SELF-INTERACTING DIFFUSIONS II:
CONVERGENCE IN LAW

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ABSTRACT. – This paper concerns convergence in law properties of self-interacting diffusions on a compact Riemannian manifold.

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RÉSUMÉ. – Cet article étudie les propriétés de convergence en loi des diffusions interagissantes sur une variété riemannienne compacte.

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1. Introduction

Self interacting diffusions (as considered here) are continuous time stochastic processes living on a Riemannian manifold \( M \) which can be typically described as solutions to a stochastic differential equation (SDE) of the form

\[
\text{d}X_t = \sum_{\alpha} F_\alpha(X_t) \circ \text{d}B^\alpha_t - \frac{1}{t} \left( \int_0^t \nabla V_{X_s}(X_t) \text{d}s \right) \text{d}t,
\]

where \((B^\alpha)\)\(\alpha\) is a family of independent Brownian motions, \((F_\alpha)\)\(\alpha\) is a family of smooth vector fields on \( M \) such that \( \sum_{\alpha} F_\alpha(F_\alpha f) = \Delta f \) (for \( f \in C^\infty(M) \)), where \( \Delta \) denotes the Laplacian on \( M \) and \( V_{\alpha}(x) \) a “potential” function.

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These processes are characterized by the fact that the drift term in Eq. (1) depends both on the position of the process $X_t$, and its empirical occupation measure up to time $t$:

$$
\mu_t = \frac{1}{t} \int_0^t \delta_{X_s} \, ds.
$$

The asymptotic behavior of $\{\mu_t\}$ is the subject of a recent paper by Benaïm, Ledoux and Raimond [1]. This paper provides tools and results which allow to describe the long term behavior of $\{\mu_t\}$ in terms of the long term behavior of a certain deterministic semi-flow $\{\Psi_t\}_{t \geq 0}$ defined on the space of probability measure on $M$. For instance, there are situations (depending on the shape of $V$) in which $\{\mu_t\}$ converges almost surely to an equilibrium point $\mu^*$ of $\Psi$ and other situations where the limit set of $\{\mu_t\}$ coincides almost surely with a periodic orbit for $\Psi$ (see the examples in Section 4 of [1] and below in Section 7). In the simple case where $\mu_t$ converges to $\mu^*$ one expects $(X_t + u, u \geq 0)$ to behave like a homogeneous diffusion of generator

$$
L_{\mu^*} = \frac{1}{2} \Delta + \langle \nabla V_{\mu^*}, \nabla \rangle,
$$

where $V_{\mu^*}(x) = \int V_y(x) \mu^*(dy)$ and $\langle \cdot, \cdot \rangle$ denotes the Riemannian inner product on $M$. The purpose of this note is to address this type of question.

In Section 2, following [1], self-interacting diffusions on a smooth compact manifold are defined. In Section 3, the basic tool of this paper is presented, namely the Girsanov transform.

In Section 4, we show that on the event “$\mu_t$ converges towards $\mu^*$”, the law of $(X_t + u, u \geq 0)$ given $B_t = \sigma(X_s, s \leq t)$ is asymptotically equal to the law of the diffusion with generator $L_{\mu^*}$ and initial condition $X_t$.

In Section 5, we show that the law of $X_t + s(t)$ given $B_t$ is asymptotically equal to $\Pi(\mu_t)$, the invariant probability measure of the diffusion with generator $L_{\mu^*}$; provided $s(t) \to \infty$ at a convenient rate. Moreover the law of $X_t$ given $\Omega = \{\mu_t \to \mu^*\}$ converges towards $E[\mu^* | \Omega]$. In particular, when $P(\Omega) = 1$. $X_t$ converges in law towards $E[\mu^*]$.

Section 6 generalizes results of Section 5 to the law of the process $(X_t + s(t) + v, v \geq 0)$.

In Section 7, examples developed in [1] and [2], for which $\mu_t$ converges a.s. are presented.

2. Background and notation

The notation and definitions here are from [1].

Throughout we let $M$ denote a $d$-dimensional, compact connected smooth ($C^\infty$) Riemannian manifold. Without loss of generality (see Nash [4]) we shall assume that $M$ is isometrically embedded in $\mathbb{R}^N$. We denote $C^r(M), 0 \leq r \leq \infty$, the space of $C^r$ real valued functions on $M$.

Given a metric space $E$ we let $\mathcal{P}(E)$ denote the space of Borel probability measures on $E$ equipped with the topology induced by the weak convergence. Recall that a sequence
\{P_n\}_{n \geq 0} of Borel probability measures on \(E\) converges weakly to \(P\) provided
\[
\lim_{n \to \infty} \int f \, dP_n = \int f \, dP
\]
for every bounded and continuous function \(f : E \to \mathbb{R}\). When \(E\) is compact (e.g., \(E = M\)), \(P(E)\) is a compact metric space.

