Brownian penalisations related to excursion lengths, VII

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Abstract. Limiting laws, as $t \to \infty$, for Brownian motion penalised by the longest length of excursions up to $t$, or up to the last zero before $t$, or again, up to the first zero after $t$, are shown to exist, and are characterized.

Résumé. Il est prouvé que les lois limites, lorsque $t \to \infty$, du mouvement brownien pénalisé par la plus grande longueur des excursions jusqu’en $t$, ou bien jusqu’au dernier zéro avant $t$, ou encore jusqu’au premier zéro après $t$, existent. Ces lois limites sont décrites en détail.

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1. Introduction

(a) Let $(\Omega, (X_t, t \geq 0), (\mathcal{F}_t, t \geq 0), (P_x, x \in \mathbb{R}))$ denote the canonical realisation of the Wiener process, i.e. $\Omega = C([0, \infty), \mathbb{R})$; $(X_t, t \geq 0)$ is the coordinate process on $\Omega$; $(\mathcal{F}_t, t \geq 0)$ denotes its natural filtration, and $\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$. $(P_x, x \in \mathbb{R})$ is the family of Wiener measures such that $P_x(X_0 = x) = 1$, for every $x \in \mathbb{R}$. We write simply $P$ for $P_0$.

(b) Let $(\Gamma_t, t \geq 0)$ denote an $\mathbb{R}_+^+$-valued, $(\mathcal{F}_t)$ adapted process defined on $\Omega$, which satisfies: $0 < E_x(\Gamma_t) < \infty$ for every $t \geq 0$, and every $x \in \mathbb{R}$. With the help of this process $\Gamma$ – which we call the penalisation process – we define the family of probabilities $P^{(t)}_x$ by:

$$P^{(t)}_x(A_t) = \frac{P_x(1_{\Lambda_t} \Gamma_t)}{E_x(\Gamma_t)} \quad (A_t \in \mathcal{F}_t). \quad (1.1)$$

In several preceding papers ([11–13,15–17], see also [14] for a survey), we have shown that for many penalisation processes $(\Gamma_t, t \geq 0)$, the following holds:

(i) $\lim_{t \to \infty} P^{(t)}_x(A_s)$ exists, for every fixed $s \geq 0$, and $A_s \in \mathcal{F}_s$. \quad (1.2)

(ii) This limit is of the form $E_x(1_{A_s} M^{\Gamma}_s)$, where $(M^{\Gamma}_s, s \geq 0)$ is a $(\mathcal{F}_s, P_x)$ positive martingale. \quad (1.3)

Here is our main tool to prove (1.2) and (1.3).

Theorem 1.1. Assume that:

$$E(\Gamma_t | \mathcal{F}_s) \quad t \to \infty \quad M_s \quad a.s.,$$
(ii) \( E(M_s) = 1 \), for every \( s \geq 0 \).

Then,

1. \[
\forall s \geq 0, \forall A_s \in \mathcal{F}_s, \quad \frac{E(1_{A_s} \Gamma_t)}{E(\Gamma_t)} \xrightarrow{t \to \infty} E(1_{A_s} M_s),
\]

2. \((M_s, s \geq 0)\) is a \((\mathcal{F}_s, P)\) positive martingale.

The proof of this Theorem 1.1 is quite elementary. It hinges on Scheffé’s lemma (see [9], p. 37, Theorem 21).

(c) Assume that the hypotheses of Theorem 1.1 are satisfied, and define, for \( s \geq 0 \), and \( \Lambda_s \in \mathcal{F}_s \):

\[
Q(\Lambda_s) := E(1_{\Lambda_s} M_s).
\]

Then, (1.4) induces a probability \( Q \) on \((\Omega, \mathcal{F}_\infty)\). In [11–13,15–17], we have described precisely the main properties of the canonical process \((X_t, t \geq 0)\) under \( Q \).

(d) The aim of the present paper is to show the existence of the limit \( P(t)(\Lambda_s) \), as \( t \to \infty \), \( s \) being fixed, and to study the canonical process \((X_t, t \geq 0)\) under \( Q \) – the Wiener measure \( P \) penalized by the process \((\Gamma_t, t \geq 0)\) – when \((\Gamma_t, t \geq 0)\) is defined in terms of lengths of excursions. Let us be more precise and fix notation: For \( t \geq 0 \), we denote by \( g_t \) (resp. \( d_t \)) the last zero of \((X_u, u \geq 0)\) before time \( t \), resp.: the first zero after time \( t \):

\[
g_t = \sup\{s \leq t; X_s = 0\}, \quad d_t = \inf\{s > t; X_s = 0\}.
\]

Hence, \((d_t - g_t)\) is the length of the excursion which straddles \( t \). We also introduce:

\[
\Sigma_t \equiv \Sigma_{g_t} = \sup\{d_s - g_s; d_s \leq t\},
\]

\[
\Sigma^*_t \equiv \Sigma_{d_t} = \sup\{d_s - g_s; g_s \leq t\}.
\]

Thus, \( \Sigma_t \) is the length of the longest excursion before \( g_t \), whereas \( \Sigma^*_t \) is the length of the longest excursion before \( d_t \). Consequently:

\[
\Sigma^*_t = \Sigma_t \vee (d_t - g_t).
\]

We denote by \((A_t, t \geq 0)\) the so-called age process:

\[
A_t := t - g_t \quad \text{and} \quad A^*_t := \sup_{s \leq t} A_s.
\]

Hence:

\[
A_t^* = \Sigma_t \vee (t - g_t) \quad \text{and} \quad \Sigma_t \leq A^*_t \leq \Sigma^*_t \leq A^*_t \vee (d_t - g_t).
\]

The aim of this article is to study the effects on Brownian motion of penalisations induced by the following processes \((\Gamma_t, t \geq 0)\):

(i) \( \Gamma_t := 1_{(\Sigma_t \leq x)} \) \((x > 0, \text{fixed})\). This is studied in Section 2.

(ii) \( \Gamma_t := h(\Sigma_t) \), where \( h \) is an “integrable function.” This study extends that made in (i), and is developed in Section 3;

(iii) \( \Gamma_t := 1_{(A_t^* \leq x)} \) \((x > 0, \text{fixed})\), and \( \Gamma_t = 1_{[\Sigma^*_t \leq x]} \) \((x > 0, \text{fixed})\). This is studied in Section 4.

However, in this case, we have not been able to obtain a full proof of the existence of the penalisised measure; we present a conjecture (4.5) upon which the existence rests.
(e) Some prerequisites relative to the Brownian meander.

As \( \Sigma_t \) is \( \mathcal{F}_{gt} \)-measurable, it turns out that certain features of the part of the trajectory of our Brownian motion \((X_s, s \geq 0)\) between times \(gt\) and \(t\) play some important role throughout the discussion. We now gather a few useful facts about the Brownian meander:

\[
\left( \tilde{m}_u^{(t)} = \frac{1}{\sqrt{t-g_t}} X_{g_t+u(t-g_t)}, 0 \leq u \leq 1 \right),
\]

which is a well-defined process, whose law, thanks to the scaling property of Brownian motion does not depend on \(t > 0\). (Here, we slightly depart from the classical Brownian terminology, for which it is \(m(t) \equiv (|\tilde{m}_u^{(t)}|, u \leq 1)\) which is called the Brownian meander.)

The simple fact, obtained by Brownian scaling, that the law of \(\tilde{m}(t)\) does not depend on \(t\), can be further extended as follows.

**Proposition 1.2.** Let \(T\) be a finite \(\{\mathcal{F}_t\}\) stopping time, such that: \(P(X_T = 0) = 0\). Then:

(i) the process \((\tilde{m}_u^{(T)}), u \leq 1\) is independent from \(\mathcal{F}_{gT}\), and its law does not depend on \(T\);

(ii) for any Borel function \(f : \mathbb{R} \rightarrow \mathbb{R}_+\), one has:

\[
E\left[ f(X_T) | \mathcal{F}_{gT} \right] = Kf(A_T),
\]

where \(K\) denotes the Markov kernel defined by:

\[
Kf(y) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{|z|}{y} \exp \left\{ -\frac{z^2}{2y} \right\} f(z) \, dz \quad (y \in \mathbb{R}_+).\]

In particular, if \(T = T^A_a = \inf\{t: A_t > a\}\), for \(a > 0\), then \(X_{T^A_a}\) and \(gT^A\) are independent; hence, \(X_{T^A_a}\) and \(T^A_a\) are independent, and:

\[
P(X_{T^A_a} \in dz) = \frac{|z|}{2a} \exp \left\{ -\frac{z^2}{2a} \right\} \, dz.\]  

(iii) Let \(\Psi_{1,x}(z) = \sqrt{\frac{2}{\pi x}} |z|\) and \(\Psi_{2,b}(z) = \Phi(\frac{|z|}{\sqrt{b}})\) for some \(x > 0\), and \(b > 0\), where

\[
\Phi(y) = \sqrt{\frac{2}{\pi}} \int_{-y}^{\infty} du \, e^{-u^2/2} = P(|G| > y),
\]

for \(G\) a standard Gaussian variable. Then,

\[
K\Psi_{1,x}(y) = \sqrt{\frac{y}{x}}, \quad K\Psi_{2,b}(y) = 1 - \sqrt{\frac{y}{b+y}}.
\]

We note that:

\[
K\Psi_{1,x}(y) + K\Psi_{2,x-y}(y) = 1 \quad \text{for all } y < x.
\]

**Proof.**

- Points (i) and (ii) are very classical; they are proven and applied in [1–3,10].
- Point (iii) follows from elementary computations. Indeed:

\[
K\Psi_{1,x}(y) = \sqrt{\frac{2}{\pi x}} \left( \frac{1}{2} \right) \int_{-\infty}^{\infty} \frac{z^2}{y} e^{-z^2/(2y)} \, dz = \sqrt{\frac{y}{x}}
\]
whilst:

\[ K\Psi_{b,2}(y) = \frac{1}{2} \int_{-\infty}^{\infty} |z| e^{-z^2/(2y)} \Phi \left( \frac{|z|}{\sqrt{b}} \right) dz \]

\[ = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} dz \frac{z}{y} e^{-z^2/(2y)} \int_{z/\sqrt{b}}^{\infty} e^{-u^2/2} du \]

\[ = \frac{1}{\sqrt{2}} \int_{0}^{\infty} e^{-u^2/2} du \int_{0}^{u/\sqrt{b}} z e^{-z^2/(2y)} dz \quad \text{(by Fubini)} \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-u^2/2} (1 - e^{-u^2/(2y)b}) du \]

\[ = 1 - \sqrt{\frac{b+y}{b+y}}. \]

\[ \square \]

2. Penalisation induced by \( \Gamma_t = 1_{[\Sigma_t \leq x]} \)

Here, \( x > 0 \) is fixed.

**Theorem 2.1.**

1. For every \( s \geq 0 \), and \( \Lambda_s \in \mathcal{F}_s \),

\[ Q(\Lambda_s) := \lim_{t \to \infty} \frac{E(1_{\Lambda_s} 1_{[\Sigma_t \leq x]})}{E(1_{[\Sigma_t \leq x]})} \text{ exists.} \]

\[ (2.1) \]

2. This limit induces a probability \( Q \) on \((\Omega, \mathcal{F}_\infty)\) such that:

\[ Q(\Lambda_s) := E(1_{\Lambda_s} \widetilde{M}_s 1_{[\Sigma_s \leq x]}) = E(1_{\Lambda_s} M_s), \]

\[ (2.2) \]

where:

\[ M_s := \widetilde{M}_s 1_{[\Sigma_s \leq x]}, \]

\[ (2.3) \]

\[ \widetilde{M}_s := |X_s| \sqrt{\frac{2}{\pi x}} + \Phi \left( \frac{|X_s|}{\sqrt{x-A_s}} \right) 1_{(A_s \leq x)} \]

\[ (2.4) \]

with \( \Phi \) given by (1.12). Moreover, \((M_s, s \geq 0)\) is a continuous, positive martingale, such that \( M_0 = 1 \).

3. Under \( Q \), the process \((X_t, t \geq 0)\) satisfies:

   (a) \( \Sigma_\infty \leq x \) a.s., and \( \sqrt{\Sigma_\infty/\Sigma_x} \) is uniformly distributed on \([0,1]\);

   (b) \( A_\infty^x = \infty \) a.s.;

   (c) Let \( g = \sup \{ t: X_t = 0 \} \). Then, \( Q(0 < g < \infty) = 1 \), and the law of \( g \) is given by:

\[ Q(g \geq t) = E \left[ \sqrt{\frac{A_t \wedge T^A_y}{x}} \right], \]

\[ (2.7) \]

where \( T^A_x := \inf \{ t \geq 0 : A_t = x \} \).

(d) Let \( 0 \leq y \leq x \), and denote \( T^A_y := \inf \{ t \geq 0 : A_t = y \} \). Then:
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(i) The process \((A_u, u \leq T^A_y)\) is identically distributed under \(P\) and \(Q\);
(ii) \((A_u, u \leq T^A_y)\) and \(X_{T^A_y}\) are independent under either \(P\) or \(Q\);
(iii) The density of the distribution of \(X_{T^A_y}\) under \(Q\) equals:

\[
f^Q_{X_{T^A_y}}(z) = \frac{|z|}{2y} e^{-z^2/(2y)} \left( |z| \sqrt{\frac{2}{\pi y}} + \Phi \left( \frac{|z|}{\sqrt{x-y}} \right) \right)
\]  

(2.8)

(iv) \(Q(g > T^A_y) = 1 - \sqrt{\frac{y}{x}}\);
(v) The process \((A_u, u \leq T^A_y)\) and the event \((g > T^A_y)\) are independent under \(Q\).

