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On the explosion of the local times along lines of Brownian sheet

Davar Khoshnevisan [a,1,2], Pál Révész [b,3], Zhan Shi [c,2]

[a] *Department of Mathematics, The University of Utah, Salt Lake City, UT 84112-0090, USA*
[b] *Institut für Statistik und Wahrscheinlichkeitstheorie, Technische Universität Wien, Wiedner Haupstraße 8-10/1071, A-1040, Wien, Austria*
[c] *Laboratoire de probabilités, Université Paris VI, 4, place Jussieu, 75252 Paris Cedex 05, France*

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Abstract

One can view a 2-parameter Brownian sheet $\{W(s,t); s,t \geqslant 0\}$ as a stream of interacting Brownian motions $\{W(s,\bullet); s \geqslant 0\}$. Given this viewpoint, we aim to continue the analysis of [J.B. Walsh, The local time of the Brownian sheet, Astérisque 52–53 (1978) 47–61] on the local times of the stream $W(s,\bullet)$ near time $s = 0$. Our main result is a kind of maximal inequality that, in particular, verifies the following conjecture of [D. Khoshnevisan, The distribution of bubbles of Brownian sheet, Ann. Probab. 23 (2) (1995) 786–805]: As $s \to 0^+$, the local times of $W(s,\bullet)$ explode almost surely. Two other applications of this maximal inequality are presented, one to a capacity estimate in classical Wiener space, and one to a uniform ratio ergodic theorem in Wiener space. The latter readily implies a quasi-sure ergodic theorem. We also present a sharp Hölder condition for the local times of the mentioned Brownian streams that refines earlier results of [M.T. Lacey, Limit laws for local times of the Brownian sheet, Probab. Theory Related Fields 86 (1) (1990) 63–85; P. Révész, On the increments of the local time of a Wiener sheet, J. Multivariate Anal. 16 (3) (1985) 277–289; J.B. Walsh, The local time of the Brownian sheet, Astérisque 52–53 (1978) 47–61].

Résumé

Le drap brownien $\{W(s,t); s,t \geqslant 0\}$ à deux paramètres peut être vu comme une famille de mouvements browniens $\{W(s,\bullet); s \geqslant 0\}$. Nous nous proposons de poursuivre l'analyse de [J.B. Walsh, The local time of the Brownian sheet, Astérisque 52–53 (1978) 47–61] sur les temps locaux de la famille $W(s,\bullet)$ au voisinage de $s = 0$. Notre résultat principal est une inégalité du type maximale, qui, en particulier, prouve la conjecture suivante de [D. Khoshnevisan, The distribution of bubbles of Brownian sheet, Ann. Probab. 23 (2) (1995) 786–805] : lorsque $s \to 0^+$, il y a une explosion presque sûre du temps local de $W(s,\bullet)$. Deux autres applications de cette inégalité sont présentées : une estimation de capacité dans l'espace de Wiener, et un théorème ergodique dans l'espace de Wiener. Ce dernier implique en fait un théorème ergodique au sens quasi-sûr. Nous obtenons également une estimation précise de la continuité höldérienne du temps local de $W(s,\bullet)$, ce qui raffine des résultats antérieurs de [M.T. Lacey, Limit laws for local times of the Brownian sheet, Probab. Theory Related Fields 86 (1) (1990) 63–85 ;

E-mail addresses: davar@math.utah.edu (D. Khoshnevisan), revesz@ci.tuwien.ac.at (P. Révész), zhan@proba.jussieu.fr (Z. Shi).
URLs: http://www.math.utah.edu/~davar (D. Khoshnevisan), http://felix.proba.jussieu.fr/pageperso/zhan (Z. Shi).
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1. Introduction

Let $W = \{W(s,t);\ \ s,t \geqslant 0\}$ denote standard 2-parameter Brownian sheet, and write $W_s(t)$ and $W(s,t)$ interchangeably. One of the many natural ways to think about the process W is as a stream $\{W_s;\ \ s \geqslant 0\}$ of interacting Brownian motions, where the interaction is in some sense governed by the temporal structure of the stochastic wave equation.

In this paper we are interested in the properties of the local times of the stream W_u at 0. Let us write $\{L_t^a(X);\ \ a \in \mathbb{R}, t \geqslant 0\}$ for the local times of process X if a measurable version of such local times exists. Formally, this means

$$L_t^a(X) = \int_0^t \delta_a\big(X(s)\big)\,\mathrm{d}s, \quad a \in \mathbb{R},\ t \geqslant 0,$$

where δ_a denotes Dirac's delta function at $a \in \mathbb{R}$. Summarily, we are interested in the properties of the process $u \mapsto L_t^0(W_u)$ where $t \geqslant 0$ is fixed. These local times are also called the local times along lines of W, and arise quite naturally in the analysis of Brownian sheet.

A notable application of local times along lines arises in the analysis of [10]. Therein, local time methods are devised that show that if $N(h)$ denotes the number of excursions of W in $[0,1]^2$ that have height greater than $h > 0$, then with probability one, $N(h) = h^{-3+o(1)}$ as $h \to 0^+$, where $o(\bullet)$ is Landau's "little o" notation.

[10] contains two open problems pertaining to the latter result and its derivation. The first is to identify the "little o" term above, and has recently been solved by T.S. Mountford (1999, personal communications). Indeed, Mountford has invented a novel method that shows that a.s., $N(h) = \Lambda h^{-3}(1 + o(1))$, where Λ is a random variable which is defined in terms of the local times along lines $u \mapsto L_t^0(W_u)$.

The second open problem in [10] is whether or not with probability one, $\lim_{u\to 0^+} L_1^0(W_u) = +\infty$. Our goal, in this paper, is to answer this in the affirmative. As pointed out in [10], the difficulty here is in proving pointwise convergence. In fact, scaling considerations show that $u^{1/2}L_1^0(W_u)$ has the same distribution as $L_1^0(W_1)$, which is Brownian local time. Consequently, as $u \to \infty$, $L_1^0(W_u)$ blows up in probability. Thus, the mentioned explosion problem amounts to the "strong law" corresponding to this weak limit theorem. Viewed as such, it should not be a great surprise that a sufficiently sharp maximal inequality is in order. This turns out to be the case, and we will indeed show the following:

$$\limsup_{h\to 0^+} \frac{\log\log(1/h)}{\log(1/h)} \log \mathrm{P}\Big\{\inf_{1\leqslant u\leqslant 2} L_1^0(W_u) < h\Big\} \leqslant -\frac{1}{2}, \tag{1.1}$$

where here and throughout, log denotes the natural logarithm. The above will appear in Theorem 3.3 below and has a number of interesting consequences one of which is the pointwise explosion of local times along lines mentioned earlier (Theorem 3.1). It also implies large-time decay for the very same local times (Theorem 3.2). The maximal inequality (1.1) also implies a capacity estimate in Wiener space (Corollary 4.2), as well as a uniform ratio ergodic theorem for Brownian motion in Wiener space that we describe in Theorem 4.3 below. Finally, let us mention that such a maximal inequality has geometric consequences for two-parameter random walks. We hope to treat this subject at a later time.

2. Local times

In this section we describe some of the basic properties of local times along lines. While some of this material is known, we will also present a new and nontrivial modulus of continuity for these local times, viewed as a function of the line in question.

Recall that by Tanaka's formula,

$$\left|W_u(t) - a\right| = |a| + M_t^a(W_u) + uL_t^a(W_u), \tag{2.1}$$

where

$$M_t^a(W_u) = \int_0^t \operatorname{sgn}\bigl(W_u(s) - a\bigr)\, W_u(\mathrm{d}s)$$

is an Itô integral, viewed as a process in $t \geqslant 0$. In Eq. (2.1), the extra u in front of $L_t^a(W_u)$ accounts for the quadratic variation of W_u which is ut at time t. Thus, with this extra multiplicative factor of u we have the usual *occupation density formula*: For all bounded Borel functions $f:\mathbb{R} \to \mathbb{R}$,

$$\int_0^t f\bigl(W_u(s)\bigr)\,\mathrm{d}s = \int_{-\infty}^{\infty} f(a) L_t^a(W_u)\,\mathrm{d}a. \tag{2.2}$$

So far, the parameter u has been fixed, and the above follows from the existing theory of Brownian motion; cf. [18, Chapter 6, Section 2] for a pedagogic treatment. Moreover, as a function of u, $L_t^a(W_u)$ is the local times of Brownian sheet at level a along the line $\{u\} \times [0, t]$. It is this process that will concern us henceforth.

According to [19], $(a, t, u) \mapsto L_t^a(W_u)$ can be chosen to be continuous on $\mathbb{R} \times [0, \infty) \times (0, \infty)$. Moreover, $u \mapsto L_t^a(W_u)$ is Hölder continuous of any order $< 1/4$. In order to better understand the structure of local times along lines, we begin our analysis with an improvement of this continuity result that we believe is sharp. Namely, in this section we will prove the following:

Theorem 2.1. *If $T > 1$ is fixed, then with probability one,*

$$\limsup_{\delta \to 0^+} \sup_{\substack{1 \leqslant u, v \leqslant T: \\ |u-v| \leqslant \delta}} \frac{|L_1^0(W_u) - L_1^0(W_v)|}{\{2\delta \log(1/\delta)\}^{1/4} \cdot \{\log(1/\delta)\}^{1/2}} \leqslant 4\sqrt{\sup_{u \in [1,T]} L_1^0(W_u)}.$$

To study the regularity of $u \mapsto L_t^0(W_u)$, we begin by refining the analysis of [19], and closely examine the quadratic variation of $t \mapsto M_t^0(W_u) - M_t^0(W_v)$, when $u \approx v$.

Lemma 2.2. *For each $0 < u < v$ and for all $t \geqslant 0$,*

$$\bigl\langle M^0(W_u) - M^0(W_v)\bigr\rangle_t \leqslant 8u \sup_{0 \leqslant r \leqslant t} \bigl|W_v(r) - W_u(r)\bigr| \times \sup L_t^a(W_u) + t(v-u),$$

where the supremum is taken over all $0 \leqslant a \leqslant \sup_{0 \leqslant r \leqslant t} |W_v(r) - W_u(r)|$.

Remark 2.3. Among other things, the previous lemma, and the modulus of continuity of W, together show that if $u \approx v$, then $\langle M^0(W_u) - M^0(W_v)\rangle_t \leqslant |u-v|^{\frac{1}{2}+o(1)}$. We believe this to be sharp. This was also noticed by an anonymous referee.

Proof. A few lines of calculation show that

$$\left\langle M^0(W_u)-M^0(W_v)\right\rangle_t = t(v-u)+4u\int_0^t \mathbf{1}\left\{W_u(r)<0,\, W_v(r)>0\right\}\mathrm{d}r$$
$$+4u\int_0^t \mathbf{1}\left\{W_u(r)>0,\, W_v(r)<0\right\}\mathrm{d}r$$
$$= t(v-u)+4uT_1+4uT_2,$$

notation being obvious. Since $0<u<v$, owing to the occupation density formula, we can write

$$T_2=\int_0^t \mathbf{1}\left\{0<W_u(r)<-(W_v-W_u)(r)\right\}\mathrm{d}r \leqslant \int_0^t \mathbf{1}\left\{0<W_u(r)<\sup_{0\leqslant s\leqslant t}\left|W_v(s)-W_u(s)\right|\right\}\mathrm{d}r$$
$$=\int_0^{\mathfrak{m}} L_t^a(W_u)\,\mathrm{d}a,$$

where $\mathfrak{m}=\sup_{0\leqslant s\leqslant t}|W_v(s)-W_u(s)|$. Similarly,

$$T_1\leqslant \int_0^t \mathbf{1}\left\{-\mathfrak{m}<W_u(r)<0\right\}\mathrm{d}r=\int_{-\mathfrak{m}}^0 L_t^a(W_u)\,\mathrm{d}a.$$

We combine these estimates for T_1 and T_2 and use the occupation density formula to finish. □

The argument used to prove Theorem 2.1 will be described shortly. However, we mention in passing that using similar methods, one can deduce the following "local" result whose proof is omitted.

Theorem 2.4. *For any fixed $T>1$ and for all fixed $u\in[1,T]$, with probability one,*

$$\limsup_{\delta\to 0^+}\ \sup_{\substack{v\in[1,T]:\\ |v-u|\leqslant\delta}} \frac{|L_1^0(W_u)-L_1^0(W_v)|}{\{2\delta\log\log(1/\delta)\}^{1/4}\cdot\{\log\log(1/\delta)\}^{1/2}}\leqslant 4\sqrt{L_1^0(W_u)}.$$

Remark 2.5. Originally, we stated and proved a weaker version of Theorem 2.1 that is now equation (2.3) below. The present improvement owes its existence to an argument devised by an anonymous referee.

