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SCIENCE DIRECT
Ann. I. H. Poincaré – PR 42 (2006) 89–101
ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.elsevier.com/locate/anihpb

Lévy's area under conditioning

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Received 17 November 2004; accepted 18 February 2005

Available online 6 September 2005

Abstract

We consider Hölder regularity for Brownian Motion and Lévy's Stochastic Area conditional on the Brownian path being uniformly small. Our motivation comes from the analysis of SDE's via the theory of "rough paths".

Résumé

Pour les chemins browniens uniformément petits, nous considérons la régularité hölderiennes de l'aire stochastique de Lévy. Notre intérêt provient de l'analyse des équations differentielles stochastiques par la théorie des « rough paths ».

MSC: 60G15; 60H05; 60J65

1. Introduction

Throughout, W will denote the space of continuous paths $w:[0,\infty)\to\mathbb{R}^d$ which start at the origin $\mathbf{0}\in\mathbb{R}^d$, and W will be thought of as a Polish space in which convergence means uniform convergence on compacts. Further, $\mathbb{P}$ will denote Wiener measure on the Borel field $\mathcal{B}_W$ over W, and, for $0\leqslant s<t<\infty$, we define the Lévy area[2] $w\in W\mapsto A_{s,t}(w)\in\mathrm{so}(d)$, the skew symmetric matrices, so that

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1 The third author acknowledges support from the NSF for support under DMS-0244991 and the British government for funds from the EPSRC to visit to Oxford University.
2 We omit a commonly used factor $\frac{1}{2}$ in the area-definition. This will eliminate the need a lot of $\frac{1}{2}$'s and $\frac{1}{4}$'s in what follows, and, of course, has no impact otherwise.

0246-0203/$ – see front matter
doi:10.1016/j.anihpb.2005.02.003

$$A_{s,t}(w)_{k,\ell} = \int_s^t \big(w_k(\tau) - w_k(s)\big)\,\mathrm{d}w_\ell(\tau) - \int_s^t \big(w_\ell(\tau) - w_\ell(s)\big)\,\mathrm{d}w_k(\tau). \tag{A}$$

More precisely, we mean by this that we have a map $w \mapsto A(w) \in C(\Delta; \mathrm{so}(d))$, where $\Delta \equiv \{(s,t) \in [0,\infty)^2\colon s \leqslant t\}$, such that

(1) The map $(s,t) \in \Delta \mapsto A_{s,t}(w) \in \mathrm{so}(d)$ is continuous for each $w \in W$.
(2) For each $(s,t) \in \Delta$, $w \in W \mapsto A_{s,t}(w) \in \mathrm{so}(d)$ is $\overline{\mathcal{B}}_{[s,t]}$-measurable when $\overline{\mathcal{B}}_{[s,t]}$ is the σ-algebra which is obtained by completing $\sigma(\{w(\tau) - w(s)\colon \tau \in [s,t]\})$ with respect to $\mathbb{P}$. Moreover, for all $1 \leqslant k < \ell \leqslant d$, $A_{s,t}(w)_{k,\ell}$ $\mathbb{P}$-almost surely satisfies (A) when the integrals are taken in the sense of Itô.
(3) If $w \in W$ has bounded variation on $[s,t]$, $A_{s,t}(w)_{k,\ell}$ is given by (A) when the integrals there are taken in the sense of Riemann–Stieltjes.
(4) For all $0 \leqslant s \leqslant t \leqslant u < \infty$ and $w \in W$
$$A_{s,t}(w)_{k,\ell} = A_{0,t}(w)_{k,\ell} - A_{0,s}(w)_{k,\ell} - w_k(s)\big(w_\ell(t) - w_\ell(s)\big) + w_\ell(s)\big(w_k(t) - w_k(s)\big).$$

Given $0 \leqslant s < t < \infty$, set

$$\|w\|_{0,[s,t]} \equiv \sup\big\{\big|w(\tau)\big|\colon \tau \in [s,t]\big\} \quad \text{and} \quad \big\|A(w)\big\|_{0,[s,t]} \equiv \sup\Big\{\sqrt{\big|A_{\sigma,\tau}(w)\big|}\colon s \leqslant \sigma \leqslant \tau \leqslant t\Big\},$$

and, for $\alpha > 0$,

$$\|w\|_{\alpha,[s,t]} \equiv \sup\left\{\frac{|w(\tau) - w(\sigma)|}{(\tau - \sigma)^\alpha}\colon s \leqslant \sigma < \tau \leqslant t\right\}$$

and

$$\big\|A(w)\big\|_{\alpha,[s,t]} = \sup\left\{\frac{\sqrt{|A_{\sigma,\tau}(w)|}}{(\tau - \sigma)^\alpha}\colon s \leqslant \sigma < \tau \leqslant t\right\}.$$

Our main result is a quantitative version (cf. (a) in Remarks, (6), (7), (9), and (10) below) of the statement that, for each $\alpha \in (0, \frac{1}{2})$, $\epsilon > 0$, $T \in (0,\infty)$, and piecewise smooth[3] $h \in W$,

$$\lim_{\delta \searrow 0} \mathbb{P}\big(\|w - h\|_{\alpha,[0,T]} \vee \big\|A(w) - A(h)\big\|_{\alpha,[0,T]} \geqslant \epsilon \;\big|\; \|w - h\|_{0,[0,T]} \leqslant \delta\big) = 0. \tag{1}$$

Our interest in (1) stems from its application to the analysis of stochastic integral equations via the theory of “rough paths”. In particular, after combining (1) with the continuity result in [9], one arrives at the statement that, when $w \rightsquigarrow X(\cdot, w)$ is the solution to a Stratonovich stochastic integral equation with sufficiently smooth coefficients[4]

$$\lim_{\delta \searrow 0} \mathbb{P}\big(\big\|X(\cdot, w) - X(\cdot, h)\big\|_{\alpha,[0,T]} \geqslant \epsilon \;\big|\; \|w - h\|_{0,[0,T]} \leqslant \delta\big) = 0. \tag{2}$$

When $\alpha = 0$ and $\|w - h\|_{0,[0,T]}$ is computed relative to the $\ell^\infty(\mathbb{R}^d)$-norm on $\mathbb{R}^d$, this Lebesgue density type result is the one proved originally in [15]. The statement in (2) for $\alpha \in (0, \frac{1}{2})$ was proved for the first time in [2] (cf. [1] as well) by a rather intricate argument.

As corollary of the density statement, one gets the “support theorem” for the distribution of $w \rightsquigarrow X(\cdot, w)$. As was shown in [11] (cf. [13] and [14] as well), the conclusion about the support which follows from (2) can be obtained by a simpler argument than the one in [2]. Furthermore, because it is a much more qualitative than a

[3] An extension to Cameron–Martin paths h is possible.

[4] For instance, C^3-regular and globally Lipschitz coefficients will do. For precise conditions see [10]. The preprint [7] shows how to include drift vector fields of minimal regularity.

density statement, the support theorem has had many extensions, and some of these extensions are unlikely to be shared by the density statement. For example, it is unlikely that there is a density version of the extension of the support theorem in [16] to diffusion coefficients which do not admit a smooth square root. The idea of deriving the support theorem via rough path considerations was discussed several times by the second and third authors, mentioned in [9], and implemented in [5], relative to p-variation, in [3] relative to the Hölder norm, and in [4] for more general moduli of continuity.

