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New scaling of Itzykson–Zuber integrals

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Abstract

We study asymptotics of the Itzykson–Zuber integrals in the scaling when one of the matrices has a small rank compared to the full rank. We show that the result is basically the same as in the case when one of the matrices has a fixed rank. In this way we extend the recent results of Guionnet and Maïda who showed that for the fixed rank scaling, the Itzykson–Zuber integral is given in terms of the Voiculescu's R-transform of the full rank matrix.

Résumé

Nous étudions les asymptotiques de l'intégrale de Itzykson–Zuber dans le cas où l'une des matrices est de rang petit. Nous prouvons que l'asymptotique obtenue est la même que lorsque la matrice est de rang fixe. De cette manière, nous étendons les récents résultats de Guionnet et Maïda, qui prouvent que en rang fixe, l'intégrale de Itzykson–Zuber s'exprime à l'aide de la R-transformée de Voiculescu de la matrice en rang plein.

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1. Introduction and the main result

For diagonal matrices $A_N, B_N \in M_N(\mathbb{C})$ we consider the Itzykson–Zuber integral

$$I_N^{(\beta)}(A_N, B_N) = \int \mathrm{e}^{N \operatorname{Tr} U A_N U^\star B_N} \, \mathrm{d}m_N^\beta(U),$$

where m_N^β denotes the Haar measure on the orthogonal group O_N when $\beta = 1$, on the unitary group U_N when $\beta = 2$, and on the symplectic group $\mathrm{Sp}(N/2)$ when $\beta = 4$ (in the latter case, N is even). Usually one is interested in the study

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of the asymptotics of the Itzykson–Zuber integral as the size N of the matrices tends to infinity and for this reason one would like to have an insight into the limit

$$\lim_{N\to\infty}\frac{1}{N^2}\log I_N^{(\beta)}(A_N,B_N)=:\tilde{I}^{(\beta)}(\mu_A,\mu_B) \tag{1}$$

when the spectral measures μ_{A_N},μ_{B_N} converge weakly towards some probability measures on $\mathbb{R}$. Existence of the limit of (1) was proved by Guionnet and Zeitouni [4].

In [1], the study of another scaling of Itzykson–Zuber integrals was initiated, namely when the rank M of the matrix A_N is small compared to the full rank N. In order to obtain a non-trivial limit one should consider rather the limit

$$\lim_{N\to\infty}\frac{1}{MN}\log I_N^{(\beta)}(A_N,B_N). \tag{2}$$

In this paper we shall say that a family of matrices $B_N\in M_N(\mathbb{C})$ *converges in distribution* if and only if for all integer k, $N^{-1}\operatorname{Tr}(B_N^k)$ tends towards a finite limit as $N\to\infty$.

The first author proved that if $A_N=\operatorname{diag}(t,0,0,\ldots)$ has rank one and spectral measures of B_N converge to some probability measure μ_B then for any integer k,

$$\lim_{N\to\infty}\frac{1}{N}\frac{\partial^k}{\partial t^k}\log I_N^{(2)}(A_N,B_N)\bigg|_{t=0}=\frac{\mathrm{d}^k}{\mathrm{d}t^k}\int_0^t R_{\mu_B}(s)\,\mathrm{d}s\bigg|_{t=0}.$$

The function R is called Voiculescu's R-transform and is of central importance in free probability theory. We recall its definition in Section 2.

In this paper, the following hypothesis and notation about the sequence of matrices $A_N\in M_N(\mathbb{C})$ will be used frequently.

Hypothesis 1. A_N is diagonal and it has rank $M(N)=\mathrm{o}(N)$. Its non-zero eigenvalues are denoted by $a_{1,N}\geqslant\cdots\geqslant a_{M,N}$.

