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SELF-INTERACTING DIFFUSIONS II: CONVERGENCE IN LAW

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ABSTRACT. – This paper concerns convergence in law properties of self-interacting diffusions on a compact Riemannian manifold.

RÉSUMÉ. – Cet article étudie les propriétés de convergence en loi des diffusions inter-agissantes sur une variété riemannienne compacte.

1. Introduction

Self interacting diffusions (as considered here) are continuous time stochastic processes living on a Riemannian manifold M which can be typically described as solutions to a stochastic differential equation (SDE) of the form

$$dX_t = \sum_\alpha F_\alpha(X_t) \circ dB_t^\alpha - \frac{1}{t}\left(\int_0^t \nabla V_{X_s}(X_t)\,ds\right)dt, \tag{1}$$

where $(B^\alpha)_\alpha$ is a family of independent Brownian motions, $(F_\alpha)_\alpha$ is a family of smooth vector fields on M such that $\sum_\alpha F_\alpha(F_\alpha f) = \Delta f$ (for $f \in C^\infty(M)$), where Δ denotes the Laplacian on M and $V_u(x)$ a "potential" function.

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These processes are characterized by the fact that the drift term in Eq. (1) depends both on the position of the process X_t, and its empirical occupation measure up to time t:

$$\mu_t = \frac{1}{t}\int_0^t \delta_{X_s}\,ds. \tag{2}$$

The asymptotic behavior of $\{\mu_t\}$ is the subject of a recent paper by Benaïm, Ledoux and Raimond [1]. This paper provides tools and results which allow to describe the long term behavior of $\{\mu_t\}$ in terms of the long term behavior of a certain deterministic semi-flow $\{\Psi_t\}_{t\geqslant 0}$ defined on the space of probability measure on M. For instance, there are situations (depending on the shape of V) in which $\{\mu_t\}$ converges almost surely to an equilibrium point μ^* of Ψ and other situations where the limit set of $\{\mu_t\}$ coincides almost surely with a periodic orbit for Ψ (see the examples in Section 4 of [1] and below in Section 7). In the simple case where μ_t converges to μ^* one expects $(X_{t+s},\ s\geqslant 0)$ to behave like a homogeneous diffusion of generator

$$L_{\mu^*} = \frac{1}{2}\Delta + \langle \nabla V_{\mu^*}, \nabla\rangle,$$

where $V_{\mu^*}(x) = \int V_y(x)\mu^*(dy)$ and $\langle\cdot,\cdot\rangle$ denotes the Riemannian inner product on M. The purpose of this note is to address this type of question.

In Section 2, following [1], self-interacting diffusions on a smooth compact manifold are defined. In Section 3, the basic tool of this paper is presented, namely the Girsanov transform.

In Section 4, we show that on the event "μ_t converges towards μ^*", the law of $(X_{t+u},\ u\geqslant 0)$ given $\mathcal{B}_t = \sigma(X_s, s\leqslant t)$ is asymptotically equal to the law of the diffusion with generator L_{μ^*} and initial condition X_t.

In Section 5, we show that the law of $X_{t+s(t)}$ given $\mathcal{B}_t$ is asymptotically equal to $\Pi(\mu_t)$, the invariant probability measure of the diffusion with generator L_{μ_t}; provided $s(t)\to\infty$ at a convenient rate. Moreover the law of X_t given $\widetilde{\Omega} = \{\mu_t \to \mu^*\}$ converges towards $\mathsf{E}[\mu^*|\widetilde{\Omega}]$. In particular, when $\mathsf{P}(\widetilde{\Omega}) = 1,\ X_t$ converges in law towards $\mathsf{E}[\mu^*]$.

Section 6 generalizes results of Section 5 to the law of the process $(X_{t+s(t)+v},\ v\geqslant 0)$.

In Section 7, examples developed in [1] and [2], for which μ_t converges a.s. are presented.

2. Background and notation

The notation and definitions here are from [1].

Throughout we let M denote a d-dimensional, compact connected smooth (C^∞) Riemannian manifold. Without loss of generality (see Nash [4]) we shall assume that M is isometrically embedded in $\mathbb{R}^N$. We denote $C^r(M),\ 0\leqslant r\leqslant\infty$, the space of C^r real valued functions on M.

Given a metric space E we let $\mathcal{P}(E)$ denote the space of Borel probability measures on E equipped with the topology induced by the weak convergence. Recall that a sequence

$\{\mathsf{P}_n\}_{n\geqslant 0}$ of Borel probability measures on E *converges weakly* to P provided

$$\lim_{n\to\infty}\int f\,d\mathsf{P}_n=\int f\,d\mathsf{P} \tag{3}$$

for every bounded and continuous function $f:E\to\mathbb{R}$. When E is compact (e.g., $E=M$), $\mathcal{P}(E)$ is a compact metric space.

Throughout we assume given a measurable mapping

$$\begin{aligned} &V:M\times M\to\mathbb{R},\\ &(u,x)\mapsto V(u,x)=V_u(x). \end{aligned} \tag{4}$$

We furthermore assume that for all $u\in M$, $V_u:M\to\mathbb{R}$ is a C^1 function whose first derivatives are bounded (in the variables u and x). For $\mu\in\mathcal{P}(M)$ we let $V_\mu\in C^1(M)$ denote the function defined by

$$V_\mu(x)=\int_M V(u,x)\mu(du), \tag{5}$$

and L_μ the operator defined on $C^\infty(M)$ by

$$L_\mu f=\frac{1}{2}\Delta f-\langle\nabla V_\mu,\nabla f\rangle, \tag{6}$$

where $\langle\cdot,\cdot\rangle$, ∇ and Δ stand, respectively, for the Riemannian inner product, the associated gradient and Laplacian on M.