Finally, the development of rough path theory itself was started in [8] and completed in [9]. Its applicability to differential equations driven by Brownian Motion goes back to unpublished thesis work [12]. See also the recent survey [6] and the monograph [10].

Remarks. (a) Results of the sort in (1) and (2) appear already in [2]. Aside from the fact that our approach leads to more quantitative statements than theirs, they proved that

$$\lim_{\delta\searrow 0}\mathbb{P}\big(\big\|A(w)-A(h)\big\|_{\alpha,[0,T]}\geqslant\epsilon \;\big|\; \|w-h\|_{0,[0,T]}\leqslant\delta\big)=0$$

only for $\alpha\in[0,\frac{1}{4})$, and for the application to the theory of rough paths it is essential to allow $\alpha>\frac{1}{4}$. Thus, although they too arrive at (2), they could not have done so as an application of rough path theory. On the other hand, because our own approach is inextricably tied to the special properties (cf. (e) below) of Brownian motion, the approach taken in [2], which is more functional analytic, may apply to situations where our own approach fails.

(b) Throughout this article, we will be taking the norm on $\mathbb{R}^d$ and $\mathrm{so}(d)$ to be the Euclidean norm. However, an examination of our argument reveals that there is only one place at which we make essential of this choice of norm. In particular, our proof shows that

$$\lim_{\delta\searrow 0}\mathbb{P}\big(\|w-h\|_{\alpha,[0,T]}\geqslant\epsilon \;\big|\; \|w-h\|_{0,[0,T]}\leqslant\delta\big)=0$$

and, for each $\alpha\in(\frac{1}{4},\frac{1}{2})$ and $R>0$,

$$\lim_{\delta\searrow 0}\mathbb{P}\big(\big\{\big\|A(w)-A(h)\big\|_{\alpha,[0,T]}\geqslant\epsilon\big\}\cap\big\{\big\|A(w)-A(h)\big\|_{\alpha',[0,T]}\leqslant R\delta^{1/2}\big\} \;\big|\; \|w-h\|_{0,[0,T]}\leqslant\delta\big)=0$$
$$\text{if } \frac{\alpha}{2}<\alpha'<\frac{1}{4}$$

no matter what norm one uses for $\mathbb{R}^d$ and $\mathrm{so}(d)$. However, we make critical use of the Euclidean norm on $\mathbb{R}^d$ in our argument that

$$\lim_{R\to\infty}\mathbb{P}\big(\big\|A(w)-A(h)\big\|_{\alpha',[0,T]}\geqslant R\delta^{1/2} \;\big|\; \|w-h\|_{0,[0,T]}\leqslant\delta\big)=0$$

uniformly in $\delta\in(0,1]$. We suspect that even here one should not require the Euclidean norm. In fact, the original result in [15] indicates that one can use the $\ell^\infty(\mathbb{R}^d)$-norm, but we do not know at present how to deal with general norms.

(c) By an obvious scaling argument, we need only prove (1) when $T=1$. More important, an elementary application of the Cameron–Martin theorem allows us to restrict our attention to the case when $h=0$. Namely, if

$$R_h(T,w)=\exp\left(-\int_0^T\big(\dot h(\tau),\mathrm{d}w(\tau)\big)_{\mathbb{R}^d}-\frac{1}{2}\int_0^T\big|\dot h(\tau)\big|^2\,\mathrm{d}\tau\right),$$

then their theorem says that

$$\mathbb{P}\big(\|w-h\|_{\alpha,[0,T]}\vee\big\|A(w-h)\big\|_{\alpha,[0,T]}\geqslant\epsilon \;\big|\; \|w-h\|_{0,[0,T]}\leqslant\delta\big)$$
$$=\frac{\mathbb{E}[R_h(T,w),\ \{\|w\|_{\alpha,[0,T]}\vee\|A(w)\|_{\alpha,[0,T]}\geqslant\epsilon\}\cap\{\|w\|_{0,[0,T]}\leqslant\delta\}]}{\mathbb{E}[R_h(T,w),\ \{\|w\|_{0,[0,T]}\leqslant\delta\}]}.$$

Hence, since

$$\left|\int_{T_1}^{T_2}\big(\dot h(\tau),\mathrm{d}w(\tau)\big)_{\mathbb{R}^d}\right|\leqslant\|w\|_{0,[T_1,T_2]}\left(\big|\dot h(T_1)\big|+\big|\dot h(T_2)\big|+\int_{T_1}^{T_2}\big|\ddot h(\tau)\big|\,\mathrm{d}\tau\right),$$

if $h\in C^2([T_1,T_2];\mathbb{R}^d)$, it is clear that we will know that

$$\lim_{\delta\searrow 0}\mathbb{P}\big(\|w-h\|_{\alpha,[0,T]}\vee\big\|A(w-h)\big\|_{\alpha,[0,T]}\geqslant\epsilon\;\big|\;\|w-h\|_{0,[0,T]}\leqslant\delta\big)=0\tag{3}$$

for all piecewise smooth $h\in W$ once we show it for $h=0$. At the same time, when $k\neq\ell$,

$$\int_s^t\big(w_k(\tau)-w_k(s)\big)\,\mathrm{d}w_\ell(\tau)-\int_s^t\big(h_k(\tau)-h_k(s)\big)\,\mathrm{d}h_\ell(\tau)-\int_s^t\big((w_k-h_k)(\tau)-(w_k-h_k)(s)\big)\,\mathrm{d}(w_\ell-h_\ell)(\tau)$$
$$=\int_s^t\big((w_k-h_k)(\tau)-(w_k-h_k)(s)\big)\,\mathrm{d}h_\ell(\tau)+\int_s^t\big((w_\ell-h_\ell)(t)-(w_\ell-h_\ell)(\tau)\big)\,\mathrm{d}h_k(\tau),$$

from which we see that (3) implies (1) for piecewise smooth $h\in W$.

Taking these considerations into account, we concentrate in this paper on proving that, for each $\alpha\in(0,\frac12)$ and $\epsilon>0$,

$$\lim_{\delta\searrow 0}\mathbb{P}\big(\|w\|_{\alpha,[0,1]}\vee\big\|A(w)\big\|_{\alpha,[0,1]}\geqslant\epsilon\;\big|\;\|w\|_{0,[0,1]}\leqslant\delta\big)=0.\tag{4}$$

(d) It is instructive to compare the Lévy area with the symmetric part of the second iterated integral. That is, for $k\neq\ell$, consider

$$\int_s^t\big(w_k(\tau)-w_k(s)\big)\,\mathrm{d}w_\ell(\tau)+\int_s^t\big(w_\ell(\tau)-w_\ell(s)\big)\,\mathrm{d}w_k(\tau)=\big(w_k(t)-w_k(s)\big)\big(w_\ell(t)-w_\ell(s)\big)$$

whose size, under the conditioning $\|w\|_{0,[0,1]}\leqslant\delta$, is obviously of order δ^2. As we will show in the Addendum, this is in stark contrast with the size

$$\sup\big\{\big|A_{s,t}(w)_{k,\ell}\big|:\;0\leqslant s\leqslant t\leqslant 1\big\}=\big\|A(w)\big\|_{0,[0,1]}^2$$

of the area, which, under the same conditioning, is (cf. (18)) of order δ.