Using the results presented in our previous article [2] it is possible to prove that the following quite general statement holds true:

Theorem 2. *Assume that B_N has a limiting distribution μ_B and that A_N is uniformly bounded and satisfies Hypothesis* 1. *Then for $\beta=1$ and* 2,

$$\left|\frac{1}{NM(N)}\frac{\partial^k}{\partial t^k}\log I_N^{(\beta)}(tA_N,B_N)\bigg|_{t=0}-\frac{\beta}{2M(N)}\sum_{i=1}^{M(N)}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\int_0^{2t/\beta}R_\mu(ta_{i,N})\,\mathrm{d}s\bigg|_{t=0}\right|=\mathrm{o}(1).$$

If $\beta=4$ then

$$\left|\frac{1}{NM(N)}\frac{\partial^k}{\partial t^k}\log I_N^{(\beta)}(tA_N,B_N)\bigg|_{t=0}+\frac{1}{M(N)}\sum_{i=1}^{M(N)}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\int_0^{t}R_\mu(-ta_{i,N})\,\mathrm{d}s\bigg|_{t=0}\right|=\mathrm{o}(1).$$

Observe that it is convenient to separate the case $\beta=4$ from the cases $\beta=1,2$.

We shall not prove this theorem since it is an immediate consequence of Theorem 5.5 of [2]. On the other hand, it has been proved recently in Theorem 7 of [3], that if the L^∞-norm of A_N is bounded by a constant depending on B_N (see Theorem 4 in this paper), under Hypothesis 3 for B_N, and provided that $M(N)=\mathrm{O}(N^{1/2-\varepsilon})$ for some $\varepsilon>0$, one has for $\beta=1,2$:

$$\left|\frac{1}{NM(N)}\log I_N^{(\beta)}(A_N,B_N)-\frac{\beta}{2M(N)}\sum_{i=1}^{M(N)}\int_0^{2t/\beta}R_\mu(sa_{i,N})\,\mathrm{d}s\right|=\mathrm{o}(1). \tag{3}$$

The main result of this paper is Theorem 6. It shows that Eq. (3) still holds true if one replaces the restriction $M(N) = \mathrm{O}(N^{1/2-\varepsilon})$ by Hypothesis 1, and that under additional assumption on B_N, it is enough to assume that the L^∞-norm of A_N is uniformly bounded by any constant, independently on B_N.

The techniques used in this paper are elementary, and substantially simplify the proofs of [3]. Better, it becomes possible to express the R-transform as a limit of Guionnet–Zeitouni's limits obtained in [4].

Let $b_{1,N} \geqslant \cdots \geqslant b_{N,N}$ be the real eigenvalues of the diagonal matrix B_N and let μ_{B_N} be the probability measure $N^{-1}\sum_{i=1}^N \delta_{b_{i,N}}$. We make the following hypothesis on the family of matrices $(B_N)_{N\in\mathbb{N}}$ with $B_N \in M_N(\mathbb{C})$:

Hypothesis 3. The probability measure μ_{B_N} converges weakly towards a compactly supported probability measure μ_B and that $b_{1,N}$ (resp. $b_{N,N}$) converge towards the upper bound $\lambda_{\max}$ (resp. the lower bound $\lambda_{\min}$) of the support of μ_B.

The Harish–Chandra formula in the case when $\beta = 2$ and when the eigenvalues have no multiplicity tells that

$$I_N^{(2)}(A_N, B_N) = \frac{\det(\mathrm{e}^{N\cdot a_{i,N} b_{j,N}})_{i,j=1}^N}{\Delta(a_{i,N})\Delta(b_{i,N})},$$

where Δ denotes Vandermonde determinant. In particular it implies that the value of $I_N^{(2)}$ depends only on the eigenvalues of A_N, B_N therefore the assumption that A_N, B_N are diagonal can be removed in the case $\beta = 2$. However a priori it cannot be removed in the other cases (although it is an open question to the authors' knowledge).

We recall some definitions that can be found in [3]. For a compactly supported probability measure μ, call H_μ its *Hilbert transform*

$$H_\mu(z) = \int \frac{1}{z-\lambda}\,\mathrm{d}\mu(\lambda).$$

It is invertible under composition in a complex neighborhood of infinity; and let us call $z^{-1} + R_\mu$ its inverse. R_μ is called the *R-transform* of Voiculescu.

