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On unique extension of time changed reflecting Brownian motions

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Abstract. Let D be an unbounded domain in $\mathbb{R}^d$ with $d \geq 3$. We show that if D contains an unbounded uniform domain, then the symmetric reflecting Brownian motion (RBM) on $\overline{D}$ is transient. Next assume that RBM X on $\overline{D}$ is transient and let Y be its time change by Revuz measure $\mathbf{1}_D(x)m(x)\,\mathrm{d}x$ for a strictly positive continuous integrable function m on $\overline{D}$. We further show that if there is some $r > 0$ so that $D \setminus \overline{B(0,r)}$ is an unbounded uniform domain, then Y admits one and only one symmetric diffusion that genuinely extends it and admits no killings.

Résumé. Notons D un domaine non borné dans $\mathbb{R}^d$ avec $d \geq 3$. Nous montrons que si D contient un domaine uniforme non borné, alors le mouvement brownien réfléchi (RBM) sur $\overline{D}$ est transient. Par ailleurs, supposons que le RBM X sur $\overline{D}$ est transient et notons Y son changement de temps par une mesure de Revuz $\mathbf{1}_D(x)m(x)\,\mathrm{d}x$ pour une fonction m strictement positive, continue et intégrable sur $\overline{D}$. Nous démontrons alors que si il existe un $r > 0$ tel que $D \setminus \overline{B(0,r)}$ soit un domaine uniformément non borné, alors Y admet une unique extension en une diffusion symétrique qui n'est jamais tuée.

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1. Introduction

Consider an unbounded domain D in $\mathbb{R}^d$ with $d \geq 3$ possessing continuous boundary. Then $(\frac{1}{2}\mathbf{D}, W^{1,2}(D))$ is a regular strongly local Dirichlet form on $L^2(\overline{D}; \mathrm{d}x)$, where $\mathbf{D}(u,v) = \int_D \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x$. The symmetric diffusion process X associated with this form is the symmetric reflecting Brownian motion on $\overline{D}$ with infinite lifetime. In this paper, we are concerned with the following two questions:

(i) When X is transient?
(ii) If X is transient, then almost all sample paths of X approach to the point at infinity ∂ of $\overline{D}$ as time goes to infinity. Let Y be its time change by Revuz measure $\mathbf{1}_D(x)m(x)\,\mathrm{d}x$ for a strictly positive continuous integrable function m on $\overline{D}$. The time-changed process Y has finite lifetime with probability one. How many symmetric diffusions are there that genuinely extends Y?

We show that if D contains an unbounded uniform domain (see Definition 2.1 below), then RBM X on $\overline{D}$ is transient. We further show that if there is some $r > 0$ so that $D \setminus \overline{B(0,r)}$ is an unbounded uniform domain, then Y

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admits a unique genuine diffusion extension that admits no killings. Here $B(x,r)$ denotes the ball in $\mathbb{R}^d$ centered at x with radius r. The extension diffusion can be obtained through one-point darning of Y at ∂. It can also be obtained through the active reflected Dirichlet form of Y. The key of our approach is to identify the reflected Dirichlet space of Y with the space

$$\mathrm{BL}(D) := \left\{u \in L^2_{\mathrm{loc}}(D)\colon \frac{\partial u}{\partial x_i} \in L^2(D), 1 \le i \le d\right\}$$

of Beppo Levi functions on D and to show that $\mathrm{BL}(D)$ is the linear space spanned by $W^{1,2}_e(D)$ and constant functions under suitable conditions on D. Here $W^{1,2}_e(D)$ is the extended Dirichlet space of $(\frac{1}{2}\mathbf{D}, W^{1,2}(D))$; see next section for its definition.

2. Transience of reflecting Brownian motion

Let E be a locally compact separable metric space and m a positive Radon measure on E with full support. Numerical functions f, g on E is said to be m-equivalent if $m(f \neq g) = 0$. We write as $f = g$ $[m]$ in this case. Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(E; m)$ and $(\mathcal{F}_e, \mathcal{E})$ be its extended Dirichlet space. A function u is in $\mathcal{F}_e$ if and only if $|u| < \infty$ $[m]$ and there exists $\{u_n\} \subset \mathcal{F}$ called an approximating sequence of u such that $\{u_n\}$ is $\mathcal{E}$-Cauchy and $\lim_{n\to\infty} u_n = u$ $[m]$. It holds then that $\mathcal{E}(u,u) = \lim_{n\to\infty} \mathcal{E}(u_n, u_n)$. We know that $\mathcal{F} = \mathcal{F}_e \cap L^2(E; m)$ and that every normal contraction operates on $(\mathcal{F}_e, \mathcal{E})$.