We end this section by proving Theorem 2.1, using the exponential martingale ideas of [15].

Proof of Theorem 2.1. We first prove the following weaker bound: For any $T, T'>1$, the following holds with probability one:

$$\limsup_{\delta\to 0^+}\ \sup_{u\in[1,T']}\ \sup_{\substack{v\in[1,T]:\\ |v-u|\leqslant\delta}} \frac{|L_1^0(W_u)-L_1^0(W_v)|}{\{2\delta\log(1/\delta)\}^{1/4}\cdot\{\log(1/\delta)\}^{1/2}}\leqslant 4\sqrt{(T\wedge T')\sup_{u\in[1,T\vee T']}L_1^0(W_u)}. \tag{2.3}$$

We will prove this for $T=T'=2$; this is not a great loss in generality, and simplifies some of the notation in our exposition.

Throughout this proof, we define the modulus of continuity,

$$\mu_\varepsilon=\sup_{\substack{u,v\in[1,2]:\\ |u-v|\leqslant\varepsilon}}\ \sup_{0\leqslant r\leqslant 1}\left|W_u(r)-W_v(r)\right|. \tag{2.4}$$

With regards to this modulus, we introduce two events. First, for any $c > 1$ and $\varepsilon > 0$, we define

$$\Theta_{c,\varepsilon} = \bigl\{\omega\colon \forall \varrho \in (0,\varepsilon),\ \mu_\varrho \leqslant c\sqrt{2\varrho\log(1/\varrho)}\bigr\}.$$

By the proof of the uniform version of the law of the iterated logarithm of [13],

$$\sum_n \mathrm{P}\bigl\{\Theta^{\complement}_{c,\mathsf{q}^{-n}}\bigr\} < \infty, \quad \forall c, \mathsf{q} > 1. \tag{2.5}$$

As for our second event, we define for all $\varepsilon, \delta \in (0,1)$,

$$\Xi_{\varepsilon,\delta} = \Bigl\{\omega\colon \sup_{0\leqslant a\leqslant \mu_\delta}\ \sup_{1\leqslant u\leqslant 2} \bigl|L_t^a(W_u) - L_t^0(W_u)\bigr| \leqslant \varepsilon\Bigr\}. \tag{2.6}$$

By combining the moment estimate of [14, Proposition 4.2]—or alternatively, Lemma 3.12 below—with Eq. (2.5) above,

$$\sum_n \mathrm{P}\bigl\{\Xi^{\complement}_{\varepsilon,\mathsf{q}^{-n}}\bigr\} < \infty, \quad \forall \varepsilon \in (0,1),\ \mathsf{q} > 1. \tag{2.7}$$

Having disposed of the needed preliminaries, we begin our proof by fixing $u, v \in [1,2]$, writing $\delta = |u - v|$, and defining

$$N_t^{u,v} = M_t^0(W_u) - M_t^0(W_v), \quad t \geqslant 0.$$

Note that $N^{u,v}$ is a martingale with respect to its own natural filtration. Moreover, by Lemma 2.2,

$$\langle N^{u,v}\rangle_t \leqslant 16cL_t^\star(\varepsilon)\sqrt{2\delta\log(1/\delta)} + \delta t, \quad \text{on } \Xi_{\varepsilon,\delta} \cup \Theta_{c,\delta}, \tag{2.8}$$

where $L_t^\star(\varepsilon) = \varepsilon + \sup_{u\in[1,2]} L_t^0(W_u)$. Now for any $\alpha, \beta > 0$,

$$\begin{aligned}
&\mathrm{P}\bigl\{N_t^{u,v} \geqslant \bigl[\alpha + \beta L_t^\star(\varepsilon)\bigr]\delta^{1/4}\log^{3/4}(1/\delta) + \zeta_\delta \text{ and } \Xi_{\varepsilon,\delta} \cup \Theta_{c,\delta}\bigr\}\\
&\quad = \mathrm{P}\bigl\{N_t^{u,v} - 16\gamma c\sqrt{2\delta\log(1/\delta)}L_t^\star(\varepsilon) - \gamma\delta t \geqslant \alpha\delta^{1/4}\log^{3/4}(1/\delta) \text{ and } \Xi_{\varepsilon,\delta} \cup \Theta_{c,\delta}\bigr\},
\end{aligned}$$

where

$$\begin{aligned}
\gamma &= \beta(16c\sqrt{2})^{-1}\delta^{-1/4}\log^{1/4}(1/\delta),\\
\zeta_\delta &= \frac{\beta t\delta^{3/4}\log^{1/4}(1/\delta)}{16c\sqrt{2}}.
\end{aligned}$$

Therefore, by (2.8)

$$\begin{aligned}
&\mathrm{P}\bigl\{N_t^{u,v} \geqslant \bigl[\alpha + \beta L_t^\star(\varepsilon)\bigr]\delta^{1/4}\log^{3/4}(1/\delta) + \zeta_\delta \text{ and } \Xi_{\varepsilon,\delta} \cup \Theta_{c,\delta}\bigr\}\\
&\qquad \leqslant \mathrm{P}\bigl\{N_t^{u,v} - \gamma\langle N^{u,v}\rangle_t \geqslant \alpha\delta^{1/4}\log^{3/4}(1/\delta)\bigr\}\\
&\qquad = \mathrm{P}\bigl\{\mathcal{E}_t^\gamma \geqslant \exp\bigl[2\alpha\gamma\delta^{1/4}\log^{3/4}(1/\delta)\bigr]\bigr\},
\end{aligned}$$

where $\mathcal{E}_t^\gamma$ is the mean 1 exponential martingale

$$\mathcal{E}_t^\gamma = \exp\biggl\{2\gamma N_t^{u,v} - \frac{1}{2}(2\gamma)^2\langle N^{u,v}\rangle_t\biggr\}, \quad t \geqslant 0.$$

By Chebyshev's inequality, for any $y > 0$, $\mathrm{P}\{\mathcal{E}_t^\gamma \geqslant y\} \leqslant y^{-1}$. Consequently, for all $u, v \in [1,2]$ such that $|u - v| \leqslant \delta$,

$$\mathrm{P}\bigl\{N_t^{u,v} \geqslant \bigl[\alpha + \beta L_t^\star(\varepsilon)\bigr]\delta^{1/4}\log^{3/4}(1/\delta) + \zeta_\delta \text{ and } \Xi_{\varepsilon,\delta} \cup \Theta_{c,\delta}\bigr\} \leqslant \exp\biggl(-\frac{\alpha\beta}{8c\sqrt{2}}\log(1/\delta)\biggr). \tag{2.9}$$

The remainder of our proof is a standard application of Eq. (2.5) and chaining; cf. [15] for references and some of the details in relation to Lévy's original chaining argument for the samples of Brownian motion. The upshot of this chaining argument, used in conjunction with Eqs. (2.5) and (2.7), is that with probability one,

$$\limsup_{\delta\to 0^+} \sup_{\substack{u,v\in[1,2]:\\ |u-v|\leqslant\delta}} \frac{|N_t^{u,v}|}{\delta^{1/4}\log^{3/4}(1/\delta)} \leqslant \alpha+\beta L_t^\star(\varepsilon), \tag{2.10}$$

for all $\alpha,\beta,\varepsilon>0$, as long as q^n times the right-hand side of Eq. (2.9) sums along $\delta=\mathsf{q}^{-n}$ for any fixed $\mathsf{q}>1$.[4] This means that $\alpha\beta>8c\sqrt{2}$, and as a result, Eq. (2.10) holds a.s. for all rational $\alpha,\beta>0$ and $c>1$ such that $\alpha\beta>8c\sqrt{2}$. The optimal choice is obtained upon choosing $c>1$, $\beta=8c\sqrt{2}\alpha^{-1}$, and a sequence of rational α's larger than, but arbitrarily close to $\{8c\sqrt{2}L_t^\star(\varepsilon)\}^{1/2}$. Finally, we let $c\downarrow 1$ along a rational sequence. In this way, we derive the following almost sure statement:

$$\limsup_{\delta\to 0^+} \sup_{\substack{u,v\in[1,2]:\\ |u-v|\leqslant\delta}} \frac{|N_t^{u,v}|}{\delta^{1/4}\log^{3/4}(1/\delta)} \leqslant 4\cdot 2^{1/4}\sqrt{2L_t^\star(\varepsilon)}. \tag{2.11}$$

On the other hand, $u\mapsto W_u(r)$ is Hölder continuous of any order $<1/2$, uniformly in $r\in[0,1]$; see [21] for instance. Consequently, (2.3) follows from (2.11) and (2.1) after taking $\varepsilon\to 0$ along a rational sequence.

Now we conclude our argument by proving that (2.3) implies the theorem. [This part is the referee's argument that we reproduce here with his/her permission.]

Choose and hold fixed some small $\varepsilon>0$, and consider the intervals

$$\mathbb{I}_n=\mathbb{I}_n^\varepsilon=\left[(1+\varepsilon)^{n-1},(1+\varepsilon)^n\right],\quad \forall n=1,2,\ldots.$$

Clearly, $[1,T]^2$ is covered by rectangles of the form $\mathbb{I}_n\times\mathbb{I}_m$, where $1\leqslant n,m\leqslant N(\varepsilon)$, and

$$N(\varepsilon)=1+\left\lfloor\frac{\log T}{\log(1+\varepsilon)}\right\rfloor.$$

This and symmetry considerations together imply that for all $\delta>0$,

$$\sup_{\substack{u,v\in[1,T]\\ |v-u|\leqslant\delta}} \left|L_1^0(W_u)-L_1^0(W_v)\right| \leqslant \max_{1\leqslant n\leqslant m\leqslant N(\varepsilon)} \sup_{u\in\mathbb{I}_m} \sup_{\substack{v\in\mathbb{I}_n:\\ |v-u|\leqslant\delta}} \left|L_1^0(W_u)-L_1^0(W_v)\right|. \tag{2.12}$$

On the other hand, since $\mathbb{I}_n=(1+\varepsilon)^{n-1}\mathbb{I}_1$ and $(1+\varepsilon)^{-n+1}\mathbb{I}_m=\mathbb{I}_{m-n+1}$, Brownian scaling shows us that for any fixed $1\leqslant n\leqslant m\leqslant N(\varepsilon)$,

$$\sup_{u\in\mathbb{I}_m} \sup_{\substack{v\in\mathbb{I}_n:\\ |v-u|\leqslant\delta}} \left|L_1^0(W_u)-L_1^0(W_v)\right| \stackrel{(d)}{=} (1+\varepsilon)^{-(n-1)/2} \sup_{u\in\mathbb{I}_{m-n+1}} \sup_{\substack{v\in\mathbb{I}_1:\\ |v-u|\leqslant\delta(1+\varepsilon)^{n-1}}} \left|L_1^0(W_u)-L_1^0(W_v)\right|,$$

where $\stackrel{(d)}{=}$ denotes equality of finite-dimensional distributions as processes in $u\in[1,T]$. Thus, we can apply (2.3) with T (respectively T') replaced by $(1+\varepsilon)$ (respectively $(1+\varepsilon)^{m-n+1}$) to deduce that for any $\varepsilon>0$ and $1\leqslant n\leqslant m\leqslant N(\varepsilon)$, almost surely,

$$\limsup_{\delta\to 0^+} \sup_{u\in\mathbb{I}_m} \sup_{\substack{v\in\mathbb{I}_n:\\ |v-u|\leqslant\delta}} \frac{|L_1^0(W_u)-L_1^0(W_v)|}{\{2\delta\log(1/\delta)\}^{1/4}\cdot\{\log(1/\delta)\}^{1/2}}$$
$$\leqslant 4(1+\varepsilon)^{-(n-1)/4}\sqrt{\sup_{u\in[1,T]} L_1^0(W_u)} \leqslant 4\sqrt{\sup_{u\in[1,T]} L_1^0(W_u)}.$$

[4] This uses the obvious fact that $\zeta_\delta=o(\delta^{1/4}\log^{3/4}(1/\delta))$, as $\delta\to 0$.

(More precisely, the above follows from the argument that led to (2.3).) The theorem follows from this and Eq. (2.12). □

3. An explosion theorem

In this section we intend to demonstrate the following blowup result for local times along lines.