Throughout we assume given a measurable mapping
\[
V : M \times M \to \mathbb{R},
\]
\((u, x) \mapsto V(u, x) = V_u(x).
\]
We furthermore assume that for all \(u \in M\), \(V_u : M \to \mathbb{R}\) is a \(C^1\) function whose first derivatives are bounded (in the variables \(u\) and \(x\)). For \(\mu \in \mathcal{P}(M)\) we let \(V_\mu \in C^1(M)\) denote the function defined by
\[
V_\mu(x) = \int_M V(u, x) \mu(du),
\]
and \(L_\mu\) the operator defined on \(C^\infty(M)\) by
\[
L_\mu f = \frac{1}{2} \Delta f - \langle \nabla V_\mu, \nabla f \rangle,
\]
where \(\langle \cdot, \cdot \rangle\), \(\nabla\) and \(\Delta\) stand, respectively, for the Riemannian inner product, the associated gradient and Laplacian on \(M\).

We let \(\Omega\) denote the space of continuous paths \(w : \mathbb{R}_+ \to M\), equipped with the topology of uniform convergence on compact intervals; \(\mathcal{B} = \mathcal{B}(\Omega)\) the Borel \(\sigma\)-field of \(\Omega\), \(X_t\) the \(M\)-valued random variable defined by \(X_t(w) = w(t)\); and \(\mathcal{B}\), the \(\sigma\)-field generated by the random variables \(\{X_s : 0 \leq s \leq t\}\).

Since \(\Omega\) is polish, \(\mathcal{P}(\Omega)\) equipped with the weak convergence is metrizable. A distance \(d\) on \(\mathcal{P}(\Omega)\) is given by
\[
d(P, Q) = \sum_{n=1}^{\infty} 2^{-n} \left| \int Z_n \, dP - \int Z_n \, dQ \right|
\]
for \(P\) and \(Q\) in \(\mathcal{P}(\Omega)\) where \(Z_n : \Omega \to \mathbb{R}\) is continuous, \(\mathcal{B}_n\)-measurable, and \(\{Z_n : n \geq 1\}\) is dense in \(\{Z \in C^0(\Omega) : \|Z\|_\infty \leq 1\}\).

For \(r > 0\), \(\mu \in \mathcal{P}(M)\) and \(w \in \Omega\), the empirical occupation measure of \(w\) with initial weight \(r\) and initial measure \(\mu\) is the sequence \(\{\mu_t(r, \mu, w) \in \mathcal{P}(M) : t \geq 0\}\) defined by
\[
\mu_t(r, \mu, w) = \frac{1}{r + t} \left( r \mu + \int_0^t \delta_{w(s)} \, ds \right),
\]
where \(\int_0^t \delta_{w(s)} \, ds(A) = \int_0^t 1_A(w(s)) \, ds\), for every Borel set \(A \subset M\). In the following we will denote by \(\mu_t(r, \mu)\) the \(\mathcal{P}(M)\)-valued random variable \(w \mapsto \mu_t(r, \mu, w)\).
A self-interacting diffusion associated to $V$ is a family

$$\{P_{x,r,\mu}: x \in M, \ r > 0, \ \mu \in \mathcal{P}(M)\} \subset \mathcal{P}(\Omega)$$

(9)

such that

(i) $P_{x,r,\mu}(X_0 = x) = 1$.

(ii) For all $f \in C^\infty(M)$,

$$M_t^f = f(X_t) - f(x) - \int_0^t (L_{\mu_t, r, \mu} f)(X_s) \, ds$$

is a $P_{x,r,\mu}$-martingale relative to $\{B_t: t \geq 0\}$.

Existence and uniqueness of the self-interacting diffusion associated to $V$ is proved in [1], Proposition 2.5. More precisely, it is shown in this paper that $P_{x,r,\mu}$ can be obtained as the law of $\{X_t\}$, a solution (unique in law) of the following SDE on $M$:

$$dX_t = \sum_{i=1}^N F_i(X_t) \circ dB^i_t - \nabla V_{\mu_t, r, \mu}(X_t) \, dt, \quad X_0 = x,$$

(10)

where $(F_1(x), \ldots, F_N(x))$ denote the orthogonal projection of the canonical basis $(e_1, \ldots, e_N)$ of $\mathbb{R}^N$ on $T_x M$ and $B_t = (B^1_t, \ldots, B^N_t)$ is an $N$-dimensional Brownian motion.

For $x \in M$ and $\mu \in \mathcal{P}(M)$ we let $P_{x,\mu} \in \mathcal{P}(\Omega)$ denote the probability measure on $\Omega$ such that

(i) $P_{x,\mu}(X_0 = x) = 1$.

(ii) For all $f \in C^\infty(M)$,

$$M_t^f = f(X_t) - f(x) - \int_0^t (L_{\mu} f)(X_s) \, ds$$

is a $P_{x,\mu}$-martingale relative to $\{B_t: t \geq 0\}$.

In other words, $P_{x,\mu}$ is the law of the diffusion process $\{Y_t\}$ with initial condition $x$ and generator $L_{\mu}$ solution to the SDE:

$$dY_t = \sum_{i=1}^N F_i(Y_t) \circ dB^i_t - \nabla V_{\mu}(Y_t) \, dt, \quad Y_0 = x.$$  

(11)

In the following $E_{x,r,\mu}$ and $E_{x,\mu}$ will respectively denote the expectation with respect to $P_{x,r,\mu}$ and to $P_{x,\mu}$. 


