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Last exit times for transient semistable processes

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Abstract

Let L_{B_a} be the last exit time from the ball $B_a = \{|x| < a\}$ for a nondegenerate transient α-semistable process $\{X_t\}$ on $\mathbb{R}^d$. The problem to determine the set $\mathfrak{T}$ defined by $\mathfrak{T} = \{0\} \cup \{\eta > 0\colon\ E[{L_{B_a}}^{\eta}] < \infty\}$ is studied. The process $\{X_t\}$ is called first-class or second-class according as it is strictly α-semistable or not. A unique location parameter $\tau \in \mathbb{R}^d$ is introduced in connection to the space–time relation of $\{X_t\}$; $\tau = 0$ if and only if $\{X_t\}$ is first-class; τ is the drift if $0 < \alpha < 1$ and the center if $1 < \alpha \leqslant 2$.

The set $\mathfrak{T}$ is determined in the case $d = 1$ and in the following cases with $d \geqslant 2$: (i) $0 < \alpha < 1$; (ii) $1 \leqslant \alpha \leqslant 2$ and $\tau = 0$; (iii) $1 \leqslant \alpha < 2$, $\tau \neq 0$, and $\sigma(\{-\tau/|\tau|\}) > 0$; (iv) $1 \leqslant \alpha < 2$, $\tau \neq 0$, and $-\tau/|\tau| \notin C_\sigma$. Here σ is the spherical component of the Lévy measure, and C_σ is a set defined by the support of σ.

Weak transience and strong transience correspond to $1 \notin \mathfrak{T}$ and $1 \in \mathfrak{T}$, respectively, and they are completely classified in terms of d, α, τ, and another parameter β.

Applications to the Spitzer type limit theorems involving capacity are given.

Résumé

Soit L_{B_a} le dernier temps de passage dans la boule $B_a = \{|x| < a\}$ pour un processus α-semi-stable transitoire non-dégénéré $\{X_t\}$ à valeur dans $\mathbb{R}^d$. On étudie le problème de déterminer l'ensemble $\mathfrak{T} = \{0\} \cup \{\eta > 0\colon\ E[{L_{B_a}}^{\eta}] < \infty\}$. Le processus $\{X_t\}$ est appelé de la première classe (resp. de la seconde classe), s'il est strictement α-semi-stable (resp. s'il n'est pas strictement α-semi-stable). Un paramètre unique de position $\tau \in \mathbb{R}^d$ est introduit par rapport à la relation de temps-espace du processus $\{X_t\}$. On montre que $\tau = 0$ si et seulement si $\{X_t\}$ est de la première classe, que τ est la dérive quand $0 < \alpha < 1$ et que τ est le centre quand $1 < \alpha \leqslant 2$.

L'ensemble $\mathfrak{T}$ est déterminé dans le cas où $d = 1$. Quant au cas où $d \geqslant 2$, il est déterminé dans les cas suivants : (i) $0 < \alpha < 1$; (ii) $1 \leqslant \alpha \leqslant 2$ et $\tau = 0$; (iii) $1 \leqslant \alpha < 2$, $\tau \neq 0$ et $\sigma(\{-\tau/|\tau|\}) > 0$; (iv) $1 \leqslant \alpha < 2$, $\tau \neq 0$ et $-\tau/|\tau| \notin C_\sigma$, où σ est la partie sphérique de la mesure de Lévy et C_σ est un ensemble défini par le support de σ.

La propriété faiblement transitoire du processus et celle fortement transitoire correspondent respectivement au cas où $1 \notin \mathfrak{T}$ et où $1 \in \mathfrak{T}$. Elles sont classées complètement en termes des paramètres d, α, τ avec un autre paramètre β.

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Des applications aux théorèmes limite de type Spitzer mettant en jeu la capacité sont données.

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Mots-clés : Processus de Lévy ; Processus stable ; Processus semi-stable ; Dernier temps de passage

1. Introduction

After the paper [20] we have been working on classification of transient stable processes on $\mathbb{R}^d$ by the properties of the last exit time L_{B_r} from the ball $B_r = \{|x| < r\}$, and its extension to transient semistable processes. Accomplishment of this objective is still far from us, but here we present in this paper some progress in our knowledge. This classification aims to express a degree of transience of those processes. It originated from the understanding of the difference between weak and strong transience in limit theorems for hitting times of the Spitzer type involving capacity by Spitzer [26], Getoor [4], Port [10–12], Port and Stone [16] in 1960s; the study was continued by Port [13,14] for stable processes around 1990. Le Gall [8] made a refinement of the limit theorem in the case of Brownian motion. Jain and Pruitt [7] also met weak and strong transience in limit theorems of the ranges of random walks and there are many subsequent works. Earlier Takeuchi [27] took up the problem of the last exit times of rotation-invariant stable processes.

In [24] we gave some results on the moments of the last exit time L_{B_r} for a general transient Lévy process $\{X_t\}$ on $\mathbb{R}^d$; among others, an analytic criterion for finiteness of $E({L_{B_r}}^{\eta})$ for a given $\eta > 0$ was given in terms of the function $\psi(z)$, the distinguished logarithm of the characteristic function of the distribution of X_1. This was an extension to a general case of Hawkes's criterion [5] in the symmetric case. However, it turned out that this analytic criterion was often difficult to apply to concrete nonsymmetric Lévy processes. Weak transience and strong transience are respectively equivalent to infiniteness and finiteness of $E(L_{B_r})$, but it is very hard to classify by the analytic criterion weak and strong transience of one-dimensional stable processes. Here in this paper we lay a basis of our study on reduction of the problem to estimation of the density function $p(t,x)$ of the distribution of X_t. It is known that $E(L_{B_r}^{\eta}) < \infty$ for all (equivalently for some) r if and only if $\int_{|x|<r} \mathrm{d}x \int_0^{\infty} t^{\eta} p(t,x)\,\mathrm{d}t < \infty$ for all (equivalently for some) r. In the case of stable or semistable processes, the problem is transformed to estimation of $p(1,x)$ when $|x|$ is large or small, by a space–time relation. Thus our objective to determine whether $E(L_{B_r}^{\eta})$ is finite or not for a general nondegenerate transient α-semistable process for any given $\eta > 0$ is completely achieved in the case of one dimension ($d = 1$). In multi-dimensional case ($d \geqslant 2$), it is achieved except in the case where $1 \leqslant \alpha < 2$, $\tau \neq 0$, $-\tau/|\tau| \in C_{\sigma}$, and $\sigma(\{-\tau/|\tau|\}) = 0$. Here the measure σ is the spherical component of the Lévy measure, the set C_{σ} is the radial projection to the unit sphere of a cone-like set spanned by the support of σ in some sense, and the vector $\tau \in \mathbb{R}^d$ is a location parameter, one of whose properties is that $\tau = 0$ if and only if the process is strictly semistable. Our results will be summarized in Theorems A and B in Section 2. The difficulty of our problem stems from the fact that we do not know in general asymptotic behaviors of the stable or semistable density $p(1,x)$ as $|x|$ is large. They delicately depend on the direction of x in relation to τ and σ. However, Theorems A and B are strong enough to give a complete classification of transient semistable processes on $\mathbb{R}^d$, $d \geqslant 1$, into weakly transient and strongly transient. This result is new even for stable processes if $d = 2$, $\alpha = 1$, and not strictly stable.

In the case of stable processes, one can get better results on tail estimates of $p(1,x)$ for $d \geqslant 2$, based on the explicit form of the radial component of the Lévy measure. These make it possible to determine the set $\mathfrak{T}$ in various situations in the remaining case. They will be given in another paper by one of the authors.

A nontrivial Lévy process $\{X_t\}$ on $\mathbb{R}^d$ is called α-semistable if, for some $a > 0$ with $a \neq 1$ and for some $c \in \mathbb{R}^d$, $\{X_{at}\}$ and $\{a^{1/\alpha} X_t + tc\}$ are identical in law. Here we necessarily have $0 < \alpha \leqslant 2$. This is an extension of α-stable processes, which are deeply studied Lévy processes. By this extension the class is much enlarged and unexpected phenomena occur. For example, if $\{X_t\}$ is a stable process on $\mathbb{R}$, then the distribution of X_t is unimodal and, as t increases, the movement of the mode m_t of X_t changes its direction at most once. But among semistable processes $\{X_t\}$ on $\mathbb{R}$, there is a case where the distribution of X_t is not unimodal for any $t > 0$; also there is a case where the distribution is unimodal for any $t > 0$ but its mode m_t oscillates as t increases; also there is a case where the distribution is unimodal for some t and not unimodal for some t and unimodality and nonunimodality appear alternately as time passes. See Choi [3], Sato [19,22], and Watanabe [30,31] for properties of semistable processes. Historically, semistable distributions were introduced in probability theory by Lévy [9].

In the last section of the paper we discuss the implication of the finiteness of $E(L^{\eta}_{B_r})$ in the Spitzer type limit theorems.

2. Main results

We use the terminology in Sato's book [23]. See also Bertoin [1] for general properties of Lévy processes. Let $\{X_t\colon t \geqslant 0\}$ be a Lévy process on $\mathbb{R}^d$, $d \geqslant 1$, with generating triplet (A, ν, γ). Here A is the Gaussian covariance matrix, ν is the Lévy measure, and γ is the location parameter. That is,

$$E\exp\bigl(\mathrm{i}\langle z, X_t\rangle\bigr) = \exp\bigl(t\psi(z)\bigr), \quad z \in \mathbb{R}^d, \tag{2.1}$$

with

$$\psi(z) = \int_{\mathbb{R}^d} \bigl(\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1 - 1_{\{|x|\leqslant 1\}}(x)\mathrm{i}\langle z, x\rangle\bigr)\nu(\mathrm{d}x) + \mathrm{i}\langle\gamma, z\rangle - \frac{1}{2}\langle Az, z\rangle, \tag{2.2}$$

where ν is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\nu(\mathrm{d}x) < \infty$, $\gamma \in \mathbb{R}^d$, and A is a nonnegative-definite matrix. The process $\{X_t\}$ is of type A if $A = 0$ and $\nu(\mathbb{R}^d) < \infty$; of type B if $A = 0$, $\nu(\mathbb{R}^d) = \infty$, and $\int_{|x|\leqslant 1}|x|\nu(\mathrm{d}x) < \infty$; of type C otherwise. If $A = 0$, $\{X_t\}$ is said to be purely non-Gaussian. If $\int_{|x|\leqslant 1}|x|\nu(\mathrm{d}x) < \infty$, then

$$\psi(z) = \int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1)\nu(\mathrm{d}x) + \mathrm{i}\langle\gamma_0, z\rangle - \frac{1}{2}\langle Az, z\rangle, \tag{2.3}$$

where γ_0 is called the drift of $\{X_t\}$. If $\int_{|x|>1}|x|\nu(\mathrm{d}x) < \infty$ (equivalently, if $E|X_1| < \infty$), then

$$\psi(z) = \int_{\mathbb{R}^d} \bigl(\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z, x\rangle\bigr)\nu(\mathrm{d}x) + \mathrm{i}\langle\gamma_1, z\rangle - \frac{1}{2}\langle Az, z\rangle, \tag{2.4}$$

where γ_1 is called the center of $\{X_t\}$ and $\gamma_1 = EX_1$.