Moreover, under \(Q\):

(a) The processes \((X_u, u \leq g)\) and \((X_{g+u}, u \geq 0)\) are independent;
(b) The process \((X_{g+u}, u \geq 0)\) is positive, resp.: negative, with probability \(\frac{1}{2}\), and \(|X_{g+u}|, u \geq 0\) is a BES(3) process;
(c) Denoting by \((L_t, t \geq 0)\) the local time process of \(X\) at level 0, then:

- \(\sqrt{\frac{x}{\pi}} L_\infty\) is exponentially distributed, with mean 1;
- Conditionally on \(L_\infty = \ell\), the process \((X_u, u \leq g)\) is a Brownian motion \(B\) stopped at \(\tau_\ell := \inf\{t: L_t > \ell\}\), and conditioned on \(|\Sigma_{\tau_\ell} \leq x\).

**Remark 2.2.** Obviously, the probability measure \(Q\) defined by (2.1) and the martingale \((M_s)\) depend on the parameter \(x\). In this section, since \(x\) is fixed, there is no ambiguity. A generalisation of Theorem 2.1 will be given in Section 3 and we shall denote \(Q^{(x)}\) (resp. \(M^{(x)}\)) the p.m. (resp. the martingale) defined by (2.1) (resp. (2.3)).

**Proof of Theorem 2.1.**

(1) We begin with the following lemma.

**Lemma 2.3.**

\[
P(\Sigma_t \leq x) = P(\Sigma_1 \leq \frac{x}{t}) \sim \sqrt{\frac{x}{t}}
\]  

(2.9)

(the equality in (2.9) follows from the scaling property).

**Proof.** We might use the computation in Exercise 4.19 of [10], p. 507, which discusses a result of Knight [7], but we shall proceed from scratch by showing that:

(a) if \(S_\beta\) denotes an independent exponential time, with parameter \(\beta\), then:

\[
\beta \int_0^\infty e^{-\beta t} P(\Sigma_t \leq x) \, dt = P(\Sigma_{S_\beta} \leq x) = \frac{E[\sinh(\sqrt{2\beta} |X_{T^A_y}|)]}{E[\cosh(\sqrt{2\beta} X_{T^A_y})]}
\]  

(2.10)

Let us admit this result for one moment; then, as \(\beta \to 0\), we obtain:

\[
P(\Sigma_{S_\beta} \leq x) \sim \sqrt{2\beta} E[|X_{T^A_y}|].
\]

Using (1.11), we have

\[
E[|X_{T^A_y}|] = \frac{1}{2} \int_{-\infty}^\infty \frac{z^2}{x} \exp \left( -\frac{z^2}{2x} \right) \, dz = \sqrt{\frac{\pi x}{2}}.
\]
Thus, we have obtained:

\[
\int_0^\infty e^{-\beta t} P(\Sigma_t \leq x) \, dt \sim \sqrt{\frac{\pi x}{\beta}}. \tag{2.11}
\]

(b) We now prove formula (2.10): we first note that:

\[
\{ \Sigma_s^{\beta} \leq x \} = \{ \{ s \leq T^A_x \} = \{ s \leq d_{T^A_x} \}, \tag{2.12}
\]

hence:

\[
P(\Sigma_s^{\beta} \leq x) = 1 - E \left[ e^{-\beta d_{T^A_x}} \right].
\]

Let \((\theta_u)\) denote the usual family of time-translation operators:

\[
X_s \circ \theta_u = X_{s+u} \quad (s, u \geq 0). \tag{2.13}
\]

Since: 

\[
d_{T^A_x} = T^A_x + T_0 \circ \theta_{T^A_x},
\]

we obtain:

\[
P(\Sigma_s^{\beta} \leq x) = 1 - E \left[ e^{-\beta T^A_x} E_{T^A_x} \left[ e^{-\beta T_0} \right] \right],
\]

\[
P(\Sigma_s^{\beta} \leq x) = 1 - E \left[ e^{-\beta T^A_x} e^{-\sqrt{2\beta} |X_{T^A_x}|} \right].
\]

Next, we use the independence of \(T^A_x\) and \(X_{T^A_x}\), recalled in Proposition 1.2(ii), which yields:

\[
P(\Sigma_s^{\beta} \leq x) = 1 - E \left[ e^{-\beta T^A_x} \right] E \left[ e^{-\sqrt{2\beta} |X_{T^A_x}|} \right].
\]

Formula (2.10) now follows from Wald’s identity:

\[
E \left[ e^{\sqrt{2\beta} X_{T^A_x}} \right] E \left[ e^{-\beta T^A_x} \right] = 1 \tag{2.14}
\]

(2) We now show relations (2.1) and (2.3).

Let \(T_0 = \inf \{ s \geq 0; X_s = 0 \}\) and \(t \geq s \geq 0\). First, we observe:

\[
\{ \Sigma_t \leq x \} = \{ \Sigma_s \leq x \} \cap \left( \{ d_s > t \} \cup \{ d_s \leq t \wedge (s - A_s + x, \Sigma_{t-d_s} \circ \theta_{d_s} \leq x) \} \right). \tag{2.15}
\]

Hence:

\[
P(\Sigma_t \leq x | \mathcal{F}_s) = (1) + (2),
\]

where

\[
(1) = 1_{\{ \Sigma_s \leq x \}} P(\{ d_s > t \} | \mathcal{F}_s),
\]

\[
(2) = 1_{\{ \Sigma_s \leq x \}} P(\{ d_s \leq t \wedge (s - A_s + x, \Sigma_{t-d_s} \circ \theta_{d_s} \leq x) \} | \mathcal{F}_s).
\]

We now study the asymptotic behavior of (1), then (2), as \(t \to \infty\).

As is well known:

\[
P_\gamma (T_0 \geq t) = P_0 \left( \frac{|G|}{\sqrt{t}} \leq \frac{|x|}{\sqrt{t}} \right) \sim \sqrt{\frac{2}{\pi}} \frac{|x|}{\sqrt{t}}, \tag{2.16}
\]

\[
\sim \frac{1}{\sqrt{\pi t}}. \tag{2.16}
\]
where \( G \) denotes a standard \( \mathcal{N}(0, 1) \) Gaussian variable.

Since \( ds = s + T_0 \circ \theta_s \), the equivalence (2.16) implies:

\[
(1) = 1_{\{\Sigma_t \leq x\}} P_{X_t}(T_0 > t - s) \sim \sqrt{\frac{2}{\pi t}} |X_s| 1_{\{\Sigma_t \leq x\}} \frac{1}{\sqrt{t}}.
\]

As for the term (2), we apply both the strong Markov property at time \( ds \) and Lemma 2.3:

\[
(2) \sim 1_{\{\Sigma_t \leq x\}} \sqrt{\frac{x}{t}} P(\{ds \leq s - A_s + x\} \mid \mathcal{F}_s).
\]

From the Markov property at time \( s \), together with (1.12) and (2.16) we deduce:

\[
(2) \sim 1_{\{\Sigma_t \leq x\}} \sqrt{\frac{x}{t}} \Phi \left( \frac{|X_s|}{\sqrt{x - A_s}} \right).
\]

Finally

\[
P(\Sigma_t \leq x \mid \mathcal{F}_s) \sim \frac{1}{\sqrt{t}} \sqrt{x} \left( |X_s| \sqrt{\frac{2}{\pi x}} + \Phi \left( \frac{|X_s|}{\sqrt{x - A_s}} \right) 1_{\{\Sigma_t \leq x\}} \right) 1_{\{\Sigma_t \leq x\}}
\]

\[
\sim \frac{1}{\sqrt{t}} \sqrt{x} \sqrt{t} M_s 1_{\{\Sigma_t \leq x\}} = \sqrt{x} M_s.
\]

It is clear that (2.9) implies that:

\[
\frac{P(\Sigma_t \leq x \mid \mathcal{F}_s)}{P(\Sigma_t \leq x)} \sim \frac{M_s}{t \rightarrow \infty E[M_s]} = M_s,
\]

since \( E[M_s] = 1 \) follows from the next point.

(3.a) We begin with the following lemma.

**Lemma 2.4.**

(1) Let \( f : (y, a) \rightarrow f(y, a) \) be a \( C_{y,a}^{2,1} \) function, from \( \mathbb{R} \times \mathbb{R}_+ \) to \( \mathbb{R} \). Then, \( (f(X_t, A_t), t \geq 0) \) is a ((\( \mathcal{F}_t \)), \( P \)) semi-martingale, which decomposes as:

\[
f(X_t, A_t) = f(0, 0) + \int_0^t \frac{\partial f}{\partial y}(X_s, A_s) dX_s + \int_0^t \left( \frac{1}{2} \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial a} \right)(X_s, A_s) ds
\]

\[+ \sum_{s \leq t} (f(0, A_s) - f(0, A_{s-})).\]

In particular, if:

(i) \( f(0, a) \) does not depend on \( a \geq 0 \);

(ii) \( \frac{1}{2} \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial a} = 0 \),

then, \( (f(X_t, A_t), t \geq 0) \) is a ((\( \mathcal{F}_t \)), \( P \)) local martingale, with Itô representation:

\[
f(X_t, A_t) = f(0, 0) + \int_0^t \frac{\partial f}{\partial y}(X_s, A_s) dX_s.
\]

(2) Likewise, let \( f : (y, a) \rightarrow f(y, a) \) be a \( C_{y,a}^{2,1} \) function defined on \( \mathbb{R}_+ \times \mathbb{R}_+ \).
Then, \((f(\langle X_t, A_t \rangle, t \geq 0)\) is a \(((\mathcal{F}_t), P)\) semimartingale, which decomposes as:

\[
f(\langle X_t, A_t \rangle, t \geq 0) = f(0, 0) + \int_0^t \frac{\partial f}{\partial y}(\langle X_s, A_s \rangle) \text{sgn}(X_s) \, dX_s + \int_0^t \left( \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(\langle X_s, A_s \rangle) + \frac{\partial f}{\partial a}(\langle X_s, A_s \rangle) \right) ds + \sum_{s \leq t} (f(0, A_s) - f(0, A_{s-})) \cdot
\]

where \((L_t, t \geq 0)\) denotes the local time of \((X_t, t \geq 0)\) at level 0.

In particular, if \(f\) satisfies (i) and (ii) above, as well as:

(iii) \(\frac{\partial f}{\partial y}(0, 0) = 0\),

then \((f(\langle X_t, A_t \rangle, t \geq 0)\) is a \(((\mathcal{F}_t), P)\) local martingale, with Itô representation:

\[
f(\langle X_t, A_t \rangle, t \geq 0) = f(0, 0) + \int_0^t \frac{\partial f}{\partial y}(\langle X_s, A_s \rangle) \text{sgn}(X_s) \, dX_s.
\]

Proof.

(a) Since the process \((A_t, t \geq 0)\) has bounded variation, we may apply Itô’s formula to \(f(X_t, A_t)\) to obtain:

\[
f(X_t, A_t) = f(0, 0) + \int_0^t \left( \frac{\partial f}{\partial y}(X_s, A_s) \, dX_s + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(X_s, A_s) \, ds \right) + \gamma_t,
\]

where

\[
\gamma_t = \int_0^t \frac{\partial f}{\partial a}(X_s, A_s) \, ds + \sum_{s \leq t} (f(0, A_s) - f(0, A_{s-}))
\]

since the continuous part \((A^c_t)\) of \((A_t)\) is equal to \(t\), and moreover if \(\Delta A_s \neq 0\), then \(X_s = 0\), and \(A_s = 0\).

(b) We use similar arguments, together with the Tanaka decomposition:

\[
\langle X_t \rangle = \int_0^t \text{sgn}(X_s) \, dX_s + L_t, \quad t \geq 0,
\]

where \((L_t, t \geq 0)\) denotes the local time of \(X\) at 0.

We then use the fact that the support of \(dL_s\) is \(\{s: X_s = 0\}\), and if \(s\) is a zero of \(X\) then: \(A_s = 0\).

Thus:

\[
\int_0^t \frac{\partial f}{\partial y}(\langle X_s \rangle, A_s) \, dL_s = \frac{\partial f}{\partial y}(0, 0) L_t.
\]

\(3.b\) We now show that \((M_s, s \geq 0)\) is a local martingale.

We define:

\[
T^A_y := \inf\{t \geq 0: A_t \geq y\} \quad \text{and} \quad T^\Sigma_y := \inf\{t \geq 0: \Sigma_t \geq y\} \quad (y > 0).
\]

Clearly, one has:

\[
T^\Sigma_x = d_{T^A_y} = T^A_x + T_0 \circ \theta_{T^A_y}.
\]
We denote:

\[
\tilde{M}_s := 1_{\{s \leq T_{\Sigma}\}} \left( \left| X_s \right| \sqrt{\frac{2}{\pi x}} + \Phi \left( \frac{|X_s|}{\sqrt{x - A_s}} \right) \right)
\]

\[
+ 1_{\{s > T_{\Sigma}\}} \sqrt{\frac{2}{\pi x}} (\text{sgn } X_{T_{\Sigma}}) X_s.
\]

(2.18)

Then, clearly:

\[
\hat{M}_s = \tilde{M}_s 1_{\{\Sigma \leq x\}} = \tilde{M}_s 1_{\{s \leq T_{\Sigma}\}} = \hat{M}_{s \land T_{\Sigma}}.
\]

Thus, it remains to prove that \( (\hat{M}_s, s \geq 0) \) is a local martingale, and, for this purpose, it suffices to apply point (2) of Lemma 2.4 to the function:

\[
f(y, a) = y \sqrt{\frac{2}{\pi x}} + \Phi \left( \frac{y}{\sqrt{x - a}} \right) \quad (y \geq 0, x > a \geq 0).
\]

We note that (i), (ii) and (iii) of Lemma 2.4 hold:

\((\alpha)\) \quad f(0, a) = \Phi(0) = 1 \quad (\text{hence, (i) is satisfied}).