3. The Girsanov transform and some lemmas

3.1. The Girsanov transform

Let \( B_t = (B^1_t, \ldots, B^N_t) \) be a standard Brownian motion on \( \mathbb{R}^N \), \( \mathcal{P} \) the law of \( (B_s; \ s \geq 0) \), \( \mathcal{E} \) the associated expectation, \( \mathcal{F}_t \) the \( \mathcal{P} \)-completion of \( \sigma(B_s; \ 0 \leq s \leq t) \) and \( \mathcal{F} = \mathcal{F}_\infty \). Let \( \{W^i_t\} \) be the solution to the SDE

\[
dW^i_t = \sum_{i=1}^N F_i(W^i_s) \circ dB^i_s, \quad W^i_0 = x \in M.
\]

Then \( W^x = (W^i_s; \ t \geq 0) \) is a Brownian motion on \( M \) starting at \( x \). We denote its law \( \mathcal{P}_x \).

Note that \( W^x: C([0, \infty) : \mathbb{R}^N) \to \kappa \Omega_m = C([0, \infty) : M) \) is measurable. Let

\[
M^{x,r,\mu}_t = \exp \left[ \int_0^t \left\langle \nabla V_{\mu x}^{r,\mu}(W^x_s), F_i(W^i_s) \right\rangle dB^i_s - \frac{1}{2} \int_0^t \| \nabla V_{\mu x}^{r,\mu}(W^x_s) \|^2 \ ds \right], \tag{13}
\]

where

\[
\mu^x(r, \mu) = \frac{1}{t+r} \left( r \mu + \int_0^t \delta_{W^x_s} \ ds \right), \tag{14}
\]

and

\[
M^{x,\mu}_t = \exp \left[ \int_0^t \left\langle \nabla V_{\mu x}(W^x_s), F_i(W^i_s) \right\rangle dB^i_s - \frac{1}{2} \int_0^t \| \nabla V_{\mu x}(W^x_s) \|^2 \ ds \right]. \tag{15}
\]

Then \( \{M^{x,r,\mu}_t\} \) and \( \{M^{x,\mu}_t\} \) are \( (\mathcal{P}, \{\mathcal{F}_t\}) \)-martingales. By the transformation of drift formula (see [3], Section IV 4.1 and Theorem IV 4.2),

\[
\begin{align*}
E_{x,r,\mu}[Z_t] &= E[M^{x,r,\mu}_t(Z_t \circ W^x)], \\
E_{x,\mu}[Z_t] &= E[M^{x,\mu}_t(Z_t \circ W^x)].
\end{align*}
\]

for every bounded \( \mathcal{B}_t \)-measurable random variable \( Z_t \). Note that this implies in particular that \( \mathcal{P}_x, \mathcal{P}_{x,\mu} \) and \( \mathcal{P}_{x,r,\mu} \) are equivalent.

3.2. Some lemmas

The next lemma is a basic tool to estimate quantities such as

\[
E_{x,r,\mu'}[Z_t] - E_{x,\mu}[Z_t],
\]

for large \( r \) and \( \mu' \) close to \( \mu \).

**Lemma 3.1.** For \( a = 1, 2 \) let \( A^a_t = (A^a_{1,t}, \ldots, A^a_{N,t}) \) be a \( \mathbb{R}^N \)-valued bounded \( \{\mathcal{F}_t\} \)-previsible process. Suppose that for all \( 0 \leq s \leq t \)

\[
\|A^a_s - A^a_t\| \leq \delta(t) \tag{17}
\]
for some deterministic function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$. Let

$$M^a_t = \exp \left[ \int_0^t \sum_i A^a_{s,i} dB^i_s - \frac{1}{2} \int_0^t \|A^a_s\|^2 ds \right], \quad a = 1, 2.$$  

Then there exists a positive constant $C$ such that for any $F_t$-measurable random variable $Z_t$ bounded by 1 in absolute value,

$$\left| E [M^1_t Z_t] - E [M^2_t Z_t] \right| \leq e^{Ct} \delta(t).$$  

The constant $C$ depends only on $\sup_{a,s} \|A^a_s\|_{\infty}$.

Lemma 3.1 will be proved in Section 8. Note that Lemma 3.1 and the Girsanov transforms given in Section 3.1 imply that $P_{x,r,\mu}$ converges weakly towards $P_{x,\mu}$ as $r \to \infty$. More precisely

**Lemma 3.2.** There exists a positive constant $C$ (depending only on $\sup_{x,y} \|\nabla V(x)\|$) such for any $B$-measurable random variable $Z_t$ bounded by 1 in absolute value,

$$\left| E_{x,r,\mu}[Z_t] - E_{x,\mu}[Z_t] \right| \leq \frac{e^{Ct}}{r + t}.\quad (19)$$