The support S_ρ of a measure ρ on $\mathbb{R}^d$ is the smallest closed set that carries the whole measure of ρ. The distribution of an $\mathbb{R}^d$-valued random variable X is denoted by $\mathcal{L}(X)$. We write $\{X_t\} \stackrel{\mathrm{d}}{=} \{Y_t\}$ for two stochastic processes $\{X_t\}$ and $\{Y_t\}$ if they have an identical system of finite-dimensional joint distributions. For a Lévy process $\{X_t\}$, we denote $\mu = \mathcal{L}(X_1)$ and $\mu^t = \mathcal{L}(X_t)$. The support of μ^t is denoted by $S(X_t)$. A set B in $\mathbb{R}^d$ is called *one-sided* if there is $c \neq 0$ in $\mathbb{R}^d$ such that $B \subset \{x\colon \langle c, x\rangle \geqslant 0\}$. A measure ρ on $\mathbb{R}^d$ is called one-sided if S_ρ is one-sided. A measure ρ on $\mathbb{R}^d$ is called *degenerate* if there are $a \in \mathbb{R}^d$ and a proper linear subspace V of $\mathbb{R}^d$ such that $S_\rho \subset a + V$; otherwise ρ is called nondegenerate. A Lévy process $\{X_t\}$ on $\mathbb{R}^d$ is called degenerate if $\mathcal{L}(X_t)$ is degenerate for every $t > 0$ (equivalently, for some $t > 0$); otherwise $\{X_t\}$ is called nondegenerate. See [23], Proposition 24.17 for conditions for nondegenerateness on the generating triplet. A Lévy process $\{X_t\}$ on $\mathbb{R}^d$ is called a trivial process if there is $c \in \mathbb{R}^d$ such that, for every t, $X_t = tc$ a. s.; otherwise $\{X_t\}$ is said to be nontrivial.

A Lévy process $\{X_t\}$ on $\mathbb{R}^d$, $d \geqslant 1$, is called *stable* if, for every $a > 0$, there are $b > 0$ and $c \in \mathbb{R}^d$ such that

$$\{X_{at}\colon t \geqslant 0\} \stackrel{\mathrm{d}}{=} \{bX_t + tc\colon t \geqslant 0\}. \tag{2.5}$$

A Lévy process $\{X_t\}$ on $\mathbb{R}^d$, $d \geqslant 1$, is called *semistable* if, for some $a > 0$ with $a \neq 1$, there are $b > 0$ and $c \in \mathbb{R}^d$ satisfying (2.5). In this case we can choose $a > 1$ without loss of generality. The following fact is found in [23] Theorems 13.11 and 13.15.

Proposition 2.1. *Let $\{X_t\}$ be a nontrivial semistable process on $\mathbb{R}^d$. Let Γ be the set of $a > 0$ such that there are $b > 0$ and $c \in \mathbb{R}^d$ satisfying (2.5). Then each $a \in \Gamma$ uniquely determines b and c; there is a unique $\alpha \in (0, 2]$ such that $b = a^{1/\alpha}$ for all $a \in \Gamma$. The set Γ is either equal to $(0, \infty)$ or expressed as $\{a_0^n\colon n \in \mathbb{Z}\}$ by a unique $a_0 > 1$.*

The process $\{X_t\}$ in Proposition 2.1 is called a semistable process with index α or α-semistable process. If $\Gamma = (0, \infty)$, $\{X_t\}$ is a stable process with index α or α-stable process. If $a \in \Gamma \cap (1, \infty)$, then a and $a^{1/\alpha}$ are called, respectively, an epoch and a span of $\{X_t\}$.

In this paper, when we say $\{X_t\}$ is semistable (or stable), we implicitly assume that $\{X_t\}$ is a nontrivial Lévy process. Let $\{X_t\}$ be an α-semistable process on $\mathbb{R}^d$. If $\alpha = 2$, then it is Gaussian. If $0 < \alpha < 2$, then it is purely non-Gaussian and

$$a\,\nu(B) = \nu(a^{-1/\alpha} B) \tag{2.6}$$

for all $a \in \Gamma$ and Borel sets B ([23], Theorem 14.3). If $0 < \alpha < 1$, then $\{X_t\}$ is of type B and $E|X_t| = \infty$ for $t > 0$. If $\alpha = 1$, then $\{X_t\}$ is of type C and $E|X_t| = \infty$ for $t > 0$. If $1 < \alpha < 2$, then $\{X_t\}$ is of type C and $E|X_t| < \infty$ for $t > 0$. ([23], Proposition 14.5)

Let $\{X_t\}$ be a semistable process on $\mathbb{R}^d$. If $c = 0$ in (2.5) for every a in the set Γ, we call $\{X_t\}$ a *first-class* semistable process. Otherwise we call $\{X_t\}$ a *second-class* semistable process. (If $c = 0$ in (2.5) for some $a \in \Gamma \setminus \{1\}$, then $\{X_t\}$ is first-class semistable. This is a consequence of Proposition 2.4 below.) Since $\{X_t\}$ is assumed to be nontrivial, it is first-class semistable if and only if it is strictly semistable in the terminology of [23]; it is second-class semistable if and only if it is semistable but not strictly semistable. Similarly we use the words first-class stable and second-class stable.

Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$. Transience and recurrence of $\{X_t\}$ are defined in [23]. Let L_B be the last exit time from an open set B, that is,

$$L_B = \sup\{t \geqslant 0\colon X_t \in B\}.$$

Let $B_r = \{x\colon |x| < r\}$. The process is recurrent if and only if $L_{B_r} = \infty$ a.s. for all $r > 0$; it is transient if and only if $L_{B_r} < \infty$ a.s. for all $r > 0$.

Proposition 2.2. *Let $\{X_t\}$ be a transient Lévy process on $\mathbb{R}^d$. Let $\eta > 0$. Then one of the following is true*:

$$E[L_{B_r}^{\eta}] < \infty \quad \textit{for all } r > 0, \tag{2.7}$$

$$E[L_{B_r}^{\eta}] = \infty \quad \textit{for all } r > 0. \tag{2.8}$$

This is Theorem 2.8 of [24]. Given a transient Lévy process on $\mathbb{R}^d$, denote

$$\mathfrak{T} = \{0\} \cup \bigl\{\eta > 0\colon \text{(2.7) is true}\bigr\}. \tag{2.9}$$

The bigger is this set $\mathfrak{T}$, the stronger is a degree of transience. The process $\{X_t\}$ is called strongly transient if $1 \in \mathfrak{T}$; it is called weakly transient if $1 \notin \mathfrak{T}$.

The purpose of this paper is to investigate the set $\mathfrak{T}$ for nondegenerate transient semistable processes on $\mathbb{R}^d$. Let us recall the following fact ([23], Theorems 37.8, 37.16, 37.18).

Proposition 2.3. *Let $\{X_t\}$ be a nondegenerate α-semistable process on $\mathbb{R}^d$. If $d \geqslant 3$, then it is transient. If $d = 1$ or 2, then it is recurrent if and only if it is first-class α-semistable with $d \leqslant \alpha \leqslant 2$.*

The first of our main results is the following.

Theorem A. *Assume that $\{X_t\}$ is a nondegenerate transient α-semistable process on $\mathbb{R}^d$ with Lévy measure ν.*

(i) *If $0 < \alpha < 1$ and ν is one-sided, then $\mathfrak{T} = [0, \infty)$.*
(ii) *If $0 < \alpha < 1$ and ν is not one-sided, then $\mathfrak{T} = [0, d/\alpha - 1)$.*
(iii) *If $1 \leqslant \alpha \leqslant 2$ and $\{X_t\}$ is first-class semistable, then $\mathfrak{T} = [0, d/\alpha - 1)$.*
(iv) *If $\alpha = 2$ and $\{X_t\}$ is second-class semistable, then $\mathfrak{T} = [0, \infty)$.*

By the assumption of transience of $\{X_t\}$, the case of (iii) is empty if $d = 1$. The remaining case of nondegenerate transient α-semistable processes on $\mathbb{R}^d$, $d \geqslant 1$, not covered by Theorem A is the case of second-class with $1 \leqslant \alpha < 2$.

In order to formulate the next result, we need the following facts.

Proposition 2.4. *Let $\{X_t\}$ be an α-semistable process on $\mathbb{R}^d$.*

(i) *There exists a unique element $\tau \in \mathbb{R}^d$ such that, for every $a \in \Gamma$,*

$$\{X_{at}\} \stackrel{\mathrm{d}}{=} \{a^{1/\alpha} X_t + t(a - a^{1/\alpha})\tau\} \quad (\textit{if } \alpha \neq 1), \tag{2.10}$$

$$\{X_{at}\} \stackrel{\mathrm{d}}{=} \{a X_t + ta(\log a)\tau\} \quad (\textit{if } \alpha = 1). \tag{2.11}$$

(ii) *If $0 < \alpha < 1$, then τ in* (i) *is the drift γ_0. If $1 < \alpha \leqslant 2$, then τ is the center γ_1. If $\alpha = 1$, then*

$$\tau = \frac{1}{\log a} \int_{1<|x|\leqslant a} x \nu(\mathrm{d}x) \quad \textit{for every } a \in \Gamma \cap (1, \infty). \tag{2.12}$$

(iii) *The process $\{X_t\}$ is first-class semistable or second-class semistable according as $\tau = 0$ or $\tau \neq 0$.*
(iv) *If $\alpha \neq 1$, then τ is a unique element in $\mathbb{R}^d$ such that $\{X_t - t\tau\}$ is first-class semistable. If $\alpha = 1$ and $\tau \neq 0$, then there is no $\tau' \in \mathbb{R}^d$ such that $\{X_t - t\tau'\}$ is first-class semistable.*

Proof. (i) For any $a \in \Gamma \setminus \{1\}$ define $\tau = \tau_a$ by the formula (2.10) or (2.11). We claim that this τ_a does not depend on a. First, it follows by induction that

$$\{X_{a^n t}\} \stackrel{\mathrm{d}}{=} \{a^{n/\alpha} X_t + t(a^n - a^{n/\alpha})\tau_a\} \quad (\text{if } \alpha \neq 1), \tag{2.13}$$

$$\{X_{a^n t}\} \stackrel{\mathrm{d}}{=} \{a^n X_t + tna^n(\log a)\tau_a\} \quad (\text{if } \alpha = 1) \tag{2.14}$$

for all $n \in \mathbb{N}$. Hence

$$\{X_t\} \stackrel{\mathrm{d}}{=} \{a^{-n/\alpha} X_{a^n t} - ta^{-n/\alpha}(a^n - a^{n/\alpha})\tau_a\} \quad (\text{if } \alpha \neq 1),$$

$$\{X_t\} \stackrel{\mathrm{d}}{=} \{a^{-n} X_{a^n t} - tn(\log a)\tau_a\} \quad (\text{if } \alpha = 1).$$

It follows that (2.13) and (2.14) hold for all $n \in \mathbb{Z}$. If a and a' in $\Gamma \setminus \{1\}$ satisfy $a' = a^n$ for some $n \in \mathbb{Z}$, then (2.13) or (2.14) shows that $\tau_{a'} = \tau_a$. If $\Gamma = \{a_0^n\colon n \in \mathbb{Z}\}$ with some $a_0 > 1$, this finishes the proof. If $\Gamma = (0, \infty)$, then we fix $a_0 > 1$ and see that $\tau_{a_0^{n/m}} = \tau_{a_0^{1/m}} = \tau_{a_0}$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. Since such points $a_0^{n/m}$ are dense in $(0, \infty)$ and since τ_a is continuous with respect to a in the intervals $(0, 1)$ and $(1, \infty)$, this finishes the proof.

(ii) If $\alpha = 2$, then $\{X_t\}$ is Gaussian and it is easy to see that $\tau = EX_1$. If $\alpha \neq 2$, then ν satisfies, by (2.6),

$$a \int_{\mathbb{R}^d} f(x)\nu(\mathrm{d}x) = \int_{\mathbb{R}^d} f(a^{1/\alpha}x)\nu(\mathrm{d}x) \quad \text{for } a \in \Gamma \tag{2.15}$$

for all nonnegative measurable functions f. Hence it is easy to see that $\tau = \gamma_0$ or $\tau = \gamma_1$ according as $0 < \alpha < 1$ or $1 < \alpha < 2$.