(e) Our proof of (3) relies heavily on Brownian scaling. For example, we make repeated use of the fact that the distribution of $w\rightsquigarrow(\|w\|_{\alpha,[T_1,T_2]},\|A(w)\|_{\alpha,[T_1,T_2]})$ is the same as the distribution of

$$w\rightsquigarrow(T_2-T_1)^{1/2-\alpha}\big(\|w\|_{\alpha,[0,1]},\big\|A(w)\big\|_{\alpha,[0,1]}\big)$$

in order to show that the probability of "bad" behavior on an interval decreases rapidly with the length of the interval. For this reason, the arguments which we have developed are incapable of proving results which would say anything so refined as, conditional on $\|w\|_{0,[0,T]}$ being small, the Lévy modulus of $w\restriction[0,T]$ is small.

2. Controlling $\|w\|_\alpha$

If $\|w\|_{0,[0,1]}\leqslant\delta$ and $\|w\|_{\alpha,[0,1]}\geqslant\epsilon$, then there exist $0\leqslant s<t\leqslant 1$ such that

$$t-s<\left(\frac{2\delta}{\epsilon}\right)^{1/\alpha}\quad\text{and}\quad\frac{|w(t)-w(s)|}{(t-s)^\alpha}\geqslant\epsilon.$$

Hence, $\mathbb{P}(\|w\|_{\alpha,[0,1]} \geqslant \epsilon \;\&\; \|w\|_{0,[0,1]} \leqslant \delta)$ is dominated by the sum of $\sim (\epsilon/\delta)^{1/\alpha}$ terms of the form

$$\mathbb{P}\big(\|w\|_{0,[0,T_1]} \vee \|w\|_{0,[T_2,1]} \leqslant \delta \;\&\; \|w\|_{\alpha,[T_1,T_2]} \geqslant \epsilon\big),$$

where $0 \leqslant T_1 < T_2 \leqslant 1$ and $T_2 - T_1 \leqslant (2\delta/\epsilon)^{1/\alpha}$.

Now set $u(t,x) \equiv \mathbb{P}(\|x + w\|_{0,[0,t]} < 1)$. Then, by scaling,

$$\mathbb{P}\big(\|x + w\|_{0,[0,T]} < \delta\big) = u\left(T\delta^{-2}, \frac{x}{\delta}\right).$$

Hence, by the Markov property, each of the above terms is dominated by

$$\mathbb{E}\big[u\big((1-T_2)\delta^{-2}, w(T_2)\delta^{-1}\big),\ \|w\|_{0,[0,T_1]} \leqslant \delta \;\&\; \|w\|_{\alpha,[T_1,T_2]} \geqslant \epsilon\big]. \tag{5}$$

Lemma 1. *Let λ be the smallest eigenvalue of $-\frac{1}{2}\Delta$ with Dirichlet boundary condition on the unit ball $B(\mathbf{0},1)$ centered at the origin in $\mathbb{R}^d$. Then*

$$\mathrm{e}^{-\lambda t} \leqslant u(t,\mathbf{0}) \quad \textit{and} \quad u(t,x) \leqslant K\,\mathrm{e}^{-\lambda t},$$

where $K = \mathrm{e}^{2\lambda}\sqrt{(4\pi)^{-d/2}\,\mathrm{vol}(B(\mathbf{0},1))}$.

Proof. Use $p_t(x,y)$ to denote the Dirichlet heat kernel for $\frac{1}{2}\Delta$ on $B(\mathbf{0},1)$. Then $u(t,x) = \int_{B(\mathbf{0},1)} p_t(x,y)\,\mathrm{d}y$. Next, use ψ to denote the L^2-normalized eigenfunction corresponding to λ, and recall that ψ can be taken to be smooth and positive. Moreover, ψ achieves its maximum at the origin. There are various ways to check this last statement. One is via explicit expressions for Bessel functions. A softer approach is to realize that, by uniqueness, ψ is symmetric in each coordinate separately, and therefore, for each $1 \leqslant k \leqslant d$, the partial derivative $\psi_{,k}$ of ψ in the kth coordinate direction is an odd function of the kth coordinate which satisfies $\Delta\psi_{,k} + 2\lambda\psi_{,k} = 0$ on $B(\mathbf{0},1)$ and is non-positive on $\partial B(\mathbf{0},1) \cap \{x\colon x_k \geqslant 0\}$. Hence, by the minimum principle, $\psi_{,k} \leqslant 0$ on $B(\mathbf{0},1) \cap \{x\colon x_k \geqslant 0\}$, and so, for each k, $\psi(x) \leqslant \psi(x')$, where $x'_\ell = x_\ell$ when $\ell \neq k$ and $x'_k = 0$. After applying this remark to each coordinate, we conclude that ψ achieves its maximum at $\mathbf{0}$. Knowing that $\psi(\mathbf{0}) \geqslant \psi$, the lower bound becomes the simple observation that

$$\psi(\mathbf{0}) = \mathrm{e}^{\lambda t} \int_{B(0,1)} p_t(\mathbf{0},y)\psi(y)\,\mathrm{d}y \leqslant \mathrm{e}^{\lambda t}\psi(\mathbf{0})u(t,\mathbf{0}).$$

To get the upper bound, assume that $t \geqslant 1$ and use Ω_d to denote the volume of $B(\mathbf{0},1)$. By the Chapman–Kolmogorov equation plus the symmetry of $p_t(x,y)$ in x and y, one can easily justify[5]

$$\begin{aligned}
u(t,x) &= \int_{B(\mathbf{0},1)} p_t(x,y)\,\mathrm{d}y = \int_{B(\mathbf{0},1)} \left(\int_{B(\mathbf{0},1)} p_1(x,\xi)p_{t-1}(\xi,y)\,\mathrm{d}\xi \right) \mathrm{d}y \\
&= \int_{B(\mathbf{0},1)} \left(\int_{B(\mathbf{0},1)} p_1(x,\xi)p_{t-1}(y,\xi)\,\mathrm{d}\xi \right) \mathrm{d}y \leqslant \sqrt{\Omega_d}\left(\int_{B(\mathbf{0},1)} \left(\int_{B(\mathbf{0},1)} p_{t-1}(y,\xi)p_1(x,\xi)\,\mathrm{d}\xi \right)^2 \mathrm{d}y \right)^{1/2} \\
&\leqslant \sqrt{\Omega_d}\,\mathrm{e}^{-\lambda(t-1)} \big\| p_1(x,\cdot\,) \big\|_{L^2(B(\mathbf{0},1))} \leqslant \sqrt{\Omega_d}\,\mathrm{e}^{-\lambda(t-1)}(4\pi)^{-d/4},
\end{aligned}$$

since $\|p_1(x,\cdot\,)\|^2_{L^2(B(\mathbf{0},1))} = p_2(x,x) \leqslant (4\pi)^{-d/2}$. In particular, by the lower bound, $\mathrm{e}^{-\lambda} \leqslant u(1,\mathbf{0}) \leqslant \sqrt{(4\pi)^{-d/2}\Omega_d}$, and so

$$\mathrm{e}^{\lambda}\sqrt{(4\pi)^{-d/2}\,\mathrm{vol}\big(B(\mathbf{0},1)\big)} \geqslant 1.$$

[5] To get line 3 from line 2 below, we think of $p_{t-1}(x,y)$ as the kernel for an operator whose norm is $\mathrm{e}^{-\lambda(t-1)}$.