Theorem 3.1 (Explosion Theorem). *With probability one,*

$$\lim_{h\to 0^+} \frac{1}{\log(1/h)} \log L_1^0(W_h) = \frac{1}{2}.$$

In particular, $\lim_{h\to 0^+} L_1^0(W_h) = +\infty$, *almost surely.*

There is a companion result to Theorem 3.1 that states that if we look at lines far away from the axis, the local time at 0 is very small almost surely. In fact, we have the following

Theorem 3.2. *With probability one,*

$$\lim_{h\to\infty} \frac{1}{\log h} \log L_1^0(W_h) = -\frac{1}{2}.$$

In particular, $\lim_{h\to\infty} L_1^0(W_h) = 0$, *almost surely.*

Theorem 3.2 follows from Theorem 3.1 and time-inversion; we omit the elementary details.

Stated somewhat informally, Theorem 3.1 states that the local time along lines at 0 explodes with probability one as we consider lines that are closer to the axes. (The stress being on "*with probability one*", for explosion in probability follows trivially from scaling considerations). Moreover, the rate of explosion is $h^{-1/2}$ upto terms that are negligible at the logarithmic scale. As is the case in many delicate limit theorems of probability and analysis, we prove this by verifying an upper and a lower bound, respectively. While each bound relies on a probability estimate, the important half is derived from the following "maximal inequality", whose proof requires most of the work toward deriving Theorem 3.1.

Theorem 3.3. *For all* $\gamma \in (0, 1/2)$, *there exists* $h_0 > 0$, *such that for every* $h \in (0, h_0)$,

$$\mathrm{P}\Big\{\inf_{1\leqslant u\leqslant 2} L_1^0(W_u) \leqslant h\Big\} \leqslant \exp\left(-\frac{\gamma \log(1/h)}{\log\log(1/h)}\right).$$

Remark 3.4. We conjecture that Theorem 3.3 is nearly sharp. This issue is discussed further in item (2) of Section 5 below.

Note that Theorem 3.3 is a reformulation of equation (1.1). Moreover, it has other consequences, one of which is the following large-time asymptotic result.

Corollary 3.5. *For any* $\eta \in (0, 1/2)$, *with probability one,*

$$\lim_{t\to\infty} t^{-\eta} \inf_{u\in[1,2]} L_t^0(W_u) = +\infty.$$

One can easily construct a proof for this by following our derivation of Theorem 3.1 below. It is also worth pointing out that there are variants of Theorem 3.3 that are different in form as well as in derivation. Let us mention one such possibility.

Theorem 3.6. *If j_1 denotes the smallest positive zero of the Bessel function J_0,*

$$\lim_{h\to 0^+} h^2 \log \mathrm{P}\Big\{ \inf_{u\in[1,2]} \sup_{a\in\mathbb{R}} L_1^a(W_u) \leqslant h \Big\} = -2j_1^2.$$

Our proof of Theorem 3.3 is somewhat long and is divided into several parts. We begin our demonstration by introducing a sequence of numbers that go to 0 a little faster than exponentially. Namely, we first hold fixed a nondecreasing sequence $\Phi_1, \Phi_2, \ldots,$ to be determined later, such that $\lim_{k\to\infty} \Phi_k = +\infty$. Then, we define

$$t_k = \Phi_k^{-k}, \quad \forall k \geqslant 1. \tag{3.1}$$

(It will turn out later on in the proof that $\Phi_k = ck$ for an approrpriate constant c. Thus, $t_k \approx \exp\{-k \log k\}$, which indeed vanishes a little faster than exponentially.)

Before discussing things further, let us record the following elementary estimate on the asymptotics of the relative gap sizes in the sequence $\{t_j\}_{j\geqslant 1}$:

$$\Phi_k \leqslant \frac{t_{k-1}}{t_k}, \quad \forall k \geqslant 2. \tag{3.2}$$

Next, we consider the following collection of measurable events: For all $\varepsilon > 0$ and all $n \geqslant 2$, define

$$\Lambda_{n,\varepsilon} = \left\{ \omega\colon \sup_{1\leqslant u\leqslant 2} \big|W_u(t_n)\big| \leqslant \frac{\varepsilon}{2}\sqrt{t_{n-1}} \right\}. \tag{3.3}$$

It turns out that for large n, $\Lambda_{n,\varepsilon}, \ldots, \Lambda_{2n,\varepsilon}$ *all* happen simultaneously, and with overwhelmingly large probability. To be more precise, we have the following:

Lemma 3.7. *For all $n \geqslant 2$,*

$$\mathrm{P}\left\{ \bigcup_{j=n}^{2n} \Lambda_{j,\varepsilon}^{\complement} \right\} \leqslant 4n \exp\left(-\frac{\varepsilon^2}{16}\Phi_n \right).$$

Proof. By Brownian scaling, $\sup_{0\leqslant u\leqslant 2} |W_u(t)|$ has the same distribution as $\sqrt{2t}\ \sup_{0\leqslant u\leqslant 1} |W_u(1)|$. Thus,

$$\mathrm{P}\left\{ \bigcup_{j=n}^{2n} \Lambda_{j,\varepsilon}^{\complement} \right\} \leqslant \sum_{j=n}^{2n} \mathrm{P}\left\{ \sup_{0\leqslant u\leqslant 1} \big|W_u(1)\big| \geqslant \frac{\varepsilon}{2\sqrt{2}} \sqrt{\frac{t_{j-1}}{t_j}} \right\} \leqslant (n+1)\mathrm{P}\left\{ \sup_{0\leqslant u\leqslant 1} \big|W_u(1)\big| \geqslant \frac{\varepsilon}{2\sqrt{2}} \Phi_n^{1/2} \right\}.$$

The lemma follows from standard Gaussian tail estimates, used in conjunction with two successive applications of the reflection principle, since $u \mapsto W_u(1)$ is a Brownian motion. □

Our next goal is to obtain uniform upcrossing estimates for Brownian sheet. To this end, we first hold fixed some $\varepsilon > 0$ and define a set of indicator variables $I_{2,\varepsilon}, I_{3,\varepsilon}, \ldots$ as follows: For all $k \geqslant 2$, define $I_{k,\varepsilon}$ to be 1 if for all $u \in [1,2]$, the random map $t \mapsto W_u(t)$ upcrosses or downcrosses $[-\varepsilon\sqrt{t_{k-1}}, \varepsilon\sqrt{t_{k-1}}]$ while $t \in [t_k, \frac{1}{10}(9t_k + t_{k-1})]$; otherwise, we set $I_{k,\varepsilon} = 0$. We make the obvious but necessary remark that $\frac{1}{10}(9t_k + t_{k-1})$ is 1/10 of the way between t_k and t_{k-1}, although the proportion 1/10 could be replaced by any $\alpha \in (0,1)$ that is sufficiently small (how small comes out of our arguments). To understand these $I_{j,\varepsilon}$'s, we start with a warmup lemma. While it is too simple to be of fundamental use to our analysis, its proof is indicative of the nature of things to come.

Lemma 3.8. *For every $\varepsilon > 0$, $\psi_0(\varepsilon) = \lim_{n\to\infty} \mathrm{E}\{I_{n,\varepsilon}\}$ exists and is nonincreasing, and $\lim_{\varepsilon\to 0^+} \psi_0(\varepsilon) = 1$.*

Proof. By scaling, $\mathrm{E}\{I_{n,\varepsilon}\}$ is the probability that for every $u \in [1,2]$, the map $t \mapsto W_u(t)$ upcrosses or downcrosses $[-\varepsilon, +\varepsilon]$ while

$$t \in \left[\frac{t_n}{t_{n-1}}, \frac{1}{10} + \frac{9}{10}\frac{t_n}{t_{n-1}}\right].$$

By (3.2) and the assumed fact that $\lim_{k\to\infty} \Phi_k = +\infty$, this interval converges, as $n \to \infty$, to $[0, 1/10]$. Having mentioned this, only a few simple lines suffice to verify that as $n \to \infty$, $\mathrm{E}\{I_{n,\varepsilon}\}$ converges to the probability $\psi_0(\varepsilon)$ that for all $u \in [1,2]$, the map $t \mapsto W_u(t)$ upcrosses or downcrosses $[-\varepsilon, +\varepsilon]$ some time in $[0, 1/10]$ (this uses continuity of Brownian sheet). Since ψ_0 is clearly nonincreasing, it remains to show that $\lim_{\varepsilon\to 0^+} \psi_0(\varepsilon)$ exists and equals 1. Existence, of course, is a consequence of monotonicity. In fact, $\lim_{\varepsilon\to 0^+} \psi_0(\varepsilon)$ is at least

$$\mathrm{P}\left\{\forall u \in [1,2],\ \overline{\lim_{t\to 0^+}} \frac{W_u(t)}{\sqrt{2ut \log\log(1/t)}} = -\underline{\lim_{t\to 0^+}} \frac{W_u(t)}{\sqrt{2ut \log\log(1/t)}} = 1\right\},$$

which is one thanks to the law of the iterated logarithm of [23]; cf. also [20]. [21] presents an elegant proof of this fact, together with related facts on the propagation of singularities of the sheet. □

Now we strive to show that with overwhelming probability, nearly all of the random variables $\{I_{j,\varepsilon};\ n \leqslant j \leqslant 2n\}$ are one as long as n is large and ε is small. To do this, we start with an elementary large deviations bound.

Lemma 3.9. *Suppose $J_1, J_2, \ldots$ are $\{0,1\}$-valued random variables that are adapted to a filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$ and satisfy the following for some $\eta > 0$:*

$$\mathrm{E}\{J_k \mid \mathcal{F}_{k-1}\} \geqslant \eta, \quad \forall k \geqslant 2.$$

Then, for all $\lambda \in (0, \eta)$ and all $n \geqslant 1$,

$$\mathrm{P}\left\{\sum_{j=1}^{n} J_i \leqslant \lambda n\right\} \leqslant \exp\left(-\frac{n}{2\eta}(\eta - \lambda)^2\right).$$

Proof. This is based on the familiar fact that $M_1, M_2, \ldots$ is a supermartingale, where

$$M_k = \exp\left(-\xi S_k + k\eta\left[\xi - \frac{\xi^2}{2}\right]\right), \quad k \geqslant 2,$$

$S_k = J_1 + \cdots + J_k$, and $\xi > 0$ is an arbitrary constant. Indeed,

$$\begin{aligned}
\mathrm{E}\{e^{-\xi S_n} \mid \mathcal{F}_{n-1}\} &= e^{-\xi S_{n-1}}\, \mathrm{E}\{e^{-\xi J_n} \mid \mathcal{F}_{n-1}\} \\
&= e^{-\xi S_{n-1}}\big[1 - (1 - e^{-\xi})\mathrm{E}\{J_n \mid \mathcal{F}_{n-1}\}\big] \\
&\leqslant e^{-\xi S_{n-1}}\big[1 - (1 - e^{-\xi})\eta\big].
\end{aligned}$$

Since for all $y \geqslant 0$, $1 - y \leqslant e^{-y} \leqslant 1 - y + \frac{1}{2}y^2$, we have found the announced supermartingale M. Moreover,

$$\mathrm{E}\{e^{-\xi S_n}\} \leqslant \exp\left(-\eta n\left[\xi - \frac{\xi^2}{2}\right]\right).$$

By Chebyshev's inequality, for all $\lambda \in (0, \eta)$,

$$\mathrm{P}\{S_n \leqslant \lambda n\} \leqslant \exp\left(-n\left[\xi(\eta - \lambda) - \frac{\eta^2}{2}\right]\right), \quad \forall \xi > 0.$$

The lemma follows from making the optimal choice of $\xi = (\eta - \lambda)/\eta$. □

We would like to apply the above to $J_i = I_{n+i-1,\varepsilon}$ $(i = 1, \ldots, n+1)$. However, a number of technical problems arise, all involving independence issues. To avoid them, we define auxiliary random variable $J_{2,\varepsilon}, J_{3,\varepsilon}, \ldots$ that are a modified version of $I_{2,\varepsilon}, I_{3,\varepsilon}, \ldots$ as follows: For each $k \geqslant 2$, we let $J_{k,\varepsilon}$ to be 1 if for all $u \in [1,2]$, the random map $t \mapsto W_u(t) - W_u(t_k)$ upcrosses or downcrosses $[-\frac{3\varepsilon}{2}\sqrt{t_{k-1}}, +\frac{3\varepsilon}{2}\sqrt{t_{k-1}}]$ while $t \in [t_k, \frac{1}{10}(9t_k + t_{k-1})]$. Recalling (3.3), we have the following.