**Proof.** There exists a constant $C$ such that $\|\nabla V\|_{\infty} \leq C$ and $\|\nabla V_{\mu_s(r,\mu)} - \nabla V_{\mu}\|_{\infty} \leq Ct/(r + t)$, for all $0 \leq s \leq t$. The result then follows from Girsanov formulas (16) and Lemma 3.1 applied with

$$\left\{ \begin{array}{l}
A^1_{s,i}(W_s^x) = \langle \nabla V_{\mu_s(r,\mu)}(W_s^x), F_i(W_s^x) \rangle,
A^2_{s,i}(W_s^x) = \langle \nabla V_{\mu}(W_s^x), F_i(W_s^x) \rangle.
\end{array} \right. \quad (20)$$

4. The asymptotic of $P_{x,t,r+t,\mu_t(r,\mu)}$

Here we shall prove:

**Theorem 4.1.** Let $\mu^* : \Omega \to \mathcal{P}(M)$ denote a $\mathcal{P}(M)$-valued random variable. Let $\Omega = \{ w \in \Omega : \lim_{t \to \infty} \mu_t(r, \mu, w) = \mu^* \}$. Then $P_{x,\mu^*}$-a.s. on $\Omega$,

$$\lim_{t \to \infty} d(P_{x,\mu_t(r,\mu)}, P_{x,\mu^*}) = 0.\quad (21)$$

**Corollary 4.2.** For every bounded and continuous function $Z : \Omega \to \mathbb{R}$, $P_{x,\mu^*}$-a.s.

$$\lim_{t \to \infty} \left| E_{x,\mu}[Z \circ \theta_t|B_t] - E_{x,\mu^*}[Z]\right|_{\Omega} = 0,\quad (22)$$

where $\theta_t : \Omega \to \Omega$ is the shift on $\Omega$ defined by $\theta_t(w)(s) = w(t + s)$.

**Proof.** By the Markov property, we have

$$E_{x,\mu}[Z \circ \theta_t|B_t] = E_{x,t+r,\mu_t(r,\mu)}[Z],\quad (23)$$
and the result follows from Theorem 4.1. □

**Proof of Theorem 4.1.** – Follows directly from the following estimate:

**Proposition 4.3.** – Let \( \{ \mu_r : r > 0 \} \subset \mathcal{P}(M) \). Assume that

\[
\lim_{r \to \infty} \mu_r = \mu^* \quad \text{in} \quad \mathcal{P}(M).
\]

Let \( Z_t \) be a random variable \( \mathcal{F}_t \)-measurable and bounded by 1 in absolute value. Then

\[
\lim_{r \to \infty} \mathbb{E}_{x,r,\mu_r}[Z_t] = \mathbb{E}_{x,\mu^*}[Z_t] \quad \text{(24)}
\]

uniformly in \( x \in M \).

More precisely, there exists \( C > 0 \) (depending only on \( \sup_{x,y} \| \nabla V_{\mu_r}(x) \| \) ) such that

\[
\left| \mathbb{E}_{x,r,\mu_r}[Z_t] - \mathbb{E}_{x,\mu^*}[Z_t] \right| \leq e^{Ct} \left( \frac{1}{r} + \varepsilon(r) \right), \quad (25)
\]

where \( \varepsilon(r) = \sup_{x} \| \nabla V_{\mu_r}(x) - \nabla V_{\mu^*}(x) \| \).

Note that \( \lim_{r \to \infty} \varepsilon(r) = 0 \) (since \( x \mapsto \nabla V_{\mu'}(x) - \nabla V_{\mu^*}(x) \) is equicontinuous in \( x \) and converges towards 0 for every \( x \)).

**Proof.** – Lemma 3.1 applied with

\[
\begin{aligned}
A_{1,i}^t &= \langle \nabla V_{\mu'}(W_{x,s}^i), F_i(W_{x,s}^i) \rangle, \\
A_{2,i}^t &= \langle \nabla V_{\mu^*}(W_{x,s}^i), F_i(W_{x,s}^i) \rangle
\end{aligned} \quad (26)
\]

implies

\[
\left| \mathbb{E}_{x,\mu'}[Z_t] - \mathbb{E}_{x,\mu^*}[Z_t] \right| \leq e^{Ct} \varepsilon(r). \quad (27)
\]

The conclusion follows from this last inequality combined with Lemma 3.2 and the triangle inequality. □

### 5. The convergence in law of \( X_t \)

For every \( \mu \in \mathcal{P}(M) \) let \( \Pi(\mu) \in \mathcal{P}(M) \) denote the invariant probability measure of the diffusion process with generator \( L_\mu \). That is

\[
\Pi(\mu)(dx) = \frac{e^{-2V_\mu(x)}}{Z(\mu)} \lambda(dx), \quad (28)
\]

where \( Z(\mu) \) is the normalization constant.