At the same time, $u(t,x) \leqslant 1$, which completes the proof our upper bound for all $t \geqslant 0$. $\square$

We now use the upper bound in Lemma 1 to estimate the expression in (5) and thereby get that it is dominated by

$$K\,\mathrm{e}^{-\lambda(1-T_2)\delta^{-2}}\mathbb{P}\big(\|w\|_{0,[0,T_1]} \leqslant \delta \,\&\, \|w\|_{\alpha,[T_1,T_2]} \geqslant \epsilon\big) = K\,\mathrm{e}^{-\lambda(1-T_2)\delta^{-2}}u\big(T_1\delta^{-2},\mathbf{0}\big)\mathbb{P}\big(\|w\|_{\alpha,[0,T_2-T_1]} \geqslant \epsilon\big) \leqslant K^2\,\mathrm{e}^{-\lambda(1-(T_2-T_1))\delta^{-2})}\mathbb{P}\big(\|w\|_{\alpha,[0,(T_2-T_1)]} \geqslant \epsilon\big),$$

since $\{\|w\|_{0,[0,T_1]} \leqslant \delta\}$ is independent of $\{\|w\|_{\alpha,[T_1,T_2]} \geqslant \epsilon\}$. We next use the lower bound in Lemma 1 to arrive at

$$\mathbb{P}\big(\|w\|_{\alpha,[T_1,T_2]} \geqslant \epsilon \bigm| \|w\|_{0,[0,1]} \leqslant \delta\big) \leqslant K^2\,\mathrm{e}^{\lambda(T_2-T_1)\delta^{-2}}\mathbb{P}\big(\|w\|_{\alpha,[0,2(2\epsilon^{-1}\delta)^{1/\alpha}]} \geqslant \epsilon\big).$$

Finally, we apply Brownian scaling and Gaussian tail estimates for $\|w\|_{\alpha,[0,1]}$ to get

$$\mathbb{P}\big(\|w\|_{\alpha,[0,1]} \geqslant \epsilon \bigm| \|w\|_{0,[0,1]} \leqslant \delta\big) \leqslant C_\alpha\big(\epsilon\delta^{-1}\big)^{1/\alpha}\exp\big[2\lambda\big(2\epsilon^{-1}\delta^{1-2\alpha}\big)^{1/\alpha} - \beta_\alpha\big(\epsilon^{-1}\delta^{1-2\alpha}\big)^{-1/\alpha}\big] \tag{6}$$

for appropriate $C_\alpha < \infty$ and $\beta_\alpha > 0$. In particular, after taking $\epsilon = \delta^\eta$, we can say that there exists a $C_{\alpha,\eta} < \infty$ such that

$$\mathbb{P}\big(\|w\|_{\alpha,[0,1]} \geqslant \delta^\eta \bigm| \|w\|_{0,[0,1]} \leqslant \delta\big) \leqslant C_{\alpha,\eta}\exp\big[-\beta_\alpha\delta^{-(1-2\alpha-\eta)/\alpha}\big]$$
$$\text{for each } \alpha \in \left(0,\frac{1}{2}\right) \,\&\, \eta \in (0, 1-2\alpha). \tag{7}$$

Notice that (7) is, at least qualitatively, what one should expect on the basis of the trivial interpolation inequality

$$\|w\|_{\alpha,[0,1]} \leqslant \|w\|_{0,[0,1]}^{1-\alpha/\beta}\|w\|_{\beta,[0,1]}^{\alpha/\beta} \quad \text{for } 0 < \alpha < \beta.$$

Indeed, we know that $\|w\|_{\beta,[0,1]} < \infty$ for each $\beta < \frac{1}{2}$, and so it is not surprising that, given $\|w\|_{0,[0,1]} \leqslant \delta$, we find that the size of $\|w\|_{\alpha,[0,1]}$ is no greater than δ^η for each $\eta < 1-2\alpha$.

3. Controlling $\|A\|_{\alpha,[0,1]}$ for small α

In this section we deal with the case when α is small in the sense that it lies in the interval $(0,\frac{1}{4})$. For this purpose, write

$$A_{s,t}(w)_{1,2} = A_{0,t}(w)_{1,2} - A_{0,s}(w)_{1,2} - w_1(s)\big(w_2(t) - w_2(s)\big) + w_2(s)\big(w_1(t) - w_1(s)\big).$$

Recall that the conditional distribution of $w \rightsquigarrow A_{0,\cdot}(w)_{1,2} \restriction [0,1]$ given $\sigma(\{\|w\|_{0,[0,1]} \leqslant \delta\})$ is that of an independent Brownian motion B run with the clock

$$T_{1,2}(t,w) \equiv \int_0^t \big(w_1(\tau)^2 + w_2(\tau)^2\big)\,\mathrm{d}\tau.$$

Therefore

$$\mathbb{P}\left(\sup_{0\leqslant s<t\leqslant 1}\frac{|A_{s,t}(w)_{1,2}|}{(t-s)^{2\alpha}} \geqslant 3\epsilon^2 \Bigm| \|w\|_{0,[0,1]} \leqslant \delta\right) \leqslant \mathbb{P}\left(\sup_{0\leqslant s<t\leqslant 1}\frac{|B(T_{1,2}(t,w)) - B(T_{1,2}(s,w))|}{(t-s)^{2\alpha}} \geqslant \epsilon^2 \Bigm| \|w\|_{0,[0,1]} \leqslant \delta\right) + 2\mathbb{P}\big(\|w\|_{2\alpha,[0,1]} \geqslant \epsilon^2\delta^{-1} \bigm| \|w\|_{0,[0,1]} \leqslant \delta\big). \tag{8}$$

Because, given $\|w\|_{0,[0,1]} \leqslant \delta$,

$$\sup_{0\leqslant s<t\leqslant 1} \frac{|B(T_{1,2}(t,w)) - B(T_{1,2}(s,w))|}{(t-s)^{2\alpha}} \leqslant \delta^{4\alpha} \sup_{0\leqslant s<t\leqslant \delta^2} \frac{|B(t)-B(s)|}{(t-s)^{2\alpha}},$$

which, by Brownian scaling, has the same distribution as

$$\delta \sup_{0\leqslant s<t\leqslant 1} \frac{|B(t)-B(s)|}{(t-s)^{2\alpha}}.$$

When we combine this with (8), we get

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leqslant s<t\leqslant 1} \frac{|A_{s,t}(w)|}{(t-s)^{2\alpha}} \geqslant 3\epsilon^2 \,\Big|\, \|w\|_{0,[0,1]} \leqslant \delta\right)\\ &\quad\leqslant \mathbb{P}\big(\|w\|_{2\alpha,[0,1]} \geqslant \epsilon^2\delta^{-1}\big) + 2\mathbb{P}\big(\|w\|_{2\alpha,[0,1]} \geqslant \epsilon^2\delta^{-1} \,\big|\, \|w\|_{0,[0,1]} \leqslant \delta\big),\end{aligned}$$

which, by standard Gaussian estimates plus (6) means that, for each $\alpha \in (0, \frac{1}{4})$, there exist $\gamma_\alpha > 0$ and $C_\alpha < \infty$ such that

$$\mathbb{P}\big(\|A\|_{\alpha,[0,1]} \geqslant R\delta^{1/2} \,\big|\, \|w\|_{0,[0,1]} \leqslant \delta\big) \leqslant C_\alpha\, \mathrm{e}^{-\gamma_\alpha R^4} \quad \text{for } \alpha \in \left(0, \frac{1}{4}\right),\ \delta < 1,\ \text{and } R \geqslant 1. \tag{9}$$