We let Ω denote the space of continuous paths $w:\mathbb{R}_+\to M$, equipped with the topology of uniform convergence on compact intervals; $\mathcal{B}=\mathcal{B}(\Omega)$ the Borel σ-field of Ω, X_t the M-valued random variable defined by $X_t(w)=w(t)$; and $\mathcal{B}_t$ the σ-field generated by the random variables $\{X_s:\ 0\leqslant s\leqslant t\}$.

Since Ω is polish, $\mathcal{P}(\Omega)$ equipped with the weak convergence is metrizable. A distance d on $\mathcal{P}(\Omega)$ is given by

$$d(\mathsf{P},\mathsf{Q})=\sum_{n=1}^{\infty}2^{-n}\left|\int Z_n\,d\mathsf{P}-\int Z_n\,d\mathsf{Q}\right| \tag{7}$$

for P and Q in $\mathcal{P}(\Omega)$ where $Z_n:\Omega\to\mathbb{R}$ is continuous, $\mathcal{B}_n$-measurable, and $\{Z_n;\ n\geqslant 1\}$ is dense in $\{Z\in C^0(\Omega);\ \|Z\|_\infty\leqslant 1\}$.

For $r>0$, $\mu\in\mathcal{P}(M)$ and $w\in\Omega$, the *empirical occupation measure of* w *with initial weight* r *and initial measure* μ is the sequence $\{\mu_t(r,\mu,w)\in\mathcal{P}(M):\ t\geqslant 0\}$ defined by

$$\mu_t(r,\mu,w)=\frac{1}{r+t}\left(r\mu+\int_0^t\delta_{w(s)}\,ds\right), \tag{8}$$

where $\int_0^t\delta_{w(s)}\,ds(A)=\int_0^t\mathbf{1}_A(w(s))\,ds$, for every Borel set $A\subset M$. In the following we will denote by $\mu_t(r,\mu)$ the $\mathcal{P}(M)$-valued random variable $w\mapsto\mu_t(r,\mu,w)$.

A *self-interacting diffusion associated to V* is a family

$$\{\mathsf{P}_{x,r,\mu}\colon x \in M,\ r > 0,\ \mu \in \mathcal{P}(M)\} \subset \mathcal{P}(\Omega) \tag{9}$$

such that

(i) $\mathsf{P}_{x,r,\mu}(X_0 = x) = 1$.

(ii) For all $f \in C^\infty(M)$,

$$M_t^f = f(X_t) - f(x) - \int_0^t (L_{\mu_s(r,\mu)} f)(X_s)\,ds$$

is a $\mathsf{P}_{x,r,\mu}$-martingale relative to $\{\mathcal{B}_t\colon t \geqslant 0\}$.

Existence and uniqueness of the self-interacting diffusion associated to V is proved in [1], Proposition 2.5. More precisely, it is shown in this paper that $\mathsf{P}_{x,r,\mu}$ can be obtained as the law of $\{X_t\}$, a solution (unique in law) of the following SDE on M:

$$dX_t = \sum_{i=1}^N F_i(X_t) \circ dB_t^i - \nabla V_{\mu_t(r,\mu)}(X_t)\,dt, \quad X_0 = x, \tag{10}$$

where $(F_1(x), \ldots, F_N(x))$ denote the orthogonal projection of the canonical basis $(e_1, \ldots, e_N)$ of $\mathbb{R}^N$ on $T_x M$ and $B_t = (B_t^1, \ldots, B_t^N)$ is an N-dimensional Brownian motion.

For $x \in M$ and $\mu \in \mathcal{P}(M)$ we let $\mathsf{P}_{x,\mu} \in \mathcal{P}(\Omega)$ denote the probability measure on Ω such that

(i) $\mathsf{P}_{x,\mu}(X_0 = x) = 1$.

(ii) For all $f \in C^\infty(M)$,

$$M_t^f = f(X_t) - f(x) - \int_0^t (L_\mu f)(X_s)\,ds$$

is a $\mathsf{P}_{x,\mu}$-martingale relative to $\{\mathcal{B}_t\colon t \geqslant 0\}$.

In other words, $\mathsf{P}_{x,\mu}$ is the law of the diffusion process $\{Y_t\}$ with initial condition x and generator L_μ solution to the SDE:

$$dY_t = \sum_{i=1}^N F_i(Y_t) \circ dB_t^i - \nabla V_\mu(Y_t)\,dt, \quad Y_0 = x. \tag{11}$$

In the following $\mathsf{E}_{x,r,\mu}$ and $\mathsf{E}_{x,\mu}$ will respectively denote the expectation with respect to $\mathsf{P}_{x,r,\mu}$ and to $\mathsf{P}_{x,\mu}$.