Let us first remark that as \( r \to \infty \), the law of \( X_t \) under \( \mathbb{P}_{x,r,\mu} \) converges weakly towards the law of \( X_t \) under \( \mathbb{P}_{x,\mu} \) (see Lemma 3.2). We also have the convergence

\[
\lim_{t \to \infty} \mathbb{E}_{x,\mu}[g(X_t)] = \Pi(\mu)g \quad \text{for all} \quad g \in C^0(M).
\]

The next proposition shows that

\[ ^1 \text{For a measure} \ \mu \text{and} \ f \in L^1(\mu) \text{we let} \ \mu f \text{denote} \ \int f \, d\mu. \]
\[ E_{x,r,\mu}[g(X_{t+s})|B_t] = E_{X_{t},r+\mu_t(r,\mu)}[g(X_s)] \text{ and } \Pi(\mu_t(r,\mu))g \text{ are close when } s \text{ and } t \text{ tends to } \infty \text{ at a certain rate.} \]

**Proposition 5.1.** – For all \( t \geq 1, r > 0, s > 0 \) and \( g \in C^0(M) \),

\[
|E_{x,r,\mu}[g(X_{t+s})|B_t] - \Pi(\mu_t)g| \leq \|g\|_\infty \left( \frac{e^{Cs}}{r + s + t} + Ce^{-s/\kappa} \right),
\]

where \( C \) and \( \kappa \) are positive constants depending only on \( V \).

The proof of Proposition 5.1 is given in Section 8.

**Corollary 5.2.** –

(i) For all positive \( s \) and all \( g \in C^0(M) \),

\[
\limsup_{t \to \infty} |E_{x,r,\mu}[g(X_{t+s})|B_t] - \Pi(\mu_t)g| \leq C\|g\|_\infty e^{-s/\kappa}. \]

(ii) Let \( s \) be a real valued positive function such that \( 1 \ll \exp(s(t)) \ll t^{1/C} \)

when \( t \) tends to \( \infty \). Then for all \( g \in C^0(M) \),

\[
\limsup_{t \to \infty} |E_{x,r,\mu}[g(X_{t+s(t)})|B_t] - \Pi(\mu_t)g| = 0. \]

**Proof.** – Straightforward. \( \square \)

**Remark 5.3.** – Let \( \mathcal{L}_t \) denote the law of \( X_{t+s(t)} \) knowing \( B_t \). Then Corollary 5.2 means that \( \mathcal{L}_t \) is asymptotically equal to \( \Pi(\mu_t) \). That is, \( \lim_{t \to \infty} \text{dist}_w(\mathcal{L}_t, \Pi(\mu_t)) = 0 \), where \( \text{dist}_w \) is a distance on \( \mathcal{P}(M) \) for the weak topology.

Remark that Proposition 5.1 and Corollary 5.2 make no assumption on the asymptotic of \( \{\mu_t\} \). Let \( \tilde{L} \) be the event that “\( \mu_t \) converges towards \( \mu^* \)”, where \( \mu^* \) is a \( \mathcal{P}(M) \)-valued random variable. In [1] and [2], several examples of self-interacting diffusions for which \( P_{x,r,\mu}(\tilde{L}) = 1 \) are given (these examples are shortly presented in Section 7).

The following theorem describes the law of \( X_{t+s(t)} \) knowing \( B_t \) on \( \tilde{L} \).

**Theorem 5.4.** – Let \( s(t) \) be as in Corollary 5.2. Then, the law of \( X_{t+s(t)} \) knowing \( B_t \) converges weakly towards \( \mu^*P_{x,r,\mu}-\text{a.s. on } \tilde{L} \). That is, for all \( g \in C^0(M) \),

\[
\lim_{t \to \infty} E_{x,r,\mu}[g(X_{t+s(t)})|B_t] = \mu^*g
\]

\( P_{x,r,\mu}-\text{a.s. on } \tilde{L} \).

**Proof.** – It follows from Theorem 3.8 in [1] that \( \mu^* \) (almost surely on \( \tilde{L} \)) a fixed point of \( \Pi \), i.e., \( \Pi(\mu^*) = \mu^* \). The proof now follows from Corollary 5.2(ii) and the fact that \( \Pi: \mathcal{P}(M) \to \mathcal{P}(M) \) is continuous. \( \square \)
COROLLARY 5.5 (Convergence in law). – For all \( g \in C^0(M) \),
\[
\lim_{t \to \infty} E_{x,r,\mu} \left[ g(X_t) \mathbf{1}_{\Omega} \right] = E_{x,r,\mu} \left[ (\mu^* g) \mathbf{1}_{\Omega} \right].
\] (34)

In particular, if \( P_{x,r,\mu}(\Omega) = 1 \) then for all \( g \in C^0(M) \),
\[
\lim_{t \to \infty} E_{x,r,\mu} \left[ g(X_t) \right] = E_{x,r,\mu} \left[ (\mu^* g) \right],
\] (35)
i.e., \( X_t \) converges in law towards \( E_{x,r,\mu} \left[ \mu^* \right] \) when \( t \) tends to \( \infty \).