\((\beta)\) \quad \frac{\partial^2 f}{\partial y^2}(y, a) = \sqrt{\frac{2}{\pi}} \frac{y}{(x - a)^{3/2}} e^{-y^2/(2(x-a))},

\quad \frac{\partial f}{\partial a}(y, a) = -\frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{y}{(x - a)^{3/2}} e^{-y^2/(2(x-a))},

\quad \text{so that: } \frac{1}{2} \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial a} = 0 \quad (\text{hence, (ii) is satisfied}).

\((\gamma)\) \quad \frac{\partial f}{\partial y}(y, a) = \sqrt{\frac{2}{\pi x}} - \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x - a}} e^{-y^2/(2(x-a))},

\quad \text{so that: } \frac{\partial f}{\partial y}(0, 0) = 0 \quad (\text{hence, (iii) is satisfied}).

Finally, one has:

\[
M_s = 1 + \int_0^{s \land T_{\Sigma}} \left\{ \sqrt{\frac{2}{\pi x}} + \frac{1}{\sqrt{(x - A_u)_+}} \Phi \left( \frac{|X_u|}{\sqrt{(x - A_u)_+}} \right) \right\} \text{sgn}(X_u) \, dX_u,
\]

(2.20)

with the natural convention:

\[
\Phi \left( \frac{y}{z} \right) = 0 \quad \text{and} \quad \frac{1}{z} \Phi' \left( \frac{y}{z} \right) = 0, \quad \text{if } z = 0 \text{ and } y > 0.
\]

(2.21)

(3.c) We now show that \( (M_s, s \geq 0) \) is a true martingale.

Since \( T_{\Sigma}^* \) is a finite stopping time, the identity (2.19) and Doob’s optional stopping theorem imply that \( (M_s, s \geq 0) \) is a martingale as soon as \( (\hat{M}_s, s \geq 0) \) is a martingale.

Due to (2.21), we have:

\[
\hat{M}_s = |X_s| \sqrt{\frac{2}{\pi s}} + \Phi \left( \frac{|X_s|}{\sqrt{(x - A_s)_+}} \right).
\]

(2.22)
From the preceding (3.3), since \( \hat{M}_s, s \geq 0 \) is a positive local martingale, in order to prove that it is a true martingale, it suffices to show: \( E[\hat{M}_s] = 1 \). One has: \( E(\hat{M}_s) = (1) + (2) \), with:

\[
(1) = E \left[ 1_{\{s \leq T^A_x\}} \left( |X_s| \sqrt{\frac{2}{\pi x}} + \Phi \left( \frac{|X_s|}{\sqrt{x-A_s}} \right) \right) \right],
\]

\[
(2) = \sqrt{\frac{2}{\pi x}} E \left[ 1_{\{s > T^A_x\}} (\text{sgn} \, X_{T^A_x}) X_s \right].
\]

Since \( T^A_x \) is a \( (\mathcal{F}_s) \) stopping time, \( \{s \leq T^A_x\} \) belongs to \( \mathcal{F}_s \). Using now Proposition 1.2, we get:

\[
(1) = E \left[ 1_{\{s \leq T^A_x\}} \left( |X_s| \sqrt{\frac{2}{\pi x}} + \Phi \left( \frac{|X_s|}{\sqrt{x-A_s}} \right) \right) \bigg| \mathcal{F}_s \right] = E \left[ 1_{\{s \leq T^A_x\}} \left( K \Psi_{1,x}(A_s) + K \Psi_{2,x-A_s}(A_s) \right) \right] = E \left[ 1_{\{s \leq T^A_x\}} \left( \sqrt{\frac{A_s}{x}} + 1 - \sqrt{\frac{A_s}{x}} \right) \right] = P(s \leq T^A_x).
\]

As for (2), we use again Proposition 1.2:

\[
(2) = \sqrt{\frac{2}{\pi x}} E \left[ 1_{\{s > T^A_x\}} (\text{sgn} \, X_{T^A_x})(X_{T^A_x} + X_s - X_{T^A_x}) \right] = \sqrt{\frac{2}{\pi x}} E \left[ 1_{\{s > T^A_x\}} |X_{T^A_x}| \right] = \sqrt{\frac{2}{\pi x}} P(s > T^A_x) E[|X_{T^A_x}|] = P(s > T^A_x) K \Psi_{1,x}(x) = P(s > T^A_x).
\]

Finally:

\[
E(\hat{M}_s) = P(s \leq T^A_x) + P(s > T^A_x) = 1. \tag{2.23}
\]

This ends the proof of point (2) in Theorem 2.1.

(4) We now show that, under \( Q \), \( \sqrt{\frac{\Sigma_s}{x}} \) is uniformly distributed on \([0, 1]\).

(4.a) We have:

\[
Q(\Sigma_s \leq x) = \lim_{t \to \infty} \frac{P(\Sigma_s \leq x, \Sigma_t \leq x)}{P(\Sigma_t \leq x)} = \frac{P(\Sigma_t \leq x)}{P(\Sigma_t \leq x)} = 1.
\]

Thus:

\[
Q(\Sigma_{\infty} \leq x) = 1. \tag{2.24}
\]

(4.b) Proof of point (3.a) (of Theorem 2.1).

Let \( y \in [0, x] \), and \( s > 0 \). According to (2.2) and Doob’s optional stopping theorem, we have:

\[
Q(\Sigma_s > y) = Q(T^\Sigma_y < s) = E[1_{\{T^\Sigma_y < s\}} M_s] = E[1_{\{T^\Sigma_y < s\}} M_{T^\Sigma_y}].
\]

Consequently:

\[
Q(\Sigma_{\infty} > y) = Q(T^\Sigma_y < \infty) = E[ M_{T^\Sigma_y}]. \tag{2.25}
\]
However:

\[ M_{T_y} = \begin{cases} 
1, & \text{if the length of the 1st excursion of length } \geq y \text{ is smaller than } x, \\
0, & \text{otherwise}. 
\end{cases} \tag{2.26} \]

To compute \( E(M_{T_y}) \), we use the excursions theory for Brownian motion (see, for instance, [10], Chapter XII). Let \( e = (e_s, s > 0) \) be the excursion process related to \((X_t)\) under \( P \). We introduce, for any excursion \( \varepsilon \), its duration \( \zeta(\varepsilon) \), and let

\[ U = \{ \varepsilon; \zeta(\varepsilon) \geq y \}, \quad \Gamma = \{ \varepsilon; y \leq \zeta(\varepsilon) \leq x \} \quad (y < x) \tag{2.27} \]

and \( N_t^U \) the number of excursions starting before time \( t \), and belonging to the set \( U \):

\[ N_t^U = \sum_{s \leq t} 1_{\{e_s \in U\}}. \]

We now consider the \((\mathcal{F}_t)\) stopping time \( S^y \):

\[ S^y = \inf\{l > 0; \zeta(e_l) \geq y\} = \inf\{t > 0; N_t^U > 0\}, \]

where \( \tau_l = \inf\{t \geq 0; L_t > l\} \).

From [10], Lemma 1.13, Chapter XII and (2.25), one has:

\[ E[M_{T_y}] = P(\zeta(e_{S^y}) \in U) = \frac{n(\Gamma)}{n(U)}, \]

with \( n \) denoting Itô’s excursion measure.

It is easy to compute \( n(\Gamma) \) and \( n(U) \):

\[ n(\Gamma) = \frac{1}{\sqrt{2\pi}} \int_y^x \frac{dr}{\sqrt{r^3}} \sqrt{\frac{x}{y}} = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\sqrt{y}} - \frac{1}{\sqrt{x}} \right), \quad n(U) = \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{y}}. \]

Consequently:

\[ Q(\Sigma_\infty > y) = 1 - \sqrt{\frac{y}{x}}. \tag{2.28} \]

This ends the proof of point (3.a) in Theorem 2.1.

(4.c) We determine the density function of \( L_\infty \) under \( Q \).

Let \( a > 0 \). We have:

\[ Q(L_\infty > a) = Q(\tau_a < \infty) = \lim_{t \to \infty} Q(\tau_a < t), \]

where \( \tau_a = \inf\{s; L_s \geq a\} \).

According to (2.2), (2.3) and the optimal stopping theorem, we get:

\[ Q(\tau_a < t) = E[1_{\{\tau_a < t\}}M_t] = E[1_{\{\tau_a < t\}}M_{\tau_a}] = P(\tau_a < t, \Sigma_{\tau_a} \leq x). \]

Consequently, using notation introduced in (4.b) above, we obtain:

\[ Q(L_\infty > a) = P(\Sigma_{\tau_a} \leq x) = \exp(-an(\zeta \geq x)). \]

Since:

\[ n(\zeta \geq x) = \int_x^\infty \frac{dr}{\sqrt{2\pi r^3}} = \sqrt{\frac{2}{\pi x}}, \]

\[ n(\zeta \geq x) = \int_x^\infty \frac{dr}{\sqrt{2\pi r^3}} = \sqrt{\frac{2}{\pi x}}, \]

\[ n(\zeta \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{dr}{\sqrt{r^3}} = \sqrt{\frac{2}{\pi x}}, \]

\[ n(\zeta \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{dr}{\sqrt{r^3}} = \sqrt{\frac{2}{\pi x}}, \]

\[ n(\zeta \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{dr}{\sqrt{r^3}} = \sqrt{\frac{2}{\pi x}}, \]

\[ n(\zeta \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{dr}{\sqrt{r^3}} = \sqrt{\frac{2}{\pi x}}, \]

\[ n(\zeta \geq x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{dr}{\sqrt{r^3}} = \sqrt{\frac{2}{\pi x}}, \]
we get:

\[
Q(L_\infty > a) = \exp\left\{-a\sqrt{\frac{2}{\pi x}}\right\} \quad (a > 0).
\]  

(2.29)

This proves the first part of point (4.c) in Theorem 2.1.

(5) We show that, for any \( y < x \), \( E[M_{T_y^A}] = 1 \).

According to identity (2.3), we have:

\[
M_{T_y^A} = |X_{T_y^A}| \sqrt{\frac{2}{\pi x}} + \Phi\left(\frac{|X_{T_y^A}|}{\sqrt{x-y}}\right).
\]  

(2.30)

From Proposition 1.2, we deduce:

\[
E[M_{T_y^A}] = K\Psi_{1,x}(y) + K\Psi_{2,x-y}(y) = \sqrt{\frac{y}{x}} + 1 - \sqrt{\frac{y}{x}} = 1.
\]  

(2.31)

We note that the preceding computation yields another proof of (2.28).

(6) We now show:

\[
Q(A^*_\infty = \infty) = 1.
\]

Let \( 0 < \eta < 1 \). We will prove that \( Q(A^*_\infty = \infty) \geq \sqrt{1-\eta} \), which will yield the result.

Similarly to the proof of point (4.b) one has: \( Q(T_y^A < \infty) = E[M_{T_y^A}] \), for any \( y \in [0, x] \). Identity (2.31) implies that \( Q(T_y^A < \infty) = 1 \).

From point (3.a) of Theorem 2.1:

\[
Q(\Sigma_{\infty} < x - x\eta) = \sqrt{1-\eta}.
\]

Consequently:

\[
Q(\{\Sigma_{\infty} < x - x\eta\} \cap \{T_y^A < \infty\}) = \sqrt{1-\eta}, \quad \text{where} \ x - x\eta < y < x.
\]

Let us observe that on the set \( \{\Sigma_{\infty} < x(1 - \eta) \cap \{T_y^A < \infty\} \), \( X_t \) does not vanish after time \( T_y^A \). In particular, one has: \( A^*_\infty = \infty \). Consequently \( Q(A^*_\infty = \infty) \geq \sqrt{1-\eta} \).

We also observe that \( \Sigma_{\infty} < \infty \) and \( A^*_\infty < \infty \), \( Q \) a.s. imply that:

\[
Q(g < \infty) = 1,
\]  

(2.32)

where

\[
g = \sup\{t : X_t = 0\}.
\]  

(2.33)

(7) We then prove simultaneously that the process \((A_u, u \leq T_y^A)\) has the same distribution under \(P\) and \(Q\), and that it is independent from the r.v. \(X_{T_y^A}\) under \(Q\) (and under \(P\)), and we compute the density of \(X_{T_y^A}\) under \(Q\).

(7.a) From the definition (2.2) of \(Q\), for every positive functional \(F\), and every Borel function \(h: \mathbb{R} \to \mathbb{R}_+\), we have:

\[
E_Q[F(A_u, u \leq T_y^A)h(X_{T_y^A})] = E[F(A_u, u \leq T_y^A)h(X_{T_y^A})M_{T_y^A}].
\]  

(2.34)

Note that \(M_{T_y^A}\) is a function of \(X_{T_y^A}\), and recall that from Proposition 1.2, \(X_{T_y^A}\), and \((A_u, u \leq T_y^A)\) are \(P\)-independent. Thus:

\[
E_Q[F(A_u, u \leq T_y^A)h(X_{T_y^A})] = E[F(A_u, u \leq T_y^A)]E[h(X_{T_y^A})M_{T_y^A}].
\]  

(2.35)

(7.b) Now, taking \(h \equiv 1\), and using (2.31), we obtain:

\[
E_Q[F(A_u, u \leq T_y^A)] = E[F(A_u, u \leq T_y^A)]
\]

(2.36)
thus proving the first point announced in (7).