Denote by $\{T_t; t > 0\}$ be the L^2-semigroup generated by the Dirichlet form $\mathcal{E}$ and define $\{S_t,\ t > 0\}$ by the Bochner integral $S_t f = \int_0^t T_s f\,\mathrm{d}s$, $f \in L^2(E; m)$. The operator S_t then extends from $L^2 \cap L^1$ to a bounded linear operator on $L^1(E; m)$ such that $0 \le S_t f \le S_{t'} f$ for $0 \le t < t'$ and $f \in L^1_+(E; m)$. Hence for $f \in L^1_+(E; m)$,

$$Gf(x) = \lim_{N\to\infty} S_N f(x) (\le \infty)$$

defines a function Gf uniquely up to the m-equivalence. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *transient* (resp. *recurrent*) if $Gf < \infty$ $[m]$ for some $f \in L^1_+(E; m)$ with $f > 0$ $[m]$ (resp. $m(0 < Gf < \infty) = 0$ for every $f \in L^1_+(E; m)$). An m-measurable set $A \subset E$ is said to be $\{T_t\}$-*invariant* if $1_A T_t 1_{A^c} = 0$ $[m]$. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called m-*irreducible* if for any $\{T_t\}$-invariant set A, we have either $m(A) = 0$ or $m(A^c) = 0$. If the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is irreducible, then it is either transient or recurrent. The following criteria are known [6]. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is transient if and only if one of the next conditions is satisfied:

$$(\mathcal{F}_e, \mathcal{E}) \text{ is a real Hilbert space;} \tag{2.1}$$

$$u \in \mathcal{F}_e, \quad \mathcal{E}(u,u) = 0 \quad \Longrightarrow \quad u = 0\,[m]. \tag{2.2}$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is recurrent if and only if

$$1 \in \mathcal{F}_e \quad \text{and} \quad \mathcal{E}(1,1) = 0. \tag{2.3}$$

Denote by $C_c(E)$ the space of all continuous functions on E with compact support. For a Dirichlet form $(\mathcal{E}, \mathcal{F})$, a subspace $\mathcal{C}$ of $\mathcal{F} \cap C_c(E)$ is called a *core* of $\mathcal{E}$ if $\mathcal{C}$ is $\mathcal{E}_1$-dense in $\mathcal{F}$ and uniformly dense in $C_c(E)$. A Dirichlet form $\mathcal{E}$ is called *regular* if it possesses a core. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is said to be *strongly local* if $\mathcal{E}(u, v) = 0$ whenever $K = \mathrm{Supp}[u \cdot m]$ is compact and v is constant in a neighborhood of K. Any strongly local regular Dirichlet form on $L^2(E; m)$ is known to admit an associated m-symmetric diffusion process on E that admits no killing inside E.

In the following, we will take the state space E to be a Euclidean domain D in $\mathbb{R}^d$. We denote by $L^p(D)$, $p \ge 1$, the L^p-space of functions on D with respect to the Lebesgue measure $\mathrm{d}x$. We focus our attention on the space

$$\mathrm{BL}(D) = \left\{T\colon \frac{\partial T}{\partial x_i} \in L^2(D), 1 \le i \le d\right\} \tag{2.4}$$

of Schwartz distributions T. It is known that any distribution $T \in \mathrm{BL}(D)$ can be identified with a function in $L^2_{\mathrm{loc}}(D)$ (cf. Schwartz [13], Deny and Lions [5]) so that

$$\mathrm{BL}(D) = \left\{ u \in L^2_{\mathrm{loc}}(D)\colon \frac{\partial u}{\partial x_i} \in L^2(D), 1 \le i \le d \right\}, \tag{2.5}$$

where the derivatives are taken in Schwartz distribution sense. Members in $\mathrm{BL}(D)$ are called BL (Beppo Levi) *functions* on D. For $u, v \in \mathrm{BL}(D)$, we put

$$\mathbf{D}(u, v) = \sum_{i=1}^{d} \int_D \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, \mathrm{d}x. \tag{2.6}$$

The space $\mathrm{BL}(D)$ is known to enjoy the following properties (cf. [5]):

(BL.1) The quotient space $\dot{\mathrm{BL}}(D)$ of $\mathrm{BL}(D)$ by the subspace of constant functions is a Hilbert space with inner product $\mathbf{D}$. Any $\mathbf{D}$-Cauchy sequence $u_n \in \mathrm{BL}(D)$ admits $u \in \mathrm{BL}(D)$ and constants c_n such that u_n is $\mathbf{D}$-convergent to u and $u_n + c_n$ is L^2_{loc}-convergent to u.