Let $H_{\min} = \lim_{z\uparrow\lambda_{\min}} H_{\mu_B}(z)$ and $H_{\max} = \lim_{z\downarrow\lambda_{\max}} H_{\mu_B}(z)$.

Definition. For $\beta \in \{1, 2, 4\}$, a compactly supported probability measure μ and a real parameter t we define $f_\mu^{(\beta)}(t)$ as follows.

If $\beta = 1, 2$, the function is

$$f_\mu^{(\beta)}(t) = tv(t) - \frac{\beta}{2}\int \log\left(1 + \frac{2}{\beta}tv(t) - \frac{2}{\beta}t\lambda\right)\mathrm{d}\mu(\lambda)$$

with

$$v(t) = \begin{cases} R_\mu(\frac{2}{\beta}t) & \text{if } H_{\min} \leqslant \frac{2t}{\beta} \leqslant H_{\max}, \\ \lambda_{\max} - \frac{\beta}{2t} & \text{if } \frac{2t}{\beta} > H_{\max}, \\ \lambda_{\min} - \frac{\beta}{2t} & \text{if } \frac{2t}{\beta} < H_{\min}. \end{cases}$$

Observe that in the particular case $t \in [H_{\min}, H_{\max}]$,

$$f_\mu^{(\beta)}(t) = \int_0^{2t/\beta} R_\mu(st)\,\mathrm{d}s.$$

In the case $\beta = 4$ the function is defined by

$$f_\mu^{(4)}(t) = -f_\mu^{(2)}(-t).$$

One can prove that this function is continuous with respect to t and also with respect to μ_B for the metric on the measures given by

$$d(\mu,\mu') = |\lambda_{\max} - \lambda'_{\max}| + |\lambda_{\min} - \lambda'_{\min}| + \sup\left\{\int f(\mathrm{d}\mu - \mathrm{d}\mu'),\ \|f\|_\infty = 1 \text{ and } f \text{ of Lipschitz constant } \leqslant 1\right\}. \tag{4}$$

The following result of [3] will be of fundamental importance for us.

Theorem 4. *([3], Theorems 1.2 and 1.6) Assume that the family of matrices B_N satisfies Hypothesis* 3. *For $\beta = 1, 2$, for any real number t,*

$$\lim_{N\to\infty} N^{-1} \log I_N\big(B_N, \operatorname{diag}(t, 0, \ldots, 0)\big) = f^{(\beta)}_{\mu_B}(t).$$

Note that although Guionnet and Maïda in [3] do not handle the case $\beta = 4$, it is easy to extend their results to this case and we will take these results for granted.

At some points, we will need to make the following assumption on the matrix B_N:

Hypothesis 5. There exists a constant $c > 0$ such that

$$b_{i+1,N} + \frac{1}{cN} \geqslant b_{i,N}.$$

This assumption implies that the distribution of μ_B has an absolutely continuous part with respect to the Lebesgue measure whose density is $\geqslant c$ almost everywhere on its support, and the support should be an interval.

With these preliminaries, we are ready to state our main result:

Theorem 6. *Let $\beta \in \{1, 2, 4\}$.*

(1) *Let A_N be a sequence of matrices satisfying Hypothesis* 1*, and*

$$H_{\min} < \liminf_N a_{N,N} \leqslant \limsup_N a_{1,N} < H_{\max}.$$

Assume that B_N satisfies Hypothesis 3. *Then*

$$\left|\frac{1}{NM(N)} \log I^{(\beta)}_N(A_N, B_N) - \frac{\beta}{2M(N)} \sum_{i=1}^{M(N)} f^{(\beta)}_{\mu_B}(a_{i,N})\right| = \mathrm{o}(1).$$

(2) *Let A_N be a sequence of matrices satisfying Hypothesis* 1. *Assume that B_N satisfies Hypotheses* 3 *and* 5. *Then*

$$\left|\frac{1}{NM(N)} \log I^{(\beta)}_N(A_N, B_N) - \frac{\beta}{2M(N)} \sum_{i=1}^{M(N)} f^{(\beta)}_{\mu_B}(a_{i,N})\right| = \mathrm{o}(1).$$

