and $A_l \leqslant 80$; we have seen, Section 3, that these circuits have a probability to be present in the configuration that is exponentially decreasing with $|\gamma|$ (for low enough temperatures); this is important to divide the configuration space in two regions separated by a zone which has a very low probability to be crossed. This has been relevant, using a sequence of $\Lambda_L \to \mathbf{Z}^2$, to prove the low speed of relaxation to the equilibrium for low temperatures. We remind that $\{C_n^+\} = \mathcal{C}^+$ are the positive vertex clusters belonging to $\mathcal{C}$ and analogously $\{C_n^-\} = \mathcal{C}^-$ are the negative vertex clusters belonging to $\mathcal{C}$.

LEMMA A.4. – *If $A_l \geqslant 80$ and l is large enough then there exists a dual circuit γ in $B_l(v)$ with $|\gamma| > 30l$ and $|\gamma_n| - |\gamma_r| \geqslant |\gamma|/10$.*

Proof. – Assume $|\mathcal{C}^+| \geqslant |\mathcal{C}^-|$. For each C_i^+ there exists a dual path $\gamma_i \in \partial C_i^+$ of ferromagnetic not satisfied edges connecting ∂B_l to $\partial B_{O,l}$. Moreover $\gamma_i \cap \gamma_j = \emptyset$ (seen as edge set) if $i \neq j$; in fact $\partial C_i^+ \cap \partial C_j^+ = \emptyset$ if $i \neq j$ and all the edges of ∂C_i^+ are ferromagnetic because they are in $B_l \setminus (\partial B_{O,l} \cup B_{O,l})$ and they are not satisfied. We can construct a dual circuit γ joining these paths with parts of ∂B_l and $\partial B_{O,l}$ and we remember that each satisfied edge belongs to $\partial B_l \cup \partial B_{O,l}$. Therefore the satisfied edges are at most $|\partial B_l| + |\partial B_{O,l}| = 20l$. The number of not satisfied edges in γ are at least $|\bigcup_i \gamma_i|$ and for every path we have $|\gamma_i| \geqslant l - 1$ because they link ∂B_l to $\partial B_{O,l}$. There are at least 40 distinct paths γ_i since $A_l > 80$ and the cardinality of the circuit γ is at least $40l - 40$. So, for l large enough, we obtain $|\gamma_r|/|\gamma_n| \leqslant 4/7$ and the thesis follows. The case $|\mathcal{C}^+| < |\mathcal{C}^-|$ can be analogously proved. □

In the proof of the next lemma we will indicate the edges of the corners in $\partial B_{O,l}$ with G_0; it is immediate to see that $|G_0| = 8$ because in every corner there are two incident edges.

LEMMA A.5. – *If $A_l < 80$, l is large enough and if there exists an interface α in $B_{O,l}$ then the dual circuit γ_3 is in B_l with $|\gamma_3| \geqslant 2l$ and $|(\gamma_3)_n| - |(\gamma_3)_r| \geqslant |\gamma_3|/10$.*

Proof. – We notice that $|\gamma_3| \geqslant 2l$ because $\alpha \subset \gamma_3$ and $|\alpha| \geqslant 2l$. It is easy to see that $(\gamma_3 \setminus \alpha_0)$ is made of one path, called ω, and eventually some circuits, called $\{\hat{\gamma}_1, \ldots, \hat{\gamma}_k\}$. Using Lemma A.3 we know that in every circuit $\hat{\gamma}_i$, with $i \in \{1, \ldots, k\}$, $|(\hat{\gamma}_i)_n| \geqslant |(\hat{\gamma}_i)_r|$ if in $\hat{\gamma}_i$ there is not any edge belonging to $F_0 \cup G_0$. Analogously for the path ω we obtain $|\omega_n| - |\omega_r| \geqslant -1$ if in ω there is not any edge belonging to $F_0 \cup G_0$. Now collecting the inequalities on the path ω and the circuits $\{\hat{\gamma}_1, \ldots, \hat{\gamma}_k\}$ we obtain, by Lemma A.3 and considering all the edges in $F_0 \cup G_0$ satisfied (the worst case), this upper bound

$$\bigl|(\gamma_3)_n\bigr| - \bigl|(\gamma_3)_r\bigr| \geqslant 2l - 2\bigl(|F_0| + |G_0|\bigr) - 2. \tag{A.2}$$