(BL.2) A function u on D is in $\mathrm{BL}(D)$ if and only if for each i $(1 \le i \le d)$, there is a version $u^{(i)}$ of u such that it is absolutely continuous on almost all straight lines parallel to x_i-axis and the derivative $\partial u^{(i)}/\partial x_i$ in the ordinary sense (which exists a.e. on D) is in $L^2(D)$. In this case, the ordinary derivatives coincide with the distribution derivatives of u.

The *Sobolev space of order* $(1, 2)$ on the domain $D \subset \mathbb{R}^d$ is defined by

$$W^{1,2}(D) = \mathrm{BL}(D) \cap L^2(D). \tag{2.7}$$

Then

$$(\mathcal{E}, \mathcal{F}) = \left(\frac{1}{2}\mathbf{D}, W^{1,2}(D) \right) \tag{2.8}$$

is a Dirichlet form on $L^2(D)$.

Let $\mathcal{D}$ denote the class of domains in $\mathbb{R}^d$ so that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of (2.8) is regular on $L^2(\overline{D})$; that is, $W^{1,2}(D) \cap C_c(\overline{D})$ is both dense in $(W^{1,2}(D), \|\cdot\|_{1,2})$ and in $(C_\infty(\overline{D}), \|\cdot\|_\infty)$. Here $C_\infty(\overline{D})$ is the space of continuous functions on $\overline{D}$ that vanishes at infinity. A domain $D \subset \mathbb{R}^d$ is in $\mathcal{D}$ if one of the following two conditions hold:

(i) D has continuous boundary (see Theorem 2 on p. 14 of [12]); that is, for every $z \in \partial D$, there is some $r > 0$ so that $B(z, r) \cap D$ is the domain lying above the graph of a continuous function;

(ii) D is an *extendable domain* in the sense that $W^{1,2}(D) = W^{1,2}(\mathbb{R}^d)|_D$; in other words, every function $u \in W^{1,2}(D)$ admits a function $\widehat{u} \in W^{1,2}(\mathbb{R}^d)$ such that $\widehat{u} = u$ a.e. on D. When D is an extendable domain, $C_c^\infty(\overline{D}) = C_c^\infty(\mathbb{R}^d)|_{\overline{D}}$ is a core for $(\mathcal{E}, \mathcal{F})$. Here $C_c^\infty(\mathbb{R}^d)$ is the space of infinitely differentiable functions with compact support on $\mathbb{R}^d$.

Definition 2.1. *A domain D is called a locally uniform domain if there are $\delta \in (0, \infty]$ and $C > 0$ such that for every $x, y \in D$ with $|x - y| < \delta$, there is a rectifiable curve γ in D connecting x and y with* $\mathrm{length}(\gamma) \le C|x - y|$ *and, moreover,*

$$\min\{|x - z|, |z - y|\} \le C \,\mathrm{dist}(z, D^c) \quad \textit{for every } z \in \gamma.$$

A domain is said to be a uniform domain if the above property holds with $\delta = \infty$.

The above definition is taken from Väisälä [15], where various equivalent definitions are discussed. Uniform domain and locally uniform domain are also called (ε, ∞)-domain and (ε, δ)-domain, respectively, in [10]. For example, the classical van Koch snowflake domain in the conformal mapping theory is a uniform domain in $\mathbb{R}^2$. Note that every bounded Lipschitz domain is uniform, and every *nontangentially accessible domain* defined by Jerison and Kenig in

[9] is a uniform domain (see (3.4) of [9]), while every Lipschitz domain is an (ε, δ)-domain. Let S^{d-1} denote the unit sphere $\{x \in \mathbb{R}^d\colon |x| = 1\}$ in $\mathbb{R}^d$ and (r, θ) the polar coordinates in $\mathbb{R}^d$. It is easy to check that a truncated infinite cone $C_{A,a} := \{(r, \theta)\colon r > a \text{ and } \theta \in A\}$ in $\mathbb{R}^d$ for any connected open set $A \subset S^{d-1}$ with Lipschitz boundary is a uniform domain. However, the boundary of a uniform domain can be highly nonrectifiable and, in general, no regularity of its boundary can be inferred (besides the easy fact that the Hausdorff dimension of the boundary is strictly less than n). It is known that every locally uniform domain D admits a bounded linear operator

$$T : \left(W^{1,2}(D), \|\cdot\|_{1,2}\right) \to \left(W^{1,2}\left(\mathbb{R}^d\right), \|\cdot\|_{1,2}\right) \tag{2.9}$$

(see [10]) and so in particular it is an extendable domain.