Let $\alpha = 1$ and $a \in \Gamma \cap (1, \infty)$. Then, by (2.1), (2.2), and (2.15), we get

$$E\mathrm{e}^{\mathrm{i}\langle z, X_{at}\rangle} = E\mathrm{e}^{\mathrm{i}\langle z, aX_t\rangle} \exp\left[\mathrm{i}t \int_{1/a<|x|\leqslant 1} \langle z, ax\rangle \nu(\mathrm{d}x)\right]$$

and hence, by (2.11), $(\log a)\tau = \int_{1/a<|x|\leqslant 1} x\nu(\mathrm{d}x)$. This means (2.12), since $\int_{1/a<|x|\leqslant 1} x\nu(\mathrm{d}x) = \int_{1<|x|\leqslant a} x\nu(\mathrm{d}x)$ by (2.15).

(iii) This is a consequence of (i).

(iv) Let $\alpha \neq 1$. We have $\{X_{at} - at\tau\} \stackrel{\mathrm{d}}{=} \{a^{1/\alpha}(X_t - t\tau)\}$ from (2.10). This shows that $\{X_t - t\tau\}$ is first-class α-semistable. Conversely, if $\{X_t - t\tau'\}$ is first-class α-semistable for some $\tau' \in \mathbb{R}^d$, then there is $a > 0$ with $a \neq 1$ such that $\{X_{at} - at\tau'\} \stackrel{\mathrm{d}}{=} \{a^{1/\alpha}(X_t - t\tau')\}$, that is, $\{X_{at}\} \stackrel{\mathrm{d}}{=} \{a^{1/\alpha}X_t + t(a - a^{1/\alpha})\tau'\}$ and hence $a \in \Gamma$ and $\tau' = \tau$. The remaining assertion for $\alpha = 1$ is found in [23] Theorem 14.8. □

Remark 2.5. For any α-semistable process on $\mathbb{R}^d$ with $0 < \alpha < 2$, define $\beta \in \mathbb{R}^d$ as

$$\beta = \int_{1<|x|\leqslant a^{1/\alpha}} x\nu(\mathrm{d}x) \Big/ \int_{1<|x|\leqslant a^{1/\alpha}} |x|\nu(\mathrm{d}x) \tag{2.16}$$

for $a \in \Gamma \cap (1, \infty)$. Then, using (2.15), we can prove that β does not depend on the choice of $a \in \Gamma \cap (1, \infty)$. Obviously, $|\beta| \leqslant 1$. If $\alpha = 1$, then $\beta = c\tau$, where c is a positive constant independent of $a \in \Gamma \cap (1, \infty)$. In the case of an α-stable process on $\mathbb{R}$, this β coincides with the parameter β in [23] Definition 14.16.

Proposition 2.6. *Let $\{X_t\}$ be an α-semistable process on $\mathbb{R}^d$ with $\alpha \neq 2$. Then there are a probability measure σ on $S^{d-1} = \{\xi \in \mathbb{R}^d\colon |\xi| = 1\}$ and measures ν_ξ on $(0, \infty)$ for $\xi \in S^{d-1}$ such that*

$$\nu_\xi(E) \text{ is measurable in } \xi \text{ for each } E \in \mathcal{B}_{(0,\infty)}, \tag{2.17}$$

$$\nu(B) = \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(\mathrm{d}r) \quad \text{for } B \in \mathcal{B}_{\mathbb{R}^d}, \tag{2.18}$$

$$\int_0^\infty (1 \wedge r^2)\nu_\xi(\mathrm{d}r) \text{ is a finite constant independent of } \xi \in S^{d-1}. \tag{2.19}$$

This σ is uniquely determined and ν_ξ is unique for σ-a.e. $\xi \in S^{d-1}$. Further, for σ-a.e. $\xi \in S^{d-1}$,

$$a\,\nu_\xi(E) = \nu_\xi(a^{-1/\alpha}E) \quad \text{for } E \in \mathcal{B}_{(0,\infty)} \text{ and } a \in \Gamma. \tag{2.20}$$

In particular, for σ-a.e. $\xi \in S^{d-1}$, $\nu_\xi((1, a^{1/\alpha}]) > 0$.

Proof. Existence and uniqueness of σ and ν_ξ are proved by the conditional distribution theorem for every infinitely divisible distribution with nonzero Lévy measure. The property (2.20) is a consequence of (2.6). □

When $\{X_t\}$ is an α-semistable process on $\mathbb{R}^d$ with $\alpha \neq 2$, let

$$C_\sigma^0 = \Biggl\{\xi \in S^{d-1}\colon\ \xi = \sum_{j=1}^{d} c_j \xi_j \text{ for some } c_j > 0 \text{ and } \xi_j \in S_\sigma,\ j = 1, \ldots, d,$$
$$\text{such that } \xi_1, \ldots, \xi_d \text{ are linearly independent}\Biggr\}, \tag{2.21}$$

where S_σ is the support of σ. This set was introduced by Hiraba [6]. Let $C_\sigma = \overline{C_\sigma^0}$, the closure of C_σ^0. The set C_σ^0 is nonempty if $\{X_t\}$ is nondegenerate.

Now we formulate the second of our main results.

Theorem B. *Let $\{X_t\}$ be a nondegenerate transient, second-class (that is, $\tau \neq 0$), α-semistable process on $\mathbb{R}^d$ with $1 \leqslant \alpha < 2$.*

(i) *If $\alpha = 1$, then $\mathfrak{T} \supset [0, d-1]$.*
(ii) *If $\alpha = 1$ and $\sigma(\{-\tau/|\tau|\}) > 0$, then $\mathfrak{T} = [0, d-1]$.*
(iii) *If $1 < \alpha < 2$, then $\mathfrak{T} \supset [0, (d-1)/\alpha + \alpha - 1)$.*
(iv) *If $1 < \alpha < 2$ and $\sigma(\{-\tau/|\tau|\}) > 0$, then $\mathfrak{T} = [0, (d-1)/\alpha + \alpha - 1)$.*
(v) *If $-\tau/|\tau| \notin C_\sigma$, then $\mathfrak{T} = [0, \infty)$.*

When $1 < \alpha < 2$ and $\tau \neq 0$, we have $X_t \sim t\tau$, $t \to \infty$, almost surely and we observe that the largeness of the last exit time L_{B_r} is determined by the relationship of the point $-\tau/|\tau|$ with the measure σ as in (iv) and (v) above. This is in similarity to the fact that the largeness of L_{B_r} of a one-dimensional transient Lévy process with finite positive mean is determined by its Lévy measure in the negative half line ([24], Theorem 5.1).

We will study in Section 3 the support of $\mathcal{L}(X_t)$ and the positivity of the density of $\mathcal{L}(X_t)$ for general Lévy processes $\{X_t\}$ and especially for semistable processes, using the works of Tortrat [29] and Sharpe [25]. Then proofs of Theorems A and B will be given in Sections 4 and 5. Now the case where we do not have an exact description of $\mathfrak{T}$ is that $1 \leqslant \alpha < 2$, $-\tau/|\tau| \in C_\sigma$, and $\sigma(\{-\tau/|\tau|\}) = 0$. If $d = 1$, then this case is void, since $S = \{1, -1\}$. But, if $d \geqslant 2$, the set $\mathfrak{T}$ is delicate in this case.

For $d = 1$, Theorems A and B completely determine the set $\mathfrak{T}$ as follows.

Corollary 2.7. *Let $\{X_t\}$ be a transient α-semistable process on $\mathbb{R}$.*

(i) *If $0 < \alpha < 1$ and $|\beta| = 1$, then $\mathfrak{T} = [0, \infty)$.*
(ii) *If $0 < \alpha < 1$ and $|\beta| \neq 1$, then $\mathfrak{T} = [0, 1/\alpha - 1)$.*
(iii) *If $\alpha = 1$ and $|\beta| = 1$, then $\mathfrak{T} = [0, \infty)$.*
(iv) *If $\alpha = 1$ and $0 < |\beta| < 1$, then $\mathfrak{T} = \{0\}$.*
(v) *If $1 < \alpha < 2$, $\tau \neq 0$, $|\beta| = 1$, and $\tau\beta < 0$, then $\mathfrak{T} = [0, \alpha - 1)$.*
(vi) *If $1 < \alpha < 2$, $\tau \neq 0$, $|\beta| = 1$, and $\tau\beta > 0$, then $\mathfrak{T} = [0, \infty)$.*
(vii) *If $1 < \alpha < 2$, $\tau \neq 0$, and $|\beta| \neq 1$, then $\mathfrak{T} = [0, \alpha - 1)$.*
(viii) *If $\alpha = 2$ and $\tau \neq 0$, then $\mathfrak{T} = [0, \infty)$.*

Since $d = 1$, we have the following: ν is one-sided if and only if $|\beta| = 1$; if $|\beta| = 1$ and $\tau \neq 0$, then $\tau\beta < 0$ is equivalent to $\sigma(\{-\tau/|\tau|\}) > 0 = \sigma(\{\tau/|\tau|\})$, and $\tau\beta > 0$ is equivalent to $\sigma(\{-\tau/|\tau|\}) = 0 < \sigma(\{\tau/|\tau|\})$. If $\alpha = 1$, then $\tau = 0$ and $\beta = 0$ are equivalent. The eight cases in the corollary above exhaust all transient semistable processes on $\mathbb{R}$, according to Proposition 2.3.

Proof of Corollary. Assertions (i), (ii), and (viii) are consequences of Theorem A (i), (ii), and (iv), respectively. Assertions (iii)–(vi), and (vii) follow from Theorem B (v), (ii), (iv), (v), and (iv), respectively. □

Although we do not have full knowledge of the set $\mathfrak{T}$, weak and strong transience of nondegenerate transient semistable processes on $\mathbb{R}^d$ is completely determined as follows.

Corollary 2.8. *Let $\{X_t\}$ be a nondegenerate transient α-semistable process on $\mathbb{R}^d$. Then it is strongly transient if and only if one of the following conditions holds*:

(1) $d = 1$ *and* $0 < \alpha < 1/2$;
(2) $d = 1$, $1/2 \leqslant \alpha \leqslant 1$, *and* $|\beta| = 1$;
(3) $d = 1$, $1 < \alpha < 2$, $\tau \neq 0$, $|\beta| = 1$, *and* $\tau\beta > 0$;
(4) $d = 1$, $\alpha = 2$, *and* $\tau \neq 0$;
(5) $2 \leqslant d \leqslant 4$ *and* $0 < \alpha < d/2$;
(6) $2 \leqslant d \leqslant 4$, $d/2 \leqslant \alpha \leqslant 2$, *and* $\tau \neq 0$;
(7) $d \geqslant 5$.

Proof. If $d = 1$, cases (1)–(4) exhaust strongly transient case by Corollary 2.7. If $2 \leqslant d \leqslant 4$, use Theorem A (i)–(iv) and Theorem B (i) and (iii) to see that the process is strongly transient if and only if (5) or (6) holds. If $d \geqslant 5$, then strong transience comes from Theorem A(i), (ii) for $0 < \alpha < 1$, from Theorem A(iii) for first-class with $1 \leqslant \alpha \leqslant 2$, and from Theorem B(i), (iii) for second-class with $1 \leqslant \alpha \leqslant 2$. But strong transience for $d \geqslant 5$ is a consequence of a general result in Theorem 2.17 of [21]. □

As a special case of Corollary 2.8, any nondegenerate second-class 1-semistable process on $\mathbb{R}^2$ is strongly transient. This is a new result even in stable case; Port [14] did not treat this case.