(7.c) We may now write (2.35) as follows:

\[
E_Q\left[ F\left( A_u, u \leq T^A \right) h\left( X_{T^A} \right) \right] = E\left[ F\left( A_u, u \leq T^A \right) \right] E\left[ h\left( X_{T^A} \right) M_{T^A} \right]
\]

\[
= E_Q\left[ F\left( A_u, u \leq T^A \right) \right] E_Q\left[ h\left( X_{T^A} \right) \right],
\]

thus proving the independence of \( (A_u, u \leq T^A) \) and \( X_{T^A} \) under \( Q \).

(7.d) We now compute the density of \( X_{T^A} \) under \( Q \).

One has:

\[
E_Q\left[ h\left( X_{T^A} \right) \right] = E\left[ h\left( X_{T^A} \right) M_{T^A} \right].
\]

(2.36)

Hence, from formulae (2.30) and (1.11) we get:

\[
Q\left( X_{T^A} \in dz \right) = \frac{|z|}{2\pi} e^{-z^2/(2y)} \left( \frac{|z|}{\sqrt{2\pi x}} + \Phi\left( \frac{|z|}{\sqrt{x-y}} \right) \right) dz.
\]

(2.37)

(8) We show the independence under \( Q \) of \( (A_u, u \leq T^A) \) and of the set \( \{ g > T^A \} \), and we compute \( Q(g > T^A) \).

For every positive functional \( F \), one has:

\[
E_Q\left[ 1_{\{ g > T^A \}} F\left( A_u, u \leq T^A \right) \right] = E\left[ F\left( A_u, u \leq T^A \right) \right] Q\left( g > T^A \right).
\]

We now admit for an instant the result of Lemma 2.5, which will be proved below:

\[
Q\left( g > t | F_t \right) = \Phi\left( \frac{|X_t|}{\sqrt{x-A_t}} \right) \frac{1}{M_t} 1_{\{ A_t \leq x \}}.
\]

(2.38)

Replacing \( t \) by the \((F_t)\)-stopping time \( T^A \), we get:

\[
Q\left( g > T^A | F_{T^A} \right) = \Phi\left( \frac{|X_{T^A}|}{\sqrt{x-y}} \right) \frac{1}{M_{T^A}}.
\]

(2.39)

Since the right-hand side of (2.39) only depends on \( X_{T^A} \), we deduce from the above point (7) that:

\[
E_Q\left[ 1_{\{ g > T^A \}} F\left( A_u, u \leq T^A \right) \right] = Q\left( g > T^A \right) E_Q\left[ F\left( A_u, u \leq T^A \right) \right].
\]

Hence the desired independence property.

Also, from (2.39), (2.34) and Proposition 1.2, we deduce:

\[
Q\left( g > T^A \right) = E_Q\left[ \Phi\left( \frac{|X_{T^A}|}{\sqrt{x-y}} \right) \frac{1}{M_{T^A}} \right] = E\left[ \Phi\left( \frac{|X_{T^A}|}{\sqrt{x-y}} \right) \right] = 1 - \sqrt{\frac{y}{x}}.
\]

(9) To end the proof of Theorem 2.1, we shall now use the technique of progressive enlargement of filtration (see [6]). We have proven (see point (6) above) that \( Q(g < \infty) = 1 \), where \( g \) is defined by (2.33). We denote by \((G_t, t \geq 0)\) the smallest filtration which contains \((F_t, t \geq 0)\), and which makes \( g \) a \((G_t, t \geq 0)\) stopping time. In order to use the technique of enlargement of filtration, we need the following lemma.

**Lemma 2.5.** (i) For any \( t > 0 \), we have:

\[
Z_t := Q(g > t | F_t) = \Phi\left( \frac{|X_t|}{\sqrt{x-A_t}} \right) \frac{1}{M_t} 1_{\{ A_t \leq x \}}.
\]

(2.40)
Let $(k_s)$ be a predictable and non-negative process; then
\begin{equation}
E_Q[k_s] = \sqrt{\frac{2}{\pi x}} E \left[ \int_{0}^{\infty} k_s 1_{\{s \leq T_{A_s}^x\}} dL_s \right]. \tag{2.41}
\end{equation}

Proof.

(i) For every $A_t \in \mathcal{F}_t$, one has:
\[
Q(A_t \cap \{g > t\}) = Q(A_t \cap \{d_t < \infty\}) = E[1_{A_t} M_{d_t}] = P(A_t \cap \{\Sigma_{d_t} \leq x, A_{d_t} \leq x\}).
\]

Observe that:
\[
\{\Sigma_{d_t} \leq x, A_{d_t} \leq x\} = \{A^{*}_{t} \leq x, d_{t} - t \leq x - A_t\} = \{A^{*}_{t} \leq x, T_0 \circ \theta_t \leq x - A_t\}.
\]

Applying the Markov property at time $t$ and identity (2.16) leads to:
\[
Q(A_t \cap \{g > t\}) = E\left[1_{A_t} 1_{\{A^{*}_{t} \leq x\}} \Phi\left(\frac{|X_t|}{\sqrt{x - A_t}}\right)\right]
= E_Q\left[1_{A_t} 1_{\{A^{*}_{t} \leq x\}} \Phi\left(\frac{|X_t|}{\sqrt{x - A_t}}\right) \frac{1}{M_t}\right].
\]

(ii) Replacing $t$ in (2.40) by any $\mathcal{F}_t$ finite stopping time $T$, taking the expectation, and using (2.3), we get:
\[
Q(g \leq T) = E_Q\left[1_{[0,T]}(g)\right] = 1 - E\left[1_{\{A^{*}_T \leq x\}} \Phi\left(\frac{|X_T|}{\sqrt{x - A_T}}\right)\right]
= \sqrt{\frac{2}{\pi x}} E\left[|X_T \wedge T_{A_T}^x|\right] = \sqrt{\frac{2}{\pi x}} E\left[\int_{0}^{\infty} 1_{[0,T]}(s) 1_{\{s \leq T_{A_s}^x\}} dL_s\right].
\]

Point (ii) now follows by an application of the monotone class theorem. \qed

We now indicate how to make use of Lemma 2.5.

Our study in (3.c) and in particular (2.20) imply that:
\begin{equation}
M_t = \mathcal{E}(J)_t := \exp\left\{\int_{0}^{t} J_s dX_s - \frac{1}{2} \int_{0}^{t} J_s^2 ds\right\}, \quad t < T_x^\Sigma,
\end{equation}
with
\begin{equation}
J_s = \left(\sqrt{\frac{2}{\pi x}} + \frac{1}{\sqrt{x - A_s}} \Phi\left(\frac{|X_s|}{\sqrt{x - A_s}}\right)\right) \text{sgn} X_s M_s, \quad s < T_x^\Sigma.
\end{equation}

From Girsanov’s theorem, there exists a $((\mathcal{F}_t, t \geq 0), Q)$ Brownian motion $(\beta_t, t \geq 0)$ such that:
\begin{equation}
X_t = \beta_t + \int_{0}^{t} \left(\sqrt{\frac{2}{\pi x}} + \frac{1}{\sqrt{(x - A_s)_+}} \Phi\left(\frac{|X_s|}{\sqrt{(x - A_s)_+}}\right)\right) \text{sgn} X_s M_s \frac{d(Z, \beta)_s}{Z_s}, \quad t \geq 0,
\end{equation}
and the enlargement formulae imply that there exists a $((\mathcal{G}_t, t \geq 0), Q)$ Brownian motion $(\tilde{\beta}_t, t \geq 0)$ such that:
\begin{equation}
\beta_t = \tilde{\beta}_t + \int_{0}^{t \wedge g} \frac{d(Z, \beta)_u}{Z_u} - \int_{t \wedge g}^{t} \frac{d(Z, \beta)_u}{1 - Z_u}.
\end{equation}

Once we shall have computed explicitly $d(Z, \beta)_u$, these formulae (2.44) and (2.45) will help us to describe the process $(X_t, t \geq 0)$ under $Q$ (see points (11) and (12) below).
Brownian penalisations related to excursion lengths

(10) We compute the law of \( g \) under \( Q \).
Note that from our convention (2.21) we have:

\[
\Phi\left(\frac{|X_t|}{\sqrt{(x - A_t)_+}}\right) = \Phi\left(\frac{|X_t|}{\sqrt{A_t}}\right)1_{\{A_t < x\}}.
\]  

(2.46)

Since \( \{A_t^* < x\} = \{A_{g_t}^* < x, t - g_t < x\} \), from (2.40), we have:

\[
Q(g \geq t) = E\left[1_{\{A_t^* < x\}}\phi\left(\frac{|X_t|}{\sqrt{A_t}}\right)\right] = E\left[1_{\{A_t^* < x\}}E\left\{\phi\left(\frac{|X_t|}{\sqrt{A_t}}\right)|\mathcal{F}_{g_t}\right\}\right].
\]

Using Proposition 1.2 we get:

\[
Q(g \geq t) = E\left[1_{\{A_t^* < x\}}K_{2,x-A_t}(A_t)\right] = E\left[1_{\{A_t^* < x\}}\left(1 - \sqrt{\frac{A_t}{x}}\right)\right].
\]

This yields formula (2.7).

(11) We now show that, under \( Q \), \((|X_{g+u}|, u \geq 0)\) is a three-dimensional Bessel process, which is independent from \((X_u, u \leq g)\).

From formulae (2.44) and (2.45), we have, for \( t \geq g \):

\[
X_t = \tilde{\beta}_t - \int_g^t \frac{d(Z, \beta)_u}{1 - Z_u} dt + \int_0^t \left(\sqrt{\frac{2}{\pi x}} + \frac{1}{\sqrt{(x - A_u)_+}}\phi\left(\frac{|X_u|}{\sqrt{x - A_u}}\right)\right) \sgn X_u \mu_u dt.
\]

(2.47)

However, due to Itô's formula, (2.20), (2.3) and Lemma 2.5, the martingale part \( \tilde{Z} \) of \( Z \), satisfies:

\[
\begin{align*}
\mathrm{d}\tilde{Z}_t &= -\frac{1}{M_t^2}\phi\left(\frac{|X_t|}{\sqrt{(x - A_t)_+}}\right) \\
&\quad \times \left\{\sqrt{\frac{2}{\pi x}} + \frac{1}{\sqrt{(x - A_t)_+}}\phi\left(\frac{|X_t|}{\sqrt{(x - A_t)_+}}\right)\right\} \sgn X_t \phi\left(\frac{|X_t|}{\sqrt{(x - A_t)_+}}\right) dt \\
&\quad + \frac{1}{M_t} \frac{1}{\sqrt{(x - A_t)_+}}\phi\left(\frac{|X_t|}{\sqrt{(x - A_t)_+}}\right) \sgn X_t \phi\left(\frac{|X_t|}{\sqrt{(x - A_t)_+}}\right) dt \\
&\quad - \phi\left(\frac{|X_t|}{\sqrt{(x - A_t)_+}}\right) \mu_u \mathrm{d}X_t
\end{align*}
\]

(2.48)

Hence, with the help of (2.44):

\[
\begin{align*}
\mathrm{d}(Z, \beta)_t &= \frac{1}{M_t^2} \sqrt{\frac{2}{\pi x}} (\sgn X_t) \left\{\frac{|X_t|}{\sqrt{(x - A_t)_+}}\phi\left(\frac{|X_t|}{\sqrt{(x - A_t)_+}}\right)
\right.
\end{align*}
\]

(2.49)
On the other hand, from Lemma 2.5, (2.3) and (2.4), we have:

\[
\frac{1}{1 - Z_t} = \frac{M_t}{M_t - \Phi(|X_t|/\sqrt{(x - A_t)})} = \sqrt{\pi x} M_t \frac{1}{2 \cdot |X_t|}. \tag{2.50}
\]

Since \( X_g = 0 \), plugging now (2.49) and (2.50) into (2.47) leads, after simplification:

\[
X_{t+g} = \tilde{\beta}_{t+g} - \tilde{\beta}_g + \int_0^t \frac{\text{sgn} X_{s+g}}{M_{s+g}} \left( \sqrt{\frac{2}{\pi x}} + \frac{1}{|X_{s+g}|} \Phi\left( \frac{|X_{s+g}|}{\sqrt{(x - A_{s+g})}} \right) \right) \, ds
\]

\[= \tilde{\beta}_{t+g} - \tilde{\beta}_g + \int_0^t \frac{\text{sgn} X_{s+g}}{|X_{s+g}|} \, ds \quad \text{(from (2.3) and (2.4)).} \tag{2.51}
\]

It now remains to note that \( \text{sgn}(X_{s+g}) \) is constant under \( Q \), and that:

\[Q(\text{sgn}(X_{s+g}) = 1, \forall s > 0) = Q(\text{sgn}(X_{s+g}) = -1, \forall s > 0) = \frac{1}{2}.
\]

We also note that the independence of \( (X_u, u \leq g) \) and \( (X_{t+g}, t \geq 0) \) follows classically from the fact that the modification of equation (2.51) written for \( |X_{t+g}|, t \geq 0 \), admits only one strong solution.

(12) We now describe the law of \( (X_u, u \leq g) \) under \( Q \).