When $D \in \mathcal{D}$, the Dirichlet form (2.8) is regular and strongly local on $L^2(\overline{D})$. Its associated diffusion process is called symmetric (or normally) reflected Brownian motion (RBM) on $\overline{D}$. When $D = \mathbb{R}^d$, then the associated process is the d-dimensional standard Brownian motion and hence (2.8) is transient if $n \geq 3$ and recurrent if $d = 1, 2$. By the following lemma, (2.8) is recurrent for any $D \in \mathcal{D}$ whenever $d = 1, 2$.

Lemma 2.2. *Suppose $D \in \mathcal{D}$ and let $X = (X_t, \mathbf{P}_x)$ be RBM on $\overline{D}$.*

(i) *RBM X on $\overline{D}$ is conservative in the sense that $\mathbf{P}_x(\zeta = \infty) = 1$ for q.e. $x \in \overline{D}$, where ζ is the lifetime of X. Moreover, X (or equivalently, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of (2.8)) is m-irreducible.*
(ii) *Suppose D_1, $D_2 \in \mathcal{D}$ and $D_1 \subset D_2$. Let $X^{(i)}$ be RBM on $\overline{D}_i$, $i = 1, 2$. If $X^{(1)}$ is transient, then so is $X^{(2)}$. If $X^{(2)}$ is recurrent, then so is $X^{(1)}$.*

Proof. (i) The conservativeness of X follows from [6], Example 5.7.1. The transition function of X dominates the transition function of the absorbed Brownian motion on D which is obtained from the n-dimensional standard Brownian motion by killing upon leaving D and is known to have a strictly positive transition density on D. Therefore X is irreducible.

(ii) If we assume the recurrence of (2.8) for D_2, there exist $u_n \in W^{1,2}(D_2), n \geq 1$, which are $\mathbf{D}$-Cauchy and convergent pointwise to 1 on D_2. Then $v_n = u_n|_{D_1}, n \geq 1$, satisfy the same properties for D_1, yielding the recurrence of (2.8) for D_1. This implies the first assertion of (ii) because of the irreducibility proven in (i). □

Let us denote by $(W_e^{1,2}(D), \mathcal{E})$ the extended Dirichlet space of (2.8). We may call $W_e^{1,2}(D)$ the *extended Sobolev space* of order $(1, 2)$ on D. We denote by $\mathbb{H}(D)$ the space of all harmonic functions on D with finite Dirichlet integral.

Lemma 2.3. *$W_e^{1,2}(D) \subset \mathrm{BL}(D)$ and $\mathcal{E}(u, u) = \frac{1}{2}\mathbf{D}(u, u)$ for $u \in W_e^{1,2}(D)$. If we denote by $\mathbb{H}^*(D)$ the collection of functions in $\mathrm{BL}(D)$ that is $\mathbf{D}$-orthogonal to all functions in $W_e^{1,2}(D)$, then*

$$\mathbb{H}^*(D) \subset \mathbb{H}(D). \tag{2.10}$$

Proof. For $u \in W_e^{1,2}(D)$, there is a sequence $\{u_n\} \subset W^{1,2}(D)$ which is $\mathbf{D}$-Cauchy and convergent to u a.e. $\mathbf{D}(u_n, u_n)$ then converges to $\mathcal{E}(u, u)$. By (BL.1), there exist $v \in \mathrm{BL}(D)$ and constants c_n such that $\{u_n\}$ is $\mathbf{D}$-convergent to v and the sequence $\{u_n + c_n\}$ is convergent to v in $L^2_{\mathrm{loc}}(D)$. By choosing a subsequence if necessary, we may assume that the latter sequence converges to v a.e. Then $\lim_{n\to\infty} c_n = c$ exists, $u = v - c$ and consequently, $u \in \mathrm{BL}(D)$ and $\mathcal{E}(u, u) = \frac{1}{2}\mathbf{D}(u, u)$.