3. The Girsanov transform and some lemmas

3.1. The Girsanov transform

Let $B_t = (B_t^1, \ldots, B_t^N)$ be a standard Brownian motion on $\mathbb{R}^N$, P the law of $(B_s;\ s \geqslant 0)$, E the associated expectation, $\mathcal{F}_t$ the P-completion of $\sigma(B_s,\ 0 \leqslant s \leqslant t)$ and $\mathcal{F} = \mathcal{F}_\infty$. Let $\{W_t^x\}$ be the solution to the SDE

$$dW_t^x = \sum_{i=1}^N F_i(W_t^x) \circ dB_t^i, \quad W_0^x = x \in M. \tag{12}$$

Then $W^x = (W_t^x,\ t \geqslant 0)$ is a Brownian motion on M starting at x. We denote its law P_x. Note that $W^x : C(\mathbb{R}^+ : \mathbb{R}^N) \to \Omega = C(\mathbb{R}^+ : M)$ is measurable. Let

$$M_t^{x,r,\mu} = \exp\left[\int_0^t \sum_i \left\langle \nabla V_{\mu_s^x(r,\mu)}\left(W_s^x\right), F_i\left(W_s^x\right)\right\rangle dB_s^i - \frac{1}{2}\int_0^t \left\|\nabla V_{\mu_s^x(r,\mu)}\left(W_s^x\right)\right\|^2 ds\right], \tag{13}$$

where

$$\mu_t^x(r,\mu) = \frac{1}{t+r}\left(r\mu + \int_0^t \delta_{W_s^x}\, ds\right), \tag{14}$$

and

$$M_t^{x,\mu} = \exp\left[\int_0^t \sum_i \left\langle \nabla V_\mu\left(W_s^x\right), F_i\left(W_s^x\right)\right\rangle dB_s^i - \frac{1}{2}\int_0^t \left\|\nabla V_\mu\left(W_s^x\right)\right\|^2 ds\right]. \tag{15}$$

Then $\{M_t^{x,r,\mu}\}$ and $\{M_t^{x,\mu}\}$ are $(\mathsf{P}, \{\mathcal{F}_t\})$-martingales. By the transformation of drift formula (see [3], Section IV 4.1 and Theorem IV 4.2),

$$\begin{cases} \mathsf{E}_{x,r,\mu}[Z_t] = \mathsf{E}\left[M_t^{x,r,\mu}\left(Z_t \circ W^x\right)\right], \\ \mathsf{E}_{x,\mu}[Z_t] = \mathsf{E}\left[M_t^{x,\mu}\left(Z_t \circ W^x\right)\right] \end{cases} \tag{16}$$

for every bounded $\mathcal{B}_t$-measurable random variable Z_t. Note that this implies in particular that P_x, $\mathsf{P}_{x,\mu}$ and $\mathsf{P}_{x,r,\mu}$ are equivalent.

3.2. Some lemmas

The next lemma is a basic tool to estimate quantities such as

$$\mathsf{E}_{x,r,\mu^r}[Z_t] - \mathsf{E}_{x,\mu}[Z_t],$$

for large r and μ^r close to μ.

LEMMA 3.1. – *For $a = 1, 2$ let $A_t^a = (A_t^{a,1}, \ldots, A_t^{a,N})$ be a $\mathbb{R}^N$-valued bounded $\{\mathcal{F}_t\}$-previsible process. Suppose that for all* $0 \leqslant s \leqslant t$

$$\left\|A_s^1 - A_s^2\right\| \leqslant \delta(t) \tag{17}$$

for some deterministic function $\delta:\mathbb{R}_+\to\mathbb{R}_+$. *Let*

$$M_t^a=\exp\left[\int_0^t\sum_i A_s^{a,i}\,dB_s^i-\frac{1}{2}\int_0^t\left\|A_s^a\right\|^2ds\right],\quad a=1,2.$$

Then there exists a positive constant C *such that for any* $\mathcal{F}_t$*-measurable random variable* Z_t *bounded by* 1 *in absolute value,*

$$\left|\mathsf{E}\left[M_t^1 Z_t\right]-\mathsf{E}\left[M_t^2 Z_t\right]\right|\leqslant \mathrm{e}^{Ct}\delta(t). \tag{18}$$

The constant C *depends only on* $\sup_{a,s}\|A_s^a\|_\infty$.

Lemma 3.1 will be proved in Section 8. Note that Lemma 3.1 and the Girsanov transforms given in Section 3.1 imply that $\mathsf{P}_{x,r,\mu}$ converges weakly towards $\mathsf{P}_{x,\mu}$ as $r\to\infty$. More precisely

LEMMA 3.2. – *There exists a positive constant* C (*depending only on* $\sup_{x,y}\|\nabla V_y(x)\|$) *such for any* $\mathcal{B}_t$*-measurable random variable* Z_t *bounded by* 1 *in absolute value,*

$$\left|\mathsf{E}_{x,r,\mu}[Z_t]-\mathsf{E}_{x,\mu}[Z_t]\right|\leqslant\frac{\mathrm{e}^{Ct}}{r+t}. \tag{19}$$

Proof. – There exists a constant C such that $\|\nabla V_\mu\|_\infty\leqslant C$ and $\|\nabla V_{\mu_s(r,\mu)}-\nabla V_\mu\|_\infty\leqslant Ct/(r+t)$, for all $0\leqslant s\leqslant t$. The result then follows from Girsanov formulas (16) and Lemma 3.1 applied with

$$\begin{cases} A_s^{1,i}=\left\langle\nabla V_{\mu_s(r,\mu)}\left(W_s^x\right),F_i\left(W_s^x\right)\right\rangle,\\ A_s^{2,i}=\left\langle\nabla V_\mu\left(W_s^x\right),F_i\left(W_s^x\right)\right\rangle.\end{cases} \tag{20}$$

□

4. The asymptotic of $\mathsf{P}_{X_t,r+t,\mu_t(r,\mu)}$

Here we shall prove:

THEOREM 4.1. – *Let* $\mu^*:\Omega\to\mathcal{P}(M)$ *denote a* $\mathcal{P}(M)$*-valued random variable. Let* $\widetilde{\Omega}=\{w\in\Omega:\ \lim_{t\to\infty}\mu_t(r,\mu,w)=\mu^*\}$. *Then* $\mathsf{P}_{x,r,\mu}$*-a.s. on* $\widetilde{\Omega}$,

$$\lim_{t\to\infty}d\left(\mathsf{P}_{X_t,r+t,\mu_t(r,\mu)},\mathsf{P}_{X_t,\mu^*}\right)=0. \tag{21}$$

COROLLARY 4.2. – *For every bounded and continuous function* $Z:\Omega\to\mathbb{R}$, $\mathsf{P}_{x,r,\mu}$*-a.s.*

$$\lim_{r\to\infty}\left|\mathsf{E}_{x,r,\mu}[Z\circ\theta_t|\mathcal{B}_t]-\mathsf{E}_{X_t,\mu^*}[Z]\right|\mathbf{1}_{\widetilde{\Omega}}=0, \tag{22}$$

where $\theta_t:\Omega\to\Omega$ *is the shift on* Ω *defined by* $\theta_t(w)(s)=w(t+s)$.

Proof. – By the Markov property, we have

$$\mathsf{E}_{x,r,\mu}[Z\circ\theta_t|\mathcal{B}_t]=\mathsf{E}_{X_t,t+r,\mu_t(r,\mu)}[Z], \tag{23}$$

and the result follows from Theorem 4.1. □

Proof of Theorem 4.1. – Follows directly from the following estimate:

PROPOSITION 4.3. – *Let* $\{\mu^r\colon r>0\}\subset\mathcal{P}(M)$. *Assume that*

$$\lim_{r\to\infty}\mu^r=\mu^*$$

in $\mathcal{P}(M)$. *Let* Z_t *be a random variable* $\mathcal{F}_t$*-measurable and bounded by* 1 *in absolute value. Then*

$$\lim_{r\to\infty}\mathsf{E}_{x,r,\mu^r}[Z_t]=\mathsf{E}_{x,\mu^*}[Z_t] \tag{24}$$

uniformly in $x\in M$.

More precisely, there exists $C>0$ (*depending only on* $\sup_{x,y}\|\nabla V_y(x)\|$) *such that*

$$\left|\mathsf{E}_{x,r,\mu^r}[Z_t]-\mathsf{E}_{x,\mu^*}[Z_t]\right|\leqslant e^{Ct}\left(\frac{1}{r}+\varepsilon(r)\right), \tag{25}$$

where $\varepsilon(r)=\sup_x\|\nabla V_{\mu^r}(x)-\nabla V_{\mu^*}(x)\|$.

Note that $\lim_{r\to\infty}\varepsilon(r)=0$ (since $x\mapsto\nabla V_{\mu^r}(x)-\nabla V_{\mu^*}(x)$ is equicontinuous in x and converges towards 0 for every x).

Proof. – Lemma 3.1 applied with

$$\begin{cases} A_s^{1,i}=\langle\nabla V_{\mu^r}(W_s^x),F_i(W_s^x)\rangle,\\ A_s^{2,i}=\langle\nabla V_{\mu^*}(W_s^x),F_i(W_s^x)\rangle\end{cases} \tag{26}$$

implies

$$\left|\mathsf{E}_{x,\mu^r}[Z_t]-\mathsf{E}_{x,\mu^*}[Z_t]\right|\leqslant e^{Ct}\varepsilon(r). \tag{27}$$

The conclusion follows from this last inequality combined with Lemma 3.2 and the triangle inequality. □

5. The convergence in law of X_t

For every $\mu\in\mathcal{P}(M)$ let $\Pi(\mu)\in\mathcal{P}(M)$ denote the invariant probability measure of the diffusion process with generator L_μ. That is

$$\Pi(\mu)(dx)=\frac{e^{-2V_\mu(x)}}{Z(\mu)}\lambda(dx), \tag{28}$$

where $Z(\mu)$ is the normalization constant.

Let us first remark that as $r\to\infty$, the law of X_t under $\mathsf{P}_{x,r,\mu}$ converges weakly towards the law of X_t under $\mathsf{P}_{x,\mu}$ (see Lemma 3.2). We also have the convergence $\lim_{t\to\infty}\mathsf{E}_{x,\mu}[g(X_t)]=\Pi(\mu)g$ [1] for all $g\in C^0(M)$. The next proposition shows that

[1] For a measure μ and $f\in L^1(\mu)$ we let μf denote $\int f\,d\mu$.

$\mathsf{E}_{x,r,\mu}[g(X_{t+s})|\mathcal{B}_t] = \mathsf{E}_{X_t,r+t,\mu_t(r,\mu)}[g(X_s)]$ and $\Pi(\mu_t(r,\mu))g$ are close when s and t tends to ∞ at a certain rate.

PROPOSITION 5.1. – *For all* $t \geqslant 1,\ r > 0,\ s > 0$ *and* $g \in C^0(M)$,

$$\left|\mathsf{E}_{x,r,\mu}\left[g(X_{t+s})|\mathcal{B}_t\right] - \Pi(\mu_t)g\right| \leqslant \|g\|_\infty \left(\frac{\mathrm{e}^{Cs}}{r+s+t} + C\,\mathrm{e}^{-s/\kappa}\right), \tag{29}$$

where C and κ are positive constants depending only on V.