From (2.41), we deduce that for any \( \mathbb{R}_+ \)-valued predictable process \( (F(X_u, u \leq s), s \geq 0) \), and any Borel function \( h \geq 0 \):

\[
E_Q[F(X_u, u \leq g)h(L_g)]
\]

\[= \sqrt{\frac{2}{\pi x}} E\left[ \int_0^\infty F(X_u, u \leq s)h(L_s)1_{\{A^*_s \leq x\}} \, dL_s \right]
\]

\[= \sqrt{\frac{2}{\pi x}} E\left[ \int_0^\infty F(X_u, u \leq \tau_{\ell})h(\ell)1_{\{A^*_\ell \leq x\}} \, d\ell \right] \tag{2.52}
\]

with \( \tau_{\ell} = \inf\{s \geq 0: L_s > \ell\} \).

Taking \( F = 1 \) in (2.52), we deduce that the density function of \( L_g = L_\infty \) under \( Q \) is

\[\sqrt{\frac{2}{\pi x}} P(A^*_\ell \leq x).\]

Consequently:

\[E_Q[F(X_u, u \leq g)h(L_g)] = \int_0^\infty E[F(X_u, u \leq \tau_{\ell})1_{\{A^*_\ell \leq x\}}]h(\ell)Q(L_\infty \in d\ell).\]

Thus, the law under \( Q \) of \( (X_u, u \leq g) \), conditioned on \( \{L_\infty = \ell\} \), is that of Brownian motion stopped at \( \tau_{\ell} \), and conditioned by the event \( \{A^*_\ell \leq x\} = \{\Sigma_{\tau_{\ell}} \leq x\} \).

This ends up the proof of Theorem 2.1. \( \square \)

**Remark 2.6.** The technique of enlargement of filtration, applied before \( g \) (see (2.44) and (2.45), with \( t \leq g \), yields:

\[X_t = \tilde{\beta}_t + \int_0^{g \wedge t} \frac{d(Z, \beta)_u}{Z_u} \]  

\[+ \int_0^{g \wedge t} \left\{ \sqrt{\frac{2}{\pi x}} + \frac{1}{\sqrt{(x - A_u)_+}} \Phi\left( \frac{|X_u|}{\sqrt{(x - A_u)_+}} \right) \right\} \frac{\text{sgn} X_u}{M_u} \, du.\]
Hence with (2.49) and (2.3)–(2.4), we obtain:

\[ X_t = \tilde{\beta}_t + \int_0^{g/t} \frac{1}{\sqrt{(x - A_u)_+}} \frac{\Phi'(\frac{|X_u|}{\sqrt{(x - A_u)_+}})}{\Phi\left(\frac{|X_u|}{\sqrt{(x - A_u)_+}}\right)} \text{sgn} X_u \, du. \quad (2.53) \]

We note that, in this equation, the drift term tends to \(-\infty\) (resp: \(+\infty\)) when \(u \to T^A_x\), with \(X_u > 0\) (resp: \(X_u < 0\)).

(2.53) shows that \((X_t, A_t, t \leq g)\) is Markov. We would like to point out that the theory of enlargement of filtrations is very helpful here since it permits to determine the law of \((X_t, A_t, t \leq g)\) as expressed in Theorem 2.1, point (4.c) – although it seems rather difficult to do it only from (2.53).

3. Penalisation with a function of \(\Sigma_t\)

The aim of this section is to extend the results of the preceding section, by replacing the penalisation functional \(1_{\{\Sigma_t \leq x\}}\) by a functional of the form \(h(\Sigma_t)\). We shall use the following notation: let \(\psi : \mathbb{R}_+ \to \mathbb{R}_+\) be a function which is almost surely differentiable and such that:

\[ \int_0^{\infty} \psi(x) \, dx = 1. \]

In particular, \(\psi\) is a probability density on \(\mathbb{R}_+\). We also introduce:

\[ h(x) := 2\sqrt{x} \psi(x) \quad (x \geq 0), \quad (3.1) \]
\[ \Psi_1(x) := \int_0^x \psi(y) \, dy \quad (x \geq 0), \quad (3.2) \]
\[ h_1(x) := 2x \psi(x) + 1 - \Psi_1(x) \quad (x \geq 0). \quad (3.3) \]

We assume that:

\[ x \psi(x) \longrightarrow 0 \quad (x \to 0) \quad (3.4) \]
as well as:

\[ x \psi(x) \longrightarrow 0 \quad (x \to \infty) \quad (3.5) \]

and

\[ \int_0^{\infty} x |\psi'(x)| \, dx < \infty. \quad (3.6) \]

We shall now take \(h(\Sigma_t)\) as our penalisation functional. Note that Theorem 2.1 corresponds to the following choice of \(\psi\):

\[ \psi(u) = \frac{1}{2\sqrt{u}} 1_{[0,x]}(u) \quad (x \text{ fixed}). \quad (3.7) \]

Remark 3.1. We note that the definition and assumptions (3.1) to (3.6) imply that:

\[ \int_0^{\infty} \sqrt{x} |h'(x)| \, dx < \infty, \quad (3.8) \]
\[ h_1(x) = -\int_x^{\infty} \sqrt{y} h'(y) \, dy \quad (3.9) \]
and
\[ h_1(0) = 1. \tag{3.10} \]

Indeed:
\[
- \int_{\gamma}^{\infty} \sqrt{y} h'(y) dy = -[2y \psi(y)]_{\gamma}^{\infty} + \int_{\gamma}^{\infty} \psi(y) dy = 2x \psi(x) + 1 - \Psi_1(x).
\]

**Theorem 3.2.**

1. For any \( s \geq 0 \), and every \( \Lambda_s \in \mathcal{F}_s \):
   \[
   \lim_{t \to \infty} E\left[ 1_{\Lambda_s} h(\Sigma_t) \right] \exists.
   \tag{3.11}
   \]

2. This limit is equal to:
   \[
   Q^\psi(\Lambda_s) := E\left[ 1_{\Lambda_s} M^\psi_s \right] \tag{3.12}
   \]
   with:
   \[
   M^\psi_s = \sqrt{\frac{2}{\pi}} h(\Sigma_s) |X_s| + h_1(\Sigma_s) \Phi\left( \frac{|X_s|}{\sqrt{\Sigma_s - A_s}} \right)
   + \sqrt{\frac{2}{\pi}} \int_{|X_s|/\sqrt{\Sigma_s - A_s}}^{\infty} h_1\left( A_s + \frac{X_s^2}{v^2} \right) e^{-v^2/2} dv;
   \tag{3.13}
   \]
   the function \( \Phi \) has been defined by (1.12) and the convention (2.21) holds.

Moreover, \((M^\psi_s, s \geq 0)\) is a continuous positive martingale such that \( M^\psi_0 = 1 \).

3. The formula (3.12) induces a probability on \((\Omega, \mathcal{F}_\infty)\).

4. Under \( Q^\psi \), the canonical process \((X_t, t \geq 0)\) satisfies:
   
   (a) \( \Sigma_\infty \) is finite a.s., and admits \( \psi \) as its density.
   
   (b) \( \Lambda_\infty \) is a.s. infinite.
   
   (c) Let \( g := \sup\{s: X_s = 0\} \). Then: \( Q^\psi(0 < g < \infty) = 1 \), and for every \( t \geq 0 \):
   \[
   Q^\psi(g > t) = -\int_0^{\infty} h'(x) E\left[ \sqrt{A_t \wedge T_{\ell}} \right] dx. \tag{3.14}
   \]
   (d) The processes \((X_t, t \leq g)\) and \((X_{g+t}, t \geq 0)\) are independent.
   
   (e) With probability 1/2, \((X_{g+t}, t \geq 0)\) is positive (resp. negative), and \((|X_{g+t}|, t \geq 0)\) is the 3-dimensional Bessel process starting from 0.
   
   (f) Conditionally on \( L_\infty = L_\gamma = \ell \), and \( \Sigma_\ell \leq x \), the law of \((X_u, u \leq g)\) is that of Brownian motion \( B \) considered until \( \tau_\ell \), i.e.: \( (B_u, u \leq \tau_\ell) \), and conditioned on \( \Sigma_\ell \leq x \).

**Remark 3.3.** It seems, when first looking at the definition of \( M^\psi_s \) (see (3.13)) that there might be some ambiguity concerning the definition of \( M^\psi_s \) at the times \( s \), which are ends of excursions; but, a careful inspection shows that there is no ambiguity.

**Remark 3.4.** (1) The martingale \( (M^\psi_s, s \geq 0) \) defined by (3.13) is equal to:
\[
M^\psi_s = -\int_0^{\infty} \sqrt{x} h'(x) M_s^x dx. \tag{3.15}
\]
where \((M^x_s, s \geq 0)\) is the martingale defined in Theorem 2.1 by (2.3) (see also Remark 2.2).

Indeed, we have:

\[- \int_0^\infty \sqrt{x} h'(x) M^x_s \, dx = (1) + (2), \tag{3.16}\]

with:

\[(1) = - \frac{2}{\pi} |X_1| \int_{\Sigma_s} h'(x) \, dx = \sqrt{\frac{2}{\pi}} h(\Sigma_s)|X_1| \quad \text{(from (3.5))} \tag{3.17}\]

and:

\[(2) = - \int_{\Sigma_s} \int_0^\infty 1_{\{A_s < x\}} \Phi\left( \frac{|X_1|}{\sqrt{x - A_s}} \right) \sqrt{x} h'(x) \, dx \]
\[= - \frac{2}{\pi} \int_{\Sigma_s \vee A_s} \int_0^\infty e^{-v^2/2} \, dv \int_{|X_1|/\sqrt{x - A_s}}^\infty \sqrt{x} h'(x) \, dx \quad \text{(by Fubini)} \]
\[= \frac{2}{\pi} \int_0^\infty e^{-v^2/2} \, dv \int_{\Sigma_s \vee \{A_s + X^2_1/v^2\}} h_1\left( \Sigma_s \vee \{A_s + X^2_1/v^2\} \right) \, dv \quad \text{(from (3.9))} \]
\[= \frac{2}{\pi} \int_0^\infty h_1(\Sigma_s) \int_{|X_1|/\sqrt{(\Sigma_s - A_s)_+}}^\infty e^{-v^2/2} \, dv \]
\[+ \frac{2}{\pi} \int_0^\infty h_1\left( A_s + \frac{X^2_1}{v^2} \right) \int_{|X_1|/\sqrt{(\Sigma_s - A_s)_+}}^\infty e^{-v^2/2} \, dv \]
\[= h_1(\Sigma_s) \Phi\left( \frac{|X_1|}{\sqrt{(\Sigma_s - A_s)_+}} \right) \]
\[+ \frac{2}{\pi} \int_{|X_1|/\sqrt{(\Sigma_s - A_s)_+}}^\infty h_1\left( A_s + \frac{X^2_1}{v^2} \right) e^{-v^2/2} \, dv. \tag{3.18}\]

Thus, the validity of relation (3.15) now follows immediately from the conjunction of (3.17) and (3.18).

(2) From (3.15), we get:

\[Q^\psi(\Lambda) = - \int Q^{(x)}(\Lambda) \sqrt{x} h'(x) \, dx \quad (\Lambda \in \mathcal{F}_\infty) \tag{3.19}\]

with \(Q^{(x)}(\Lambda)\) the probability measure defined via (2.2) (see Remark 2.2). The reader may have been surprised to find that, in the disintegration formula (3.15) of the martingale \(M^\psi\), the representing measure \((-\sqrt{x} h'(x) \, dx)\) is not in general positive; nonetheless, it has total integral equal to 1, i.e.:

\[- \int_0^\infty \sqrt{x} h'(x) \, dx = 1. \tag{3.20}\]

We shall discuss the positivity of \(Q^{(x)}(\Lambda)\) and the martingale \(M^\psi\) in Theorem 3.5 below.

**Proof of Theorem 3.2.** (1) To prove this theorem, it suffices, thanks to Remark 3.4, to prove point (1) of the theorem (which we shall do in the second step in this proof). From now, we use the representation (3.19) to compute
\(Q^\psi(\Sigma_\infty > a)\):

\[
Q^\psi(\Sigma_\infty > a) = -\int_0^\infty \sqrt{x} h'(x) Q^\psi\left(\frac{\Sigma_\infty}{x} > \frac{a}{x}\right) dx
\]

\[
= -\int_0^\infty \sqrt{x} h'(x) Q^\psi\left(\sqrt{\frac{\Sigma_\infty}{x}} > \sqrt{\frac{a}{x}}\right) dx
\]

(from point (3.a) in Theorem 2.1)

\[
= -\int_a^\infty h'(x) \left(\sqrt{x} - \sqrt{a}\right) dx.
\]

(3.21)

(3.22)

Hence, \(\Sigma_\infty\) is a finite r.v. under \(Q^\psi\) with density function:

\[
-\frac{1}{2\sqrt{a}} \int_a^\infty h'(x) dx = \frac{1}{2\sqrt{a}} h(a) = \psi(a),
\]

from (3.5) and (3.1).

(2) We now show point (1) in Theorem 3.2.