It is interesting to observe that the estimate in (9) is better than the one which is predicted by interpolation. Namely, as we will see in the Addendum, given $\|w\|_{0,[0,1]} \leqslant \delta$, $\|A(w)\|_{0,[0,1]}$ is of order $\delta^{1/2}$. At the same time, (11) says that $\|A(w)\|_{\beta,[0,1]}$ is finite for each $\beta < \frac{1}{2}$. Hence, interpolation would lead one to believe that $\|A(w)\|_{\alpha,[0,1]}$ ought to be of order $\delta^{\eta/2}$ for each $\eta < 1 - 2\alpha$. That we are getting a better conclusion here indicates that the estimate in (9) is more subtle than interpolation.

4. Controlling $\|A\|_{\alpha,[0,1]}$ for large α

We now want to handle large α, those from $[\frac{1}{4}, \frac{1}{2})$. Then, for each $\alpha' \in [0, \frac{1}{4})$ and $R > 0$,

$$\begin{aligned}\mathbb{P}\big(\|A\|_{\alpha,[0,1]} \geqslant \epsilon \,\big|\, \|w\|_{0,[0,1]} \leqslant \delta\big) &\leqslant \mathbb{P}\big(\|A\|_{\alpha,[0,1]} \geqslant \epsilon \ \&\ \|A\|_{\alpha',[0,1]} \leqslant R\delta^{1/2} \,\big|\, \|w\|_0 \leqslant \delta\big)\\ &\quad + \mathbb{P}\big(\|A\|_{\alpha',[0,1]} \geqslant R\delta^{1/2} \,\big|\, \|w\|_{0,[0,1]} \leqslant \delta\big).\end{aligned}$$

By the estimate in (9), we know that the second term is dominated by a constant times $\mathrm{e}^{-\gamma_{\alpha'} R^4}$.

The first term is handled very much the same way as $\|w\|_{\alpha,[0,1]}$ was. Namely, because

$$\begin{aligned}&\big\|A(w)\big\|_{\alpha,[0,1]} \geqslant \epsilon \quad \text{and}\\ &\big\|A(w)\big\|_{\alpha',[0,1]} \leqslant R\delta^{1/2} \Longrightarrow \exists 0 < t-s \leqslant \left(\frac{R\delta^{1/2}}{\epsilon}\right)^{1/(\alpha-\alpha')} \sqrt{\big|A_{s,t}(w)\big|} \geqslant \epsilon(t-s)^\alpha,\end{aligned}$$

this term is dominated by $\sim (\epsilon^{-1}R\delta^{1/2})^{1/(\alpha'-\alpha)}$ terms of the form

$$\frac{\mathbb{P}(\|w\|_{0,[0,T_1]} \vee \|w\|_{0,[T_2,1]} \leqslant \delta \ \&\ \|A\|_{\alpha,[T_1,T_2]} \geqslant \epsilon)}{\mathbb{P}(\|w\|_0 \leqslant \delta)},$$

where $T_2 - T_1 \leqslant 2(\epsilon^{-1}R\delta^{1/2})^{1/(\alpha-\alpha')}$, and, just as before, each of these is dominated by $K^2 \exp[\lambda(T_2 - T_1)\delta^{-2}]$ times

$$\mathbb{P}\big(\|A\|_{\alpha,[0,T_2-T_1]} \geqslant \epsilon\big) = \mathbb{P}\big(\|A\|_{\alpha,[0,1]} \geqslant \epsilon(T_2-T_1)^{\alpha-1/2}\big)$$
$$\leqslant \mathbb{P}\left(\|A\|_{\alpha,[0,1]} \geqslant 2^{\alpha-1/2}\epsilon\left(\frac{R\delta^{1/2}}{\epsilon}\right)^{(2\alpha-1)/(2(\alpha-\alpha'))}\right),$$

which, by the estimate in (11) below, is dominated by

$$K_\alpha \exp\left[-\frac{2^{2\alpha-1}\epsilon^2}{4d(d-1)}\left(\frac{R\delta^{1/2}}{\epsilon}\right)^{(2\alpha-1)/(\alpha-\alpha')}\right].$$

Hence, we now know that $\mathbb{P}(\|A\|_{\alpha,[0,1]} \geqslant \epsilon \mid \|w\|_{0,[0,1]} \leqslant \delta)$ is dominated by a constant times

$$\exp\big[-\gamma_{\alpha'}R^4\big] + \left(\frac{\epsilon}{R\delta^{1/2}}\right)^{1/(\alpha-\alpha')}\exp\left[2\lambda\left(\frac{R\delta^{1/2}}{\epsilon}\right)^{1/(\alpha-\alpha')}\delta^{-2} - \frac{2^{2\alpha-1}\epsilon^2}{4d(d-1)}\left(\frac{R\delta^{1/2}}{\epsilon}\right)^{(2\alpha-1)/(\alpha-\alpha')}\right].$$

When we take $R = (\epsilon^{1-2\alpha'}\delta^{(2\alpha-1)/2})^{1/(1+2\alpha-4\alpha')}$, the preceding leads to

$$\begin{aligned}
&\mathbb{P}\big(\|A\|_{\alpha,[0,1]} \geqslant \epsilon \bigm| \|w\|_{0,[0,1]} \leqslant \delta\big)\\
&\quad\leqslant C_{\alpha,\alpha'}\big[1 + \big(\epsilon\delta^{-1}\big)^{2/(1+2\alpha-4\alpha')}\big]\\
&\qquad\times \exp\left[2\lambda\big(\epsilon^{-1}\delta^{2(2\alpha'-\alpha)}\big)^{2/(1+2\alpha-4\alpha')} - \left(\gamma_{\alpha'} \wedge \frac{1}{8d^2}\right)\big(\epsilon^{2-4\alpha'}\delta^{2\alpha-1}\big)^{2/(1+2\alpha-4\alpha')}\right]
\end{aligned}$$

for each $\alpha \in [\frac{1}{4}, \frac{1}{2})$ and $\alpha' \in [0, \frac{1}{4})$.

Finally, the preceding guarantees the existence of $K_{\alpha,\eta,\mu} < \infty$ and $\gamma_{\alpha,\eta,\mu} > 0$ such that

$$\begin{aligned}
&\mathbb{P}\big(\|A\|_{\alpha,[0,1]} \geqslant \delta^\eta \bigm| \|w\|_{0,[0,1]} \leqslant \delta\big) \leqslant K_{\alpha,\eta,\mu}\exp\big[-\gamma_{\alpha,\eta,\mu}\delta^{-\mu}\big]\\
&\quad\text{for all } \alpha \in \left[\frac{1}{4}, \frac{1}{2}\right),\ \eta \in (0, 1-2\alpha),\ \text{and } 0 < \mu < \frac{1-2\alpha-\eta}{\alpha}.
\end{aligned} \tag{10}$$

Observe that, once again, this is more or less what one would expect on the basis of naïve interpolation.