The proof of Proposition 5.1 is given in Section 8.

COROLLARY 5.2. –
(i) *For all positive s and all $g \in C^0(M)$,*

$$\limsup_{t\to\infty} \left|\mathsf{E}_{x,r,\mu}\left[g(X_{t+s})|\mathcal{B}_t\right] - \Pi(\mu_t)g\right| \leqslant C\|g\|_\infty\,\mathrm{e}^{-s/\kappa}. \tag{30}$$

(ii) *Let s be a real valued positive function such that*

$$1 \ll \exp\bigl(s(t)\bigr) \ll t^{1/C} \tag{31}$$

when t tends to ∞. Then for all $g \in C^0(M)$,

$$\limsup_{t\to\infty} \left|\mathsf{E}_{x,r,\mu}\left[g(X_{t+s(t)})|\mathcal{B}_t\right] - \Pi(\mu_t)g\right| = 0. \tag{32}$$

Proof. – Straightforward. □

Remark 5.3. – Let $\mathcal{L}_t$ denote the law of $X_{t+s(t)}$ knowing $\mathcal{B}_t$. Then Corollary 5.2 means that $\mathcal{L}_t$ is asymptotically equal to $\Pi(\mu_t)$. That is, $\lim_{t\to\infty} \mathrm{dist}_w(\mathcal{L}_t, \Pi(\mu_t)) = 0$, where dist_w is a distance on $\mathcal{P}(M)$ for the weak topology.

Remark that Proposition 5.1 and Corollary 5.2 make no assumption on the asymptotic of $\{\mu_t\}$. Let $\widetilde{\Omega} \in \mathcal{B}$ be the event that "μ_t converges towards μ^*", where μ^* is a $\mathcal{P}(M)$-valued random variable. In [1] and [2], several examples of self-interacting diffusions for which $\mathsf{P}_{x,r,\mu}(\widetilde{\Omega}) = 1$ are given (these examples are shortly presented in Section 7). The following theorem describes the law of $X_{t+s(t)}$ knowing $\mathcal{B}_t$ on $\widetilde{\Omega}$.

THEOREM 5.4. – *Let $s(t)$ be as in Corollary* 5.2. *Then, the law of $X_{t+s(t)}$ knowing $\mathcal{B}_t$ converges weakly towards μ^* $\mathsf{P}_{x,r,\mu}$-a.s. on $\widetilde{\Omega}$. That is, for all $g \in C^0(M)$,*

$$\lim_{t\to\infty} \mathsf{E}_{x,r,\mu}\left[g(X_{t+s(t)})|\mathcal{B}_t\right] = \mu^* g \tag{33}$$

$\mathsf{P}_{x,r,\mu}$*-a.s. on* $\widetilde{\Omega}$.

Proof. – It follows from Theorem 3.8 in [1] that μ^* is (almost surely on $\widetilde{\Omega}$) a fixed point of Π, i.e., $\Pi(\mu^*) = \mu^*$. The proof now follows from Corollary 5.2(ii) and the fact that $\Pi : \mathcal{P}(M) \to \mathcal{P}(M)$ is continuous. □

COROLLARY 5.5 (Convergence in law). – *For all* $g \in C^0(M)$,

$$\lim_{t\to\infty} \mathsf{E}_{x,r,\mu}\big[g(X_t)\mathbf{1}_{\widetilde{\Omega}}\big] = \mathsf{E}_{x,r,\mu}\big[(\mu^* g)\mathbf{1}_{\widetilde{\Omega}}\big]. \tag{34}$$

In particular, if $\mathsf{P}_{x,r,\mu}(\widetilde{\Omega}) = 1$ *then for all* $g \in C^0(M)$,

$$\lim_{t\to\infty} \mathsf{E}_{x,r,\mu}\big[g(X_t)\big] = \mathsf{E}_{x,r,\mu}[\mu^* g], \tag{35}$$

i.e., X_t *converges in law towards* $\mathsf{E}_{x,r,\mu}[\mu^*]$ *when* t *tends to* ∞.

Proof. – In view of Theorem 5.4

$$\lim_{t\to\infty} \mathsf{E}_{x,r,\mu}\big[\mathsf{E}_{x,r,\mu}\big[g(X_{t+s(t)})|\mathcal{B}_t\big]\mathbf{1}_{\widetilde{\Omega}}\big] = \mathsf{E}_{x,r,\mu}\big[(\mu^* g)\mathbf{1}_{\widetilde{\Omega}}\big]. \tag{36}$$

It then suffices to prove that $\lim_{t\to\infty} a_t = 0$ where

$$a_t = \mathsf{E}_{x,r,\mu}\big[\mathsf{E}_{x,r,\mu}\big[g(X_{t+s(t)})|\mathcal{B}_t\big]\mathbf{1}_{\widetilde{\Omega}} - \mathsf{E}_{x,r,\mu}\big[g(X_{t+s(t)})\mathbf{1}_{\widetilde{\Omega}}|\mathcal{B}_t\big]\big]. \tag{37}$$

Let $\Delta_t = \mathbf{1}_{\widetilde{\Omega}} - \mathsf{E}_{x,r,\mu}[\mathbf{1}_{\widetilde{\Omega}}|\mathcal{B}_t]$. Then

$$a_t = \mathsf{E}_{x,r,\mu}\big[\mathsf{E}_{x,r,\mu}\big[g(X_{t+s(t)})|\mathcal{B}_t\big]\Delta_t - \mathsf{E}_{x,r,\mu}\big[g(X_{t+s(t)})\Delta_t|\mathcal{B}_t\big]\big]. \tag{38}$$