(a) We first estimate \(E[h(\Sigma_t)]\). We have:

\[
E[h(\Sigma_t)] = -E\left[\int_{\Sigma_t} h'(u) du\right] \quad \text{(since } h(\infty) = 0)\]

\[
= -\int_0^\infty h'(u) P(\Sigma_t < u) du
\]

\[
= -\int_0^\infty h'(u) P(t \Sigma_1 < u) du
\]

\[
= -\frac{1}{\sqrt{t}} \int_0^\infty h'(u) \rho\left(\frac{u}{t}\right) \rho\left(\frac{u}{t}\right) du,
\]

where \(\rho(y) = \frac{P(\Sigma_1 < y)}{\sqrt{y}}, y > 0\), and we have used the scaling property:

\[
\Sigma_t \mathop{=}^{(d)} t \Sigma_1 \quad \text{(under } P).\]

From (2.9), \(\lim_{y \to 0} \rho(y) = 1\) and \(\rho\) is bounded on \([0, \infty[\). The dominated convergence theorem implies that:

\[
E[h(\Sigma_t)] \sim -\frac{1}{\sqrt{t}} \int_0^\infty h'(u) \sqrt{u} du = \frac{h_1(0)}{\sqrt{t}} = \frac{1}{\sqrt{t}} \quad \text{(from (3.10)).}
\]

(3.25)

(b) We now write:

\[
E[1_{\Lambda_s} h(\Sigma_t)] = E\left[1_{\Lambda_s} E(h(\Sigma_t)|\mathcal{F}_s)\right].
\]

In order to estimate:

\[
\Delta := E[h(\Sigma_t)|\mathcal{F}_s] \quad (s < t)
\]

two cases need to be studied:

(b.i): \(d_s = s + T_0 \circ \theta_s > t\) and (b.ii): \(s + T_0 \circ \theta_s < t\).
We then write: \( \Delta = \Delta_1 + \Delta_2 \) with:

\[
\Delta_1 := E\left[h(\Sigma_t)1_{\{s+T_0 \circ \theta_s > t\}}|\mathcal{F}_s\right], \quad \Delta_2 := E\left[h(\Sigma_t)1_{\{s+T_0 \circ \theta_s < t\}}|\mathcal{F}_s\right],
\]

which we study separately.

(b.i) On the set: \( \{s + T_0 \circ \theta_s > t\} \), one has: \( \Sigma_s = \Sigma_t \); hence relation (2.16) implies:

\[
\Delta_1 = h(\Sigma_s)P_{\{X_s\}}(T_0 > t-s) \sim h(\Sigma_s)\frac{2|X_s|}{\pi \sqrt{t}}.
\]

(b.ii) On \( \{s + T_0 \circ \theta_s < t\} = \{d_s < t\} \), the r.v. \( \Sigma_t \) may be decomposed as follows: \( \Sigma_t = \Sigma + \Sigma' \), with \( \Sigma := \Sigma_s \vee (A_s + d_s - s) \) and \( \Sigma' := \Sigma_{t-d_s} \circ \theta_{d_s} \).

We first take the conditional expectation with respect to \( \mathcal{F}_{d_s} \):

\[
E\left[h(\Sigma_t)1_{\{d_s < t\}}|\mathcal{F}_{d_s}\right] = H(\tilde{\Sigma}, t-d_s)1_{\{d_s < t\}}
\]

with

\[
H(u, v) = E\left[h(u \vee \Sigma_v)\right], \quad u, v \geq 0.
\]

Let \( \tilde{h}(y) := h(u \vee y) \), \( y \geq 0 \) with \( u > 0 \) fixed. Then

\[
\tilde{h}_1(x) := -\int_x^\infty \sqrt{y}\tilde{h}'(y)\,dy = -\int_x^\infty \sqrt{y}h'(y)\,dy.
\]

Using (3.25) we get, with the help of (3.9):

\[
H(u, v) \sim \frac{1}{\sqrt{v}}\tilde{h}_1(0) = -\frac{1}{\sqrt{v}}\int_u^\infty \sqrt{y}h'(y)\,dy = \frac{h_1(u)}{\sqrt{v}}.
\]

Then we have successively:

\[
E\left[h(\Sigma_t)1_{\{d_s < t\}}|\mathcal{F}_{d_s}\right] \sim \frac{h_1(\tilde{\Sigma})}{\sqrt{t}}1_{\{d_s < t\}},
\]

\[
\Delta_2 \sim \frac{1}{\sqrt{t}}E\left[h_1(\Sigma_t \vee (A_s + d_s - s))1_{\{d_s < t\}}|\mathcal{F}_s\right] = \frac{1}{\sqrt{t}}E\left[h_1(\Sigma_t \vee (A_s + T_0 \circ \theta_s))1_{\{s+T_0 \circ \theta_s < t\}}|\mathcal{F}_s\right].
\]

According to the well-known identity:

\[
P_y(T_0 \in du) = \frac{y}{\sqrt{2\pi u^3}}\exp\left\{-\frac{y^2}{2u}\right\}1_{\{u > 0\}}\,du,
\]

we deduce:

\[
\Delta_2 \sim \frac{1}{\sqrt{t}}\int_0^\infty h_1(\Sigma_t \vee (A_s + u))\frac{|X_s|}{\sqrt{2\pi u^3}}e^{-X_s^2/(2u)}\,du.
\]

We then write the integral on the RHS, as:

\[
\int_0^{(\Sigma_t - A_s)_+} h_1(\Sigma_s)\frac{|X_s|}{\sqrt{2\pi u^3}}e^{-X_s^2/(2u)}\,du + \int_0^\infty h_1(\Sigma_t \vee (A_s + u))\frac{|X_s|}{\sqrt{2\pi u^3}}e^{-X_s^2/(2u)}\,du
\]

\[
= h_1(\Sigma_s)\Phi\left(\frac{|X_s|}{\sqrt{(\Sigma_s - A_s)_+}}\right) + \sqrt{2 \over \pi} \int_0^{(\Sigma_t - A_s)_+} h_1\left(A_s + X_s^2 / u^2\right)e^{-u^2/2}\,du.
\]

(3.33)
Thus, we finally obtain:

\[
\frac{E[h(\Sigma_t)|F_{\gamma}]}{E[h(\Sigma_t)]} = \Delta_1 + \Delta_2 \sim M^{\psi}_t,
\]

from (3.13), (3.27), (3.32), (3.33) and (3.25).

(3) End of the proof of Theorem 3.2.

We observe that identity (3.15) of Remark 3.4 and (3.12) imply item (3) of Theorem 3.2. We claim that point (4) of Theorem 3.2 may be directly deduced from this property and Theorem 2.1. Indeed as an illustration we prove that \((X_t, t \leq g)\) and \((X_{g+t}, t \geq 0)\) are independent under \(Q^{\psi}\).

Let \(F_1\) and \(F_2\) denote two positive functionals. Then:

\[
E_{Q^{\psi}}[F_1(X_t, t \leq g)F_2(X_{g+t}, t \geq 0)]
= -\int_0^\infty \sqrt{x}h'(x)dx E_{Q^{(\psi)}}[F_1(X_t, t \leq g)F_2(X_{g+t}, t \geq 0)]
= -\int_0^\infty \sqrt{x}h'(x)dx E_{Q^{(\psi)}}[F_1(X_t, t \leq g)]E_{Q^{(\psi)}}[F_2(X_{g+t}, t \geq 0)]
\]

from Theorem 2.1, (4.a).

But, from Theorem 2.1, (4.b), the law of \((X_{g+t}, t \geq 0)\), under \(Q^{(x)}\), does not depend on \(x\), hence it is equal to its law under \(Q^{\psi}\) since \(-\sqrt{x}h'(x)dx\) is a finite measure on \([0, \infty)\), whose integral is equal to 1.

Consequently, we deduce from the preceding identity that:

\[
E_{Q^{\psi}}[F_1(X_t, t \leq g)F_2(X_{g+t}, t \geq 0)]
= E_{Q^{\psi}}[F_1(X_t, t \leq g)]E_{Q^{\psi}}[F_2(X_{g+t}, t \geq 0)].
\]

We now take care of the drawback of positivity of \(Q^{\psi}\) and \(M^{\psi}\) by “changing the parametrisation.” Indeed, as in the proof of Theorem 3.5 below, it is more convenient to write the penalisation process \(h(\Sigma_t)\) as \(h_0(\sqrt{\Sigma_t})\). Note that in the context of Theorem 2.1, which is a particular case of Theorem 3.2, the distribution of \(\sqrt{\Sigma_\infty}\) is simpler than that of \(\Sigma_\infty\) since it is equal to a uniform distribution.

This leads us to present more naturally point (3) of Theorem 3.2.

**Theorem 3.5.** The hypotheses and notation are those found in Theorem 3.2. Consider, for any probability density \(\psi\) on \(\mathbb{R}_+\), the disintegration of \(Q^{\psi}\) with respect to the random variable \(\Sigma_\infty\), which admits the density \(\psi\):

\[
Q^{\psi}(\Lambda) = \int_0^\infty Q^{\psi}(\Lambda | \Sigma_\infty = y)\psi(y)dy \quad (\Lambda \in \mathcal{F}_\infty). \tag{3.34}
\]

Then: dy a.e., \(Q^{\psi}(\Lambda | \Sigma_\infty = y)\) does not depend on \(\psi\).

Thus, if one defines, for \(y > 0\),

\[
\hat{Q}^{(y)}(A) := Q^{\psi_0}(A | \Sigma_\infty = y) \tag{3.35}
\]

for some probability density \(\psi_0(y) > 0\) everywhere, one obtains:

\[
Q^{\psi}(\Lambda) = \int_0^\infty \hat{Q}^{(y)}(\Lambda)\psi(y)dy \quad (\Lambda \in \mathcal{F}_\infty), \tag{3.36}
\]

\[
\hat{Q}^{(y)}(\Sigma_\infty = y) = 1 \quad \forall y \geq 0, \tag{3.37}
\]

and, furthermore:

\[
Q^{(x)} = \frac{1}{\sqrt{x}} \int_0^\infty \frac{dy}{2\sqrt{y}} \hat{Q}^{(y)}. \tag{3.38}
\]
Brownian penalisations related to excursion lengths

Proof. (1) In order to take into account the penalisation by \( \sqrt{\Sigma_t} \), we set:

\[
h(x) = h_0(\sqrt{x}), \quad x \geq 0.
\]

Consequently the penalisation process \( h(\Sigma_t) \) equals \( h_0(\sqrt{\Sigma_t}) \).

It is clear that our assumptions and notation related to \( h \) and \( \psi \) may be interpreted in terms of \( h_0 \) in the following way:

\[
\psi(x) = \frac{1}{2\sqrt{x}} h_0(\sqrt{x}), \quad x \geq 0,
\]

\[
\int_0^\infty h_0(x) \, dx = 1,
\]

\[
x h_0(x) \to 0, \quad \text{as } x \to 0 \text{ or } x \to \infty,
\]

\[
\int_0^\infty y |h_0'(y)| \, dy < \infty.
\]

Consequently:

\[
Q^\psi(\Lambda) = -\int_0^\infty \sqrt{x} h'(x) Q^{(x)}(A) \, dx = -\frac{1}{2} \int_0^\infty h_0'(\sqrt{x}) Q^{(x)}(A) \, dx
\]

\[
= -\int_0^\infty h'_0(y) y Q^{(y^2)}(A) \, dy,
\]

for any \( \Lambda \in \mathcal{F}_\infty \).

Let \( k : \mathbb{R}_+ \to \mathbb{R}_+ \), with compact support and of \( C^\infty \) class; applying (3.44) with \( h_0 = k/c \) and \( c := \int_0^\infty k(y) \, dy \), we get:

\[
-\int_0^\infty k'(y) y Q^{(y^2)}(A) \, dy = \left( \int_0^\infty k(y) \, dy \right) Q^\psi(A),
\]

\[
\int_0^\infty k(y) d(\gamma Q^{(y^2)}(A)) = \left( \int_0^\infty k(y) \, dy \right) Q^\psi(A),
\]

where \( d(\gamma Q^{(y^2)}(A)) \) denotes the differential in the distribution sense of \( \gamma Q^{(y^2)}(A) \).

The relation (3.46) implies that \( d(\gamma Q^{(y^2)}(A)) \) is a non-negative measure which is absolutely continuous with respect to the Lebesgue measure. We set:

\[
\hat{Q}^{(y^2)}(A) := \frac{d}{dy}(\gamma Q^{(y^2)}(A)).
\]

We observe that, in this definition, \( \hat{Q}^{(y^2)}(A) \) is only defined a.e. in \( y \). But, it follows from [4] that the quasi-kernel \( (\Lambda, y) \to \hat{Q}^{(y^2)}(A) \) may be “regularized” as a kernel, so that the definition (3.47) holds for every \( \Lambda \), a.e.

(2) Coming back to (3.44) and using (3.47) and (3.40) we obtain:

\[
Q^\psi(A) = \int_0^\infty h_0(y) \hat{Q}^{(y^2)}(A) \, dy = 2 \int_0^\infty \psi(y^2) \hat{Q}^{(y^2)}(A) \, dy
\]

\[
= \int_0^\infty \psi(z) \hat{Q}^{(z)}(A) \, dz.
\]
Let $x > 0$ fixed and $\psi \!: (y) := \frac{1}{2\sqrt{xy}} 1_{[0,x]}(y)$. Then

\[
Q^{\psi}(A) = - \int_0^\infty \sqrt{y} h'_y(y) Q^{(y)}(A) \, dy
= - \int_0^\infty \sqrt{y} \frac{d}{dy} \left(2 \sqrt{y} \psi(y)\right) Q^{(y)}(A) \, dy
= Q^{(x)}(A).
\]

Using (3.48) we obtain:

\[
\sqrt{x} Q^{(x)}(A) = \int_0^x \hat{Q}^{(y)}(A) \frac{dy}{2\sqrt{y}}, \quad x > 0.
\]

(3) Let us prove that: $\hat{Q}^{(x)}(\Sigma_\infty = x) = 1$.

Recall that, from item (3.a) of Theorem 2.1, we have:

\[
Q^{(x)}(\Sigma_\infty > a) = 1 - \sqrt{\frac{a}{x}}, \quad \forall a \in [0, x].
\]

Hence, for any $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, Borel, we have:

\[
\sqrt{x} Q^{(x)}[\varphi(\Sigma_\infty)] = \int_0^x \frac{dy}{2\sqrt{y}} \varphi(y).
\]

Comparing with (3.49), we deduce: $\hat{Q}^{(y)}[\varphi(\Sigma_\infty)] = \varphi(y)$, dy a.e. and it follows that $\hat{Q}^{(y)}(\Sigma_\infty = y) = 1$, dy a.e.