In the proofs of Theorems A and B we will use the following facts.

Proposition 2.9. *Let $\{X_t\}$ be a transient Lévy process on $\mathbb{R}^d$. Let $\eta > 0$. Then $\eta \in \mathfrak{T}$ if and only if*

$$\int_0^\infty t^\eta P[X_t \in B_\varepsilon]\,\mathrm{d}t < \infty \quad \textit{for all } \varepsilon > 0; \tag{2.22}$$

$\eta \notin \mathfrak{T}$ if and only if

$$\int_0^\infty t^\eta P[X_t \in B_\varepsilon]\,\mathrm{d}t = \infty \quad \textit{for all } \varepsilon > 0. \tag{2.23}$$

This is Lemma 2.3 of [24].

Proposition 2.10. *Let $\{X_t\}$ be a nondegenerate α-semistable process on $\mathbb{R}^d$. Then, for any $t > 0$, μ^t has a C^∞ density $p(t,x)$ on $\mathbb{R}^d$ and*

$$p(a^n t, x) = a^{-nd/\alpha} p\big(t, a^{-n/\alpha} x + (1 - a^{(1-1/\alpha)n}) t\tau\big) \quad (\textit{if } \alpha \neq 1), \tag{2.24}$$

$$p(a^n t, x) = a^{-nd} p\big(t, a^{-n} x - n(\log a) t\tau\big) \quad (\textit{if } \alpha = 1) \tag{2.25}$$

for $a \in \Gamma$ and $n \in \mathbb{Z}$.

Proof. As in [23] Proposition 24.20 there is a constant $c > 0$ such that

$$\big|\widehat{\mu^t}(z)\big| \leqslant \mathrm{e}^{-ct|z|^\alpha} \quad \text{for } t > 0,\ z \in \mathbb{R}^d, \tag{2.26}$$

where $\widehat{\mu^t}(z)$ is the characteristic function of μ^t. Hence μ^t has a continuous density $p(t,x)$ expressed as

$$p(t,x) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle x,z\rangle} \widehat{\mu^t}(z)\,\mathrm{d}z \quad \text{for } t>0,\ x\in\mathbb{R}^d. \tag{2.27}$$

Further, $p(t,x)$ is of C^∞ in x. By (2.10) or (2.11) $p(t,x)$ satisfies (2.24) or (2.25) for $a \in \Gamma$ and $n = 1$. But, since $a \in \Gamma$ implies $a^n \in \Gamma$ for $n \in \mathbb{Z}$, (2.24) or (2.25) is true for $a \in \Gamma$ and $n \in \mathbb{Z}$. □

3. Supports of semistable processes

A subset H of $\mathbb{R}^d$ is said to be a closed additive semigroup if H is a closed set such that $H + H \subset H$. If moreover $-H \subset H$, then H is called a closed additive group. Given a Lévy process $\{X_t\}$ on $\mathbb{R}^d$ with generating triplet (A,ν,γ), let $V = A(\mathbb{R}^d)$. If we denote by $\{X'_t\}$ the Lévy process with triplet $(A,0,0)$, that is, the centered Gaussian component of $\{X_t\}$, then $V = S(X'_t)$ for $t > 0$.

Tortrat [29] and Sharpe [25] prove the following remarkable result.

Proposition 3.1. *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure ν. Let M be the linear subspace defined by*

$$M = \left\{ y \in \mathbb{R}^d \colon \int\limits_{|x|\leqslant 1} |\langle y,x\rangle|\,\nu(\mathrm{d}x) < \infty \right\} \tag{3.1}$$

and let Π_M be the orthogonal projection from $\mathbb{R}^d$ onto M. Let $\{X_t^M\}$ be the Lévy process defined by $X_t^M = \Pi_M X_t$. Then the Lévy measure ν_M of $\{X_t^M\}$ satisfies

$$\int\limits_{|x|\leqslant 1} |x|\,\nu_M(\mathrm{d}x) < \infty. \tag{3.2}$$

Denote the drift of $\{X_t^M\}$ by γ_0^M. Then, $S(X_t) - t\gamma_0^M$, which we denote by H, does not depend on $t \in (0,\infty)$. That is,

$$S(X_t) = t\gamma_0^M + H \quad \textit{for } t > 0. \tag{3.3}$$

The set H has an expression

$$H = \overline{\Pi_M^{-1}\,\mathrm{Sgp}(\nu_M) + V}, \tag{3.4}$$

where $\mathrm{Sgp}(\nu_M)$ is the smallest closed additive semigroup containing $\{0\}$ and S_{ν_M}.

It follows from (3.4) that H is a closed additive semigroup containing $\{0\}$. Following [25], we call H the *invariant semigroup* of $\{X_t\}$. If $\{X_t\}$ is of type A or B, then $M = \mathbb{R}^d$, $V = \{0\}$, and $H = \mathrm{Sgp}(\nu)$. If $M = \{0\}$, then $S(X_t) = H = \mathbb{R}^d$ for $t > 0$.

Sharpe [25] writes (3.4) without taking the closure in the right-hand side by an oversight, but $\Pi_M^{-1}\,\mathrm{Sgp}(\nu_M) + V$ may not be closed. The assertion (3.3) is written by him as $S(X_t) = tb + H$ with some $b \in \mathbb{R}^d$, but we can choose $b = \gamma_0^M$ in his proof.

Remark 3.2. Let $\{X_t\}$ be a nondegenerate Lévy process on $\mathbb{R}^d$.

(i) Suppose that $\{X_t\}$ is of type A or B. Then the invariant semigroup H is a closed additive group if and only if ν is not one-sided.

(ii) Suppose that $\{X_t\}$ is of type C and purely non-Gaussian. Then H is a closed additive group if and only if either $M = \{0\}$ or ν_M is not one-sided in M.

Proof follows from the fact proved in [24] that any closed additive semigroup which is not one-sided is an additive group. Note that $\{X_t^M\}$ is nondegenerate in M, since $\{X_t\}$ is nondegenerate.

Sharpe [25] gives another nice result. For stable processes this is shown by Taylor [28] and Port and Vitale [17].

Proposition 3.3. *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ such that, for $t > 0$, $\mathcal{L}(X_t)$ is absolutely continuous and has density $p(t,x)$ measurable as a function of (t,x) and satisfying*

$$p(t+s,x) = \int_{\mathbb{R}^d} p(t,x-y)p(s,y)\,\mathrm{d}y \quad \text{for all } x \in \mathbb{R}^d,\ t > 0,\ s > 0. \tag{3.5}$$

Let $G(X_t) = \{x \in \mathbb{R}^d\colon\ p(t,x) > 0\}$ for $t > 0$. Then, using the notation in Proposition 3.1*,*

$$G(X_t) = \operatorname{int} S(X_t) \quad \text{for } t > 0, \tag{3.6}$$

$$\operatorname{int} S(X_t) = t\gamma_0^M + \operatorname{int} H \quad \text{for } t > 0, \tag{3.7}$$

where int *means "the interior of".*

Remark 3.4. Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ satisfying the conditions in Proposition 3.3.

(i) Suppose that $\{X_t\}$ is of type A or B. Then $G(X_t) = \mathbb{R}^d$ for all $t > 0$ if and only if ν is not one-sided.
(ii) Suppose $\{X_t\}$ is of type C and purely non-Gaussian. Then $G(X_t) = \mathbb{R}^d$ for all $t > 0$ if and only if either $M = \{0\}$ or ν_M is not one-sided in M.

To see (i), note that $\operatorname{int} H \neq \emptyset$ by Proposition 3.3, use Remark 3.2, and conclude that $0 \in \operatorname{int} H$ if ν is not one-sided. The proof of (ii) is similar.

Let us apply these results to semistable processes.

Theorem 3.5. *Let $\{X_t\}$ be a nondegenerate α-semistable process on $\mathbb{R}^d$ with $0 < \alpha \leqslant 2$.*

(i) *Suppose that $0 < \alpha < 1$ and ν is one-sided. Then the invariant semigroup H equals* $\operatorname{Sgp}(\nu)$ *and is one-sided, convex, and closed under multiplication by nonnegative reals, and*

$$S(X_t) = t\gamma_0 + H \quad \text{and} \quad G(X_t) = t\gamma_0 + \operatorname{int} H \quad \text{for } t > 0, \tag{3.8}$$

where $G(X_t) = \{x\colon\ p(t,x) > 0\}$.
(ii) *Suppose that $1 \leqslant \alpha \leqslant 2$, or suppose that $0 < \alpha < 1$ and ν is not one-sided. Then*

$$S(X_t) = G(X_t) = \mathbb{R}^d \quad \text{for } t > 0. \tag{3.9}$$

Proof. The estimate (2.26) and the expression (2.27) show that $p(t,x)$ is continuous in (t,x) and bounded for $(t,x) \in [\varepsilon,\infty) \times \mathbb{R}^d$ for every $\varepsilon > 0$. Hence the conditions in Proposition 3.3 are satisfied.

(i) In this case $\{X_t\}$ is of type B. Thus $H = \operatorname{Sgp}(\nu)$, $S(X_t) = t\gamma_0 + H$, $G(X_t) = t\gamma_0 + \operatorname{int} H$ for $t > 0$. Let $x \in H$ and $c \geqslant 0$. We claim that $cx \in H$. This is obvious if $x = 0$ or $c = 0$. Let $x \neq 0$ and $c > 0$. There is a sequence $x_k \to x$ such that each x_k is the sum of a finite number of elements of S_ν. Thus $x_k = \sum_{j=1}^{n_k} x_k^j$, $x_k^j \in S_\nu$. Let $a \in \Gamma$ and $b = a^{1/\alpha}$, a span. Choose $h = 2$ or 3 such that the ratio $\log h/\log b$ is irrational. Then there are positive integers l_k, m_k increasing to ∞ such that $l_k \log h - m_k \log b \to \log c$ as $k \to \infty$. Since $b^{-m_k} x_k^j \in S_\nu$ by the property (2.6), we have $h^{l_k} b^{-m_k} x_k \in H$ and this tends to cx as $k \to \infty$. Hence $cx \in H$. It follows that H is convex.

(ii) If $0 < \alpha < 1$ and ν is not one-sided, then (3.9) follows from Remark 3.4(i). If $\alpha = 2$, then the assertion is obvious. Suppose $1 \leqslant \alpha < 2$. Use the decomposition of ν in Proposition 2.6. Then,

$$\int\limits_{|x|\leqslant 1} \bigl|\langle y, x\rangle\bigr| \nu(\mathrm{d}x) = \int\limits_{S^{d-1}} \bigl|\langle y, \xi\rangle\bigr| \sigma(\mathrm{d}\xi) \int\limits_{(0,1]} r \nu_\xi(\mathrm{d}r) = \begin{cases} 0 & \text{if } y \perp S_\sigma, \\ \infty & \text{otherwise,} \end{cases}$$

since we get $\int_{(0,1]} r\nu_\xi(\mathrm{d}r) = \infty$ from (2.20) as in [23] Proposition 14.5. Now it follows from nondegeneracy that $M = \{0\}$, and hence $G(X_t) = \mathbb{R}^d$ by Remark 3.4(ii). □

Corollary 3.6. *Let $\{X_t\}$ be a nondegenerate α-semistable process on $\mathbb{R}^d$ with $0 < \alpha \leqslant 2$. Then the following conditions are equivalent*:

$$0 < \alpha < 1 \quad \textit{and} \quad -\gamma_0 \notin \operatorname{int} H, \tag{3.10}$$

$$p(t, 0) = 0 \quad \textit{for all } t > 0. \tag{3.11}$$

$$p(t, 0) = 0 \quad \textit{for some } t > 0. \tag{3.12}$$

Proof. If (3.10) holds, then ν is one-sided and H is closed under multiplication by nonnegative reals, which shows that $-t\gamma_0 \notin \operatorname{int} H$ for all $t > 0$ and (3.11). If (3.12) holds, then $0 < \alpha < 1$, ν is one-sided, and $-t\gamma_0 \notin \operatorname{int} H$ for some $t > 0$, from which follows (3.10). □

For stable processes the dichotomy resulting from the corollary above was found by Taylor [28]; the processes satisfying (3.10)–(3.12) were called by him of type B and the other processes were called of type A, but we do not use his terminology. For semistable processes, the part concerning $S(X_t)$ in Theorem 3.5 was given also by Rajput et al. [18], but they did not study $G(X_t)$.