Hence $|a_t| \leqslant 2\|g\|_\infty \mathsf{E}_{x,r,\mu}[|\Delta_t|]$ and consequently $\lim_{t\to\infty} a_t = 0$ because $\lim_{t\to\infty} \Delta_t = 0$ a.s. □

6. The convergence in law of $(X_{t+u}, u \geqslant 0)$

In the previous section we were only interested by the asymptotic of the law of X_{t+s} knowing $\mathcal{B}_t$. These results can be extended to the law of $(X_{t+s+u};\ u \geqslant 0)$ knowing $\mathcal{B}_t$. The following proposition is analogous to Proposition 5.1 (and implies Proposition 5.1).

PROPOSITION 6.1. – *For all* $t \geqslant 1,\ s > 0,\ u > 0$ *and* Z_u *a* $\mathcal{B}_u$*-measurable random variable bounded by* 1 *in absolute value, then*

$$\big|\mathsf{E}_{x,r,\mu}[Z_u \circ \theta_{t+s}|\mathcal{B}_t] - \mathsf{E}_{\Pi(\mu_t),\mu_t}[Z_u]\big| \leqslant \left(\frac{\mathrm{e}^{C(s+u)}}{r+s+u+t} + C\,\mathrm{e}^{-s/\kappa}\right), \tag{39}$$

where C *and* κ *are positive constants depending only on* V.

The proof of Proposition 6.1 is given in Section 8.

COROLLARY 6.2. – *For any positive* u *and* Z_u *a* $\mathcal{B}_u$*-measurable random variable bounded by* 1 *in absolute value, we have*

(i) *For any positive* s,

$$\limsup_{t\to\infty} \big|\mathsf{E}_{x,r,\mu}[Z_u \circ \theta_{t+s}|\mathcal{B}_t] - \mathsf{E}_{\Pi(\mu_t),\mu_t}[Z_u]\big| \leqslant C\,\mathrm{e}^{-s/\kappa}. \tag{40}$$

(ii) *Let s be a function like in Corollary* 5.2*, then*

$$\limsup_{t\to\infty}\left|\mathsf{E}_{x,r,\mu}[Z_u\circ\theta_{t+s(t)}|\mathcal{B}_t]-\mathsf{E}_{\Pi(\mu_t),\mu_t}[Z_u]\right|=0. \tag{41}$$

Proof. – Straightforward. □

This corollary shows that the law of $(X_{t+s(t)+v};\ v\geqslant 0)$ knowing $\mathcal{B}_t$ is asymptotically equal to the law of a diffusion with generator L_{μ_t} and initial distribution $\Pi(\mu_t)$. In particular, (ii) says that

$$\lim_{t\to\infty} d(\mathsf{P}_t,\mathsf{P}_{\Pi(\mu_t),\mu_t})=0, \tag{42}$$

where P_t is the law of $(X_{t+s(t)+u};\ u\geqslant 0)$ knowing $\mathcal{B}_t$.

Like in the previous section, we now focus on $\widetilde{\Omega}$. The following theorem shows that on $\widetilde{\Omega}$, given $\mathcal{B}_t$, $(X_{t+s(t)+u};\ u\geqslant 0)$ converges in law towards a diffusion with generator L_{μ^*} and initial distribution μ^* (note that μ^* satisfies $\mu^*=\Pi(\mu^*)$ so that μ^* is the invariant probability measure of this diffusion).

THEOREM 6.3. – *For any positive u and Z_u a bounded $\mathcal{B}_u$-measurable random variable,*

$$\lim_{t\to\infty}\mathsf{E}_{x,r,\mu}[Z_u\circ\theta_{t+s(t)}|\mathcal{B}_t]=\mathsf{E}_{\mu^*,\mu^*}[Z_u] \tag{43}$$

almost surely on $\widetilde{\Omega}$, where $s(t)$ is as in Corollary 5.2*.*

Proof. – The proof is the same as the one of Theorem 5.4. □

Note that Theorem 6.3 implies that on $\widetilde{\Omega}$, P_t converges weakly towards P_{μ^*,μ^*}.

COROLLARY 6.4 (Convergence in law). – *For any positive u and Z_u a bounded $\mathcal{B}_u$-measurable random variable,*

$$\lim_{t\to\infty}\mathsf{E}_{x,r,\mu}\left[(Z_u\circ\theta_t)\mathbf{1}_{\widetilde{\Omega}}\right]=\mathsf{E}_{x,r,\mu}\left[\mathsf{E}_{\mu^*,\mu^*}[Z_u]\mathbf{1}_{\widetilde{\Omega}}\right]. \tag{44}$$

In particular, if $\mathsf{P}_{x,r,\mu}(\widetilde{\Omega})=1$ then

$$\lim_{t\to\infty}\mathsf{E}_{x,r,\mu}[Z_u\circ\theta_t]=\mathsf{E}_{x,r,\mu}\left[\mathsf{E}_{\mu^*,\mu^*}[Z_u]\right]. \tag{45}$$

Proof. – The proof is the same as the one of Corollary 5.5. □

Note that (44) and (45) respectively imply that the law of $(X_{t+u};\ u\geqslant 0)$ given $\widetilde{\Omega}$ converges weakly towards $\mathsf{E}_{x,r,\mu}[\mathsf{P}_{\mu^*,\mu^*}|\widetilde{\Omega}]$ and that $\mathsf{E}_{x,r,\mu}[\mathsf{P}_{X_t,r+t,\mu_t(r,\mu)}]$ converges weakly towards $\mathsf{E}_{x,r,\mu}[\mathsf{P}_{\mu^*,\mu^*}]$ provided $\mathsf{P}_{x,r,\mu}(\widetilde{\Omega})=1$.