By Lemma A.2 we know that $(\gamma_3)_r \subset B_{O,l}$, so $|(\gamma_3)_r| \leqslant 8l$. The thesis follows by these calculations

$$\begin{aligned} \frac{|(\gamma_3)_n|}{|(\gamma_3)_r|} &\geqslant \frac{2l - 2(|F_0| + |G_0|) - 2 + |(\gamma_3)_r|}{|(\gamma_3)_r|} \\ &\geqslant \frac{2l - 2(|F_0| + |G_0|) - 2 + 8l}{8l} = \frac{5}{4} - \frac{9 + |F_0|}{4l}. \end{aligned} \tag{A.3}$$

What we want to prove is equivalent to $|(\gamma_3)_n|/|(\gamma_3)_r| \geqslant 11/10$ and, by formula (A.3), this is true for all l large enough. This complete the proof. □

We remark that to prove the previous lemma needs the existence of the dual circuit $\partial B_{O,l}$ with interactions that are alternatively positive and negative; in fact in the ferromagnetic Ising model at zero temperature if we consider Dobrushin boundary conditions (positive spins on the superior half box and negative spins on the inferior half box) there is, with probability one, an interface that divides the box Λ and in our case this does not happen. At zero temperature the probability that there exists an interface α in a box B_l is zero because the configuration with an interface has not the minimum of the energy.

This result has not to be confused with the result in [12] that states that for every edge in $\mathbf{Z}^2$ the probability that there is an interface is zero for every boundary condition. Instead, taking the interactions alternatively 1 and -1 make the configuration in $B_{O,l}(v)$ *almost* independent from the configuration out $B_l(v)$; so the equilibrium configuration in $B_{O,l}(v)$ is almost independent from the configuration out of $B_l(v)$ and trivially from the boundary conditions.

In the following lemma we find a relation between the absence of an interface in $B_l(v)$ and the event that v is surrounded by a circuit with all spins of the same sign. We call $L_i(B_{O,l}(v))$ *the vertices sides of* $B_{O,l}(v)$ with $i = 1, \ldots, 4$. We say that a vertex $s \in V(B_{O,l}(v))$ is *connected to a side* L_i *of* $B_{O,l}(v)$ if there exists a positive (resp. negative) vertex cluster $A \subset B_{O,l}(v)$ such that $s \in A$ and $V(A) \cap V(L_i) \neq \emptyset$. Notice that the cluster A can have a part out of $B_{O,l}(v)$ but we consider only the maximal connected component in $B_{O,l}(v)$. The same definition can be used for a positive vertex cluster in place of a single vertex. In Lemma A.6 we use the concept of $\star$-connection and $\star$-cluster; we say that two vertices $u=(u_1,u_2),\ v=(v_1,v_2)\in\mathbf{Z}^2$ are $\star$-connected if $\sigma_u=\sigma_v$ and $\sup_{i=1,2}|u_i-v_i|=1$, and a $\star$-cluster is a maximal $\star$-connected component, i.e. it is a cluster in which we use the $\star$-connection in place of the connection.

LEMMA A.6. – *If there is not an interface in $B_{O,l}(v)$ for all $t\leqslant t_0$ then for all times $t\leqslant t_0$ there exists at least a circuit of vertices $\tau^+(v,t)\subset V(B_{O,l}(v))$ (or $\tau^-(v,t)$) of positive (negative) sign that is connected to the four sides of $B_{O,l}(v)$; moreover $\tau^+(v,t)$ ($\tau^-(v,t)$) surrounds v.*

Proof. – We prove that if there is not a circuit of vertices with constant sign surrounding v then there exists an interface α with an edge $b\in\alpha$ and with $d(b,v)=1$. At least an edge b with $d(b,v)=1$ is not satisfied otherwise the smaller circuit surrounding v should have all the vertices with the same sign, against the hypotheses. We find a dual path γ_1 of not satisfied edges that is connected to $\partial B_{O,l}$ and starting from the edge b (see [17]), otherwise there is a circuit surrounding v with constant sign. If there was γ_1 that connects b to $\partial B_{O,l}$ then there would exist a second dual path γ_2 composed by not satisfied edges that connects b to $\partial B_{O,l}$, with $E(\gamma_1)\cap E(\gamma_2)=\emptyset$ and with the property that $\gamma_1\cup\gamma_2$ is an interface.