Lemma 3.10. *For any $n \geqslant 2$, $\varepsilon > 0$, and for every $\omega \in \bigcap_{j=n}^{2n} \Lambda_{j,\varepsilon}$,*

$$I_{j,\varepsilon}(\omega) \geqslant J_{j,\varepsilon}(\omega), \quad \forall j = n, \ldots, 2n.$$

Furthermore,

$$\lim_{n\to\infty} \mathrm{E}\{J_{n,\varepsilon}\} = \psi(\varepsilon), \quad \forall \varepsilon > 0,$$

where ψ is nonincreasing, and $\lim_{\varepsilon\to 0^+} \psi(\varepsilon) = 1$.

Proof. The first part is a direct consequence of the triangle inequality. For example, if $W_u(t) - W_u(t_k) < -\frac{3}{2}\varepsilon\sqrt{t_{k-1}}$, by the triangle inequality, $W_u(t) < -\varepsilon\sqrt{t_{k-1}}$, as long as $W_u(t_k) \leqslant \frac{1}{2}\varepsilon\sqrt{t_{k-1}}$, a fact that holds on $\Lambda_{k,\varepsilon}$. The second part is proved *exactly* as Lemma 3.8 was. □

Now the J's are independent from one another and we can apply Lemma 3.9 to *them* in order to present the following uniform up/downcrossing result. Roughly speaking, it states that with overwhelming probability, *nearly all* of the variables $I_{n,\varepsilon}, \ldots, I_{2n,\varepsilon}$ are equal to one as long as ε (n) is chosen to be small (large).

Proposition 3.11. *For all $\delta \in (0,1)$, there exists $n_0 \geqslant 2$ and $\varepsilon_0 > 0$, such that for all $n \geqslant n_0$, $\varepsilon \in (0, \varepsilon_0)$, and $\zeta \in (\delta, 1)$,*

$$\mathrm{P}\left\{\sum_{j=n}^{2n} I_{j,\varepsilon} < (1-\zeta)n\right\} \leqslant \exp\left(-\frac{n(\zeta-\delta)^2}{2(1-\delta)}\right) + 4n\exp\left(-\frac{\varepsilon^2\Phi_n}{16}\right).$$

Proof. By Lemmas 3.7 and 3.10,

$$\mathrm{P}\left\{\sum_{j=n}^{2n} I_{j,\varepsilon} < (1-\zeta)n\right\} \leqslant \mathrm{P}\left\{\sum_{j=n}^{2n} J_{j,\varepsilon} < (1-\zeta)n\right\} + 4n\exp\left(-\frac{\varepsilon^2\Phi_n}{16}\right).$$

The second portion of Lemma 3.10 assures us that $\lim_{\varepsilon\to 0^+}\lim_{n\to\infty} \mathrm{E}\{J_{n,\varepsilon}\} = 1$. In particular, by choosing n_0 (ε_0) large enough (small enough), we can ensure that for all $m \geqslant n_0$ and all $\varepsilon \in (0, \varepsilon_0)$, $\mathrm{E}\{I_{m,\varepsilon}\} \geqslant 1 - \delta$. Since ψ is nonincreasing, n_0 depends on ε_0 but not on the value of $\varepsilon \in (0, \varepsilon_0)$, and the announced result follows from Lemma 3.9. □

We will also need an estimate for the modulus of continuity of $u \mapsto L_1^0(W_u)$ although we will not require anything as delicate as the results of Section 2. In fact, the following moment estimate suffices; it can be proved by combining the $L^p(\mathrm{P})$-estimates of [19, line −6, p. 53], together with Kolmogorov's continuity theorem [18, p. 18], and the Burkholder–Davis–Gundy inequality [18, p. 151]. The details of this derivation, and more, can be found in [14, proof of Proposition 4.2].

Lemma 3.12. *There exists a positive and finite constant $C > 0$ such that for all $p > 1$, $\varepsilon \in (0, 1/2)$, and $T > 0$,*

$$\Big\| \sup_{a\in\mathbb{R}} \sup_{0\leqslant t\leqslant T} \sup_{\substack{u,v\in[1,2]:\\ |u-v|\leqslant\varepsilon}} \big|L_t^a(W_u) - L_t^a(W_v)\big| \Big\|_{L^p(\mathrm{P})} \leqslant C(p!)^{1/p}\, T^{1/2}\, \{\varepsilon\log(1/\varepsilon)\}^{1/4}.$$

Remark. In the literature, the more cumbersome $C(p!)^{1/p}$ is usually replaced by an asymptotically equivalent term of form $O(p)$. However, this formulation is more convenient for our needs.

Our next lemma is a technical result about the ordinary Brownian motion $t \mapsto W_u(t)$, where $u \in [1,2]$ is held fixed. It is a useful way to quantify the idea that whenever Brownian motion hits zero often, then it generates a very large amount of local time with overwhelming probability.

Lemma 3.13. *Consider the event*

$$\Upsilon_{n,\varepsilon}^{\zeta} = \left\{ \omega\colon \sum_{j=n}^{2n} I_{j,\varepsilon} \geqslant (1-\zeta)n \right\}, \quad \forall n \geqslant 2,\ \varepsilon > 0,\ \zeta \in (0,1).$$

Then, for all $\delta \in (0,1)$, there exists $n_1 \geqslant n_0 \geqslant 2$ and $\varepsilon_0 > 0$ such that for all $n \geqslant n_1$, and for each $\zeta \in (\delta, 1)$ and $h \in (0,1)$,

$$\sup_{u \in [1,2]} \mathrm{P}\left\{ L_1^0(W_u) < h,\ \Upsilon_{n,\varepsilon_0}^{\zeta} \right\} \leqslant \left(\frac{h}{\sqrt{t_{2n}}} \right)^{(1-\zeta)n}.$$

Remarks. (1) It is possible to prove a slightly better estimate by using large deviations. However, we will not require a very sharp inequality for this estimate, and the simpler argument used in the proof below suffices. Our argument is inspired by some of the ideas of [17].

(2) The constants ε_0 and n_0 are given to us by Proposition 3.11.

Proof. Throughout this proof, $u \in [1,2]$ and $n \geqslant n_0$ are held fixed. With this in mind, define

$$S_u(1) = \sup\left\{ n \leqslant j \leqslant 2n \;\middle|\; \exists r \in \left[t_j, \frac{1}{10}(9t_j + t_{j-1}) \right]\colon\ W_u(r) = 0 \right\},$$
$$T_u(1) = \inf\left\{ r > t_{2n} \mid W_u(r) = 0 \right\}.$$

As usual, $\inf \varnothing = +\infty$, and $\sup \varnothing = 0$. Of course, $T_u(1)$ is a stopping time with respect to the natural filtration of the Brownian motion W_u. Having defined $(S_u(k), T_u(k))$ $(k \geqslant 1)$, we inductively define $(S_u(k+1), T_u(k+1))$ as:

$$S_u(k+1) = \sup\left\{ n \leqslant j < S_u(k) \;\middle|\; \exists r \in \left[t_j, \frac{1}{10}(9t_j + t_{j-1}) \right]\colon\ W_u(r) = 0 \right\},$$
$$T_u(k+1) = \inf\left\{ r > S_u(k) \mid W_u(r) = 0 \right\}.$$

It is not too difficult to verify that $T_u(1), T_u(2), \ldots$ are all stopping times with respect to the natural filtration of W_u and that the cardinality of $\{k\colon\ T_u(k) < \infty\}$ is greater than or equal to $\sum_{j=n}^{2n} I_{j,\varepsilon}$ for *any* $\varepsilon > 0$. (This is due to the fact that whenever W_u upcrosses or downcrosses $[-\varepsilon x, +\varepsilon x]$ for some $x > 0$, then by continuity, W_u hits a zero somewhere in the up- or downcrossing interval: i.e., Rolle's theorem of calculus). For all k such that $T_u(k) < \infty$ define

$$\Delta_k = L^0_{t_{S_u(k)-1}}(W_u) - L^0_{t_{S_u(k)}}(W_u),$$

otherwise, $\Delta_k = +\infty$. Note that $\Delta_{k+1} < \infty$ implies that $\Delta_k < \infty$. Moreover, since $t \mapsto L_t^0(W_u)$ grows only when W_u is at the origin,

$$\Delta_k = L^0_{t_{S_u(k)-1}}(W_u) - L^0_{T_u(k)}(W_u), \quad \text{on } \{\Delta_k < \infty\}.$$

Therefore, by the strong Markov property, $\Delta_1, \Delta_2, \ldots$ are independent although they are *not* identically distributed. Let us temporarily concentrate on Δ_1, for simplicity. On $\{\Delta_1 < \infty\}$, Δ_1 is the amount of local time of the process W_u at 0 accrued in the left-most interval of type

$$\mathcal{I}_k = \left[t_k, \frac{1}{10}(9t_k + t_{k-1})\right], \quad k \in \{n, \ldots, 2n\}.$$

Define $\beta = \frac{9}{10}t_{2n}(\Phi_{2n} - 1)$, and note that thanks to (3.2), on $\{\Delta_1 < \infty\}$,

$$t_{S_u(1)-1} - T_u(1) \geqslant t_{S_u(1)-1} - \frac{1}{10}(9t_{S_u(1)} + t_{S_u(1)-1}) \geqslant \beta.$$

Thus, by Brownian scaling and the strong Markov property applied at the stopping time $T_u(1)$, $\Delta_1 \mathbf{1}_{\{\Delta_1<\infty\}}$ is stochastically larger than $L^0_\beta(W_u)$. The latter has, in turn, the same distribution as $\beta^{1/2}|W_u(1)|$, thanks to Lévy's theorem. Thus, another application of scaling yields the following.

$$\mathrm{P}\{\Delta_1 < h\} \leqslant \mathrm{P}\left\{\left|W_1(1)\right| \leqslant \frac{h}{\sqrt{u\frac{9}{10}t_{2n}(\Phi_{2n}-1)}}\right\} \leqslant \sqrt{\frac{20}{9\pi u t_{2n}(\Phi_{2n}-1)}}\, h.$$

Since $u \in [1,2]$ and $\lim_{n\to\infty}\Phi_{2n} = +\infty$, there exists $n_1 \geqslant n_0$ so large that for all $n \geqslant n_1$,

$$20 \leqslant 9\pi(\Phi_{2n} - 1).$$

Since $u \in [1,2]$, for all $n \geqslant n_1$,

$$\mathrm{P}\{\Delta_1 < h\} \leqslant \frac{h}{\sqrt{t_{2n}}}. \tag{3.4}$$

We have already seen that the cardinality of $\{k\colon \Delta_k < \infty\}$ is at least $\sum_{j=n}^{2n} I_{j,\varepsilon_0}$. Thus, on $\Upsilon^\zeta_{n,\varepsilon_0}$, the cardinality of $\{k\colon \Delta_k < \infty\}$ is at least $(1-\zeta)n$. By (3.4) and its obvious extension to Δ_j $(j \geqslant 2)$, and using the independence of Δ's, we can apply induction to deduce the lemma. □

We are ready to present the following.