(3) *Let μ, ν be compactly supported real probability measures. Let ν be supported in a compact subset of $(H_{\min}, H_{\max})$. For $a \in [0, 1]$, let $\nu_a = a\nu + (1-a)\delta_0$. Then*

$$\lim_{a\to 0} a^{-1} \tilde{I}^{(\beta)}(\nu_a, \mu) = \int_{t\in\mathbb{R}} f^{(\beta)}_\mu(t)\, \mathrm{d}\nu(t).$$

(4) *Let μ, ν be compactly supported real probability measures. Assume that μ has a connected support $[\lambda_{\min}, \lambda_{\max}]$, and that there exists a constant $c > 0$ such that*

$$c \cdot \mathrm{d}x\, 1_{[\lambda_{\min},\lambda_{\max}]} \leqslant \mu.$$

Then

$$\lim_{a\to 0} a^{-1} \tilde{I}^{(\beta)}(\nu_a, \mu) = \int_{t\in\mathbb{R}} f^{(\beta)}_\mu(t)\, \mathrm{d}\nu(t).$$

Remark. In Theorem 6, we do not handle the limit $\lim_{a\to 0} a^{-1}\tilde{I}^{(\beta)}(\nu_a,\mu)$ under the most general assumption that μ,ν are compactly supported real probability measures. It is tempting to conjecture that Hypothesis 5 is irrelevant but we could not settle this question.

The next section is devoted to proving Theorem 6.

2. Proof of the main result

$I_N^{(\beta)}$ satisfies the following obvious translation invariance property:

$$I_N^{(\beta)}(A,B) = \mathrm{e}^{-Nx\operatorname{Tr}A} I_N^{(\beta)}(A, B + x\mathrm{Id}), \tag{5}$$

therefore there is no loss of generality in assuming that B_N is positive.

We shall need the following technical results:

Lemma 7. *Let $B \in M_N(\mathbb{C})$ be Hermitian, with eigenvalues $b_1 \geqslant \cdots \geqslant b_d \geqslant 0$. Let Π be a projector of rank $N-1$ and $B' = \Pi B \Pi$ and let $b'_1 \geqslant \cdots \geqslant b'_d = 0$ its eigenvalues. Then one has*

$$b_1 \geqslant b'_1 \geqslant b_2 \geqslant \cdots \geqslant b_d \geqslant b'_d = 0.$$

Proof. Let V be the set of Hermitian projections of rank $n+1-i$ and V' be the subset of projections of V dominating $1-\Pi$. According to the "minimax" theorem, $b_i = \min_{\pi\in V} \|\pi B\pi\|$, and $b'_{i-1} = \min_{\pi\in V'} \|\pi B\pi\|$. Since $V' \subset V$ this implies that $b'_{i+1} \geqslant b_i$. Replacing B by $\|B\|\mathrm{Id} - B$ shows that $b_i \geqslant b'_i$, which concludes the proof. □

The following lemma is the keystone towards the proof of Theorem 6:

Lemma 8. *One has*

$$\prod_{i=1}^{M(N)} \inf_{C_{i,N}} I_{N+1-i}\left(\frac{N}{N+1-i}\operatorname{diag}(a_{i,N},0,\ldots,0), C_{i,N}\right)$$
$$\leqslant I_N(A_N,B_N) \leqslant \prod_{i=1}^{M(N)} \sup_{C_{i,N}} I_{N+1-i}\left(\frac{N}{N+1-i}\operatorname{diag}(a_{i,N},0,\ldots,0), C_{i,N}\right), \tag{6}$$

where the inf *and* sup *are taken over all diagonal matrices*

$$C_{i,N} = \operatorname{diag}\left(c_1^{(i)},\ldots,c_{N+1-i}^{(i)}\right) \quad \text{such that } b_k \leqslant c_k^{(i)} \leqslant b_{k+i-1}. \tag{7}$$

Proof. We make this proof by induction on $M(N)$. For $M(N)=1$ there is nothing to prove.