Proof. – In view of Theorem 5.4
\[
\lim_{t \to \infty} E_{x,r,\mu} \left[ E_{x,r,\mu} \left[ g(X_{t+s(t)}) | B_t \right] \mathbf{1}_{\Omega} \right] = E_{x,r,\mu} \left[ (\mu^* g) \mathbf{1}_{\Omega} \right].
\] (36)

It then suffices to prove that \( \lim_{t \to \infty} a_t = 0 \) where
\[
a_t = E_{x,r,\mu} \left[ E_{x,r,\mu} \left[ g(X_{t+s(t)}) | B_t \right] \mathbf{1}_{\Omega} \right] - E_{x,r,\mu} \left[ g(X_{t+s(t)}) \mathbf{1}_{\Omega} \right].
\] (37)

Let \( \Delta_t = \mathbf{1}_{\Omega} - E_{x,r,\mu} \left[ \mathbf{1}_{\Omega} | B_t \right]. \) Then
\[
a_t = E_{x,r,\mu} \left[ E_{x,r,\mu} \left[ g(X_{t+s(t)}) | B_t \right] \Delta_t \right] - E_{x,r,\mu} \left[ g(X_{t+s(t)}) \Delta_t | B_t \right].
\] (38)

Hence \( |a_t| \leq 2 \|g\| \|E_{x,r,\mu}[|\Delta_t|] \) and consequently \( \lim_{t \to \infty} a_t = 0 \) because \( \lim_{t \to \infty} \Delta_t = 0 \) a.s. \( \square \)

6. The convergence in law of \( (X_{t+s}, u \geq 0) \)

In the previous section we were only interested by the asymptotic of the law of \( X_{t+s} \) knowing \( B_t \). These results can be extended to the law of \( (X_{t+s+u}, u \geq 0) \) knowing \( B_t \).

The following proposition is analogous to Proposition 5.1 (and implies Proposition 5.1).

**PROPOSITION 6.1.** – For all \( t \geq 1, s > 0, u > 0 \) and \( Z_{u} \) a \( B_{u} \)-measurable random variable bounded by 1 in absolute value, then
\[
|E_{x,r,\mu}[Z_{u} \circ \theta_{t+s}|B_t] - E_{\Pi(\mu_{t}),\mu_{t}}[Z_{u}]| \leq \left( \frac{e^{C(s+u)}}{r + s + u + t} + C e^{-t/\kappa} \right),
\] (39)

where \( C \) and \( \kappa \) are positive constants depending only on \( V \).

The proof of Proposition 6.1 is given in Section 8.

**COROLLARY 6.2.** – For any positive \( u \) and \( Z_{u} \) a \( B_{u} \)-measurable random variable bounded by 1 in absolute value, we have
(i) For any positive \( s \),
\[
\limsup_{t \to \infty} |E_{x,r,\mu}[Z_{u} \circ \theta_{t+s}|B_t] - E_{\Pi(\mu_{t}),\mu_{t}}[Z_{u}]| \leq C e^{-t/\kappa}.
\] (40)
(ii) Let \( s \) be a function like in Corollary 5.2, then
\[
\lim_{t \to \infty} \left| E_{x,r,\mu}[Z_u \circ \theta_{t+s(t)}|B_t] - E_{\Pi(\mu_t),\mu_t}[Z_u] \right| = 0. \tag{41}
\]

**Proof.** – Straightforward. \( \square \)

This corollary shows that the law of \((X_{t+s(t)}+v; \ v \geq 0)\) knowing \( B_t \) is asymptotically equal to the law of a diffusion with generator \( L_{\mu_\ast} \) and initial distribution \( \Pi(\mu_t) \). In particular, (ii) says that
\[
\lim_{t \to \infty} d(P_t, P_{\Pi(\mu_t),\mu_t}) = 0, \tag{42}
\]
where \( P_t \) is the law of \((X_{t+s(t)}+u; \ u \geq 0)\) knowing \( B_t \).

Like in the previous section, we now focus on \( \tilde{\Omega} \). The following theorem shows that on \( \tilde{\Omega} \), given \( B_t \), \((X_{t+s(t)}+u; \ u \geq 0)\) converges in law towards a diffusion with generator \( L_{\mu_\ast} \) and initial distribution \( \mu_\ast \) (note that \( \mu_\ast \) satisfies \( \mu_\ast = \Pi(\mu_\ast) \) so that \( \mu_\ast \) is the invariant probability measure of this diffusion).

**THEOREM 6.3.** – For any positive \( u \) and \( Z_u \) a bounded \( B_u \)-measurable random variable,
\[
\lim_{t \to \infty} E_{x,r,\mu}[Z_u \circ \theta_{t+s(t)}|B_t] = E_{\mu_\ast,\mu_\ast}[Z_u], \tag{43}
\]
almost surely on \( \tilde{\Omega} \), where \( s(t) \) is as in Corollary 5.2.

**Proof.** – The proof is the same as the one of Theorem 5.4. \( \square \)

Note that Theorem 6.3 implies that on \( \tilde{\Omega} \), \( P_t \) converges weakly towards \( P_{\mu_\ast,\mu_\ast} \).