5. Unconditioned estimate on $\|A\|_{\alpha,[0,1]}$

In this section we will show that, for each $\alpha \in (0, \frac{1}{2})$, there exists a $C_\alpha < \infty$ such that

$$\mathbb{P}\big(\|A\|_{\alpha,[0,1]} \geqslant R\big) \leqslant K_\alpha\, \mathrm{e}^{-R^2/(4d(d-1))}. \tag{11}$$

For this purpose, set

$$X(s,t) = \int_s^{s+t} \big(w_1(\tau) - w_1(s)\big)\,\mathrm{d}w_2(\tau).$$

When $q \geqslant 2$,

$$\mathbb{E}\big[\big|X(s,t)\big|^q\big]^{1/q} = G_q\mathbb{E}\left[\left(\int_0^t w_1(\tau)^2\,\mathrm{d}\tau\right)^{q/2}\right]^{1/q} \leqslant G_q\left(t^{q/2-1}\int_0^t \mathbb{E}\big[\big|w_1(\tau)\big|^q\big]\,\mathrm{d}\tau\right)^{1/q} \leqslant G_q^2 t,$$

where G_q is the L^q norm of a standard normal r.v. Thus,

$$\mathbb{E}\big[\big|X(s_2,t) - X(s_1,t)\big|^q\big]^{1/q} \leqslant 2G_q^2 t.$$

Moreover, if $t \geqslant s_2 - s_1$, then

$$X(s_2,t) - X(s_1,t) = \int_{s_1+t}^{s_2+t} \big(w_1(\tau) - w_1(s_2)\big)\,\mathrm{d}w_2(\tau) - \big(w_1(s_2) - w_1(s_1)\big)\big(w_2(s_1+t) - w_2(s_2)\big) + \int_{s_1}^{s_2} \big(w_1(\tau) - w_1(s_1)\big)\,\mathrm{d}w_2(\tau),$$

and so

$$\mathbb{E}\big[\big|X(s_2,t) - X(s_1,t)\big|^q\big]^{1/q} \leqslant 3G_q^2 t^{1/2}(s_2 - s_1)^{1/2}.$$

Hence, in any case,

$$\mathbb{E}\big[\big|X(s_2,t) - X(s_1,t)\big|^q\big]^{1/q} \leqslant 3G_q^2\big(t^{1/2} + (s_2 - s_1)^{1/2}\big)(s_2 - s_1)^{1/2}. \tag{12}$$

Next, for each $M \in \mathbb{N}$, let $X_{1,M}(\,\cdot\,,t)$ and $X_{2,M}(s,\,\cdot\,)$ denote the polygonalization of $X(\,\cdot\,,t)$ and $X(s,\,\cdot\,)$, respectively, between points of the form $m2^{-M}$. Then, since

$$\big\|X_{1,M+1}(\,\cdot\,,t) - X_{1,M}(\,\cdot\,,t)\big\|_{0,[0,1]}^q \leqslant \sum_{m=1}^{2^{M+1}} \big|X\big(m2^{-(M+1)},t\big) - X\big((m-1)2^{-M},t\big)\big|^q,$$

(12) says that

$$\mathbb{E}\big[\big\|X_{1,M+1}(\,\cdot\,,t) - X_{1,M}(\,\cdot\,,t)\big\|_{0,[0,1]}^q\big]^{1/q} \leqslant \left(\sum_{m=1}^{2^{M+1}} \mathbb{E}\big[\big|X\big(m2^{-(M+1)},t\big) - X\big((m-1)2^{-(M+1)},t\big)\big|^q\big]\right)^{1/q} \leqslant 6G_q^2\big(t^{1/2} + 2^{-M/2}\big)2^{-M(1/2-1/q)},$$

and so, for $q > 2$,

$$\mathbb{E}\big[\big\|X(\,\cdot\,,t) - X_{1,M}(\,\cdot\,,t)\big\|_{0,[0,1]}^q\big]^{1/q} \leqslant \frac{6G_q^2}{1 - 2^{-1/2+1/q}}\big(t^{1/2} + 2^{-M/2}\big)2^{-M(1/2-1/q)}. \tag{13}$$

Next note that

$$\mathbb{E}\big[\big|X(s,t_2) - X(s,t_1)\big|^q\big]^{1/q} \leqslant G_q^2 t_2^{1/2}(t_2 - t_1)^{1/2} \tag{14}$$

and, from (13),

$$\mathbb{E}\big[\big\|X(\,\cdot\,,t_2) - X(\,\cdot\,,t_1)\big\|_{0,[0,1]}^q\big]^{1/q} \leqslant \frac{12G_q^2}{1 - 2^{-1/2+1/q}}\big(t_2^{1/2} + 2^{-M/2}\big)2^{-M(1/2-1/q)} + \mathbb{E}\big[\big\|X_{1,M}(\,\cdot\,,t_2) - X_{1,M}(\,\cdot\,,t_1)\big\|_{0,[0,1]}^q\big]^{1/q}.$$

Since

$$\big\|X_{1,M}(\,\cdot\,,t_2) - X_{1,M}(\,\cdot\,,t_1)\big\|_{0,[0,1]}^q \leqslant \sum_{m=0}^{2^M} \big|X\big(m2^{-M},t_2\big) - X\big(m2^{-M},t_1\big)\big|^q,$$

we can use (14) to arrive at

$$\mathbb{E}\big[\big\|X(\,\cdot\,,t_2) - X(\,\cdot\,,t_1)\big\|_{0,[0,1]}^q\big]^{1/q} \leqslant \frac{56G_q^2}{1 - 2^{-1/2+1/q}} t_2^{1/2}(t_2 - t_1)^{1/2-1/q} \quad \text{when } q > 2, \tag{15}$$

first for $2^{-M-1} \leqslant t_2 - t_1 \leqslant 2^{-M}$ and then for all $0 \leqslant t_1 < t_2 \leqslant 1$.

Starting from (15) and proceeding in the same way as we passed from (12) to (13), we find that

$$\mathbb{E}\big[\|X - X_{2,N}\|^q_{0,[0,1]\times[0,2^{-N}]}\big]^{1/q} \leqslant \frac{112G_q^2}{(1-2^{-1/2+2/q})^2} 2^{-N(1-1/q)} \quad \text{for } q > 4. \tag{16}$$