7. Examples

Set $\delta_V(x,y)=\sup_{u\in M}(V_u(x)-V_u(y))-\inf_{u\in M}(V_u(x)-V_u(y))$. In [1], Corollary 4.4, it is proved that when $\sup_{(x,y)\in M^2}\delta_V(x,y)<1$, then Π has a unique fixed point μ^*

and $\lim_{t\to\infty} \mu_t(r,\mu) = \mu^*$ $\mathsf{P}_{x,r,\mu}$-a.s. The associated self-interacting diffusions produce examples for which $\mathsf{P}_{x,r,\mu}(\widetilde{\Omega}) = 1$, but the limit μ^* is not random.

From the different interactions, we distinguish those such that V is symmetric and defines a positive or a negative self-adjoint operator acting on $L^2(\lambda)$, that can be written in the form $V = \alpha \int_C G(u,x)G(u,y)\nu(du)$, where C is compact, ν is a Borel probability measure, $G : C \times M \to \mathbb{R}$ is continuous and $\alpha \in \mathbb{R}$. We call them gradient interactions. These interactions produce examples for which $\mathsf{P}_{x,r,\mu}(\widetilde{\Omega}) = 1$ and the limit μ^* may be random (see [2]).

When α is positive, we say it is a self-repelling interaction and when α is negative, we say it is a self-attracting interaction. It can be proved (see [2]) that, if $V1$ is a constant function, for all repelling cases or weakly attracting cases ($\alpha > -\alpha_G$, with $\alpha_G > 0$), the empirical occupation measure of the associated self-interacting diffusion converges towards λ a.s. But, when $\alpha < -\alpha_G$, this is not the case, and μ_t may converge towards $\mu^* \neq \lambda$.

The interaction, on the n-dimensional sphere $\mathbf{S}^n$,

$$V(x,y) = 2\alpha \cos\bigl(d(x,y)\bigr) \tag{46}$$

is a gradient interaction. This example is developed in [1], Section 4.2. When $\alpha \geqslant -(n+1)/4$, μ_t converges towards λ a.s. and when $\alpha < -(n+1)/4$, there exists a $\mathbf{S}^n$-valued random variable v such that μ_t converges a.s. towards $\exp[\beta_n(\alpha)\cos(d(x,v))]\lambda(dx)/Z_{n,\alpha}$, where $Z_{n,\alpha}$ is the normalization constant and $\beta_n(\alpha)$ is a constant depending only on n and α. In [1], Section 4.2, an example of interaction on $\mathbf{S}^n$ (which is not a gradient interaction) for which $\mathsf{P}_{x,r,\mu}(\widetilde{\Omega}) = 0$ is given.

8. Proofs

8.1. Proof of Lemma 3.1

Let C be a constant such that both $\|A_t^a\|^2$ and $\|A_t^a\|$ are lower than C. Let

$$E_t = \exp\left[\int_0^t \langle A_s^1, A_s^1 - A_s^2\rangle\, ds\right], \tag{47}$$

and $N_t = M_t^2(M_t^1 E_t)^{-1}$. Observe that M_t^a and N_t are exponential martingales solutions of the SDEs

$$\begin{cases} dM_t^a = M_t^a\left(\sum_i A_t^{a,i}\, dB_t^i\right), \\ dN_t = N_t\left(\sum_i (A_t^{2,i} - A_t^{1,i}) dB_t^i\right). \end{cases} \tag{48}$$

Therefore

$$\begin{cases} \frac{d}{ds}\mathsf{E}\left[\left(M_s^a\right)^2\right] = \mathsf{E}\left[\left(M_s^a\right)^2\|A_s^a\|^2\right] \leqslant C\mathsf{E}\left[\left(M_s^a\right)^2\right], \\ \frac{d}{ds}\mathsf{E}\left[(N_s)^2\right] = \mathsf{E}\left[(N_s)^2\|A_s^1 - A_s^2\|^2\right] \leqslant \delta^2(t)\mathsf{E}\left[(N_s)^2\right], \end{cases} \tag{49}$$

for $s \leqslant t$. Hence, by Gronwall's lemma, for $a \in \{1, 2\}$

$$\begin{cases} \mathsf{E}\left[\left(M_t^a\right)^2\right] \leqslant e^{Ct}, \\ \mathsf{E}\left[(N_t)^2\right] \leqslant \exp\left(t\delta^2(t)\right). \end{cases} \tag{50}$$

Notice that we also have

$$|E_t - 1| \leqslant \exp\left(Ct\delta(t)\right) - 1. \tag{51}$$

Using these estimates and Schwartz inequality, we get

$$\begin{aligned} &\left|\mathsf{E}\left[M_t^2 Z_t\right] - \mathsf{E}\left[M_t^1 Z_t\right]\right| = \left|\mathsf{E}\left[Z_t(N_t E_t - 1)M_t^1\right]\right| \\ &\quad \leqslant \mathsf{E}\left[\left(N_t(E_t - 1) + N_t - 1\right)^2\right]^{1/2}\mathsf{E}\left[\left(M_t^1\right)^2\right]^{1/2} \\ &\quad \leqslant \mathrm{e}^{Ct/2}\left[\left(\exp\left(Ct\delta(t)\right) - 1\right)\exp\left(\frac{t\delta^2(t)}{2}\right) + \left(\exp\left(t\delta^2(t)\right) - 1\right)^{1/2}\right]. \end{aligned}$$