**COROLLARY 6.4 (Convergence in law).** – For any positive \( u \) and \( Z_u \) a bounded \( B_u \)-measurable random variable,
\[
\lim_{t \to \infty} E_{x,r,\mu}[Z_u \circ \theta_{t}|\tilde{\Omega}] = E_{x,r,\mu}[E_{\mu_\ast,\mu_\ast}[Z_u]|\tilde{\Omega}], \tag{44}
\]
In particular, if \( P_{x,r,\mu}(\tilde{\Omega}) = 1 \) then
\[
\lim_{t \to \infty} E_{x,r,\mu}[Z_u \circ \theta_{t}] = E_{x,r,\mu}[E_{\mu_\ast,\mu_\ast}[Z_u]]. \tag{45}
\]

**Proof.** – The proof is the same as the one of Corollary 5.5. \( \square \)

Note that (44) and (45) respectively imply that the law of \((X_{t+u}; \ u \geq 0)\) given \( \tilde{\Omega} \) converges weakly towards \( E_{x,r,\mu}[P_{\mu_\ast,\mu_\ast}|\tilde{\Omega}] \) and that \( E_{x,r,\mu}[P_{x,r+\mu_t}|\tilde{\Omega}] \) converges weakly towards \( E_{x,r,\mu}[P_{\mu_\ast,\mu_\ast}] \) provided \( P_{x,r,\mu}(\tilde{\Omega}) = 1 \).

7. Examples

Set \( \delta_V(x, y) = \sup_{u \in M}(V_u(x) - V_u(y)) - \inf_{u \in M}(V_u(x) - V_u(y)) \). In [1], Corollary 4.4, it is proved that when \( \sup_{(x,y) \in M^2} \delta_V(x, y) < 1 \), then \( \Pi \) has a unique fixed point \( \mu_\ast \)
and \( \lim_{t \to \infty} \mu_t(r, \mu) = \mu^* \mathbb{P}_{x,r,\mu} \) a.s. The associated self-interacting diffusions produce examples for which \( \mathbb{P}_{x,r,\mu}(\Omega) = 1 \), but the limit \( \mu^* \) is not random.

From the different interactions, we distinguish those such that \( V \) is symmetric and defines a positive or a negative self-adjoint operator acting on \( L^2(\lambda) \), that can be written in the form \( V = \alpha \int_C G(u, x) G(u, y) \nu(du) \), where \( C \) is compact, \( \nu \) is a Borel probability measure, \( G : C \times M \to \mathbb{R} \) is continuous and \( \alpha \in \mathbb{R} \). We call them gradient interactions. These interactions produce examples for which \( \mathbb{P}_{x,r,\mu}(\tilde{\kappa}\Omega G) = 1 \), but the limit \( \mu^* \) is not random.

When \( \alpha \) is positive, we say it is a self-repelling interaction and when \( \alpha \) is negative, we say it is a self-attracting interaction. It can be proved (see [2]) that, if \( V \) is a constant function, for all repelling cases or weakly attracting cases (\( \alpha > -\alpha_G \), with \( \alpha_G > 0 \)), the empirical occupation measure of the associated self-interacting diffusion converges towards \( \lambda \) a.s. But, when \( \alpha < -\alpha_G \), this is not the case, and \( \mu_t \) may converge towards \( \mu^* \neq \lambda \).

The interaction, on the \( n \)-dimensional sphere \( S^n \),

\[
V(x, y) = 2\alpha \cos(d(x, y))
\]

is a gradient interaction. This example is developed in [1], Section 4.2. When \( \alpha \geq -(n + 1)/4 \), \( \mu_t \) converges towards \( \lambda \) a.s. and when \( \alpha < -(n + 1)/4 \), there exists a \( S^n \)-valued random variable \( v \) such that \( \mu_t \) converges a.s. towards \( \exp[\beta_n(\alpha) \cos(d(x,v))]\lambda(dx)/Z_{n,\alpha} \), where \( Z_{n,\alpha} \) is the normalization constant and \( \beta_n(\alpha) \) is a constant depending only on \( n \) and \( \alpha \). In [1], Section 4.2, an example of interaction on \( S^n \) (which is not a gradient interaction) for which \( \mathbb{P}_{x,r,\mu}(\tilde{\kappa}\Omega G) = 0 \) is given.

8. Proofs

8.1. Proof of Lemma 3.1

Let \( C \) be a constant such that both \( \|A^\alpha_t\|^2 \) and \( \|A^\alpha_t\| \) are lower than \( C \). Let

\[
E_t = \exp \left[ \int_0^t \langle A^1_s, A^1_s - A^2_s \rangle \, ds \right],
\]

and \( N_t = M^2_t (M^1_t E_t)^{-1} \). Observe that \( M^\alpha_t \) and \( N_t \) are exponential martingales solutions of the SDEs