(4) We may now conclude. Indeed, for $\psi$ a probability density, one has

\[
Q^{\psi}(A) = \int_0^\infty Q^{\psi}(A | \Sigma_\infty = y) \psi(y) \, dy
\]

and, from (3.48):

\[
Q^{\psi}(A) = \int_0^\infty \hat{Q}^{(y)}(A) \psi(y) \, dy.
\]

Since $\hat{Q}^{(x)}$ charges only $\{\Sigma_\infty = x\}$, we deduce from (3.51) and (3.52) that:

\[
Q^{\psi}(A | \Sigma_\infty = y) = \hat{Q}^{(y)}(A).
\]

This ends the proof of Theorem 3.5. \hfill $\square$

4. Penalisation by $1_{[A^*_t \leq x]}$

Let $x > 0$ be fixed. The aim of this section is to study the penalisation of Wiener measure with the functional $\Gamma_t = 1_{[A^*_t \leq x]}$.

In fact, because of the non-availability of a Tauberian argument in this case, we need to make a conjecture (see (4.5)), which, if valid, implies the existence of the penalised probability (see Theorem 4.3).

We recall that:

\[
A^*_t = \sup_{s \leq t} A_s = \Sigma_t \lor (t - g_t) = \Sigma_t \lor A_t.
\]

(4.1)
To prepare for the statement of our conjecture, let, for $\lambda \geq 0$:

$$\theta(\lambda) = \lambda e^{-\lambda} \int_0^1 e^{\lambda z} \frac{dz}{\sqrt{z}}.$$  \hfill (4.2)

From [8], p. 266, formula (9.11.1), the function $\theta$ defined above may be expressed in terms of a confluent hypergeometric function, namely:

$$\theta(\lambda) = 2\lambda e^{-\lambda} \Phi\left(\frac{1}{2}, \frac{3}{2}, \lambda\right).$$  \hfill (4.3)

We begin with the following lemma.

**Lemma 4.1.** The function $1 - \theta(\lambda)$ admits a first strictly positive zero $\lambda_0$.

**Proof.** Setting $z = 1 - \frac{u}{\lambda}$ in (4.2) and integrating by parts, we get:

$$\theta(\lambda) = 2\lambda \left(1 - \int_0^{\lambda} \sqrt{1 - \frac{u}{\lambda}} e^{-u} \frac{du}{\lambda}\right).$$

From the asymptotic expansion:

$$\sqrt{1 - \frac{u}{\lambda}} = 1 - \frac{u}{2\lambda} - \frac{u^2}{8\lambda^2} + o\left(\frac{1}{\lambda^2}\right) \quad (u \text{ fixed}),$$

it is easy to deduce:

$$\theta(\lambda) = 1 + \frac{1}{2\lambda} + o\left(\frac{1}{\lambda}\right) \quad (\lambda \to \infty).$$

As $1 - \theta(0) = 1$, the lemma is proven. \qed

Before we state the conjecture, we note that, from the scaling property of Brownian motion:

$$T^{\text{law}}_A = y T^{\text{law}}_1, \quad A^*_t = t A^*_1, \quad T^{\text{law}}_1 = \frac{1}{A^*_1}.$$  \hfill (4.4)

**Conjecture.** There exists a constant $C$, such that:

$$P\left(T^{\text{law}}_1 \geq \frac{t}{x}\right) = P\left(A^*_1 \leq x\right) \sim C e^{-\lambda_0 t/x},$$  \hfill (4.5)

where $\lambda_0$ denotes the first positive zero of $1 - \theta(\cdot)$, as defined in Lemma 4.1.

**Remark 4.2.** We note that Theorem 3.1 of Hu–Shi [5] which contains – among other results – an asymptotic estimate of $P(A^*_1 \leq x)$ as $x \to 0$ is in agreement with our conjecture. However, Hu–Shi use a refinement of the Tauberian theorem (their Theorem 3.2) which, in all rigor, does not imply the equivalence (4.5) above.

We are now in a position to state the following theorem.

**Theorem 4.3.** Let $x > 0$ be fixed, and $\lambda_0 > 0$ be defined as in Lemma 4.1.

1. Assuming the validity of the conjecture (4.5),
Lemma 4.4. For every $s > 0$, and $A_s \in \mathcal{F}_s$, the limit:

$$Q^*(A_s) := \lim_{t \to \infty} \frac{E[1_{A_s} 1_{(A^*_s \leq x)}]}{P(A^*_s \leq x)}$$

exists. (4.6)

(b) This limit is equal to:

$$Q^*(A_s) := E[1_{A_s} M^*_s],$$

where:

$$M^*_s = e^{\lambda s / x} 1_{(A^*_s \leq x)} \int_0^{x-A_s} \frac{|X_s|}{\sqrt{2\pi u^3}} e^{-(x^2/2u) + \lambda s / x} du.$$ (4.8)

(2) (a) $(M^*_s, s \geq 0)$ is a positive continuous martingale such that $M^*_0 = 1$.

(b) $Q^*$ induces a probability measure on $(\Omega, \mathcal{F}_\infty)$, such that, under $Q^*$, the process $(X_t, t \geq 0)$ satisfies:

$$A^*_x = x \quad \text{a.s. and} \quad T^A_x = \inf\{t: A_t = x\} = \infty \quad \text{a.s.}$$ (4.9)

We begin with a preliminary result.

**Lemma 4.4.** For every $\lambda \geq 0$ and $y > 0$:

$$\lambda \int_0^\infty e^{-\lambda t} P(A^*_t \leq y) dt = \frac{e^{\lambda y / \sqrt{y}} \int_0^\infty e^{-\lambda u} du}{1 + e^{\lambda y / \sqrt{y}} \int_0^\infty e^{-\lambda u} du}. \quad \text{(4.10)}$$

Equivalently, the complement to 1 of the above expression equals:

$$E[e^{-\lambda T^A_y}] = \frac{1}{1 + e^{\lambda y / \sqrt{y}} \int_0^\infty e^{-\lambda u} du} = \frac{1}{1 - \theta(-\lambda y)}. \quad \text{(4.11)}$$

**Proof.** We compute $E[e^{-\lambda T^A_y}]$, with the help of the independence property of $X^A_t$ and $T^A_y$. Namely according to Wald’s identity (2.14) we have:

$$E[e^{\lambda X^A_y}] E[e^{-\lambda^2 / 2 T^A_y}] = 1. \quad \text{(4.12)}$$

As for the calculation of $E[e^{\lambda X^A_y}]$ we use Proposition 1.2, and:

$$E[e^{\lambda X^A_y}] = \frac{1}{2} \int_{-\infty}^\infty \left| z \right| e^{-\left| z^2 / 2 \right| + \lambda z \sqrt{y}} dz = \frac{1}{2} \int_{-\infty}^\infty \left| z \right| e^{-\left| z^2 / 2 \right| - \lambda z \sqrt{y}} dz$$

$$= e^{\lambda^2 / 2 \sqrt{y}} \int_{-\infty}^\infty \left| z \right| e^{-\left| z^2 / 2 \right| + \lambda z \sqrt{y}} dz = e^{\lambda^2 / 2 \sqrt{y}} \int_{-\infty}^\infty \left| z - \lambda \sqrt{y} \right| e^{-z^2 / 2} dz$$

$$= e^{\lambda^2 / 2 \sqrt{y}} \left[ \int_{-\infty}^{\lambda \sqrt{y}} (z - \lambda \sqrt{y}) e^{-z^2 / 2} dz + \int_{\lambda \sqrt{y}}^\infty (z - \lambda \sqrt{y}) e^{-z^2 / 2} dz \right]$$

$$= e^{\lambda^2 / 2 \sqrt{y}} \left[ \lambda \sqrt{y} \left\{ \int_{-\infty}^{\lambda \sqrt{y}} e^{-z^2 / 2} dz - \int_{\lambda \sqrt{y}}^\infty e^{-z^2 / 2} dz \right\} + 2e^{-\lambda^2 y / 2} \right]$$

$$= 1 + \lambda \sqrt{ye^{\lambda^2 / 2 \sqrt{y}}} \int_0^{\lambda \sqrt{y}} e^{-z^2 / 2} dz. \quad \text{(4.13)}$$
Hence, from (4.12) and (4.13), and after changing $\lambda$ in $\sqrt{2\lambda}$, we get:

$$E[e^{-\lambda T^A_y}] = \frac{1}{1 + e^{\lambda y} \sqrt{2\lambda y} \int_0^{\sqrt{2\lambda y}} e^{-z^2/2} \, dz}$$

(4.14)

and

$$E\left[ e^{-\lambda T_y A} \right] = \frac{1}{1 + e^{\lambda y} \lambda \sqrt{y} \int_0^{\lambda \sqrt{y}} e^{-u^2/2} \, du}$$

(4.15)

after making the change of variables: $z^2 = 2\lambda u$. Now, Lemma 4.4 follows from (4.15).

It follows from the scaling property of Brownian motion (and this is confirmed by (4.15)) that:

$$T^A_y \overset{(\text{law})}{=} y T^A_1$$

(4.16)

and

$$A^*_t \overset{(\text{law})}{=} t A^*_1.$$  

(4.17)

Proof of Theorem 4.3. (1) We prove point (1).

For every $s + x \leq t$, one has:

$$1_{[A^*_s \leq x]} = 1_{[A^*_s \leq x]} 1_{[A_{s+d}-s \leq x, dt < t]} 1_{[A^*_r \leq x]}.$$  

(4.18)

Hence, from conjecture (4.5), after conditioning on $F_{d_t}$, we get:

$$E\left[ 1_{[A^*_s \leq x]} | F_{d_t} \right] \sim \frac{1}{P(A^*_s \leq x)} e^{-\lambda_0 s/2} \int_0^\infty \frac{|X_s|}{\sqrt{2\pi u} e^{-(X^2_s/(2u)) + \lambda_0 u/x}} \, du.$$  

(4.19)

It is clear that

$$\lim_{t \to \infty} \frac{E\left[ 1_{[A^*_s \leq x]} | F_{d_t} \right]}{P(A^*_s \leq x)} = M^*_s$$

(4.22)

follows from (4.21) and conjecture (4.5) (recall that $M^*_s$ has been defined by (4.8)).

It will be shown below (see (2)(e)) that $E[M^*_s] = 1$. Therefore point (1) of Theorem 4.3 is a direct consequence of Theorem 1.1.

(2) We now show that $(M^*_s, s \geq 0)$, as defined in (4.8) is a martingale. Note that the results stated in point (2) of Theorem 4.3 hold without assuming (4.5).

(a) We first remark that we may write $M^*_s$, with the help of the change of variables: $\frac{X^2}{u} = v^2$, as:

$$M^*_s = 1_{[A^*_s \leq x]} e^{\lambda_0 s/x} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-v^2/2 + [\lambda_0 X^2/(xu^2)]} \, dv$$

(4.23)

without assuming the validity of the conjecture.
We also note that \(1_{\{A_s^* \leq x\}} = 1_{\{T_s^A \geq s\}}\), and that, for \(s < T_s^A\); \(M^*_s > 0\). Thus, \((M^*_s, s \geq 0)\) is stopped at its first zero. Moreover, we have:

\[
0 \leq M^*_s \leq \sqrt{\frac{2}{\pi}} e^{\lambda_0 s/x} 1_{\{A^*_s \leq x\}} \int_{|X_s|/\sqrt{x-A_s}}^{\infty} \exp\left(-\frac{u^2}{2} + \frac{\lambda_0 X_s^2 (x - A_s)}{X_s^2 x}\right) \, du \\
\leq e^{\lambda_0 s/x} e^{\lambda_0}.
\]

(4.24)

Thus, to show that \((M^*_s, s \geq 0)\) is a martingale (under \(P\)), it suffices to see that it is a local martingale.

Note that the relation (4.23) implies that \(M^*_0 = 1\) a.s.

(b) Define:

\[
f(y, a) = \int_{y/\sqrt{x-a}}^{\infty} e^{-v^2/2 + \{\lambda_0/(x v^2)\}} y^2 \, dv \quad (y \geq 0, 0 < a \leq x)
\]

(4.25)

so that:

\[
M^*_s = e^{\lambda_0 s/x} 1_{\{A^*_s \leq x\}} \sqrt{\frac{2}{\pi}} f(|X_s|, A_s).
\]

(4.26)

To prove that \((M^*_s, s \geq 0)\) is a local martingale, we shall apply Lemma 2.4 or more precisely, a slight variant of Lemma 2.4, due to the presence of the factor \(e^{\lambda_0 s/x}\).

In fact, it suffices to show that the function \(f\) considered as a function of \((y, a)\) satisfies:

(i) \(f(0, a)\) does not depend on \(a\) \((a > 0)\); in fact, from (4.25): \(f(0, a) = \sqrt{\pi/2}\);

(ii) \(\frac{\partial f}{\partial a} + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} + \frac{\lambda_0}{x} f = 0\);

(iii) \(\frac{\partial f}{\partial y}(0, 0) = 0\).

(c) We show (iii).