Since $\mathrm{e}^u - 1 \leqslant u\,\mathrm{e}^u$ we easily obtain

$$\left|\mathsf{E}_{x,r,\mu}[Z_t] - \mathsf{E}_{x,\mu}[Z_t]\right| \leqslant \mathrm{e}^{Ct}\delta(t), \tag{52}$$

for C large enough. This proves the lemma. □

8.2. Proof of Propositions 5.1 and 6.1

Let $P^\mu = (P_t^\mu)_{t\geqslant 0}$ denote the semigroup of the diffusion with generator L_μ.

LEMMA 8.1. – *Let $g : M \to \mathbb{R}$ be a bounded continuous function, then for $t \geqslant 1$,*

$$\left|P_t^\mu g(x) - \Pi(\mu)g\right| \leqslant C\|g\|_\infty\,\mathrm{e}^{-t/\kappa}, \tag{53}$$

for some constant C and κ depending only on $\|V\|_\infty$.

Proof. – Let $\|\cdot\|_2$ be the L^2-norm defined by

$$\|f\|_2^2 = \int_M f^2(x)\Pi(\mu)(dx). \tag{54}$$

Then, by standard semigroup inequalities (see [1], Section 5.2)

$$\left\|P_t^\mu g - \Pi(\mu)g\right\|_2 \leqslant \mathrm{e}^{-t/\kappa}\left\|g - \Pi(\mu)g\right\|_2, \quad t > 0, \tag{55}$$

$$\left\|P_t^\mu g - \Pi(\mu)g\right\|_\infty \leqslant Ct^{-n/2}\left\|g - \Pi(\mu)g\right\|_2, \quad 0 < t \leqslant 1, \tag{56}$$

for some constant $\kappa > 0$ and $0 < C < \infty$ depending only on $\|V\|_\infty$. Combining (55) and (56) leads to

$$\begin{aligned}\left\|P_s^\mu g - \Pi(\mu)g\right\|_\infty &= \left\|P_1^\mu\left(P_{s-1}^\mu\left(g - \Pi(\mu)g\right)\right)\right\|_\infty \\ &\leqslant C\,\mathrm{e}^{-(s-1)/\kappa}\left\|g - \Pi(\mu)g\right\|_2 \\ &\leqslant 2C\,\mathrm{e}^{-(s-1)/\kappa}\|g\|_\infty\end{aligned}$$

for all $s > 1$. □

Proof of Proposition 5.1. – By the Markov property

$$\mathsf{E}_{x,r,\mu}\left[g(X_{t+s})|\mathcal{B}_t\right] = \mathsf{E}_{X_t,r+t,\mu_t(r,\mu)}\left[g(X_s)\right]. \tag{57}$$

Hence

$$\begin{aligned}&\left|\mathsf{E}_{x,r,\mu}\left[g(X_{t+s})|\mathcal{B}_t\right] - \Pi(\mu_t)g\right| \\ &\quad\leqslant \left|\mathsf{E}_{X_t,r+t,\mu_t(r,\mu)}\left[g(X_s)\right] - \mathsf{E}_{X_t,\mu_t}\left[g(X_s)\right]\right| + \left|\mathsf{E}_{X_t,\mu_t}\left[g(X_s)\right] - \Pi(\mu_t)g\right|\end{aligned}$$

and the result follows from Lemmas 3.2 and 8.1. □

Proof of Proposition 6.1. – This is almost the same proof. By the Markov property

$$\mathsf{E}_{x,r,\mu}[Z_u \circ \theta_{t+s}|\mathcal{B}_t] = \mathsf{E}_{X_t,r+t,\mu_t(r,\mu)}[Z_u \circ \theta_s].$$

Hence

$$\begin{aligned}&\left|\mathsf{E}_{x,r,\mu}[Z_u \circ \theta_{t+s}|\mathcal{B}_t] - \mathsf{E}_{\Pi(\mu_t),\mu_t}[Z_u]\right| \\ &\quad\leqslant \left|\mathsf{E}_{X_t,r+t,\mu_t(r,\mu)}[Z_u \circ \theta_s] - \mathsf{E}_{X_t,\mu_t}[Z_u \circ \theta_s]\right| + \left|\mathsf{E}_{X_t,\mu_t}[Z_u \circ \theta_s] - \mathsf{E}_{\Pi(\mu_t),\mu_t}[Z_u]\right|.\end{aligned}$$

The first term of the right-hand side of preceding equation can be dominated using Lemma 3.2. For the domination of the second term, let $\varphi(x) = \mathsf{E}_{x,\mu_t}[Z_u]$, then

$$\begin{cases}\mathsf{E}_{X_t,\mu_t}[Z_u \circ \theta_s] = P_s^{\mu_t}\varphi(X_t), \\ \mathsf{E}_{\Pi(\mu_t),\mu_t}[Z_u] = \Pi(\mu_t)\varphi.\end{cases} \tag{58}$$

We then conclude using Lemma 8.1. □

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