\[
\begin{align*}
\frac{dM^\alpha_t}{M^\alpha_t} &= \sum_i A^{\alpha,i}_t \, dB^i_t, \\
\frac{dN_t}{N_t} &= N_t \left( \sum_i (A^{2,i}_t - A^{1,i}_t) \, dB^i_t \right).
\end{align*}
\]
Therefore
\[
\begin{align*}
\frac{d}{ds} E[(M_s^a)^2] &= E[(M_s^a)^2] \leq C E[(M_s^a)^2], \\
\frac{d}{ds} E[(N_s)^2] &= E[(N_s)^2] \leq \delta^2(t) E[(N_s)^2],
\end{align*}
\] (49)
for \( s \leq t \). Hence, by Gronwall’s lemma, for \( a \in \{1, 2\} \)
\[
\begin{align*}
E[(M_t^a)^2] &\leq e^{Ct}, \\
E[(N_t)^2] &\leq \exp(t \delta^2(t)).
\end{align*}
\] (50)
Notice that we also have
\[
|E_t - 1| \leq \exp(Ct \delta(t)) - 1. 
\] (51)
Using these estimates and Schwartz inequality, we get
\[
|E[M_t^2 Z_t] - E[M_t^2 Z_t]| = |E[Z_t(N_t E_t - 1) M_t^2]|
\leq E[(N_t(E_t - 1) + N_t - 1)^2]^{1/2} E[(M_t^2)^2]^{1/2}
\leq e^{Ct/2} \left( (\exp(Ct \delta(t)) - 1) \exp\left(\frac{t \delta^2(t)}{2}\right) + (\exp(t \delta^2(t)) - 1)^{1/2} \right).
\]
Since \( e^u - 1 \leq e^u \) we easily obtain
\[
|E_{x,\mu}(Z_t) - E_{x,\mu}(Z_t)| \leq e^{Ct \delta(t)},
\] (52)
for \( C \) large enough. This proves the lemma. \( \square \)

8.2. Proof of Propositions 5.1 and 6.1

Let \( P_t^\mu = (P_t^\mu)_{t \geq 0} \) denote the semigroup of the diffusion with generator \( L_\mu \).

**Lemma 8.1.** – Let \( g : M \to \mathbb{R} \) be a bounded continuous function, then for \( t \geq 1 \),
\[
|P_t^\mu g(x) - \Pi(\mu) g| \leq C \|g\|_{\infty} e^{-t/\kappa},
\] (53)
for some constant \( C \) and \( \kappa \) depending only on \( \|V\|_{\infty} \).

**Proof.** – Let \( \|\cdot\|_2 \) be the \( L^2 \)-norm defined by
\[
\|f\|_2^2 = \int_M f^2(x) \Pi(\mu)(dx).
\] (54)
Then, by standard semigroup inequalities (see [1], Section 5.2)
\[
\|P_t^\mu g - \Pi(\mu) g\|_2 \leq e^{-t/\kappa} \|g - \Pi(\mu) g\|_2, \quad t > 0,
\] (55)
\[
\|P_t^\mu g - \Pi(\mu) g\|_{\infty} \leq C t^{-\kappa/2} \|g - \Pi(\mu) g\|_2, \quad 0 < t \leq 1,
\] (56)
for some constant \( \kappa > 0 \) and \( 0 < C < \infty \) depending only on \( \|V\|_{\infty} \). Combining (55) and (56) leads to
\[ \| P_s \mu g - \Pi(\mu) g \|_\infty = \| P_{s-1} \left( P_s \mu - \Pi(\mu) g \right) \|_\infty \leq C e^{-s那} \| g - \Pi(\mu) g \|_2 \leq 2C e^{-s那} \| g \|_\infty \]

for all \( s > 1 \). □

Proof of Proposition 5.1. – By the Markov property

\[ E_{x,r,\mu} \left[ g(X_{t+s}) | B_t \right] = E_{X_t,r,t,\mu t(r,\mu)} \left[ g(X_s) \right]. \] (57)

Hence

\[ \left| E_{x,r,\mu} \left[ g(X_{t+s}) | B_t \right] - \Pi(\mu) g \right| \leq \left| E_{X_t,r,t,\mu t(r,\mu)} \left[ g(X_s) \right] - E_{X_t,\mu t} \left[ g(X_s) \right] \right| + \left| E_{X_t,\mu t} \left[ g(X_s) \right] - \Pi(\mu) g \right| \]

and the result follows from Lemmas 3.2 and 8.1. □

Proof of Proposition 6.1. – This is almost the same proof. By the Markov property

\[ E_{x,r,\mu} \left[ Z_u \circ \theta_{t+s} | B_t \right] = E_{X_t,r,t,\mu t(r,\mu)} \left[ Z_u \circ \theta_s \right]. \]

Hence

\[ \left| E_{x,r,\mu} \left[ Z_u \circ \theta_{t+s} | B_t \right] - E_{\Pi(\mu),\mu} \left[ Z_u \right] \right| \leq \left| E_{X_t,r,t,\mu t(r,\mu)} \left[ Z_u \circ \theta_s \right] - E_{X_t,\mu t} \left[ Z_u \circ \theta_s \right] \right| + \left| E_{X_t,\mu t} \left[ Z_u \circ \theta_s \right] - E_{\Pi(\mu),\mu} \left[ Z_u \right] \right|. \]

The first term of the right-hand side of preceding equation can be dominated using Lemma 3.2. For the domination of the second term, let \( \varphi(x) = E_{x,\mu} \left[ Z_u \right] \), then

\[ \left\{ \begin{array}{l} E_{X_t,\mu t} \left[ Z_u \circ \theta_s \right] = P_s \mu \varphi(X_t), \\ E_{\Pi(\mu),\mu} \left[ Z_u \right] = \Pi(\mu) \varphi. \end{array} \right. \] (58)

We then conclude using Lemma 8.1. □

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