Finally, let $\alpha \in (0, \frac{1}{2})$ be given. Since for $2^{-N-1} \leqslant t \leqslant 2^{-N}$,

$$\frac{\|X(\cdot,t)\|_{0,[0,1]}}{t^{2\alpha}} \leqslant 2^{(N+1)2\alpha}\|X - X_{2,N}\|_{0,[0,1]\times[0,2^{-N}]} + 2^{(N+1)2\alpha}\big\|X\big(\cdot, 2^{-N}\big)\big\|_{0,[0,1]},$$

we can combine (15) and (16) to get

$$\mathbb{E}\bigg[\sup_{2^{-N-1}\leqslant t\leqslant 2^{-N}}\bigg(\frac{\|X(\cdot,t)\|_{0,[0,1]}}{t^{2\alpha}}\bigg)^q\bigg]^{1/q} \leqslant \frac{A_q G_q^2}{(1-2^{-1/2+2/q})^2} 2^{-N(1-2\alpha-2/q)},$$

where $A_{\alpha,q} < \infty$ for $q > (1-2\alpha)^{-1}$. Finally, if $q > 4$ satisfies $q > (1-2\alpha)^{-1}$, we can pass from the preceding to

$$\mathbb{E}\bigg[\sup_{t\in(0,1]}\bigg(\frac{\|X(\cdot,t)\|_{0,[0,1]}}{t^{2\alpha}}\bigg)^q\bigg]^{1/q} \leqslant \frac{A_{\alpha,q} G_q^2}{(1-2^{2/q-1/2})^2(1-2^{2/q+\alpha-1})}. \tag{17}$$

To complete the proof of (11) from here, note that

$$\sum_{n=0}^{\infty}\frac{G_n^{2n}}{4^n n!} \leqslant \sum_{n=0}^{\infty}\frac{G_{2n}^{2n}}{4^n n!} = \sqrt{2},$$

and use (17) to conclude that, for each $\alpha \in (0, \frac{1}{2})$

$$K_\alpha \equiv \mathbb{E}\big[\mathrm{e}^{1/4\|A\|^2_{\alpha,[0,1]}}\big] < \infty,$$

from which (11) is a trivial step.

Acknowledgement

The authors thank the referee for his careful reading of the manuscript.

Addendum

It may be helpful to realize that one cannot substantially improve the dependence of our estimates on δ in (9). To be precise, we will show here that, for each $R \geqslant 0$,

$$\lim_{\delta\searrow 0}\mathbb{P}\big(\delta^{-1/2}\big\|A(w)_{1,2}\big\|_{0,[0,1]} \leqslant R \,\big|\, \|w\|_{0,[0,1]} \leqslant \delta\big) = F(R), \tag{18}$$

where $F(R)$ is the probability that

$$\sup_{0\leqslant s<t\leqslant\sigma}\big|B(t) - B(s)\big| \leqslant R^2$$

when $B(\cdot)$ is a one-dimensional Brownian motion and (cf. the notation in Lemma 1)

$$\sigma \equiv 2\int_{B(\mathbf{0},1)} y_1^2\psi(y)^2\,\mathrm{d}y.$$

In particular, this means that, for each $\eta > \frac{1}{2}$,

$$\lim_{\delta\searrow 0} \mathbb{P}\big(\big\|A(w)\big\|_{0,[0,1]} \leqslant \delta^\eta \bigm| \|w\|_{0,[0,1]} \leqslant \delta\big) = 0.$$

A proof of (18) can be based on the following line of reasoning. First, one observes that it suffices[6] to show that

$$\lim_{\delta\searrow 0} \mathbb{P}\Big(\delta^{-1} \sup_{0\leqslant s<t\leqslant 1} \big|A_{0,t}(w)_{1,2} - A_{0,s}(w)_{1,2}\big| \leqslant R^2 \bigm| \|w\|_{0,[0,1]} \leqslant \delta\Big) = F(R). \tag{19}$$

Second, because, given $\sigma(\{|w(t)|\colon\ t \geqslant 0\})$, $t \rightsquigarrow A_{0,t}(w)_{1,2}$ an independent, one-dimensional Brownian motion run with the clock $t \rightsquigarrow \int_0^t (w_1(\tau)^2 + w_2(\tau)^2)\,\mathrm{d}\tau$, (19) is tantamount to the statement that, for each $\epsilon > 0$,

$$\lim_{\delta\searrow 0} \mathbb{P}\left(\left|\delta^{-2} \int_0^1 \big(w_1(\tau)^2 + w_2(\tau)^2\big)\,\mathrm{d}\tau - \sigma\right| \geqslant \epsilon \Bigm| \|w\|_{0,[0,1]} \leqslant \delta\right) = 0.$$

Moreover, by using Brownian scaling, this is equivalent to proving

$$\lim_{\delta\searrow 0} \mathbb{P}\left(\left|\delta^{2} \int_0^{\delta^{-2}} \big(w_1(\tau)^2 + w_2(\tau)^2\big)\,\mathrm{d}\tau - \sigma\right| \geqslant \epsilon \Bigm| \|w\|_{0,[0,\delta^{-2}]} \leqslant 1\right) = 0.$$

Hence, we need only prove the following ergodic theorem.

Proposition. *Set $\mu(\mathrm{d}x) = \psi(x)^2\,\mathrm{d}x$, where ψ is the first Dirichlet eigenfunction for $-\frac{1}{2}\Delta$ on $B(\mathbf{0},1)$. Then, for any measurable and bounded function V,*

$$\lim_{T\to\infty} \mathbb{P}\left(\left|\frac{1}{T}\int_0^T V\big(w(\tau)\big)\,\mathrm{d}\tau - \langle V\rangle\right| \geqslant \epsilon \Bigm| \|w\|_{0,[0,T]} \leqslant 1\right) = 0,$$

where $\langle V\rangle \equiv \int_{B(\mathbf{0},1)} V(x)\mu(\mathrm{d}x)$.

Proof. First note that it suffices to show that

$$\lim_{T\to\infty} \mathbb{P}\left(\left|\frac{1}{T}\int_1^T V\big(w(\tau)\big)\,\mathrm{d}\tau - \langle V\rangle\right| \geqslant \epsilon \Bigm| \|w\|_{0,[0,T+2]} \leqslant 1\right) = 0,$$

and apply the Markov property to see that (cf. the notation in the proof of Lemma 1)

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{1}{T}\int_1^{T+1} V\big(w(\tau)\big)\,\mathrm{d}\tau - \langle V\rangle\right| \geqslant \epsilon \Bigm| \|w\|_{0,[0,T+2]} \leqslant 1\right) \\
&\quad = \frac{1}{u(T+2,\mathbf{0})} \int_{B(\mathbf{0},1)} p_1(\mathbf{0},x)\mathbb{E}\big[u\big(1, x+w(T)\big), \Gamma(T,x,\epsilon) \cap \big\{\|x+w\|_{0,[0,T]} \leqslant 1\big\}\big]\,\mathrm{d}x,
\end{aligned}$$

where

$$\Gamma(T,x,\epsilon) \equiv \left\{w\colon \left|\frac{1}{T}\int_0^T V\big(x+w(\tau)\big)\,\mathrm{d}\tau - \langle V\rangle\right| \geqslant \epsilon\right\}.$$

[6] To be completely kosher, one has to check here that F is continuous. However, this follows from the fact that the joint distribution of $\min_{t\in[0,1]} B(t)$ and $\max_{t\in[0,1]} B(t)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$.