Proof of Theorem 3.3. Consider the following finite subset of $(0,1)$:

$$\mathcal{Q}_\varrho(h) = \left\{jh^\varrho\colon 0 < j < h^{-\varrho}\right\}, \quad h \in (0,1),$$

where $\varrho > 1$ is fixed. Of course, the cardinality of $\mathcal{Q}_\varrho(h)$ is no more than $2h^{-\varrho}$ and it has the property that for any $x \in [0,1]$, there exists $y \in \mathcal{Q}_\varrho(h)$ such that $|x - y| \leqslant h^\varrho$. Therefore, we can begin our bounds by approximating $[0,1]$ with the elements of $\mathcal{Q}_\varrho(h)$. Indeed, for any $p > 1$, Lemma 3.12 assures us of the veracity of the following.

$$\begin{aligned}
\mathrm{P}\Big\{\inf_{1\leqslant u\leqslant 2} L^0_1(W_u) < h\Big\} &\leqslant \mathrm{P}\Big\{\min_{u\in\mathcal{Q}_\varrho(h)} L^0_1(W_u) < 2h\Big\} + \mathrm{P}\Big\{\sup_{\substack{u,v\in[1,2]:\\ |u-v|\leqslant h^\varrho}} \left|L^0_1(W_u) - L^0_1(W_v)\right| \geqslant h\Big\}\\
&\leqslant \mathrm{P}\Big\{\min_{u\in\mathcal{Q}_\varrho(h)} L^0_1(W_u) < 2h\Big\} + C^p p! h^{p(\varrho/4-1)} \log^{p/4}(1/h^\varrho)\\
&= P_1 + P_2.
\end{aligned}\tag{3.5}$$

While P_2 is explicit enough, we need to bound P_1 which is done as follows: By Proposition 3.11, and using the notation there, for all $n \geqslant n_0$ and all $\zeta \in (\delta, 1)$,

$$P_1 \leqslant P_{1,1} + \exp\left(-n\frac{(\zeta-\delta)^2}{2(1-\delta)}\right) + 4n\exp\left(-\frac{\varepsilon_0^2\Phi_n}{16}\right), \tag{3.6}$$

where

$$P_{1,1} = \mathrm{P}\Big\{\min_{u \in \mathcal{Q}_\varrho(h)} L_1^0(W_u) < 2h,\ \Upsilon_{n,\varepsilon_0}^{\zeta}\Big\}.$$

We recall that the above display holds for all $h \in (0,1)$ and all $n \geqslant n_0$. We also recall that ε_0 and n_0 depend only on δ. On the other hand, by picking n even larger (in fact, if $n \geqslant n_1$), Lemma 3.13 guarantees us that

$$P_{1,1} \leqslant 2h^{-\varrho}\left(\frac{2h}{\sqrt{t_{2n}}}\right)^{(1-\zeta)n}, \tag{3.7}$$

since the cardinality of $\mathcal{Q}_\varrho(h)$ is no more than $2h^{-\varrho}$. Now we combine equations (3.5), (3.6), and (3.7) to obtain an upper bound for the distribution function $\mathrm{P}\{\inf_{1\leqslant u\leqslant 2} L_1^0(W_u) < h\big\}$. To make it useful, we now choose the parameters involved carefully so that for the γ of the statement of the theorem,

$$\gamma = \frac{(\zeta-\delta)^2}{2(1-\delta)}. \tag{3.8}$$

That is, pick $\delta > 0$ so small and $\zeta \in (\delta, 1)$ so large that Eq. (3.8) holds. Next, we define

$$\Phi_j = 16\varepsilon_0^{-2} j, \quad j \geqslant 1.$$

By Eq. (3.6), there exists $n_2 \geqslant n_1$ so large that for all $n \geqslant n_2$,

$$P_1 \leqslant P_{1,1} + 2e^{-\gamma n}. \tag{3.9}$$

It suffices to properly estimate $P_{1,1}$; this is achieved by choosing the parameter n in terms of h. Fix some $\nu \in (0,1)$ and choose

$$n = (1-\nu)\frac{\log(1/h)}{\log\log(1/h)},$$

to see that for all $h \in (0,1)$ small enough (how small depends on γ),

$$P_{1,1} \leqslant \exp\left(-K\frac{\log^2(1/h)}{\log\log(1/h)}\right) = o(e^{-\gamma n}),$$

where K is an uninteresting positive and finite constant. In light of equation (3.9), for all $h \in (0,1)$ sufficiently small, P_1 is bounded above by $3e^{-\gamma n}$, which equals $3\exp\{-\gamma(1-\nu)\log(1/h)/\log\log(1/h)\}$. The Theorem easily follows from this and equations (3.5), (3.7), and (3.9), by choosing $\varrho > 4$, since γ and ν can be chosen arbitrarily close to $1/2$ and 0, respectively. □

Having verified Theorem 3.3, we are ready to prove the difficult half of Theorem 3.1.

Proof of Theorem 3.1 (*Lower bound*). Theorem 3.3 and scaling, together, show us that for all $\gamma \in (0, 1/2)$ and all $\kappa \in (0, 1/2)$, there exists n_3 such that for all $n \geqslant n_3$,

$$\begin{aligned}\mathrm{P}\Big\{\inf_{2^{-n}\leqslant u\leqslant 2^{-n+1}} L_1^0(W_u) < 2^{n\kappa}\Big\} &= \mathrm{P}\Big\{\inf_{1\leqslant u\leqslant 2} L_1^0(W_u) < 2^{-n(\frac{1}{2}-\kappa)}\Big\} \\ &\leqslant \exp\left(-\gamma\log(2)\left(\frac{1}{2}-\kappa+o(1)\right)\frac{n}{\log n}\right),\end{aligned}$$

where $o(1)$ is Landau's notation, and goes to 0 as $n \to \infty$. Since this sums, the Borel–Cantelli lemma shows that almost surely,

$$\inf_{2^{-n}\leqslant u\leqslant 2^{-n+1}} L_1^0(W_u) \geqslant 2^{n\kappa},$$

eventually. We complete the lower bound by a standard monotonicity argument. Namely, if $h \in [2^{-n}, 2^{-n+1}]$,

$$L_1^0(W_h) \geqslant \inf_{2^{-n} \leqslant u \leqslant 2^{-n+1}} L_1^0(W_u) \geqslant 2^{n\kappa}, \quad \text{eventually, a.s.}$$
$$\geqslant h^{-\kappa}.$$

Since $\kappa \in (0, \frac{1}{2})$ is arbitrary, this shows that

$$\liminf_{h \to 0^+} \frac{1}{\log(1/h)} \log L_1^0(W_u) \geqslant \frac{1}{2}, \quad \text{a.s.},$$

which is the desired lower bound. □

The corresponding upper bound relies on the following large deviations result, which is a consequence of [14, Theorem 4.1] in its present formulation:

Lemma 3.14 (Lacey, 1990). *As* $x \to \infty$,

$$x^{-2} \log \mathrm{P}\Bigl\{ \sup_{u \in [1,2]} L_1^0(W_u) > x \Bigr\} \to -\frac{1}{2}.$$

The remainder of Theorem 3.1 follows the given argument for the lower bound closely, except that Lemma 3.14 is used in place of Theorem 3.3. □

We close this section with our

Proof of Theorem 3.6. We will derive this by establishing an upper and a lower bound, respectively. According to [2, Theorem 2.1],

$$\mathrm{P}\Bigl\{ \sup_{a \in \mathbb{R}} L_1^a(W_1) \leqslant h \Bigr\} = \exp\left\{ -\frac{2j_1^2}{h^2}\bigl(1 + o(1)\bigr) \right\}, \quad h \to 0^+, \tag{3.10}$$

since W_1 is standard Brownian motion. For our lower bound, we need only note that

$$\mathrm{P}\Bigl\{ \inf_{1 \leqslant u \leqslant 2} \sup_{a \in \mathbb{R}} L_1^a(W_u) \leqslant h \Bigr\} \geqslant \mathrm{P}\Bigl\{ \sup_{a \in \mathbb{R}} L_1^a(W_1) \leqslant h \Bigr\}, \quad h > 0. \tag{3.11}$$

On the other hand, for any finite set $F \subset [1, 2]$,

$$\mathrm{P}\Bigl\{ \inf_{1 \leqslant u \leqslant 2} \sup_{a \in \mathbb{R}} L_1^a(W_u) \leqslant h \Bigr\} \leqslant \mathrm{P}\Bigl\{ \inf_{u \in F} \sup_{a \in \mathbb{R}} L_1^a(W_u) \leqslant h + h^2 \Bigr\} + \mathrm{P}\bigl\{ \omega(F) \geqslant h^2 \bigr\}, \tag{3.12}$$

where

$$\omega(F) = \sup_{\substack{u, v \in F: \\ u \neq v}} \sup_{a \in \mathbb{R}} \bigl| L_1^a(W_u) - L_1^a(W_v) \bigr|.$$

Now we choose F as (the closest possible candidate to) an equipartition of $[1, 2]$ of mesh h^{1000}. Clearly, the cardinality of F is bounded above by Γh^{-1000} for some absolute $\Gamma > 0$. Thus, thanks to (3.10) and scaling, we can bound, from the above, the first term on the right-hand side of Eq. (3.12) by

$$\begin{aligned}
&\Gamma h^{-1000} \sup_{u \in [1,2]} \mathrm{P}\Bigl\{ \sup_{a \in \mathbb{R}} L_1^a(W_u) \leqslant h + h^2 \Bigr\} \\
&\quad = \Gamma h^{-1000} \sup_{u \in [1,2]} \mathrm{P}\Bigl\{ \sup_{a \in \mathbb{R}} u^{-1/2} L_1^a(W_1) \leqslant h + h^2 \Bigr\} \\
&\quad = \Gamma h^{-1000} \exp\bigl\{ -\bigl(1 + o(1)\bigr) 2 j_1^2 h^{-2} \bigr\} \\
&\quad = \exp\bigl\{ -\bigl(1 + o(1)\bigr) 2 j_1^2 h^{-2} \bigr\}.
\end{aligned} \tag{3.13}$$

On the other hand, by Lemma 3.12, for all $p > 1$,

$$\|\omega(F)\|_{L^p(\mathrm{P})}^p \leqslant D^p p!\{h^{1000}\log(1/h)\}^{p/4},$$

where D is a universal constant. In particular,

$$\Lambda = \sup_{h>0} \mathrm{E}\left[\exp\left(\frac{D\omega(F)}{2\{h^{1000}\log(1/h)\}^{1/4}}\right)\right] < \infty.$$

This yields the following bound on the second term on the right-hand side of (3.12):

$$\mathrm{P}\{\omega(F) \geqslant h^2\} \leqslant \Lambda \exp\left(-\frac{D}{2h^{998/4}\log^{1/4}(1/h)}\right) \leqslant \exp\{-(1+o(1))2j_1^2h^{-2}\}.$$

Together with (3.13) and (3.12), we can see that (3.11) is sharp at a logarithmic scale. This completes our proof. □

4. Applications

In this section we explore two applications of Theorems 3.1 and 3.3. In particular, we present in turn:

(1) an estimate for the capacity of paths in Wiener space that have small local times; and
(2) a uniform ratio ergodic theorem.

These will be discussed, in order, in the proceeding subsections.

4.1. A capacity estimate

Define the Ornstein–Uhlenbeck process on Wiener space $O_u(t) = e^{-u/2}W_{e^u}(t)$. The process O_u also has local times at 0. In fact,

Lemma 4.1. *The random field O has continuous local times along lines given by*

$$L_t^x(O_u) = \exp(u/2)L_t^{\exp(u/2)x}(W_{e^u}), \quad x \in \mathbb{R},\ t \geqslant 0,\ u \geqslant 0.$$

Proof. It suffices to show that for all bounded, measurable functions $f : \mathbb{R} \to \mathbb{R}$,

$$\int_0^t f(O_u(s))\,\mathrm{d}s = \int_{-\infty}^{\infty} f(a)\ \exp(u/2)L_t^{\exp(u/2)a}(W_{\exp(u)})\,\mathrm{d}a, \quad \forall t \geqslant 0,$$

which follows readily from the definition of the process O_u in terms of Brownian sheet. □

Recall that $O = \{O_u;\ u \geqslant 0\}$ is a diffusion on the space $C[0,1]$ of real continuous functions on $[0,1]$ endowed with the compact-open topology. Moreover, by Hunt's theorem, the hitting probabilities of O killed at rate one determine a natural Choquet capacity Cap on the classical Wiener space $C[0,1]$. This connection to capacities has received some attention in infinite-dimensional stochastic analysis, and a formal definition of the said capacity can be given as follows: For all measurable $A \subset C[0,1]$,

$$\mathsf{Cap}(A) = \int_0^{\infty} e^{-t}\mathrm{P}\{\exists u \in [0,t]:\ O_u \in A\}\,\mathrm{d}t.$$

Theorem 3.3 has the following ready capacitary translation.

Corollary 4.2. *If* Cap *denotes capacity in the classical Wiener space, then*

$$\limsup_{h\to 0^+} \frac{\log\log(1/h)}{\log(1/h)} \cdot \log \mathsf{Cap}\big\{X \in C[0,1]:\ L_1^0(X) \leqslant h\big\} \leqslant -\frac{1}{2}.$$

Proof. Define the incomplete 1-capacity Cap_1 for the OU-process as

$$\mathsf{Cap}_1(A) = \mathrm{P}\big\{\exists u \in [0,1]:\ O_u \in A\big\},$$

for all measurable $A \subset C[0,1]$. According to [4, Lemma 2.2], there exists a finite $K > 1$ such that for all measurable $A \subset C[0,1]$, $K^{-1}\mathsf{Cap}_1(A) \leqslant \mathsf{Cap}(A) \leqslant K\mathsf{Cap}_1(A)$. Thus, it suffices to prove our result with Cap replaced by Cap_1.