Suppose that the result holds at the level $M(N)-1$. In the case $\beta=2$, we identify U_{N-1} with a subgroup of U_N via the morphism

$$U_{N-1} \to \begin{pmatrix} 1 & 0 \\ 0 & U_{N-1} \end{pmatrix} \subset U_N.$$

In the case $\beta=1$ we consider instead the embedding of O_{N-1} into O_N

$$O_{N-1} \to \begin{pmatrix} 1 & 0 \\ 0 & O_{N-1} \end{pmatrix} \subset O_N,$$

and in the case $\beta=4$, the embedding SP_{N-1} into SP_N

$$\mathrm{SP}_{N-1} \to \begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & \mathrm{SP}_{N-1} \end{pmatrix} \subset \mathrm{SP}_N.$$

We shall only deal with the case $\beta=2$, the cases $\beta=1,4$ being similar.

Through the above identification, and using the invariance of Haar measure by convolution, we have

$$I_N^{(\beta)}(A_N,B_N)=\int\limits_{U\in U_N}\int\limits_{V\in U_{N-1}}\exp\bigl(N\operatorname{Tr}A_NVUB_NU^*V^*\bigr)\,\mathrm{d}m_N^\beta(U)\,\mathrm{d}m_{N-1}^\beta(V). \tag{8}$$

In the remainder of the proof, N is fixed, so in order to lighten the notation we omit the subscript N for the matrices A_N and B_N, and replace $a_{i,N}$ by a_i (resp. $b_{i,N}$ by b_i).

Since a unitary conjugation of A by a unitary leaves $I_N^{(\beta)}(A,B)$ invariant, one can assume without loss of generality that

$$A=\operatorname{diag}(a_1,\ldots,a_N)=\begin{pmatrix}a_1&0\\0&A_-\end{pmatrix}$$

with $a_i\geqslant a_j$ if $i<j$. We shall also adopt the following notation:

$$\tilde{B}=UBU^*=\begin{pmatrix}\tilde{b}_{11}&\tilde{B}_{1-}\\\tilde{B}_{-1}&\tilde{B}_{--}\end{pmatrix}.$$

Observe that $\tilde{B}_{--}\in M_{N-1}(\mathbb{C})$ and $\tilde{b}_{11}\in\mathbb{C}$ are random variables depending on the matrix random variable U. With these notations we have

$$\begin{aligned}I_N^{(\beta)}(A,B)&=\int\limits_{U\in U_N}\int\limits_{V\in U_{N-1}}\exp\bigl(N\operatorname{Tr}AV\tilde{B}V^*\bigr)\,\mathrm{d}m_N^\beta(U)\,\mathrm{d}m_{N-1}^\beta(V)\\&=\int\limits_{U\in U_N}\int\limits_{V\in U_{N-1}}\exp\bigl(Na_1\tilde{b}_{11}+N\operatorname{Tr}A_-V\tilde{B}_{--}V^*\bigr)\,\mathrm{d}m_N^\beta(U)\,\mathrm{d}m_{N-1}^\beta(V)\\&=\int\limits_{U\in U_N}\exp\bigl(Na_1\tilde{b}_{11}\bigr)\int\limits_{V\in U_{N-1}}\exp\bigl(N\operatorname{Tr}A_-V\tilde{B}_{--}V^*\bigr)\,\mathrm{d}m_N^\beta(U)\,\mathrm{d}m_{N-1}^\beta(V)\\&=\int\limits_{U\in U_N}\exp\bigl(Na_1\tilde{b}_{11}\bigr)\,I_{N-1}^{(\beta)}\left(\frac{N}{N-1}A_-,\tilde{B}_{--}\right)\mathrm{d}m_N^\beta(U).\end{aligned}\tag{9}$$