The following fact will be useful.

Proposition 3.7. *Let $\{X_t\}$ be a nondegenerate α-semistable process on $\mathbb{R}^d$, $0 < \alpha < 2$, with Lévy measure ν. Let $\{Y_t\}$ and $\{Z_t\}$ be independent Lévy processes such that $\{X_t\} \stackrel{\mathrm{d}}{=} \{Y_t + Z_t\}$ and $\{Z_t\}$ is a compound Poisson process with Lévy measure equal to ν restricted to $\{|x| > \theta\}$ for some $\theta > 0$. Then $S(Y_t) = S(X_t)$ and $G(Y_t) = G(X_t)$ for $t > 0$.*

Proof. Denote $\mu_Z = \mathcal{L}(Z_1)$. Then $|\log \hat{\mu}_Z(z)|$ is bounded in z. Hence $\mathcal{L}(Y_t)$ is absolutely continuous for each $t > 0$ and the density $p_Y(t, x)$ is continuous in (t, x) as in the case of $\{X_t\}$. We express the objects related to $\{Y_t\}$ by putting subscript Y.

Let $0 < \alpha < 1$. Then $M = M_Y = \mathbb{R}^d$. Similarly to the proof of Theorem 3.5, $\mathrm{Sgp}(\nu_Y)$ is closed under multiplication by nonnegative reals. Hence, by (2.6), $\mathrm{Sgp}(\nu) = \mathrm{Sgp}(\nu_Y)$. Since $\{X_t\}$ and $\{Y_t\}$ have an identical drift, we have $S(Y_t) = S(X_t)$ and $G(Y_t) = G(X_t)$.

Let $1 \leqslant \alpha < 2$. Then $M = M_Y = \{0\}$ as in the proof of Theorem 3.5. Hence $S(Y_t) = G(Y_t) = \mathbb{R}^d$ like for $\{X_t\}$. □

4. Proof of Theorem A

Let $\{X_t\}$ be a nondegenerate transient α-stable process on $\mathbb{R}^d$, $0 < \alpha \leqslant 2$. Let $a \in \Gamma$. We denote $X_t^0 = X_t - t\tau$. Let $p^0(t, x)$ be the continuous density of $\mathcal{L}(X_t^0)$ for $t > 0$.

Lemma 4.1. *Assume that $0 < \alpha < 1$ and ν is one-sided. Then, for any integer $n \geqslant 1$ and any $\varepsilon > 0$,*

$$\sup_{t \geqslant \varepsilon} \sup_{x \in \mathbb{R}^d} |x|^{-n} p^0(t,x) < \infty. \tag{4.1}$$

Proof. There is $c \neq 0$ in $\mathbb{R}^d$ such that $S(X_t^0) \subset F = \{x\colon \langle c, x\rangle \geqslant 0\}$ for any $t > 0$, by Theorem 3.5(i). Thus all partial derivatives of $p^0(t,\cdot)$ vanishes on $(\operatorname{int} F)^c$. It follows from (2.26) and (2.27) that any partial derivative of $p^0(t,\cdot)$ is bounded on $\mathbb{R}^d$ uniformly in $t \geqslant \varepsilon$. Hence, using Taylor's theorem around 0, we get (4.1). □

Lemma 4.2. *Assume that $0 < \alpha < 1$ and ν is one-sided. Then, for any $c > 0$ and any $\varepsilon > 0$,*

$$\sup_{|x| \leqslant \varepsilon} \int_{a^n}^{a^{n+1}} p(t,x)\,\mathrm{d}t = \mathrm{o}(a^{-cn}), \quad n \to \infty. \tag{4.2}$$

Proof. Use (2.24). Then

$$\int_{a^n}^{a^{n+1}} p(t,x)\,\mathrm{d}t = \int_1^a p^0(a^n u, x - a^n u\tau) a^n\,\mathrm{d}u = \int_1^a p^0(u, a^{-n/\alpha}x - a^{(1-1/\alpha)n}u\tau) a^{(1-d/\alpha)n}\,\mathrm{d}u. \tag{4.3}$$

Choose a positive integer l such that $c + 1 - d/\alpha + (1-1/\alpha)l < 0$. It follows from Lemma 4.1 that there are c_1, c_2 such that

$$p^0(u, a^{-n/\alpha}x - a^{(1-1/\alpha)n}u\tau) \leqslant c_1 |a^{-n/\alpha}x - a^{(1-1/\alpha)n}u\tau|^l \leqslant c_2 a^{(1-1/\alpha)nl}$$

for $|x| \leqslant \varepsilon$ and $1 \leqslant u \leqslant a$. Hence

$$\int_{a^n}^{a^{n+1}} p(t,x)\,\mathrm{d}t \leqslant c_2(a-1)a^{(1-d/\alpha+(1-1/\alpha)l)n} = \mathrm{o}(a^{-cn})$$

as in (4.2) □

Lemma 4.3. *Suppose that $0 < \alpha < 1$ and ν is not one-sided, or suppose that $1 \leqslant \alpha \leqslant 2$ and first-class semistable. Then, for any $\varepsilon > 0$,*

$$\int_{a^n}^{a^{n+1}} p(t,x)\,\mathrm{d}t \sim a^{(1-d/\alpha)n} \int_1^a p^0(u,0)\,\mathrm{d}u \tag{4.4}$$

uniformly in $|x| \leqslant \varepsilon$ as $n \to \infty$.

Proof. We use (4.3). We have $a^{-n/\alpha}x - a^{(1-1/\alpha)n}u\tau \to 0$ uniformly in $|x| \leqslant \varepsilon$ and $1 \leqslant u \leqslant a$ as $n \to \infty$, recalling that $\tau = 0$ in the case of first-class. Thus we get (4.4). Notice that $\int_1^a p^0(u,0)\,\mathrm{d}u > 0$ by virtue of Theorem 3.5. □

Proof of Theorem A. Let

$$I(\eta) = \int_{|x| \leqslant 1} \mathrm{d}x \int_1^\infty t^\eta p(t,x)\,\mathrm{d}t. \tag{4.5}$$

(i) We assume that $0 < \alpha < 1$ and ν is one-sided. Given $\eta > 0$, choose $c > \eta$ and apply Lemma 4.2. Then

$$I(\eta) = \sum_{n=0}^{\infty} \int_{|x|\leqslant 1} \mathrm{d}x \int_{a^n}^{a^{n+1}} t^{\eta} p(t,x)\,\mathrm{d}t \leqslant \mathrm{const} \sum_{n=0}^{\infty} a^{(\eta-c)n} < \infty.$$

It follows from Proposition 2.9 that $\eta \in \mathfrak{T}$.

(ii) and (iii) We assume that $0 < \alpha < 1$ and ν is not one-sided, or that $1 \leqslant \alpha \leqslant 2$ and $\{X_t\}$ is first-class. By Lemma 4.3, $I(\eta) < \infty$ if and only if $\sum_{n=1}^{\infty} a^{(\eta-d/\alpha+1)n} < \infty$. Hence $\mathfrak{T} = [0, d/\alpha - 1)$.

(iv) Assumption is that $\alpha = 2$ and $\{X_t\}$ is of second-class. Thus $\{X_t\}$ is Gaussian with nonzero center. There is a constant $c > 0$ such that

$$\sup_{|x|\leqslant 1} p(t,x) = \mathrm{o}(\mathrm{e}^{-ct}), \quad t \to \infty.$$

Hence $I(\eta) < \infty$ for all $\eta > 0$. □

5. Proof of Theorem B

Let $\{X_t\}$ be a nondegenerate α-semistable process on $\mathbb{R}^d$ with $0 < \alpha < 2$. Let $a \in \Gamma$. Our basic technique is to decompose $\{X_t\}$ into the sum of independent Lévy processes $\{Y_t\}$ and $\{Z_t\}$ as in Proposition 3.7. We take $\theta = 1$ in that proposition; thus their Lévy measures ν_Y and ν_Z are the restrictions of ν to $\{|x| \leqslant 1\}$ and $\{|x| > 1\}$, respectively, and $\{Z_t\}$ is a compound Poisson process. Let $\mu_Y = \mathcal{L}(Y_1)$ and $\mu_Z = \mathcal{L}(Z_1)$. Since $|\log \hat{\mu}_Z(z)|$ is bounded, we get $|\hat{\mu}_Y^t(z)| \leqslant \mathrm{e}^{-ct|z|^{\alpha}+c_1 t}$ with some $c > 0$ and $c_1 > 0$ from (2.26). Thus $\mathcal{L}(Y_t)$, $t > 0$, has a density $p_Y(t,x)$, which is continuous in (t,x), of class C^{∞} in x, and bounded for $(t,x) \in [\varepsilon_1, \varepsilon_2] \times \mathbb{R}^d$ for every $0 < \varepsilon_1 < \varepsilon_2 < \infty$. We write $p_Y(t,x) = q(t,x)$ and $\mathcal{L}(Z_t) = \mu_Z^t = \lambda^t$.

Lemma 5.1. *There are positive constants c_1, c_2 such that*

$$\sup_{1\leqslant t\leqslant a} q(t,x) \leqslant c_1 \exp\bigl(-c_2|x|\log|x|\bigr) \quad \textit{for } |x| > 0. \tag{5.1}$$

Proof. Let $c_3 = \sup_{1\leqslant t\leqslant a} \sup_{x\in\mathbb{R}^d} q(t/2, x)$. Theorem 26.1 of [23] tells us that

$$\int_{|y|>r} q(a/2, y)\,\mathrm{d}y \leqslant c_4 \exp(-c_5 r \log r) \quad \text{for } r > 0$$

with some positive constants c_4, c_5. Let $c_6 = P(\sup_{1\leqslant t\leqslant a} |Y_{a/2} - Y_{t/2}| \leqslant N)$, choosing N so large that $c_6 > 0$. Then

$$P\bigl(|Y_{a/2}| > r - N\bigr) \geqslant c_6 P\bigl(|Y_{t/2}| > r\bigr)$$

for $r > N$ and $1 \leqslant t \leqslant a$. Now,

$$\begin{aligned} q(t,x) &= \int_{|x-y|\leqslant|x|/2} q(t/2, x-y) q(t/2, y)\,\mathrm{d}y + \int_{|x-y|>|x|/2} q(t/2, x-y) q(t/2, y)\,\mathrm{d}y \\ &\leqslant 2c_3 \int_{|y|\geqslant|x|/2} q(t/2, y)\,\mathrm{d}y \leqslant 2c_3 c_4 c_6^{-1} \exp\bigl(-c_5\bigl(|x|/2 - N\bigr)\log\bigl(|x|/2 - N\bigr)\bigr) \\ &\leqslant c_7 \exp\bigl(-c_8|x|\log|x|\bigr) \end{aligned}$$

for large $|x|$ with some positive constants c_7, c_8. This gives (5.1). □

Lemma 5.2. *There are positive constants c_1, c_2 such that, for every positive integer k and every $r > 0$,*

$$\nu_Z^{k*}\big(\{|x| > r\}\big) \leqslant c_1 c_2^k k^{1+\alpha}(1+r)^{-\alpha}. \tag{5.2}$$