By its definition, the above incomplete 1-capacity equals

$$\begin{aligned}\mathrm{P}\Big\{\inf_{0\leqslant u\leqslant 1} L_1^0(O_u) < h\Big\} &= \mathrm{P}\Big\{\inf_{1\leqslant v\leqslant e} v^{1/2} L_1^0(W_v) < h\Big\} \leqslant \mathrm{P}\Big\{\inf_{1\leqslant v\leqslant e} L_1^0(W_v) < h\Big\}\\ &\leqslant \mathrm{P}\Big\{\inf_{1\leqslant v\leqslant 2} L_1^0(W_v) < h\Big\} + \mathrm{P}\Big\{\inf_{2\leqslant v\leqslant e} L_1^0(W_v) < h\Big\}.\end{aligned}$$

We have used Lemma 4.1 in the above. By scaling,

$$\mathrm{P}\Big\{\inf_{2\leqslant v\leqslant e} L_1^0(W_v) < h\Big\} = \mathrm{P}\Big\{\inf_{1\leqslant v\leqslant e/2} L_1^0(W_v) < 2^{1/2}h\Big\}.$$

Thus,

$$\mathsf{Cap}_1\big\{X \in C[0,1]:\ L_1^0(X) < h\big\} \leqslant 2\mathrm{P}\Big\{\inf_{1\leqslant v\leqslant 2} L_1^0(W_v) < 2^{1/2}h\Big\}.$$

The corollary easily follows from Theorem 3.3, since $\gamma \in (0, 1/2)$ is otherwise arbitrary. □

4.2. A uniform ratio ergodic theorem

The ratio ergodic theorem for Brownian motion states that for all $f \in \mathcal{L}^2(\mathrm{d}x)$ and for each fixed $u \in [1,2]$,

$$\lim_{t\to\infty} \frac{1}{L_t^0(W_u)} \int_0^t f\big(W_u(s)\big)\,\mathrm{d}s = \int_{-\infty}^{\infty} f(v)\,\mathrm{d}v, \quad \text{a.s.}; \tag{4.1}$$

see, for example, [7]. As a consequence of this, one obtains the more familiar form of the ratio ergodic theorem that states that for $f, g \in \mathcal{L}^2(\mathrm{d}x)$ with $\int_{-\infty}^{\infty} g(v)\,\mathrm{d}v \neq 0$,

$$\lim_{t\to\infty} \frac{\int_0^t f(W_u(s))\,\mathrm{d}s}{\int_0^t g(W_u(s))\,\mathrm{d}s} = \frac{\int_{-\infty}^{\infty} f(v)\,\mathrm{d}v}{\int_{-\infty}^{\infty} g(v)\,\mathrm{d}v}, \quad \text{a.s.}$$

It is instructive to consider the following quick derivation of the above. (More details will be supplied in the course of our proof of Theorem 4.3 below.) Let $t \mapsto \tau_u(t)$ denote the inverse to $t \mapsto L_t^0(W_u)$. That is,

$$\tau_u(t) = \inf\big\{s > 0:\ L_s^0(W_u) > t\big\}, \quad t \geqslant 0. \tag{4.2}$$

Then, by the strong Markov property, $t \mapsto \int_0^{\tau_u(t)} f(W_u(s))\,\mathrm{d}s$ is a Lévy process. Thus, (4.1) follows at once from Kolmogorov's law of large numbers, once we verify that the mean of $\int_0^{\tau_u(t)} f(W_u(s))\,\mathrm{d}s$ exists and equals $t \cdot \int_{-\infty}^{\infty} f(v)\,\mathrm{d}v$. On the other hand, by the occupation density formula (Eq. (2.2)),

$$\mathrm{E}\left\{\int_0^{\tau_u(t)} f\big(W_u(s)\big)\,\mathrm{d}s\right\} = \int_{-\infty}^{\infty} f(x)\mathrm{E}\big\{L_{\tau_u(t)}^x(W_u)\big\}\,\mathrm{d}x,$$

which equals $t \cdot \int_{-\infty}^{\infty} f(v)\,dv$, since standard methods of excursion theory show that $E\{L^x_{\tau_u(t)}(W_u)\} = t$; for instance, see Lemma 4.4 below. This argument is a streamlined modification of the classical methods of [16].

As an application of Theorem 3.1, we propose to show that under a slightly more stringent condition than $f \in \mathcal{L}^2(dx)$, the ratio ergodic theorem (4.1) holds uniformly in $u \in [1,2]$. This is closely-related to the quasi-sure ergodic theorem of [5].

Theorem 4.3. *If $f \in \mathcal{L}^1(\{1+|x|\}\,dx)$, then with probability one,*

$$\lim_{t\to\infty} \sup_{u\in[1,2]} \left| \frac{1}{L_t^0(W_u)} \int_0^t f\big(W_u(s)\big)\,ds - \int_{-\infty}^{\infty} f(v)\,dv \right| = 0.$$

Theorem 4.3 is proved in a few stages using some of the methods and calculations of [11].

Lemma 4.4. *For any $u > 0$, and for all $x \in \mathbb{R}$, $E\{L^x_{\tau_u(t)}(W_u) \mid W_u(0) = 0\} = t$. Moreover, $E\{L^x_{T_u(0)}(W_u) \mid W_u(0) = x\} = 2|x|u^{-1}$, if $T_u(0) = \inf\{s > 0\colon W_u(s) = 0\}$.*

Proof. By Brownian scaling, if $u > 0$ is held fixed, the stochastic process

$$\big\{\big(W_u(t), u^{-1/2} L_s^{au^{-1/2}}(W_u), T_1(0), \tau_1(ru^{1/2})\big);\ r,s,t \geqslant 0,\ a \in \mathbb{R}\big\}$$

has the same finite dimensional distributions as the process

$$\big\{\big(W_1(t), L_s^a(W_u), T_u(0), \tau_u(r)\big);\ r,s,t \geqslant 0,\ a \in \mathbb{R},\ u > 0\big\}.$$

From this one gathers that

$$E_0\big\{L^x_{\tau_u(t)}(W_u)\big\} = u^{-1/2} E\big\{L^{x/\sqrt{u}}_{\tau_1(t\sqrt{u})}(W_1) \mid W_1(0) = 0\big\}, \tag{4.3}$$

$$E_x\big\{L^x_{T_u(0)}(W_u)\big\} = u^{-1/2} E\big\{L^{x/\sqrt{u}}_{T_1(0)}(W_1) \mid W_1(0) = xu^{-1/2}\big\}, \tag{4.4}$$

for all $u > 0$, where P_x and E_x are the conditional probability measure and the expectation integral given $W_u(0) = x$ for the u in question. To be more precise, we should write $P_{u,x}$, but this would confound the notation more than our present admittedly relaxed notation. Thus, the problem is reduced to one about the *standard* Brownian motion W_1.

Since the second calculation is needed to make the first, we start with it. Without loss of generality, we assume $x > 0$ and use Tanaka's formula in the following form:

$$\big(W_1(t) - x\big)^- = M_t + \frac{1}{2} L_t^x(W_1), \quad P_x\text{-a.s.},$$

where $M_t = \int_0^t \mathbf{1}\{W_1(r) < x\}\, W_1(dr)$. Replace t by $T_1(0) \wedge n$, and let $n \to \infty$ to see that the left-hand side remains in $[0,x]$ and hence, by the optional stopping theorem, and by the bounded convergence theorem, for all $x > 0$,

$$x = \frac{1}{2} E_x\big\{L^x_{T_1(0)}(W_1)\big\}.$$

This, used in conjunction with (4.4), implies the second assertion of our lemma. To verify the first one, we use excursion theory, still assuming that $x > 0$ (without loss of generality). Let D_x denote the number of downcrossings of the interval $[0,x]$ made by $s \mapsto W_1(s)$ while $s \in [0, \tau_1(t)]$. By Itô's excursion theory, D_x is a Poisson random variable with $E_0\{D_x\}$ equaling the reciprocal of $E_x\{L^x_{T_1(0)}(W_1)\} = 2|x|$, thanks to the previous calculation. Also by excursion theory, under P_0, $L^x_{\tau_1(0)}(W_1)$ is the sum of D_x many exponential random variables each of which has the same law as the P_x-law of $L^x_{T_1(0)}(W_1)$. Finally, these exponential random variables, together with D_x, are all mutually independent. The lemma follows readily from these observations used in conjunction with Eq. (4.3). □

Lemma 4.5. *For every integer $k \geqslant 1$, every $u > 0$, and all $x \in \mathbb{R}$,*

$$\mathrm{E}\big\{\big|L^x_{\tau_u(1)}(W_u)\big|^k\big\} \leqslant k!2^k\bigg[1+\frac{|x|}{u}\bigg]^k.$$

Proof. By scaling, we can reduce to the $u = 1$ case; see the argument leading to (4.3) for instance. Keeping this in mind, and applying the strong Markov property to the first hitting time of x, we can see that

$$\mathrm{E}_0\big\{\big|L^x_{\tau_1(1)}(W_1)\big|^k\big\} \leqslant \mathrm{E}_x\big\{\big|L^x_{\tau_1(1)}(W_1)\big|^k\big\}. \tag{4.5}$$

We have used the additivity property of local times. On the other hand, under P_x,

$$L^x_{\tau_1(1)}(W_1) = L^x_{T_1(0)}(W_1) + L^x_{\tau_1(1)-T_1(0)}(W_1)\circ\theta_{T_1(0)},$$

where θ is the shift functional on the paths of W_1. In particular, it follows immediately from this that

$$L^x_{\tau_1(1)}(W_1) \leqslant L^x_{T_1(0)}(W_1) + L^x_{\tau_1(1)}(W_1)\circ\theta_{T_1(0)}, \quad \mathrm{P}_x\text{-a.s.}$$

Thanks to Lemma 4.4, this gives

$$\mathrm{E}_x\big\{L^x_{\tau_1(1)}(W_1)\big\} \leqslant 1+2|x| \leqslant 2\big\{1+|x|\big\}. \tag{4.6}$$

In light of Eq. (4.5), it suffices to show the following type of hypercontractivity: For all $k \geqslant 1$,

$$\mathrm{E}_x\big\{\big|L^x_{\tau_1(1)}(W_1)\big|^k\big\} \leqslant k!\big[\mathrm{E}_x\big\{L^x_{\tau_1(1)}(W_1)\big\}\big]^k.$$

But this follows from [12, Lemma (A.2)], since by the strong Markov property, $L^x_{\tau_1(1)}(W_1)$ is NBU (New Better than Used) under the measure P_x. That is, for all $a, b > 0$,

$$\mathrm{P}_x\big\{L^x_{\tau_1(1)}(W_1) > a+b\big\} \leqslant \mathrm{P}_x\big\{L^x_{\tau_1(1)}(W_1) > a\big\}\cdot\mathrm{P}_x\big\{L^x_{\tau_1(1)}(W_1) > b\big\}.$$

This is proved in a similar manner as (4.5) was, and completes our proof. For a similar inequality see [12, Lemma (A.4)]. □

Lemma 4.6. *For any $\nu > 0$ and for all $\varrho > 4(1+\nu)$, there exists a finite $c_{\varrho,\nu} > 0$ such that for all $t > e^e$,*

$$\mathrm{P}\Big\{\sup_{x\in\mathbb{R}}\sup_{\substack{u,v\in[1,2]:\\|u-v|\leqslant t^{-\varrho}}}\big|L^x_{\tau_u(t)}(W_u) - L^x_{\tau_u(t)}(W_v)\big| \geqslant t^{-\nu}\Big\} \leqslant \exp\bigg(-c_{\varrho,\nu}\frac{\log t}{\log\log t}\bigg).$$

Proof. Note that for any $s, t > 0$,

$$\sup_{u\in[1,2]}\tau_u(t) \leqslant s \iff \inf_{u\in[1,2]} L^0_s(W_u) \geqslant t.$$

We use this with $s = t^{2+\eta}$, where η is a small positive number (to be chosen shortly), in order to deduce the following:

$$\begin{aligned}
&\mathrm{P}\Big\{\sup_{x\in\mathbb{R}}\sup_{\substack{u,v\in[1,2]:\\|u-v|\leqslant t^{-\varrho}}}\big|L^x_{\tau_u(t)}(W_u) - L^x_{\tau_u(t)}(W_v)\big| \geqslant t^{-\nu}\Big\}\\
&\quad\leqslant \mathrm{P}\Big\{\inf_{u\in[1,2]} L^0_{t^{2+\eta}}(W_u) \leqslant t\Big\} + \mathrm{P}\Big\{\sup_{\substack{u,v\in[1,2]:\\|u-v|\leqslant t^{-\varrho}}}\sup_{x\in\mathbb{R}}\sup_{0\leqslant s\leqslant t^{2+\eta}}\big|L^x_s(W_u) - L^x_s(W_v)\big| \geqslant t^{-\nu}\Big\}\\
&\quad= P_1 + P_2,
\end{aligned}$$

using obvious notation. By Theorem 3.3 and by scaling,

$$P_1 \leqslant \exp\left(-\frac{\eta \log t}{4 \log\log t}\right),$$

as long as $\eta < 1/2$. On the other hand, by combining Lemma 3.12 with Chebyshev's inequality, we can see that for any integer $k \geqslant 1$,

$$P_2 \leqslant C_k^k \varrho^{k/4} t^{(2+\eta)k/2} t^{-\varrho k/4} t^{k\nu} \log^{k/4} t.$$

Thus, as long as we choose η strictly between 0 and the minimum of $1/2$ and $\frac{1}{2}(\varrho - 4 - 4\nu)$, it follows that $P_2 = o(P_1)$ as $t \to \infty$, from which the lemma follows. □

Before presenting our proof of Theorem 4.3 we will need to develop one final technical estimate. Roughly speaking, it states that if u and v are close, so are $\tau_u(t)$ and $\tau_v(t)$, and with overwhelming probability. However, since the latter are jump processes, we need to "smudge" time (i.e., the variable t) a little bit in order for such a statement to actually hold. This amounts to tightness in an appropriately chosen Skorohod topology (for weak convergence of random functions with jumps), and a formal statement follows.