Next, remember that ψ is strictly positive on $B(\mathbf{0},1)$. Hence, for each $r \in (0,1)$, there is a K_r such that $p_1(\mathbf{0},x)u(1,y) \leqslant K_r \psi(x)\psi(y)$ for $(x,y) \in B(\mathbf{0},r)^2$. At the same time, we know (cf. Lemma 1) that, for $r \in (0,1)$,

$$\begin{aligned}
&\int\limits_{B(\mathbf{0},1)\setminus B(\mathbf{0},r)} p_1(\mathbf{0},x)\mathbb{E}\big[u\big(1,x+w(T)\big), \Gamma(T,x,\epsilon)\cap\big\{\|x+w\|_{0,[0,T]}\leqslant 1\big\}\big]\,\mathrm{d}x \\
&\quad + \int\limits_{B(\mathbf{0},1)} p_1(\mathbf{0},x)\mathbb{E}\big[u\big(1,x+w(T)\big), \Gamma(T,x,\epsilon)\cap\big\{\|x+w\|_{0,[0,T]}\leqslant 1\big\}\cap\big\{\big|x+w(T)\big|\geqslant r\big\}\big]\,\mathrm{d}x \\
&\leqslant K\,\mathrm{e}^{-\lambda} \int\limits_{B(\mathbf{0},1)\setminus B(\mathbf{0},r)} p_1(\mathbf{0},x)u(T,x)\,\mathrm{d}x + K\,\mathrm{e}^{-\lambda}\mathbb{P}\big(\big\{\|w\|_{0,[0,T+1]}\leqslant 1\big\}\cap\big\{\big|W(T+1)\big|\geqslant r\big\}\big) \\
&\leqslant \frac{K^2\Omega_d}{(2\pi)^{d/2}}\,\mathrm{e}^{-\lambda(T+1)}\big(1-r^d\big) + K\,\mathrm{e}^{-\lambda}\int\limits_{B(\mathbf{0},1)} p_T(\mathbf{0},x)\Bigg(\int\limits_{B(\mathbf{0},1)\setminus B(\mathbf{0},r)} p_1(x,y)\,\mathrm{d}y\Bigg)\mathrm{d}x \\
&\leqslant 2\frac{K^2\Omega_d}{(2\pi)^{d/2}}\,\mathrm{e}^{-\lambda(T+1)}\big(1-r^d\big).
\end{aligned}$$

Hence, since $u(T+2,\mathbf{0}) \geqslant \mathrm{e}^{-\lambda(T+2)}$, we can find for each $\eta > 0$ we a $C_\eta < \infty$ such that

$$\begin{aligned}
&\frac{1}{u(T+2,\mathbf{0})}\int\limits_{B(\mathbf{0},1)} p_1(\mathbf{0},x)\mathbb{E}\big[u\big(1,x+w(T)\big), \Gamma(T,x,\epsilon)\cap\big\{\|x+w\|_{0,[0,T]}\leqslant 1\big\}\big] \\
&\quad\leqslant \eta + C_\eta\,\mathrm{e}^{\lambda T}\int\limits_{B(\mathbf{0},1)} \psi(x)\mathbb{E}\big[\psi\big(x+w(T)\big),\ \Gamma(T,x,\epsilon)\cap\big\{\|x+w\|_{0,[0,1]}\leqslant 1\big\}\big]
\end{aligned}$$

for all $T \geqslant 1$. Therefore, all that remains is to show that

$$\lim_{T\to\infty} \mathrm{e}^{\lambda T}\int\limits_{B(\mathbf{0},1)} \psi(x)\mathbb{E}\big[\psi\big(x+w(T)\big),\ \Gamma(T,x,\epsilon)\cap\big\{\|x+w\|_{0,[0,1]}\leqslant 1\big\}\big] = 0 \quad \text{for each } \epsilon > 0. \tag{20}$$

The proof of (20) requires us to know[7] that there is a Borel probability measure $\mathbb{Q}$ on $C([0,\infty); B(\mathbf{0},1))$ with the property that, for each $T > 0$ and $\Gamma \subseteq C([0,\infty); B(\mathbf{0},1))$ which is measurable with respect to the path on $[0,T]$,

$$\mathbb{Q}(\Gamma) = \mathrm{e}^{\lambda T}\int\limits_{B(\mathbf{0},1)} \psi(x)\mathbb{E}\big[\psi\big(x+w(T)\big),\ \Gamma\cap\big\{\|x+w\|_{0,[0,T]}\leqslant 1\big\}\big]\,\mathrm{d}x.$$

To see this, all that one has to do is check the consistency statement

$$\begin{aligned}
&\mathrm{e}^{\lambda T_1}\int\limits_{B(\mathbf{0},1)} \psi(x)\mathbb{E}\big[\psi\big(x+w(T_1)\big),\ \Gamma\cap\big\{\|x+w\|_{0,[0,T_1]}\leqslant 1\big\}\big] \\
&\quad = \mathrm{e}^{\lambda T_2}\int\limits_{B(\mathbf{0},1)} \psi(x)\mathbb{E}\big[\psi\big(x+w(T_2)\big),\ \Gamma\cap\big\{\|x+w\|_{0,[0,T_2]}\leqslant 1\big\}\big]
\end{aligned}$$

for $0 \leqslant T_1 < T_2$ and Γ's which are $\sigma(\{w(t)\colon t\in[0,T_1]\})$-measurable, and this comes down to

[7] Those familiar with the term will recognize what we are doing here is as a parabolic Doob h-transform.

$$\begin{aligned}
&\mathbb{E}\big[\psi\big(x+w(T_2)\big),\ \Gamma\cap\big\{\|x+w\|_{0,[0,T_2]}\leqslant 1\big\}\big]\\
&\quad=\mathbb{E}\Big[\int\limits_{B(\mathbf{0},1)}\psi(y)p_{T_2-T_1}\big(x+w(T_1),y\big)\,\mathrm{d}y,\ \Gamma\cap\big\{\|x+w\|_{0,[0,T_1]}\leqslant 1\big\}\Big]\\
&\quad=\mathrm{e}^{-\lambda(T_2-T_1)}\mathbb{E}\big[\psi\big(x+w(T_1)\big),\ \Gamma\cap\big\{\|x+w\|_{0,[0,T_1]}\leqslant 1\big\}\big].
\end{aligned}$$

In fact, what this calculation shows is that $\mathbb{Q}$ is a time-homogeneous Markov process with a transition probability function given by $q_t(x,y)\,\mathrm{d}y$, where $q_t(x,y)=\mathrm{e}^{\lambda t}\psi(x)^{-1}p_t(x,y)\psi(y)$. Moreover, we have started this process with initial distribution $\mu(\mathrm{d}x)=\psi(x)^2\,\mathrm{d}x$, and, because $\psi(x)^2q_t(x,y)=\mathrm{e}^{\lambda t}\psi(x)p_t(x,y)\psi(y)$ is symmetric in (x,y), this process is reversible and therefore stationary. Finally, because $q_1(x,y)$ is positive on $B(\mathbf{0},1)^2$, it follows that μ is the only stationary measure and therefore that the process must be ergodic. Hence, since

$$\mathrm{e}^{\lambda T}\int\limits_{B(\mathbf{0},1)}\psi(x)\mathbb{E}\big[\psi\big(x+w(T)\big),\ \Gamma(T,x,\epsilon)\cap\big\{\|w\|_{0,[0,1]}\leqslant 1\big\}\big]\,\mathrm{d}x=\mathbb{Q}\left(\left|\frac{1}{T}\int\limits_0^T V\big(p(\tau)\big)\,\mathrm{d}\tau-\langle V\rangle\right|\geqslant\epsilon\right),$$

(20) is now a simple application of the ergodic theorem. □

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