Proof. Let $c_2 = \nu(|x| > 1)$ and let W_j, $j = 1, 2, \ldots$, be i. i. d. sequence of random variables on R^d each with distribution $c_2^{-1}\nu_Z$. By (2.6) we have

$$P\big(|W_1| > a^{n/\alpha}\big) = c_2^{-1}\nu\big(|x| > a^{n/\alpha}\big) = c_2^{-1}a^{-n}\nu\big(|x| > 1\big) = a^{-n}$$

for all integer $n \geqslant 0$. Thus there is c_1 such that

$$P\big(|W_1| > s\big) \leqslant c_1(1+s)^{-\alpha} \quad \text{for } s > 0.$$

Hence

$$\begin{aligned}\nu_Z^{k*}\big(|x| > r\big) = c_2^k P\left(\left|\sum_{j=1}^k W_j\right| > r\right) &\leqslant c_2^k P\big(|W_j| > r/k \text{ for some } j \leqslant k\big) \leqslant c_2^k k P\big(|W_1| > r/k\big)\\ &\leqslant c_2^k c_1 k(1+r/k)^{-\alpha} \leqslant c_1 c_2^k k^{1+\alpha}(1+r)^{-\alpha},\end{aligned}$$

as asserted. □

We need a lemma to estimate the integral of $\rho(B_1 + y + t\xi)$ with respect to t for a measure ρ. As before let B_r be the open ball with center 0 and radius r.

Lemma 5.3. *Let ρ be a measure on $\mathbb{R}^d$ and let $0 \leqslant b_1 < b_2 < \infty$, $\xi \in S^{d-1}$, and $y \in \mathbb{R}^d$. Then*

$$\begin{aligned}\big((b_2 - b_1) \wedge 4^{-1}\big)\rho\bigg(\bigcup_{b_1 \leqslant t \leqslant b_2}(B_{1/2} + y + t\xi)\bigg) &\leqslant \int_{b_1}^{b_2} \rho(B_1 + y + t\xi)\,\mathrm{d}t\\ &\leqslant 4\rho\bigg(\bigcup_{b_1 \leqslant t \leqslant b_2}(B_1 + y + t\xi)\bigg).\end{aligned} \tag{5.3}$$

Proof. The second inequality is obvious if $b_2 - b_1 \leqslant 1$. So, suppose that $b_2 - b_1 > 1$. Since

$$\int_b^{b+1} \rho(B_1 + y + t\xi)\,\mathrm{d}t \leqslant \rho\bigg(\bigcup_{b \leqslant t \leqslant b+1}(B_1 + y + t\xi)\bigg),$$

we have

$$\begin{aligned}\int_{b_1}^{b_2} \rho(B_1 + y + t\xi)\,\mathrm{d}t &\leqslant \sum_{n=0}^{[b_2-b_1]-1} \rho\bigg(\bigcup_{b_1+n \leqslant t \leqslant b_1+n+1}(B_1 + y + t\xi)\bigg) + \rho\bigg(\bigcup_{b_2-1 \leqslant t \leqslant b_2}(B_1 + y + t\xi)\bigg)\\ &\leqslant 4\rho\bigg(\bigcup_{b_1 \leqslant t \leqslant b_2}(B_1 + y + t\xi)\bigg),\end{aligned}$$

where $[b_2 - b_1]$ is the integer part of $b_2 - b_1$.

In order to see the first inequality in (5.3), note that

$$\int_b^{b+1/2} \rho(B_1 + y + t\xi)\,\mathrm{d}t \geqslant \frac{1}{2}\rho\bigg(\bigcup_{b \leqslant t \leqslant b+1/2}(B_{1/2} + y + t\xi)\bigg).$$

If $b_2 - b_1 \geqslant 1/2$, then

$$\int_{b_1}^{b_2} \rho(B_1 + y + t\xi)\,\mathrm{d}t \geqslant \frac{1}{4} \sum_{n=0}^{[2(b_2-b_1)]-1} \rho\Bigg(\bigcup_{b_1+n/2\leqslant t\leqslant b_1+(n+1)/2} (B_{1/2} + y + t\xi)\Bigg)$$
$$+ \frac{1}{4}\rho\Bigg(\bigcup_{b_2-1/2\leqslant t\leqslant b_2} (B_{1/2} + y + t\xi)\Bigg)$$
$$\geqslant \frac{1}{4}\rho\Bigg(\bigcup_{b_1\leqslant t\leqslant b_2} (B_{1/2} + y + t\xi)\Bigg).$$

If $b_2 - b_1 \leqslant 1/2$, then

$$\int_{b_1}^{b_2} \rho(B_1 + y + t\xi)\,\mathrm{d}t \geqslant (b_2 - b_1)\rho\Bigg(\bigcup_{b_1\leqslant t\leqslant b_2} (B_{1/2} + y + t\xi)\Bigg).$$

This completes the proof. □

Lemma 5.4. *Suppose that $1 \leqslant \alpha < 2$ and $\{X_t\}$ is of second-class. Let $\varepsilon > 0$. Let $x \in \mathbb{R}^d$ be such that $|x| \leqslant \varepsilon$ and let*

$$u_{t,x,n} = \begin{cases} a^{-n/\alpha}x + (1 - a^{(1-1/\alpha)n})t\tau & (1 < \alpha < 2), \\ a^{-n}x + n(\log a)t\tau & (\alpha = 1). \end{cases} \tag{5.4}$$

Let $y_{x,n} \in \mathbb{R}^d$ satisfy $|y_{x,n}| \leqslant \inf_{1\leqslant t\leqslant a} |u_{t,x,n}|/2$. Then, there is a constant $c_1 > 0$ independent of the choice of x and $y_{x,n}$ such that

$$\int_1^a \lambda^t(B_1 + y_{x,n} + u_{t,x,n})\,\mathrm{d}t \leqslant \begin{cases} c_1 a^{-(1-1/\alpha)(1+\alpha)n} & (1 < \alpha < 2), \\ c_1 n^{-2} & (\alpha = 1) \end{cases} \tag{5.5}$$

for all sufficiently large integer n. If, moreover, $\sigma(\{-\tau/|\tau|\}) > 0$, then there is a constant $c_2 > 0$ independent of the choice of x such that

$$\int_1^a \lambda^t(B_1 + u_{t,x,n})\,\mathrm{d}t \geqslant \begin{cases} c_2 a^{-(1-1/\alpha)(1+\alpha)n} & (1 < \alpha < 2), \\ c_2 n^{-2} & (\alpha = 1) \end{cases} \tag{5.6}$$

for all sufficiently large integer n.

Proof. We give the proof only in the case $1 < \alpha < 2$, as the discussion in the case $\alpha = 1$ is quite similar. Let $c_3 = \nu(|x| > 1)$. There is $c_4 > 0$ such that

$$|y_{x,n} + u_{t,x,n}| \geqslant |u_{t,x,n}|/2 \geqslant c_4 a^{(1-1/\alpha)n}$$

for all large n uniformly in $|x| \leqslant \varepsilon$ and $t \in [1, a]$. Hence, for large n,

$$\int_1^a \lambda^t(B_1 + y_{x,n} + u_{t,x,n})\,\mathrm{d}t = \int_1^a \mathrm{d}t \sum_{k=1}^{\infty} \mathrm{e}^{-c_3 t} t^k (k!)^{-1} \nu_Z^{k*}(B_1 + y_{x,n} + u_{t,x,n})$$
$$\leqslant \mathrm{e}^{-c_3} \sum_{k=1}^{\infty} a^k (k!)^{-1} \int_1^a \nu_Z^{k*}(B_1 + y_{x,n} + u_{t,x,n})\,\mathrm{d}t.$$

By Lemmas 5.2 and 5.3,

$$\int_1^a \nu_Z^{k*}(B_1+y_{x,n}+u_{t,x,n})\,\mathrm{d}t \leqslant 4\nu_Z^{k*}\Big(\bigcup_{1\leqslant t\leqslant a}(B_1+y_{x,n}+u_{t,x,n})\Big)\big(a^{(1-1/\alpha)n}-1\big)^{-1}|\tau|^{-1}$$
$$\leqslant c_5 k^{1+\alpha} c_3^k a^{-(1-1/\alpha)n} a^{-(1-1/\alpha)n\alpha}$$

for large n with some c_5. Thus we get (5.5).

To see (5.6), let n be sufficiently large. By Lemma 5.3 and (2.18),

$$\int_1^a \lambda^t(B_1+u_{t,x,n})\,\mathrm{d}t \geqslant \int_1^a \mathrm{e}^{-c_3 t} t\nu_Z(B_1+u_{t,x,n})\,\mathrm{d}t \geqslant c_6\nu_Z\Big(\bigcup_{1\leqslant t\leqslant a}(B_{1/2}+u_{t,x,n})\Big)a^{-(1-1/\alpha)n}$$
$$\geqslant c_6\sigma\big(\{\xi\}\big)\int_{(a^{(1-1/\alpha)n}-1)|\tau|}^{(a^{(1-1/\alpha)n}-1)|\tau|a} \nu_\xi(\mathrm{d}r)a^{-(1-1/\alpha)n}$$

with some $c_6>0$, where $\xi=-\tau/|\tau|$. Let $a_n=(a^{(1-1/\alpha)n}-1)|\tau|$. Since $a>a^{1/\alpha}$, we have

$$\nu_\xi\big((a^{(1-1/\alpha)n}-1)|\tau|(1,a]\big)\geqslant \nu_\xi\big(a_n(1,a^{1/\alpha}]\big).$$

Choose $k=k(n)$ such that $a^{(k-1)/\alpha}<a_n\leqslant a^{k/\alpha}$. Using (2.20), we get

$$\begin{aligned}\nu_\xi\big(a_n(1,a^{1/\alpha}]\big) &= \nu_\xi\big((a_n,a^{k/\alpha}]\big)+\nu_\xi\big((a^{k/\alpha},a_n a^{1/\alpha}]\big)\\ &= a\nu_\xi\big((a_n a^{1/\alpha},a^{(k+1)/\alpha}]\big)+\nu_\xi\big((a^{k/\alpha},a_n a^{1/\alpha}]\big)\geqslant \nu_\xi\big((a^{k/\alpha},a^{(k+1)/\alpha}]\big)\\ &= a^{-k}\nu_\xi\big((1,a^{1/\alpha}]\big)\geqslant a_n^{-\alpha}a^{-1}\nu_\xi\big((1,a^{1/\alpha}]\big)=c_7 a^{-(1-1/\alpha)\alpha n}\end{aligned}$$

with some $c_7>0$. This completes the proof. □

Proof of Theorem B. Now we assume that $\{X_t\}$ is a nondegenerate transient second-class α-semistable process on $\mathbb{R}^d$ with $1\leqslant\alpha<2$. We begin with the case $1<\alpha<2$.