If $u \in \mathrm{BL}(D)$ is $\mathbf{D}$-orthogonal to all functions in $W_e^{1,2}(D)$, then, since $C_c^\infty(D) \subset W^{1,2}(D)$, we have

$$(\Delta u, f) = -\mathbf{D}(u, f) = 0 \quad \text{for every } f \in C_c^\infty(D),$$

which implies that $\Delta u = 0$, namely, (a version of) u is harmonic on D. □

Informally, $\mathbb{H}^*(D)$ is the space of harmonic functions on D having finite Dirichlet energy with zero normal derivative on the boundary ∂D.

If (2.8) is transient, then we see from (2.1) that the extended Sobolev space $(W_e^{1,2}(D), \frac{1}{2}\mathbf{D})$ is a real Hilbert space and in particular,

$$u \in W_e^{1,2}(D) \text{ having } \quad \mathbf{D}(u,u) = 0 \quad \Longrightarrow \quad u = 0 \quad \text{a.e.,} \tag{2.11}$$

which means that the Hilbert space $(W_e^{1,2}(D), \frac{1}{2}\mathbf{D})$ is isometrically imbedded into a closed subspace of $(\dot{\mathrm{BL}}(D), \frac{1}{2}\mathbf{D})$ by the canonical map $W_e^{1,2}(D) \mapsto \dot{\mathrm{BL}}(D)$. Accordingly, we have

Proposition 2.4. *Assume that the Dirichlet form* (2.8) *on $L^2(D)$ is transient. Then $(W_e^{1,2}(D), \frac{1}{2}\mathbf{D})$ can be regarded as a closed linear subspace of the Hilbert space $(\dot{\mathrm{BL}}(D), \frac{1}{2}\mathbf{D})$ by the canonical map $W_e^{1,2}(D) \mapsto \dot{\mathrm{BL}}(D)$.*

$\mathrm{BL}(D)$ *is a linear space spanned by $W_e^{1,2}(D)$ and $\mathbb{H}^*(D)$.*

The next lemma is well known (cf. [1]).

Lemma 2.5. *When $D = \mathbb{R}^d$, $\mathbb{H}^*(\mathbb{R}^d)$ consists of all constant functions. If $d \geq 3$, then*

$$\mathrm{BL}(\mathbb{R}^d) \textit{ is a linear space spanned by } W_e^{1,2}(\mathbb{R}^d) \textit{ and constant functions.} \tag{2.12}$$

See [6], Example 1.5.3 for a proof, where however an incorrect statement that "the space $W_e^{1,2}(\mathbb{R}^d)$ is obtained from $\mathrm{BL}(\mathbb{R}^d)$ by removing nonzero constant functions" was made. It should be corrected in the manner of (2.12). See also Remark 2.9 below.

We shall be concerned with a finite measure $m(\mathrm{d}x) = m(x)\,\mathrm{d}x$ on D with a density function $m(x)$ satisfying

(A.1) $m(x) > 0$ for every $x \in \overline{D}$ and $m \in C_b(\overline{D}) \cap L^1(D)$.

We then consider the form defined by

$$(\mathcal{E}^*, \mathcal{F}^*) = \left(\frac{1}{2}\mathbf{D}, \mathrm{BL}(D) \cap L^2(D;m)\right), \tag{2.13}$$

which is obtained just by replacing $L^2(D)$ with $L^2(D;m)$ in (2.7) and (2.8).

Proposition 2.6. *The symmetric form $(\mathcal{E}^*, \mathcal{F}^*)$ of* (2.13) *is a recurrent Dirichlet form on $L^2(D;m)$. Its extended Dirichlet space $(\mathcal{E}^*, \mathcal{F}_e^*)$ coincides with the space $(\frac{1}{2}\mathbf{D}, \mathrm{BL}(D))$.*

Proof. Since the convergence in $L^2(D;m)$ implies the convergence in $L^2_{\mathrm{loc}}(D)$, (2.13) can be readily seen to be a closed symmetric form on $L^2(D;m)$. Its Markovian property is an immediate consequence of (BL.2). It satisfies the recurrence condition (2.3) because m is a finite measure on D. We further have

$$u \in \mathcal{F}_e^* \text{ having } \quad \mathcal{E}^*(u,u) = 0 \quad \Longrightarrow \quad u \text{ is constant} \quad \text{a.e.} \tag{2.14}$$

To see this, suppose $u \in \mathcal{F}_e^*$ with $\mathcal{E}^*(u,u) = 0$ and put $u_\ell = \varphi^\ell \circ u$ with the normal contraction

$$\varphi^\ell(t) := \big((-\ell) \vee t\big) \wedge \ell, \quad \ell \in \mathbb{N}. \tag{2.15}$$

Then $u_\ell \in L^2(E;m) \cap \mathcal{F}_e^* = \mathcal{F}^*$ and hence $\mathcal{E}^*(u_\ell, u_\ell) = \frac{1}{2}\mathbf{D}(u_\ell, u_\ell) = 0$. Therefore u_ℓ is a constant and we get (2.14) by letting $\ell \to \infty$.