This follows from:

\[
\frac{\partial f}{\partial y}(0, 0) = -\frac{1}{\sqrt{x}} e^{\lambda_0} + \sqrt{\frac{2}{\pi}} \int_{|y/\sqrt{x-a}|}^{\infty} \frac{2 \lambda_0 y}{x v^2} e^{-v^2/2 + \{\lambda_0/(x v^2)\}} \, dv \bigg|_{y=0}
\]

\[
= -\frac{1}{\sqrt{x}} e^{\lambda_0} + \frac{\lambda_0}{\sqrt{x}} \int_{0}^{1} \frac{du}{\sqrt{u}} e^{-u^2/2 + \{\lambda_0 u\} + \lambda_0 u} \bigg|_{y=0}
\]

\[
\left(\text{after making the change of variables } \frac{y^2}{2v^2} = u\right)
\]

\[
= -\frac{e^{\lambda_0}}{\sqrt{x}} (1 - \theta(\lambda_0)) = 0,
\]

from the definition of \(\lambda_0\).

(d) We show point (ii) above. We start computing the two first derivatives \(\frac{\partial f}{\partial a}(y, a)\) and \(\frac{\partial f}{\partial y}(y, a)\).

From (4.25), we have:

\[
\frac{\partial f}{\partial a}(y, a) = -\frac{y}{2(x - a)^{3/2}} e^{-\{y^2/2(x-a)\} + \{\lambda_0/(x - a)\} (x-a)}
\]

(4.27)

and

\[
\frac{\partial f}{\partial y}(y, a) = -\frac{1}{\sqrt{x-a}} e^{-\{y^2/(2(x-a))\} + \{\lambda_0/(x-a)\} (x-a)} + \int_{y/\sqrt{x-a}}^{\infty} \frac{2 \lambda_0 y}{x v^2} e^{-v^2/2 + \{\lambda_0 y^2/(x v^2)\}} \, dv
\]

(4.28)
so that, after making the change of variables \( y^2/xv = u \):

\[
\frac{\partial f}{\partial y}(y, a) = -\frac{1}{\sqrt{x-a}} e^{-y^2/(2(x-a))} + |\lambda_0/x|(x-a) + \frac{\lambda_0}{\sqrt{x}} \int_0^{1-a/x} \frac{1}{\sqrt{u}} e^{-y^2/(2ux)} + \lambda_0 u \ du.
\] (4.29)

Consequently:

\[
\frac{\partial^2 f}{\partial y^2}(y, a) = -\frac{1}{(x-a)^{3/2}} e^{-y^2/(2(x-a))} + \frac{\lambda_0 y}{x^{3/2}} \int_0^{1-a/x} e^{-y^2/(2ux)} + \lambda_0 u \ du.
\] (4.30)

On the other hand, we write \( f \), after making the change of variables:

\[
y^2/xv^2 = u \text{ in (4.25), as:}
\]

\[
f(y, a) = \frac{y}{2\sqrt{x}} \int_0^{1-a/x} e^{-y^2/(2ux)} + \lambda_0 u \ du.
\] (4.31)

We then gather (4.27), (4.29), (4.30) and (4.31) to deduce (ii).

Moreover:

\[
M^*_s = 1 + \sqrt{\frac{2}{\pi}} \int_0^{s \land T^A_y} e^{\lambda_0 u/x} \frac{\partial f}{\partial y}(|X_u|, A_u) \ sgn(X_u) \ dX_u.
\] (4.32)

Hence, \( (M^*_s) \) is a local martingale, which, as we have seen earlier, implies it is a martingale; in particular, \( E[M^*_s] = 1 \).

(3) We now prove point (2)(b) of Theorem 4.3.

(a) It is clear that, \( Q \) a.s.: \( A^*_\infty \leq x \), since, for every \( s \geq 0 \):

\[
Q^*(A^*_s \leq x) = \lim_{t \to \infty} \frac{P((A^*_s \leq x) \cap \{A^*_t \leq x\})}{P(A^*_t \leq x)} = 1.
\] (4.33)

On the other hand, we get, for every \( y < x \):

\[
Q^*(A^*_s > y) = Q^*(T^A_y < t) = E[1_{(T^A_y < t)}] M^*_t = E[1_{(T^A_y < t)} M^*_{T^A_y}].
\]

Taking the limit \( t \to \infty \), we get:

\[
Q^*(A^*_\infty > y) = E[M^*_{T^A_y}].
\] (4.34)

However, from (4.32):

\[
M^*_{T^A_y \land s} = 1 + \sqrt{\frac{2}{\pi}} \int_0^{T^A_y \land s} e^{\lambda_0 u/x} \frac{\partial f}{\partial y}(|X_u|, A_u) \ sgn(X_u) \ dX_u, \quad s \geq 0, \quad 0 < y < x.
\] (4.35)

Next, we deduce from the form of \( \frac{\partial f}{\partial y} \) given by (4.29) the existence of a constant \( k \) such that:

\[
\left| \frac{\partial f}{\partial y}(|X_u|, A_u) \right| \leq k, \quad 0 \leq u \leq T^A_y.
\] (4.36)

Identity (4.11) implies that, for \( \lambda > 0 \), small enough:

\[
E[e^{\lambda T^A_y}] = \frac{1}{1 - \lambda y e^{-\lambda y}} \int_0^1 e^{\lambda yu} \ du = \frac{1}{1 - \theta(\lambda y)}.
\] (4.37)

From Lemma 4.1 we deduce:

\[
E[e^{\lambda T^A_y}] < \infty \quad \forall \lambda < \frac{\lambda_0}{y}.
\] (4.38)
Combining Burkholder–Davis–Gundy inequality, (4.35), (4.36) and the fact that 
\( E[e^{\lambda t_a T_1}/x] < \infty \), shows that 
\((M^*_r, T^*_1)\) belongs to 
\( H^1 \) (i.e., \( E[\sup_{s\geq 0}|M^*_r, y|] < \infty \)). Hence: \( E[M^*_r] = 1 \) and finally, from (4.34): \( Q^*(A^*_r \geq y) = 1 \) for any \( 0 \leq y < x \). Using moreover (4.33) we can conclude that \( A^*_r = x, \) \( Q^* \) a.s.

(b) We show that: \( T^*_r = \infty, \) \( Q^* \) a.s.

Since \( M^*_r = 0 \), for any \( t \geq T^*_r \), then, for any \( y < x \), one has:

\[
Q^*(T^*_r > t) = E\left[1_{\{T^*_r > t\}} M^*_r\right] \xrightarrow{y \uparrow x} E\left[1_{\{T^*_r \geq t\}} M^*_r\right] = E[M^*_r] = 1.
\]

Hence,

\[
\lim_{y \uparrow x} Q^*(T^*_r \leq t) = 0.
\]

This implies:

\[
Q^*(T^*_r \leq t) = \lim_{y \uparrow x} Q^*(T^*_r \leq t) = 0 \quad \forall t \geq 0
\]

which proves that \( T^*_r = \infty, \) \( Q^* \) a.s.

\[ \square \]

**Remark 4.5.** Let us come back to (4.10). By an analytic continuation argument, relation (4.10) may be extended as follows:

\[
\int_{0}^{\infty} e^{\lambda t} P(A^*_t \leq x) \, dt = \frac{e^{-\lambda x} \sqrt{x} \int_{0}^{x} e^{\lambda u} \frac{du}{\sqrt{v}}}{1 - \lambda \sqrt{x} e^{-\lambda x} \int_{0}^{x} e^{\lambda u} \frac{dv}{\sqrt{v}}}
\]

\[
= \frac{e^{-\lambda x} \sqrt{x} \int_{0}^{x} e^{\lambda u} \frac{du}{\sqrt{v}}}{1 - \theta(\lambda x)},
\]

(4.39)

for any \( \lambda < \lambda_0 \). Since the first positive zero \( \lambda_0 \) of \( 1 - \theta \) is simple, a formal application of classical results about the Mellin–Fourier transformation yield to (4.5). However we have not been able to justify this approach.

**Remark 4.6.** Let \( \varphi(y) := P(A^*_1 < \frac{1}{y}) = P(T^*_1 > y) \) \((y > 0)\). Since \( A^*_t \leq t \), we have:

\[
P(A^*_t < y) = P(tA^*_1 < y) = \varphi\left(\frac{t}{y}\right),
\]

\[
\varphi(a) = 1 \quad \text{if } a < 1,
\]

\[
\varphi(a) \rightarrow 0 \quad \text{as } a \rightarrow \infty.
\]

The aim of the present remark is to show that \( \varphi \) satisfies the following equation:

\[
\varphi(a) = -\int_{0}^{1} \varphi'(a + v) \frac{dv}{\sqrt{1 - v}}, \quad a \geq 1.
\]

(4.40)

Since \((-\varphi')\) is the density function of \( T^*_1 \), (4.40) may equivalently be presented as:

\[
P(T^*_1 > a) = E\left[\frac{1}{\sqrt{a + 1 - T^*_1}} 1_{\{a < T^*_1 < a + 1\}}\right].
\]

(4.41)
Indeed, to prove (4.40), we rewrite (4.10) in the form:

\[
\left( \frac{e^{-\lambda x}}{\lambda} + \sqrt{x} \int_0^x \frac{e^{-\lambda u}}{\sqrt{u}} \, du \right) \left( \int_0^\infty e^{-\lambda t} \varphi \left( \frac{t}{x} \right) \, dt \right) = \frac{1}{\lambda} \left( \sqrt{x} \int_0^x \frac{e^{-\lambda u}}{\sqrt{u}} \, du \right).
\] (4.42)

We denote:

\[
\mu_x(s) = \sqrt{x} 1_{[0,x]}(s) \frac{1}{\sqrt{s}} + 1_{[x,\infty]}(s) \quad (4.43)
\]

so that (4.42) becomes, for every \( t \geq 0 \):

\[
\varphi \left( \frac{t-x}{x} \right) = \sqrt{x} \int_0^{t-x} 1_{[0,x]}(s) \frac{ds}{\sqrt{s}} = 2 \sqrt{x} \sqrt{x \wedge t}.
\] (4.44)

In particular, when \( t \geq 2x \) we have:

\[
2x = \int_0^t \varphi \left( \frac{t-s}{x} \right) \mu_x(s) \, ds.
\]

Since \( \varphi \left( \frac{t-x}{x} \right) = \mu_x(s) = 1 \) for \( s \geq t - x \), then the previous relation can be reduced to:

\[
x = \int_0^{t-x} \varphi \left( \frac{t-s}{x} \right) \mu_x(s) \, ds.
\] (4.45)

Hence, differentiating with respect to \( t \), one gets:

\[
\varphi(1) \mu_x(t-x) + \frac{1}{x} \int_0^{t-x} \varphi' \left( \frac{t-s}{x} \right) \mu_x(s) \, ds = 0,
\]

which implies, since \( \varphi(1) = 1 \) and \( \mu_x(t-x) = 1 \) for \( t \geq 2x \):

\[
0 = 1 + \frac{1}{\sqrt{x}} \int_0^x \varphi' \left( \frac{t-s}{x} \right) \frac{ds}{\sqrt{s}} + \frac{1}{x} \int_x^{t-x} \varphi' \left( \frac{t-s}{x} \right) \, ds.
\]

However:

\[
\frac{1}{x} \int_x^{t-x} \varphi' \left( \frac{t-s}{x} \right) \, ds = \varphi \left( \frac{t-x}{x} \right) - \varphi(1) = \varphi \left( \frac{t-x}{x} \right) - 1.
\]

Hence:

\[
\varphi \left( \frac{t-x}{x} \right) = - \frac{1}{\sqrt{x}} \int_0^x \varphi' \left( \frac{t-s}{x} \right) \frac{dx}{\sqrt{s}} = - \int_0^1 \varphi' \left( \frac{t-u}{x} \right) \frac{du}{\sqrt{u}}.
\]

Closely related computations are found in [7].

**Remark 4.7.** With the help of Girsanov’s theorem, together with (4.7) and (4.32), there exists a \( Q^* \)-Brownian motion \( (\beta_t, t \geq 0) \) such that:

\[
X_t = \beta_t + \int_0^t \frac{\partial f}{\partial y} \left( \left| X_s \right|, A_s \right) \text{sgn}(X_s) \, ds.
\] (4.46)

From (4.25), we deduce that for \( y > 0 \):

\[
f(y, a) \sim \frac{\sqrt{x-a}}{y} e^{-y^2/(2(x-a))}, \quad \frac{\partial f}{\partial y} (y, a) \sim -\frac{1}{\sqrt{x-a}} e^{-y^2/(2(x-a))}.
\]
Consequently:

\[
\lim_{a \uparrow x} \frac{\partial f}{\partial y}(y, a) = -\infty \quad \text{and}
\]

\[
\frac{\partial f}{\partial y}(|X_s|, A_s) \operatorname{sgn}(X_s)
\]

goes to \(-\infty\), resp.: \(+\infty\), as \(A_s\) approaches \(x\), and \(X_s > 0\), resp.: \(X_s < 0\). In other terms, when the age of \(X\) approaches \(x\), the process \(X\) is strongly pushed towards 0.

This explains intuitively why, under \(Q^x\), \(A^*_\infty = x\) and \(T^A_x = \infty\), a.s.

**Remark 4.8.** The rates of decay of \(P(A^*_t \leq x)\) and \(P(\Sigma_t \leq x)\) as \(t \to \infty\) are radically different. The first one is exponential (see (4.5)), whereas the second one is polynomial (see Lemma 2.3). We already observed such a difference when we studied (see [13]) the penalisations of Wiener measure by, on one hand, \(1_{\{X^*_t \leq x\}}\) (exponential decay) and, on the other hand, \(1_{\{S_t < x\}}\) (polynomial decay), where we denote \(X^*_t = \sup_{s \leq t} |X_s|\) and \(S_t = \sup_{s \leq t} X_s\).

**References**