(iii) Let $|x|\leqslant\varepsilon$. We claim that there is a constant $c_1>0$ independent of x such that

$$\int_{a^n}^{a^{n+1}} p(t,x)\,\mathrm{d}t \leqslant c_1 a^{-(\alpha-1+(d-1)/\alpha)n} \tag{5.7}$$

for all large n. Indeed, from (2.24),

$$\int_{a^n}^{a^{n+1}} p(t,x)\,\mathrm{d}t = a^{-(d/\alpha-1)n}\int_1^a p(t,u_{t,x,n})\,\mathrm{d}t, \tag{5.8}$$

where $u_{t,x,n}$ is of (5.4). Let $b_{x,n}=\inf_{1\leqslant t\leqslant a}|u_{t,x,n}|/2$ and consider

$$p(t,u_{t,x,n})=\int_{|y|>b_{x,n}} q(t,-y)\lambda^t(u_{t,x,n}+\mathrm{d}y)+\int_{|y|\leqslant b_{x,n}} q(t,-y)\lambda^t(u_{t,x,n}+\mathrm{d}y).$$

Denote by J_1 and J_2 the first and second terms in the right-hand side. As $n\to\infty$, $b_{x,n}\to\infty$ uniformly in x and, from Lemma 5.1, there is $c_2>0$ such that

$$J_1=\mathrm{o}\big(\exp(-c_2 b_{x,n}\log b_{x,n})\big). \tag{5.9}$$

Choose $y_{x,n,j}$ for $j = 1, \ldots, N = N(x,n)$ such that

$$|y_{x,n,j}| \leqslant b_{x,n}, \quad \bigcup_{j=1}^{N} (B_1 + y_{x,n,j}) \supset \{y\colon |y| \leqslant b_{x,n}\},$$

$$\sum_{j=1}^{N} Q_{x,n,j} \leqslant c_3 \quad \text{where } Q_{x,n,j} = \sup_{y \in B_1 + y_{x,n,j},\, t \in [1,a]} q(t,-y)$$

with c_3 independent of x and n. This is possible because of Lemma 5.1. Then,

$$J_2 \leqslant \sum_{j=1}^{N} Q_{x,n,j} \lambda^t (B_1 + y_{x,n,j} + u_{t,x,n}).$$

It follows from Lemma 5.4 that

$$\int_1^a J_2 \, dt \leqslant c_4 a^{-(1-1/\alpha)(1+\alpha)n} \tag{5.10}$$

with a constant c_4. Thus we get (5.7) from (5.9) and (5.10). Define $I(\eta)$ by (4.5). It follows that $I(\eta) < \infty$ if $\eta < \alpha + (d-1)/\alpha - 1$.

(iv) Assumption is that $1 < \alpha < 2$ and $\sigma(\{-\tau/|\tau|\}) > 0$. We claim that there is a constant $c_5 > 0$ independent of x such that

$$\int_{a^n}^{a^{n+1}} p(t,x) \, dt \geqslant c_5 a^{-(\alpha-1+(d-1)/\alpha)n} \tag{5.11}$$

for all large n. We have

$$\int_1^a p(t, u_{t,x,n}) \, dt \geqslant \int_1^a dt \int_{B_1} q(t,-y) \lambda^t (u_{t,x,n} + dy) \geqslant c_6 \int_1^a \lambda^t (B_1 + u_{t,x,n}) \, dt,$$

where $c_6 = \inf_{y \in B_1, 1 \leqslant t \leqslant a} q(t,-y) > 0$ by Proposition 3.7 and by the continuity of $q(t,x)$ in (t,x). Thus we can use (5.6) to get (5.11). We see that $I(\eta) = \infty$ for $\eta \geqslant \alpha + (d-1)/\alpha - 1$.

(i) Assume $\alpha = 1$. Then we can find $c_7 > 0$ independent of $x \in \{|x| \leqslant \varepsilon\}$ such that

$$\int_{a^n}^{a^{n+1}} p(t,x) \, dt \leqslant c_7 n^{-2} a^{-(d-1)n} \tag{5.12}$$

for all large n. Proof is similar to that of (5.7), using (5.5) of Lemma 5.4. Then, $I(\eta) < \infty$ for $\eta \leqslant d-1$, as in the proof of (iii).

(ii) Assumption is that $\alpha = 1$ and $\sigma(\{-\tau/|\tau|\}) > 0$. Now there exists $c_8 > 0$ independent of $x \in \{|x| \leqslant \varepsilon\}$ such that

$$\int_{a^n}^{a^{n+1}} p(t,x) \, dt \geqslant c_8 n^{-2} a^{-(d-1)n} \tag{5.13}$$

for all large n. Proof of this and the remaining discussion is similar, using (5.6).

(v) Assumption is that $-\tau/|\tau| \notin C_\sigma$. Let Π be the radial projection from $\mathbb{R}^d \setminus \{0\}$ onto S^{d-1}, that is, $\Pi x = x/|x|$. For any integer $k \geqslant 1$, we claim that

$$\text{if } x_j \in S_\nu \text{ and } a_j \geqslant 0 \text{ for } j = 1, \ldots, k, \text{ then } \sum_{j=1}^{k} a_j x_j \in \Pi^{-1} C_\sigma \cup \{0\}. \tag{5.14}$$

In the case where $x_1, \ldots, x_k$ are linearly independent, the assertion (5.14) is evident from the definition of C_σ. In the case where $x_1, \ldots, x_k$ are linearly dependent, (5.14) is shown by induction in k since, if $\sum_{j=1}^{k} b_j a_j x_j = 0$ with $b_k = 1 \geqslant b_j$ for $j = 1, \ldots, k-1$, then $\sum_{j=1}^{k} a_j x_j = \sum_{j=1}^{k-1} (1 - b_j) a_j x_j$, in which the coefficients are nonnegative.

Next we claim that there is a constant $c_9 > 0$ such that

$$\int_{a^n}^{a^{n+1}} p(t, x)\, \mathrm{d}t = \begin{cases} \mathrm{o}\big(\exp(-c_9 a^{(1-1/\alpha)n} n)\big) & (1 < \alpha < 2), \\ \mathrm{o}\big(\exp(-c_9 n \log n)\big) & (\alpha = 1), \end{cases} \tag{5.15}$$

uniformly in $x \in \{|x| \leqslant \varepsilon\}$ as $n \to \infty$. Let $1 < \alpha < 2$ (the case $\alpha = 1$ is similarly handled). Define $u_{t,x,n}$ by (5.4). There are $\delta > 0$ and n_0 such that

$$\big(u_{t,x,n} + \big\{y\colon |y| \leqslant \delta |u_{t,x,n}|\big\}\big) \cap \Pi^{-1} C_\sigma = \emptyset \quad \text{if } n \geqslant n_0, |x| \leqslant \varepsilon, t \in [1, a]. \tag{5.16}$$

Indeed, writing $a_n = a^{(1-1/\alpha)n} - 1$ and $u = u_{t,x,n} = a^{-n/\alpha} x - a_n t \tau$, consider $z = u + y$ with $|y| \leqslant \delta |u|$. We have $a_n \to \infty$ and

$$|u| \sim a_n t |\tau| \quad \text{uniformly in } |x| \leqslant \varepsilon \text{ and } t \in [1, a]. \tag{5.17}$$

Then

$$|\Pi z + \Pi \tau| \leqslant \frac{|y + a^{-n/\alpha} x|}{|y + u|} + \left| \frac{-a_n t \tau}{|y + u|} + \frac{\tau}{|\tau|} \right|.$$

Let I_1 and I_2 be the first and second terms in the right-hand side. Then $I_1 \leqslant (1 + \delta|u|)/((1 - \delta)|u|) \to \delta/(1 - \delta)$ and $I_2 = |a_n t \tau| |1/|y + u| - 1/(a_n t |\tau|)| \leqslant |y + u + a_n t \tau|/|y + u| = I_1$. Thus (5.16) follows from $-\Pi \tau \notin C_\sigma$ for small δ and large n_0. Now (5.14) and (5.16) yield

$$\nu_Z^{k*}\big(u_{t,x,n} + \big\{y\colon |y| \leqslant \delta |u_{t,x,n}|\big\}\big) = 0 \quad \text{for } k \geqslant 1.$$

Hence,

$$p(t, u_{t,x,n}) = \int_{|y| > \delta |u_{t,x,n}|} q(t, -y) \lambda^t (u_{t,x,n} + \mathrm{d}y) \leqslant c_{10} \exp\big(-c_{11} \delta |u_{t,x,n}| \log\big(\delta |u_{t,x,n}|\big)\big)$$

with some $c_{10} > 0$ and $c_{11} > 0$ by Lemma 5.1. This combined with (5.8) and (5.17) shows (5.15). Now we get $I(\eta) < \infty$ for all $\eta > 0$. $\square$

6. Applications to Spitzer type limit theorems involving capacity

We study implications of our results on the set $\mathfrak{T}$ in the Spitzer type limit theorems mentioned in the first paragraph of Section 1. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be the Hunt process in the sense of Blumenthal and Getoor [2] induced by a Lévy process $\{X_t\}$ on $\mathbb{R}^d$. That is, $P^x(X_t \in B) = P(x + X_t \in B)$ for any $x \in \mathbb{R}^d$, $t \geqslant 0$, and Borel set B. Let T_B be the hitting time of a Borel set B defined by $T_B = \inf\{t > 0\colon X_t \in B\}$, where we understand $T_B = \infty$ if $X_t \notin B$ for all $t > 0$. Let

$$E_B(t) = \int_{\mathbb{R}^d} P^x(T_B \leqslant t)\, \mathrm{d}x. \tag{6.1}$$

Asymptotic expansion of $E_B(t)$ as $t \to \infty$ for a bounded Borel set B is a subject of research in many papers since Spitzer [26] in 1964. The first order term is connected to the capacity $C(B)$ of B. Assume that the original Lévy process $\{X_t\}$ on $\mathbb{R}^d$ is transient and nondegenerate. Denote the dual process of X by $\widetilde{X} = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \widetilde{P}^x)$, which is the Hunt process induced by the Lévy process $\{-X_t\}$. The equilibrium measures m_B and $\widetilde{m}_B$ of B for the processes X and $\widetilde{X}$, respectively, have a common total mass, which is the capacity $C(B)$ of Port and Stone [16] (see also [23], Chapter 8). We have $0 \leqslant C(B) < \infty$ for any bounded Borel set B. Fix a nonnegative continuous function $f(x)$ with compact support such that $f(0) > 0$ and $\int f(x)\,\mathrm{d}x = 1$. Let

$$r(t) = \int_t^\infty \mathrm{d}s \int_{\mathbb{R}^d} f(x)\big(E^x f(X_s)\big)\,\mathrm{d}x, \tag{6.2}$$

where E^x is the expectation under P^x. We have $r(t) < \infty$ for $t \geqslant 0$ because of the transience. For any $\eta > 0$, it follows from Proposition 2.9 that $\int_0^\infty t^{\eta-1} r(t)\,\mathrm{d}t < \infty$ if and only if $\eta \in \mathfrak{T}$. *Throughout this section, let B be a bounded Borel set in $\mathbb{R}^d$. In the case where X_t has a purely singular distribution for every $t > 0$, we make an additional assumption that $P^x(T_{\overline{B}} = T_{\mathrm{int}\,B}) = 1$ for almost every x.* It follows from the boundedness of B that $E_B(t) < \infty$ (use (3.18) of [16] and the duality formula). Let

$$\varphi_B(x) = P^x(T_B < \infty), \qquad \tilde{\varphi}_B(x) = \widetilde{P}^x(T_B < \infty). \tag{6.3}$$

If $1 \in \mathfrak{T}$, then $\int_{\mathbb{R}^d} \tilde{\varphi}_B(x)\varphi_B(x)\,\mathrm{d}x < \infty$ (see Theorem 14.2 of [16]).

The following two propositions are among the results obtained by a series of works [26,4,10–12,16].