We are in position to apply the recursion hypothesis to

$$I_{N-1}^{(\beta)}\left(\frac{N}{N-1}A_-,\tilde{B}_{--}\right).$$

Lemma 7 implies that

$$\begin{aligned}&\prod_{i=2}^{M(N)}\inf_{C_{i,N}}I_{N+1-i}\left(\frac{N}{N+1-i}\operatorname{diag}(a_i,0,\ldots,0),C_{i,N}\right)\\&\quad\leqslant I_{N-1}^{(\beta)}\left(\frac{N}{N-1}A_-,\tilde{B}_{--}\right)\leqslant\prod_{i=2}^{M(N)}\sup_{C_{i,N}}I_{N+1-i}\left(\frac{N}{N+1-i}\operatorname{diag}(a_i,0,\ldots,0),C_{i,N}\right),\end{aligned}\tag{10}$$

where the inf and sup are taken over the same set of matrices $C_{i,N}$ as in (7).

In addition, it is obvious that

$$\int\limits_{U\in U_N}\exp\bigl(Na_1\tilde{b}_{11}\bigr)\,\mathrm{d}m_N^\beta(U)=I_N^{(\beta)}\bigl(\operatorname{diag}(a_1,0,\ldots,0),B\bigr).$$

The above remark and the inequality (10) plugged into Eq. (9) validate the recursion hypothesis at rank $M(N)$ and complete the proof of the lemma. □

We rephrase a continuity statement obtained by [3] suited for our purposes

Lemma 9. *Let K be a compact interval contained in $(H_{\min}, H_{\max})$. There exists a function $g:\mathbb{R}_+ \to \mathbb{R}_+$ continuous around* 0 *and such that* $g(0)=0$ *satisfying the following property*:

For all $\varepsilon > 0$, for all $t \in K$, and for all Hermitian matrices B_N, B'_N whose distributions have distance d (defined at Eq. (4)) *less than ε to μ,*

$$\left|N^{-1}\log I_N\big(B_N, \operatorname{diag}(t,0,\ldots,0)\big) - N^{-1}\log I_N\big(B'_N, \operatorname{diag}(t,0,\ldots,0)\big)\right| \leqslant g(\varepsilon).$$

Now we are in position to prove the first two statements of Theorem 6.

Proof of Theorem 6, statements (1) and (3). Under Hypothesis 1, one observes that

$$d(\mu_{C_{N,i}}, \mu) \to 0,$$

where d is the distance defined at Eq. (4) and $C_{N,i}$ is any sequence of matrices fulfilling the condition (7).

Taking the logarithm in the inequality of Lemma 8, and using the continuity result quoted in Lemma 9 concludes the proof of statement (1).

A minor modification of the above reasoning also shows statement (3). □

Remark. Lemma 8 and the strategy used in the above proof allow us to state and prove a version of Lemma 9 where one replaces $\operatorname{diag}(t,0,\ldots,0))$ by a matrix A_N satisfying Hypothesis 1. We skip these details.

The following lemma is a direct observation from the definitions, and we skip its proof.

Lemma 10. *Let $C_{N,i}$ and $C'_{N,i}$ be a pair of matrices fulfilling the condition* (7)*. Under Hypothesis* 5*, one has*

$$C_{N,i} - \mathrm{Id}\cdot\frac{i}{cN} \leqslant C'_{N,i} \leqslant C_{N,i} + \mathrm{Id}\cdot\frac{i}{cN}.$$

Now we are in position to prove statements involving Hypothesis 5.

Proof of Theorem 6, statements (2) and (4).

We prove statement (4), statement (2) being a straightforward adaptation. Fix a in the open interval $(0,1)$ and choose sequences of random matrices A_N, B_N such that $\mu_{A_N} \to \nu_a$ and $\mu_{B_N} \to \mu$.

In agreement with statement (4) of Theorem 6, it is possible to chose B_N such that it satisfies Hypotheses 3 and 5. We chose A_N such that it satisfies Hypothesis 3, and such that its rank $M(N) \sim aN$.