Lemma 4.7. *For all $\mu > 0$ and $\varrho > 4(1+\mu)$, there exists a finite constant $d_{\varrho,\mu} > 0$ such that for all $t > e^e$,*

$$\mathrm{P}\bigl\{\exists u, v \in [1,2]\colon\ |u-v| \leqslant t^{-\varrho},\ \tau_u(t) > \tau_v(t+t^{-\mu})\bigr\} \leqslant \exp\left(-d_{\varrho,\mu}\frac{\log t}{\log\log t}\right).$$

Proof. To expedite the presentation, we fix some $\nu \in (\mu, \infty)$ such that $\varrho > 4(1+\nu)$, and define two events E_t and F_t as

$$\mathsf{E}_t = \Bigl\{\omega\colon \sup_{\substack{u,v\in[1,2]:\\ |u-v|\leqslant t^{-\varrho}}} \bigl|L^0_{\tau_v(t+t^{-\mu})}(W_u) - L^0_{\tau_v(t+t^{-\mu})}(W_v)\bigr| \leqslant t^{-\nu}\Bigr\},$$

$$\mathsf{F}_t = \bigl\{\omega\colon \forall u, v \in [1,2]:\ |u-v| \leqslant t^{-\varrho},\ \tau_u(t) < \tau_v(t+t^{-\mu})\bigr\}.$$

We claim that for all $t > e^e$,

$$\mathrm{P}\{\mathsf{E}_t\} \leqslant \mathrm{P}\{\mathsf{F}_t\}. \tag{4.7}$$

This follows from the continuity of local times. Indeed, if (4.7) were false for some $t > e^e$, on $\mathsf{E}_t \cap \mathsf{F}_t^{\complement}$, we could always find $u, v \in [1,2]$ such that $|u-v| \leqslant t^{-\varrho}$, and

$$t = L^0_{\tau_u(t)}(W_u) \geqslant L^0_{\tau_v(t+t^{-\mu})}(W_u) \geqslant L^0_{\tau_v(t+t^{-\mu})}(W_v) - t^{-\nu} = t + t^{-\mu} - t^{-\nu}.$$

Since this contradicts our choice of $\mu < \nu$, by (4.7), $\mathrm{P}\{F_t^{\complement}\} \leqslant \mathrm{P}\{\mathsf{E}_t^{\complement}\}$, and our lemma follows from this and Lemma 4.6. □

We are ready to present our

Proof of Theorem 4.3. By considering f^+ and f^- separately, we may assume, without any loss in generality, that f is a nonnegative function. This assumption will be tacitly made throughout.

For each $u \in [1,2]$, define

$$S_t^u = \int_0^{\tau_u(t)} f\bigl(W_u(s)\bigr)\,\mathrm{d}s, \quad t \geqslant 0.$$

By the strong Markov property, the process $\{S_n^u;\ n \geqslant 1\}$ is a random walk for each fixed $u \in [1,2]$. Now we hold such a u fixed, and proceed to estimate the moments of the increments of the corresponding walk $n \mapsto S_n^u$. First, to the mean: By Lemma 4.4, for all $u \in [1,2]$,

$$\mathrm{E}\{S_1^u\} = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x. \tag{4.8}$$

We have used the following consequence of the occupation density formula (cf. Eq. (2.2)):

$$S_1^u = \int_{-\infty}^{\infty} f(x) L^x_{\tau_u(1)}(W_u)\,\mathrm{d}x.$$

To estimate the higher moments, we use the occupation density formula once more, this time in conjunction with Lemma 4.5 and Minkowski's inequality, to see that for all $k \geqslant 1$,

$$\|S_1^u\|_{L^k(\mathrm{P})} \leqslant \int_{-\infty}^{\infty} f(x) \big\| L^x_{\tau_u(1)}(W_u)\big\|_{L^k(\mathrm{P})}\,\mathrm{d}x \leqslant 2(k!)^{1/k} \int_{-\infty}^{\infty} f(x)\big\{1+|x|\big\}\,\mathrm{d}x.$$

In particular, for all $k \geqslant 1$ and all $u \in [1,2]$,

$$\|S_1^u\|^k_{L^k(\mathrm{P})} \leqslant k!\, 2^k \|f\|^k_{\mathcal{L}^1(\{1+|x|\}\mathrm{d}x)}. \tag{4.9}$$

This, (4.8), and Rosenthal's inequality [6], all used in conjunction, give us the following: For each $k \geqslant 1$, there exists a (universal) finite constant $A_k > 0$ such that for all $u \in [1,2]$ all $n \geqslant 1$, and all $k \geqslant 1$,

$$\mathrm{E}\left\{ \max_{1 \leqslant i \leqslant n} \left| S_i^u - i \cdot \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \right|^k \right\} \leqslant A_k n^{k/2}. \tag{4.10}$$

Next we choose and hold fixed $\mu > 0$ and $\varrho > 4(1+\mu)$, in agreement with the conditions of Lemma 4.7. We also introduce the following equipartition of $[1,2]$ of mesh $n^{-\varrho}$:

$$\mathfrak{S}_\varrho(n) = \big\{1 + jn^{-\varrho};\ 0 \leqslant j \leqslant n^\varrho\big\}, \quad n \geqslant 1.$$

We need to introduce four more parameters as follows:

$$\nu > 0, \quad R > 1, \quad \frac{1}{2} < \delta < 1, \quad k > \varrho\left(\delta - \frac{1}{2}\right)^{-1}. \tag{4.11}$$

The remainder of our proof concerns monotonicity arguments used in conjunction with the Borel–Cantelli lemma. By Chebyshev's inequality, Eq. (4.10), and using the fact that $\#\mathfrak{S}_\varrho(m) \leqslant (1 + m^\varrho)$,

$$\sum_n \mathrm{P}\left\{ \exists u \in \mathfrak{S}_\varrho(R^n)\colon \max_{1 \leqslant i \leqslant R^n} \left| S_i^u - i \int_{-\infty}^{\infty} f(w)\,\mathrm{d}w \right| > R^{\delta n} \right\} \leqslant A_k \sum_n (1 + R^{n\varrho}) R^{-(\delta - \frac{1}{2})kn},$$

which is finite by (4.11). Thus, the Borel–Cantelli lemma assures us of the existence of a finite random variable N_1 such that a.s. for all $n \geqslant N_1$,

$$\max_{u \in \mathfrak{S}_\varrho(R^n)} \left| \int_0^{\tau_u(R^n)} f\big(W_u(r)\big)\,\mathrm{d}r - R^n \int_{-\infty}^{\infty} f(w)\,\mathrm{d}w \right| \leqslant R^{\delta n}. \tag{4.12}$$

Similarly, Lemma 4.7 and a Borel–Cantelli argument, together imply the existence of a finite random variable N_2 such that a.s. for all $n \geqslant N_2$,

$$\forall u, v \in [1,2]\colon\ |u-v| \leqslant R^{-n\varrho}, \quad \tau_u(R^{n-1}) \leqslant \tau_v(R^n) \leqslant \tau_u(R^{n+1}). \tag{4.13}$$

Finally, another Borel–Cantelli argument, this time involving Lemma 4.6, shows the existence of a finite random variable N_3 such that a.s. for all $n \geqslant N_3$,

$$\sup_{x\in\mathbb{R}} \sup_{\substack{u,v\in[1,2]:\\ |u-v|\leqslant R^{-n\varrho}}} \left| L^x_{\tau_u(R^n)}(W_u) - L^x_{\tau_v(R^n)}(W_v)\right| \leqslant R^{-n\nu}. \tag{4.14}$$

Let $N = \max_{1\leqslant i\leqslant 3} N_i$ to see from (4.12) that for all $n \geqslant N$ and $u \in \mathfrak{S}_\varrho(R^n)$,

$$R^n \int_{-\infty}^{\infty} f(w)\,\mathrm{d}w - R^{\delta n} \leqslant \int_0^{\tau_u(R^n)} f\big(W_u(r)\big)\,\mathrm{d}r \leqslant R^n \int_{-\infty}^{\infty} f(w)\,\mathrm{d}w + R^{\delta n}. \tag{4.15}$$

On the other hand, if we choose an arbitrary $v \in [1,2]$ we can always find $u \in \mathfrak{S}_\varrho(R^n)$ such that $|u-v| \leqslant R^{-\varrho n}$. Thus, by (4.13), for any such $v \in [1,2]$, and for all $n \geqslant N$,

$$\begin{aligned}
\int_0^{\tau_v(R^n)} f\big(W_v(s)\big)\,\mathrm{d}s &\leqslant \int_0^{\tau_u(R^{n+1})} f\big(W_v(s)\big)\,\mathrm{d}s = \int_{-\infty}^{\infty} f(x) L^x_{\tau_u(R^{n+1})}(W_v)\,\mathrm{d}x \\
&\leqslant \int_{-\infty}^{\infty} f(x) L^x_{\tau_u(R^{n+1})}(W_u)\,\mathrm{d}x + R^{-n\nu} \int_{-\infty}^{\infty} f(w)\,\mathrm{d}w \\
&= \int_0^{\tau_u(R^{n+1})} f\big(W_u(s)\big)\,\mathrm{d}s + R^{-n\nu} \int_{-\infty}^{\infty} f(w)\,\mathrm{d}w.
\end{aligned}$$

We have used Eq. (4.14) in the penultimate line. Consequently, equation (4.15) implies that a.s. for all $v \in [1,2]$ and all $n \geqslant N$,

$$\int_0^{\tau_v(R^n)} f\big(W_v(r)\big)\,\mathrm{d}r \leqslant \{R^{n+1} + R^{-n\nu}\} \int_{-\infty}^{\infty} f(w)\,\mathrm{d}w + R^{\delta n}.$$

A similar lower bound ensues analogously from which it follows that a.s. as $n \to \infty$,

$$\sup_{1\leqslant v\leqslant 2} \left| \int_0^{\tau_v(R^n)} f\big(W_v(r)\big)\,\mathrm{d}r - R^n \int_{-\infty}^{\infty} f(w)\,\mathrm{d}w \right| \leqslant (R^{n+1} - R^n) \int_{-\infty}^{\infty} f(w)\,\mathrm{d}w + o(R^n).$$

Consequently, a.s.,

$$\limsup_{n\to\infty} \left| R^{-n} \int_0^{\tau_v(R^n)} f\big(W_v(r)\big)\,\mathrm{d}r - \int_{-\infty}^{\infty} f(w)\,\mathrm{d}w \right| \leqslant (R-1) \int_{-\infty}^{\infty} f(w)\,\mathrm{d}w.$$

By sandwiching $t \in [R^n, R^{n+1}]$ and appealing to yet another monotonicity argument we can deduce that

$$\frac{2-R}{R}\int_{-\infty}^{\infty} f(w)\,\mathrm{d}w \leqslant \liminf_{n\to\infty}\frac{1}{R^{n+1}}\int_0^{\tau_v(R^n)} f\big(W_v(r)\big)\,\mathrm{d}r \leqslant \liminf_{t\to\infty}\frac{1}{t}\int_0^{\tau_v(t)} f\big(W_v(r)\big)\,\mathrm{d}r$$
$$\leqslant \limsup_{t\to\infty}\frac{1}{t}\int_0^{\tau_v(t)} f\big(W_v(r)\big)\,\mathrm{d}r \leqslant \limsup_{n\to\infty}\frac{1}{R^n}\int_0^{\tau_v(R^{n+1})} f\big(W_v(r)\big)\,\mathrm{d}r$$
$$\leqslant R^2\int_{-\infty}^{\infty} f(w)\,\mathrm{d}w,$$

where the convergences, as well as the inequalities, hold uniformly over all $v \in [1,2]$. Since $R > 1$ is arbitrary (cf. (4.11)), we can let $R \downarrow 1$ along a rational sequence to see that with probability one,

$$\lim_{t\to\infty}\sup_{v\in[1,2]}\left|\frac{1}{t}\int_0^{\tau_v(t)} f\big(W_v(r)\big)\,\mathrm{d}r - \int_{-\infty}^{\infty} f(w)\,\mathrm{d}w\right| = 0.$$

A final monotonicity argument used together with Corollary 3.5 concludes our proof. □

Remarks. (a) The above demonstration makes very heavy use of the notion of monotonocity which is known to be a key idea in classical ergodic theory as well. In particular, see [8] and its standard-analysis interpretation by [9].