Proposition 6.1. *Let*

$$\Delta_B^{(1)}(t) = E_B(t) - tC(B). \tag{6.4}$$

If $1 \notin \mathfrak{T}$, then

$$\Delta_B^{(1)}(t) = \big(C(B)\big)^2 \int_0^t r(s)\,\mathrm{d}s + \mathrm{o}\left(\int_0^t r(s)\,\mathrm{d}s\right), \quad t \to \infty. \tag{6.5}$$

Proof is given in Theorem 14.2 of [16].

Proposition 6.2. *Assume $1 \in \mathfrak{T}$ and let*

$$\Delta_B^{(2)}(t) = E_B(t) - tC(B) - \int_{\mathbb{R}^d} \tilde{\varphi}_B(x)\varphi_B(x)\,\mathrm{d}x. \tag{6.6}$$

Then

$$\Delta_B^{(2)}(t) = -\int_{\mathbb{R}^d} \tilde{\varphi}_B(x) P^x(t < T_B < \infty)\,\mathrm{d}x = \mathrm{O}\left(\int_t^\infty r(s)\,\mathrm{d}s\right), \quad t \to \infty. \tag{6.7}$$

If, in addition, we assume

$$\sup_{t>0} \frac{r(t/2)}{r(t)} < \infty, \tag{6.8}$$

then

$$\Delta_B^{(2)}(t) = -\big(C(B)\big)^2 \int_t^\infty r(s)\,\mathrm{d}s + \mathrm{o}\left(\int_t^\infty r(s)\,\mathrm{d}s\right), \quad t \to \infty. \tag{6.9}$$

Proof. The assertion (6.7) is obtained from (3.19) and (14.13) of [16]. The property (6.8) is the so-called dominated variation and it follows from (6.8) that $r(t) \geqslant b_2 t^{-b_1}$ for large t with some positive constants b_1, b_2. Therefore, if (6.8) holds, then

$$\limsup_{t\to\infty} \big(\mu^t(K)\big)^{1/t} = 1 \quad \text{for some compact set } K. \tag{6.10}$$

Indeed, if X does not satisfy (6.10), then there is a constant $c > 0$ such that $r(t) = \mathrm{o}(\mathrm{e}^{-ct})$ as $t \to \infty$. Hence the second half of the proposition is a consequence of Lemma 3.3 of [15]. □

Combining the limit theorems above and our estimates in Sections 4 and 5, we get the following results.

Proposition 6.3. *Let $\{X_t\}$ be a nondegenerate transient Lévy process on $\mathbb{R}^d$.*
(i) *Suppose that $1 \notin \mathfrak{T}$ and $b \in \mathfrak{T}$ for some $0 < b < 1$. Then*

$$\Delta_B^{(1)}(t) = \mathrm{o}(t^{1-b}), \quad t \to \infty. \tag{6.11}$$

(ii) *Suppose that $1 \in \mathfrak{T}$ and $b \in \mathfrak{T}$ for some $b \geqslant 1$. Then*

$$\Delta_B^{(2)}(t) = \begin{cases} \mathrm{o}(t^{1-b}) & \text{if } b > 1, \\ \mathrm{o}(1) & \text{if } b = 1 \end{cases} \tag{6.12}$$

as $t \to \infty$.

Proof. (i) Since

$$r(t) \leqslant t^{-b} \int_t^\infty s^b \, \mathrm{d}s \int f(x)\big(E^x f(X_s)\big)\, \mathrm{d}x = \mathrm{o}(t^{-b}),$$

we have $\int_0^t r(s)\,\mathrm{d}s = \mathrm{o}(t^{1-b})$. Hence (6.11) follows from Proposition 6.1.
(ii) Since we have from $1 \in \mathfrak{T}$ that

$$\mathrm{tr}(t) \leqslant \int_t^\infty s \, \mathrm{d}s \int f(x)\big(E^x f(X_s)\big)\, \mathrm{d}x \to 0, \quad t \to \infty,$$

we see that

$$\int_t^\infty r(s)\,\mathrm{d}s = -\,\mathrm{tr}(t) - \int_t^\infty s r'(s)\,\mathrm{d}s \leqslant -\,\mathrm{tr}(t) + t^{1-b} \int_t^\infty s^b\,\mathrm{d}s \int f(x)\big(E^x f(X_s)\big)\,\mathrm{d}x = \mathrm{o}(t^{1-b}).$$

Thus (6.12) follows from Proposition 6.2. □

In the case of stable processes, Port [14] gives detailed analysis of asymptotics of $\Delta_B^{(1)}(t)$ and $\Delta_B^{(2)}(t)$. Our Theorems A and B make it possible to give some asymptotics of these quantities for semistable processes.

Proposition 6.4. *Let $\{X_t\}$ be a nondegenerate transient α-semistable process on $\mathbb{R}^d$. Suppose that either it is the one treated in* (i)–(iii), *and* (iv) *of Theorem* A, *or it is in the case $d = 1$ treated in Corollary* 2.7, *but the one in* (iv) *of Corollary* 2.7 *is excluded. Then $\mathfrak{T} = [0, b)$, with some $0 < b \leqslant \infty$, and the following are true.*

(i) *If $0 < b < 1$, then*

$$\Delta_B^{(1)}(t) \asymp t^{1-b}, \quad t \to \infty. \tag{6.13}$$

(ii) *If* $b = 1$, *then*

$$\Delta_B^{(1)}(t) \asymp \log t, \quad t \to \infty. \tag{6.14}$$

(iii) *If* $1 < b < \infty$,

$$-\Delta_B^{(2)}(t) \asymp t^{-(b-1)}, \quad t \to \infty. \tag{6.15}$$

(iv) *If* $b = \infty$, *then, for any* $c > 0$,

$$\Delta_B^{(2)}(t) = \mathrm{o}(t^{-c}), \quad t \to \infty. \tag{6.16}$$

Here, in general for two positive functions $A(t)$ and $B(t)$, we write $A(t) \asymp B(t)$, $t \to \infty$, if there are constants $0 < c_1 \leqslant c_2 < \infty$ such that $c_1 B(t) \leqslant A(t) \leqslant c_2 B(t)$ for all sufficiently large t.

Proof of Proposition. Theorem A or Corollary 2.7 says that $\mathfrak{T} = [0, b)$, with some $0 < b \leqslant \infty$. Let us consider the cases treated in (ii) and (iii) of Theorem A. Thus $0 < b = d/\alpha - 1 < \infty$. We can show that condition (6.8) is satisfied. In order to see this, it is enough to show

$$\sup_{t>0} \frac{r(t/a)}{r(t)} < \infty \tag{6.17}$$

for $a \in \Gamma \cap (1, \infty)$. Let K be a compact set. By Lemma 4.3

$$\int_{a^n}^{\infty} p(s, x)\,\mathrm{d}s \sim \mathrm{const} \sum_{k=n}^{\infty} a^{-kb} = \mathrm{const}\, a^{-nb}$$

uniformly in $x \in K$, and hence for $a^n < t \leqslant a^{n+1}$

$$\frac{\int_{t/a}^{\infty} p(s, x)\,\mathrm{d}s}{\int_t^{\infty} p(s, x)\,\mathrm{d}s} \leqslant \frac{\int_{a^{n-1}}^{\infty} p(s, x)\,\mathrm{d}s}{\int_{a^{n+1}}^{\infty} p(s, x)\,\mathrm{d}s} \sim a^{2b}$$

uniformly in $x \in K$. Thus the definition (6.2) of $r(t)$ shows that (6.17) is satisfied. We also have

$$\int_t^{\infty} p(s, x)\,\mathrm{d}s \asymp t^{-b}$$

and hence

$$r(t) \asymp t^{-b}, \quad t \to \infty.$$

Thus (i), (ii), and (iii) follow from Propositions 6.1 and 6.2.

Among the cases treated in Corollary 2.7, (ii) is included in (ii) of Theorem A; (v) and (vii) have $0 < b < \infty$ and a similar argument works, using the proofs of (iii) and (iv) of Theorem B.

The remaining cases have $b = \infty$ and our assertion (iv) is a straightforward consequence of Proposition 6.3(ii). □

Proposition 6.5. *Let* $\{X_t\}$ *be a nondegenerate transient second-class* α*-semistable process on* $\mathbb{R}^d$ *with* $1 \leqslant \alpha < 2$ *treated in Theorem* B. *Assume that* $d \geqslant 2$. *Then the following hold for* $t \to \infty$.

(i) *If* $\alpha = 1$, *then*

$$\Delta_B^{(2)}(t) = \begin{cases} \mathrm{O}\big((\log t)^{-1}\big) & \textit{for } d = 2, \\ \mathrm{O}\big(t^{2-d}(\log t)^{-2}\big) & \textit{for } d \geqslant 3. \end{cases} \tag{6.18}$$

(ii) *If* $\alpha = 1$ *and* $\sigma(\{-\tau/|\tau|\}) > 0$, *then*

$$-\Delta_B^{(2)}(t) \asymp \begin{cases} (\log t)^{-1} & \text{for } d = 2, \\ t^{2-d}(\log t)^{-2} & \text{for } d \geqslant 3. \end{cases} \tag{6.19}$$

(iii) *If* $1 < \alpha < 2$, *then*

$$\Delta_B^{(2)}(t) = \mathrm{O}(t^{2-\alpha-(d-1)/\alpha}). \tag{6.20}$$

(iv) *If* $1 < \alpha < 2$ *and* $\sigma(\{-\tau/|\tau|\}) > 0$, *then*

$$-\Delta_B^{(2)}(t) \asymp t^{2-\alpha-(d-1)/\alpha}. \tag{6.21}$$

(v) *If* $-\tau/|\tau| \notin C_\sigma$, *then, for every* $c > 0$,

$$\Delta_B^{(2)}(t) = \mathrm{o}(t^{-c}). \tag{6.22}$$

Proof. (i) Denote positive constants by $c_1, c_2, \ldots$. It follows from (5.12) that

$$\int_{a^n}^{\infty} p(t,x)\,\mathrm{d}t \leqslant c_1 \sum_{k=n}^{\infty} k^{-2} a^{-(d-1)k} \leqslant c_2 \int_{a^{n-1}}^{\infty} (\log t)^{-2} t^{-d}\,\mathrm{d}t \leqslant c_3 n^{-2} a^{-(d-1)n}$$

uniformly in x in any compact set. Thus

$$\int_t^{\infty} p(s,x)\,\mathrm{d}s \leqslant c_4 (\log t)^{-2} t^{-(d-1)}$$

uniformly in x in any compact set. Hence

$$\int_t^{\infty} r(s)\,\mathrm{d}s \leqslant \begin{cases} c_5(\log t)^{-1} & \text{for } d = 2, \\ c_6(\log t)^{-2} t^{-(d-2)} & \text{for } d \geqslant 3. \end{cases}$$

Now use Proposition 6.2(i).

(ii) The estimate from below is obtained similarly, by (5.13).

(iii), (iv) Similarly use (5.7) and (5.11).

(v) Since $\mathfrak{T} = [0, \infty)$, this is a consequence of Proposition 6.3(ii). $\square$

Remark 6.6. If $d = 1$, $\alpha = 1$, and $0 < |\beta| < 1$ (the case (iv) of Corollary 2.7), then

$$\Delta_B^{(1)}(t) \asymp t(\log t)^{-1} \quad \text{as } t \to \infty. \tag{6.23}$$

Indeed, since this is in the case (ii) of Theorem B, we have

$$\int_{a^n}^{\infty} p(t,x)\,\mathrm{d}t \asymp \sum_{k=n}^{\infty} k^{-2} \asymp n^{-1}$$

uniformly in x in any compact set by (5.12) and (5.13). Thus $r(t) \asymp (\log t)^{-1}$ and $\int_0^t r(s)\,\mathrm{d}s \asymp t(\log t)^{-1}$. Then use Proposition 6.1.

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