It is evident that for every circuit $\pi\subset B_{O,l}$ the number of not satisfied edges is an even number; in fact

$$(-1)^{|\pi_n|}=\prod_{b\in\pi}\eta_b=\operatorname{sign}\left(\prod_{b\in\pi}J_b\prod_{x\in\pi}\sigma_x^{2n}\right)>0$$

since all the interactions $J_b>0$ for $b\in B_{O,l}$. If we consider a graph in which the edges of γ_1 are contracted (see for a formal definition [6]) then, again, there is not any circuit composed of satisfied edges that intersect only one contracted vertex, otherwise there should be a single not satisfied edge in a circuit of that kind, but all these circuits surround the same dual vertex x^* that is adjacent to b and there exists γ_1 that connects x^* to $\partial B_{O,l}$ because in these circuits there is at least a not satisfied edge. But γ_2 does not intersect γ_1 since γ_1 is contracted. Therefore $|\gamma_1|\geqslant l$ and $|\gamma_2|\geqslant l$ because they connect a dual vertex that is distant at least l form $\partial B_{O,l}$. So $\gamma_1\cup\gamma_2$ is an interface in fact γ_1 and γ_2 have only a dual vertex and no edge in common, they are connected with the two end-vertices to $\partial B_{O,l}$ and $|\gamma_1\cup\gamma_2|\geqslant 2l$. It is trivial to order by inclusion the vertex circuits with constant sign, in fact they cannot intersect, they could intersect if we considered a $\star$-connection (see [17]). We call τ the most external circuit surrounding v that has all the vertices with constant sign in $V(B_{O,l})$; we suppose that the sign is positive but in the other case the proof is similar. We consider the positive vertex region S connected to the circuit τ (with $\tau=\tau^+(v,t)$) and $S^\star$ the region $\star$-connected to τ. We claim that if there is not an interface then $S\cup\tau$ has not empty intersection with all the sides of $B_{O,l}$ and that therefore τ is connected to the four sides of $B_{O,l}$. If τ is the most external circuit in $V(B_{O,l})$ then $(S^\star\cup\tau)\cap(\bigcup_{i=1}^4 L_i)\neq\emptyset$, in fact if this was false then there would be a circuit of opposite sign respect to τ having τ in its interior. We note that we can write $S^\star$ as the union of $\star$-clusters A_i in $V(B_{O,l})\setminus\tau$. If $(S\cup\tau)\cap(\bigcup_{i=1}^4 L_i)=\emptyset$ we have an interface made by the boundary of τ in the region that is external to the circuit τ and from the boundary of one of the sets A_i. So if there is not an interface then we have $(S \cup \tau) \cap (\bigcup_{i=1}^{4} L_i) \neq \emptyset$. The boundary $\partial(S \cup \tau)$ is made of not satisfied edges and the region external to τ is the union of dual paths ς_i with all the edges not satisfied and the end-vertices connected to $\partial B_{O,l}$. We have $|\varsigma_i| < 2l$ for all i because there is not an interface and this is possible only if $\bigcup \varsigma_i$ has not empty intersection with the four sides L_i which have length equal to $2l$. We deduce that the most external circuit τ has the sign equal to the initial sign because the Glauber process is local and cannot change, at a single time, the spins of $(S \cup \tau) \cap (\bigcup_{i=1}^{4} L_i)$ that we know to be at least four and the spins on the opposite sides that have distance at least $2l$. So it is impossible to change the sign of τ without creating an interface. $\square$

Lemma A.6 is the unique point in which we have used that our Markov chain is local.

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