(b) In the above proof we used the $\mathcal{L}^1(\{1+|x|\}\,\mathrm{d}x)$ condition to show that the random walk $n \mapsto S_n^u$ has finite moments of all order; for instance, see equation (4.9). While this may seem extravagant, we now argue that in most interesting cases, the mentioned random walk has finite moments of all orders *if and only if* it has a finite variance. Moreover, in such cases, the condition that $f \in \mathcal{L}^1(\{1+|x|\}\,\mathrm{d}x)$ is *equivalent* to the finiteness of the variance of each such random walk. Indeed, suppose $f \geqslant 0$, and without loss of generality $u = 1$. Then, W_1 is standard Brownian motion, and we claim that $\chi_f = \int_0^{\tau_1(1)} f(W_1(r))\,\mathrm{d}r$ has a finite variance if and only if $f \in \mathcal{L}^1(\{1+|x|\}\,\mathrm{d}x)$. To show this we begin by recalling that $\chi_f = \int_{-\infty}^{\infty} f(x) L^x_{\tau_1(1)}(W_1)\,\mathrm{d}x$. Hence, thanks to Lebesgue's monotone convergence theorem, the following always holds.

$$\mathrm{E}\big\{\chi_f^2\big\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(y)\,\mathrm{E}\big\{L^x_{\tau_1(1)}(W_1)L^y_{\tau_1(1)}(W_1)\big\}\,\mathrm{d}x\,\mathrm{d}y.$$

While there are various ways of computing this "energy integral," perhaps the most elegant one uses the Ray–Knight theorem; cf. [18, Ch. XI]. Namely, we recall that if $Z_x = L^x_{\tau_1(1)}(W_1)$ $(x \in \mathbb{R})$, then (i) $\{Z_x;\ x \geqslant 0\}$ and $\{Z_{-x};\ x \geqslant 0\}$ are independent copies of one another; and (ii) $\{Z_x;\ x \geqslant 0\}$ is a squared Bessel process of dimension 0, starting at 1. In other words, $\{Z_x;\ x \geqslant 0\}$ solves the SDE $Z_x = 1 + 2\int_0^x \sqrt{Z_s}\,\mathrm{d}\beta_s$ $(x \geqslant 0)$, where β is a standard Brownian motion. From this it follows readily that for all $x \in \mathbb{R}$, $\mathrm{E}\{Z_x\} = 1$ (cf. also Lemma 4.4), and

$$\mathrm{E}\{Z_x Z_y\} = \begin{cases} 1 + 4(|x| \wedge |y|), & \text{if } xy \geqslant 0,\\ 1, & \text{otherwise.}\end{cases}$$

Thus, whenever $f \geqslant 0$, then χ_f has two finite moments *if and only if* $f \in \mathcal{L}^1(\{1+|x|\}\,\mathrm{d}x)$ in which case it has all finite moments thanks to Lemma 4.5. In this regard, see also Eq. (4.9).

(c) With very little extra effort it is possible to extend Theorem 4.3 to cover more general "continuous additive functionals" (CAFs). Consider a signed measure μ on $\mathbb{R}$, and the parametrized CAFs,

$$\mathfrak{A}^\mu_u(t) = \int_{-\infty}^{\infty} L^x_t(W_u)\mu(\mathrm{d}x).$$

Then, our proof of Theorem 4.3 goes through with no essential changes to show that as long as (i) $\int_{-\infty}^{\infty}\{1+|x|\}|\mu|(\mathrm{d}x)<+\infty$; and (ii) $|\mu|(\mathbb{R})<\infty$ with probability one, then

$$\lim_{t\to\infty}\sup_{u\in[1,2]}\left|\frac{\mathfrak{A}_u^{\mu}(t)}{L_t^0(W_u)}-\mu(\mathbb{R})\right|=0.$$

As an example of its use, we mention the choice of $\mathfrak{A}_u^{\mu}(t)=L_t^a(W_u)$, in which case we obtain the following: For each $a\in\mathbb{R}$,

$$\lim_{t\to\infty}\sup_{u\in[1,2]}\left|\frac{L_t^a(W_u)}{L_t^0(W_u)}-1\right|=0, \tag{4.16}$$

almost surely.

5. Concluding remarks and open problems

We conclude this paper with some remarks and a few open problems that we have been unable to resolve. These problems are potentially difficult, but we believe that their resolution is well worth the effort:

(1) With regards to our results on the modulus of continuity of local times along lines, we mention two difficult open problems. Since there are very few methods for analyzing $u\mapsto L_1^0(W_u)$, the resolution of the following would invariably require a better understanding of $u\mapsto L_1^0(W_u)$, which is a non-Dirichlet, non-semimartingale, non-Markov process with a very complicated evolution structure:
   (a) Are there corresponding lower bounds to the upper bound in Theorem 2.1?
   (b) Can the lim sup be replaced by a true limit?

(2) Although in this article we have no practical need for discussing lower bounds that correspond to Theorem 3.3, let us mention a few words on this topic for the sake of completeness. Recall that $L_1^0(W_1)$ is standard Brownian local time at 0 by time 1 which, by Lévy's theorem, has the same distribution as $|W_1(1)|$; cf. [18]. The explicit form of the probability density function of the latter random variable easily yields

$$\mathrm{P}\Big\{\inf_{1\leqslant u\leqslant 2}L_1^0(W_u)\leqslant h\Big\}\geqslant\big(1+o(1)\big)\sqrt{\frac{2}{\pi}}\,h,\quad \text{as } h\to 0^+.$$

There is an obvious gap between this simple estimate and the inequality of Theorem 3.3. To illustrate this, we conclude this remark with an open problem: "*Does the decay rate of the distribution function of* $\inf_{1\leqslant u\leqslant 2}L_1^0(W_u)$ *near* 0 *satisfy a power law?*" In other words, does the following exist as a positive and finite constant:

$$\Lambda=\lim_{h\to 0^+}\frac{1}{\log h}\log\mathrm{P}\Big\{\inf_{1\leqslant u\leqslant 2}L_1^0(W_u)\leqslant h\Big\}?$$

If so, what is the numerical value of Λ?

(3) Can the condition of Theorem 4.3 be reduced to $f\in\mathcal{L}^2(\mathrm{d}x)$, or are there counterexamples for the sole condition of $\mathcal{L}^2(\mathrm{d}x)$? It should be pointed out that if the limit and the supremum are interchanged, then the main result of [5] implies the existence of a ratio ergodic theorem only under the condition that $f\in\mathcal{L}^2(\mathrm{d}x)$. Nevertheless, it should be pointed out that the $\mathcal{L}^1(\{1+|x|\}\,\mathrm{d}x)$ condition is a very natural one; see the Remarks at the end of Section 4.2.

(4) In the context of Eq. (4.16), consider the process $a\mapsto R_t(a)$, which is defined by

$$a\mapsto\sup_{u\in[1,2]}\left|\frac{L_t^a(W_u)}{L_t^0(W_u)}-1\right|.$$

Is there a normalization $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n \to \infty$ such that $a \mapsto \alpha_n R_n(a)$ has a nontrivial limiting law as $n \to \infty$? For some related works, see [1,3,22].

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References

[1] E. Csáki, M. Csörgő, A. Földes, P. Révész, Brownian local time approximated by a Wiener sheet, Ann. Probab. 17 (2) (1989) 516–537.
[2] E. Csáki, A. Földes, How small are the increments of the local time of a Wiener process?, Ann. Probab. 14 (2) (1986) 533–546.
[3] E. Csáki, A. Földes, A note on the stability of the local time of a Wiener process, Stochastic Process. Appl. 25 (2) (1987) 203–213.
[4] E. Csáki, D. Khoshnevisan, Z. Shi, Capacity estimates, boundary crossings and the Ornstein–Uhlenbeck process in Wiener space, Electronic Communications in Probability 4 (1999) 103–109, http://www.math.washington.edu/~ejpecp/EcpVol4/paper13.abs.html.
[5] P.J. Fitzsimmons, The quasi-sure ratio ergodic theorem, Ann. Inst. H. Poincaré Probab. Statist. 34 (3) (1998) 385–405.
[6] P. Hall, C.C. Heyde, Martingale Limit Theory and its Applications, in: Probability Theory and Mathematical Statistics, Academic Press, New York, 1980.
[7] K. Itô, H.P. McKean Jr., Diffusion Processes and Their Sample Paths, second printing, corrected, in: Die Grundlehren der mathematischen Wissenschaften, Band 125, Springer-Verlag, Berlin, 1974.
[8] T. Kamae, A simple proof of the ergodic theorem using nonstandard analysis, Israel J. Math. 42 (4) (1982) 284–290.
[9] Y. Katznelson, B. Weiss, A simple proof of some ergodic theorems, Israel J. Math. 42 (4) (1982) 291–296.
[10] D. Khoshnevisan, The distribution of bubbles of Brownian sheet, Ann. Probab. 23 (2) (1995) 786–805.
[11] D. Khoshnevisan, Rate of convergence in the ratio ergodic theorem for Markov processes, 1995, Unpublished.
[12] D. Khoshnevisan, Lévy classes and self-normalization, Electron. J. Probab. 1 (1) (1996) 1–18, http://www.math.washington.edu/~ejpecp/EjpVol1/paper1.abs.html.
[13] J. Kuelbs, R. Le Page, The law of the iterated logarithm for Brownian motion in a Banach space, Trans. Amer. Math. Soc. 185 (1973) 253–265.
[14] M.T. Lacey, Limit laws for local times of the Brownian sheet, Probab. Theory Related Fields 86 (1) (1990) 63–85.
[15] H.P. McKean Jr., A Hölder condition for Brownian local time, J. Math. Kyoto Univ. 1 (1961/1962) 195–201.
[16] M. Motoo, Proof of the law of iterated logarithm through diffusion equation, Ann. Inst. Statist. Math. 10 (1958) 21–28.
[17] M.D. Penrose, Quasi-everywhere properties of Brownian level sets and multiple points, Stochastic Process. Appl. 36 (1) (1990) 33–43.
[18] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, third ed., Springer-Verlag, Berlin, 1999.
[19] J.B. Walsh, The local time of the Brownian sheet, Astérisque 52–53 (1978) 47–61.
[20] J.B. Walsh, Propagation of singularities in the Brownian sheet, Ann. Probab. 10 (2) (1982) 279–288.
[21] J.B. Walsh, An introduction to stochastic partial differential equations, in: École d'été de probabilités de Saint-Flour, XIV–1984, Springer, Berlin, 1986, pp. 265–439.
[22] M. Yor, Le drap brownien comme limite en loi de temps locaux linéaires, in: Séminaire de Probabilités, XVII, Springer, Berlin, 1983, pp. 89–105.
[23] G.J. Zimmerman, Some sample function properties of the two-parameter Gaussian process, Ann. Math. Statist. 43 (1972) 1235–1246.