Taking the logarithm of the inequality of Lemma 8 yields:

$$\begin{aligned}&\frac{1}{MN}\sum_{i=1}^{M(N)} \inf_{C_{N,i}} \log I_{N+1-i}\left(\frac{N}{N+1-i}\operatorname{diag}(a_i,0,\ldots,0), C_{N,i}\right)\\ &\quad\leqslant \frac{1}{MN}\log I_N(A,B) \leqslant \frac{1}{MN}\sum_{i=1}^{M(N)} \sup_{C_{N,i}} \log I_{N+1-i}\left(\frac{N}{N+1-i}\operatorname{diag}(a_i,0,\ldots,0), C_{N,i}\right),\end{aligned} \tag{11}$$

where inf and sup are taken over $C_{N,i}$ as in (7).

Let $C^+_{i,N} = \sup\{C, C \in C_{i,N}\}$ and $C^-_{i,N} = \inf\{C, C \in C_{i,N}\}$. The notions of sup and inf are well-defined because $C_{i,N}$ consists of diagonal matrices. Consider the function

$$\phi_{i,N} : C \to \log I_{N+1-i}\left(\frac{N}{N+1-i}\operatorname{diag}(a_i,0,\ldots,0), C_{N,i}\right).$$

Assume that $a_i > 0$. It is straightforward to check that $\phi_{i,N}$ is increasing on $C_{i,N}$ for any i, N in the sense that $C > C'$ implies that $\phi(C') \geqslant \phi(C)$. Therefore its sup is reached on $C^+_{i,N}$ and its inf on $C^-_{i,N}$. If $a_i < 0$, equivalent statements hold if one replaces "increasing" by "decreasing" and sup by inf.

Let us introduce

$$C^+_N = \max_{i:i\leqslant M(N+i)} C^+_{N+i,i} \quad\text{and}\quad C^-_N = \min_{i:i\leqslant M(N+i)} C^-_{N+i,i},$$

where the maximum is taken pointwise (the matrices are diagonal and the index set is finite).

It is a simple observation that C_N^+ and C_N^- have a limiting distribution and that this distribution can be expressed in simple terms as a function of μ and a. Let us call respectively μ_a^+ and μ_a^- these distributions.

According to Theorem 4, the function

$$t \to N^{-1} \log I_N\bigl(\operatorname{diag}(t, 0, \ldots, 0), C_N^+\bigr)$$

is increasing and converges pointwise towards $f_{\mu_a^+}$ as N tends towards ∞. Dini's lemma shows that this convergence is uniform and Cesaro's theorem proves that one has

$$\tilde{I}^{(\beta)}(\nu_a, \mu) \leqslant \int\limits_{t \in \mathbb{R}_+} f_{\mu_a^+}^{(\beta)}(t)\, \mathrm{d}\nu_a(t) + \int\limits_{t \in \mathbb{R}_-} f_{\mu_a^-}^{(\beta)}(t)\, \mathrm{d}\nu_a(t).$$

In the same way,

$$\int\limits_{t \in \mathbb{R}_+} f_{\mu_a^-}^{(\beta)}(t)\, \mathrm{d}\nu_a(t) + \int\limits_{t \in \mathbb{R}_-} f_{\mu_a^+}^{(\beta)}(t)\, \mathrm{d}\nu_a(t) \leqslant \tilde{I}^{(\beta)}(\nu_a, \mu).$$

In addition, Lemma 10 imply that there exists a constant K depending only on ν and μ such that for all t in the spectrum of ν,

$$\bigl| f_{\mu_a^+}(t) - f_{\mu_a^-}(t) \bigr| \leqslant caK$$

This concludes the proof of statement (4). $\square$

3. Concluding remarks

In comparison to [3], our method uses much weaker assumptions on the rank and norm of the matrices. Better, it allows us to state a continuity result about the limit of IZ in a new scaling and thus validate an "inversion of limit" phenomenon.

Last, it shows that Hypotheses 3 and 5 are relevant one to perform computations outside the phase transition zone (that is, to say something about the limit outside the open interval $(H_{\min}, H_{\max})$). This is a substantial improvement to the paper [3] in which nothing was proved at "low temperature" for the finite scaling of rank $M > 1$.

Unfortunately, our approach heavily relies on real-number inequalities and we believe that finer estimates, as well as complex valued estimates of [3] cannot be established with our methods.

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