Denote by $\dot{\mathcal{F}}_e^*$ the quotient space of $\mathcal{F}_e^*$ by the subspace of constant functions. Just as in the proof of the preceding proposition but using (2.14) in place of (2.11), we conclude that the space $(\dot{\mathcal{F}}_e^*, \mathcal{E}^*)$ is isometrically embedded into the space $(\dot{\mathrm{BL}}(D), \frac{1}{2}\mathbf{D})$.

Take any $u \in \mathrm{BL}(D)$ and put $u_\ell = \varphi^\ell \circ u$ as above. By (BL.2), $u_\ell \in \mathrm{BL}(D)$ and

$$\mathbf{D}(u - u_\ell, u - u_\ell) = \int_{\{x:|u(x)|>\ell\}} |\nabla u|^2\,\mathrm{d}x \to 0, \quad \ell \to \infty. \tag{2.16}$$

Since $u_\ell \in \mathcal{F}^*$ and u_ℓ converges to u pointwise, u must be an element of $\mathcal{F}^*_e$. Hence the above isometric embedding is an onto map and so $\mathcal{F}^*_e = \mathrm{BL}(D)$. □

When the Lebesgue measure of the domain D is finite, then we can take the function $m \equiv 1$ in (2.13) in reducing $\mathcal{F}^*$ to $W^{1,2}(D)$. Hence

Corollary 2.7. *If the domain D is of finite Lebesgue measure, then $(\frac{1}{2}\mathbf{D}, W^{1,2}(D))$ is a recurrent Dirichlet form on $L^2(D)$ and*

$$W^{1,2}_e(D) = \mathrm{BL}(D). \tag{2.17}$$

We will show in Theorem 3.1 below that $\mathrm{BL}(D)$ is the reflected Dirichlet space of $(\frac{1}{2}\mathbf{D}, W^{1,2}(D))$ for any $D \in \mathcal{D}$ in any dimension $d \geq 1$.

In what follows, we are concerned with the following condition on the domain $D \subset \mathbb{R}^d$:

(A.2) $D \in \mathcal{D}$ and the Dirichlet form (2.8) on $L^2(D)$ is transient.

This condition forces dimension d to be greater than or equal to 3 and D to have infinite Lebesgue measure in view of Lemma 2.2 and Corollary 2.7. We shall see from Theorem 2.10 below that, if $d \geq 3$ and $D \in \mathcal{D}$ contains an unbounded uniform domain, then D satisfies the condition (A.2); in other words, RBM on $\overline{D}$ is transient. This result is almost sharp since RBM X on an infinite cylinder $D = \{x = (x_1, x_2, \cdot, x_d) \in \mathbb{R}^d\colon \sum_{i=2}^d x_k^2 < 1\}$ where $d \geq 3$ is recurrent because X is direct product of the one-dimensional Brownian motion and the reflecting Brownian motion on the closed unit ball. Note that an infinite cylinder is *not* a uniform domain.

We first prepare a proposition.

Proposition 2.8. *Let D be an unbounded uniform domain in $\mathbb{R}^d$ with $d \geq 3$ and $f \in W^{1,2}_e(\mathbb{R}^d)$. If $f = c$ a.e. on D for some constant c, then $c = 0$.*

Proof. Without loss generality, we assume that the origin $0 \in D$. Since D is unbounded, there is a sequence $\{x_k, k \geq 0\}$ so that $|x_k| = 2Ck$, where $C \geq 1$ is the constant in the Definition 2.1 for the uniform domain D. By definition of uniform domain, for every $k \geq 1$, there is a rectifiable curve γ_k connecting 0 and x_k with $2Ck = |x_k| \leq \mathrm{length}(\gamma_k) \leq C|x_k| = 2C^2k$ and

$$\min\bigl\{|z|, |z - x_k|\bigr\} \leq C \operatorname{dist}\bigl(z, D^c\bigr) \quad \text{for every } z \in \gamma_k.$$

Choose $y_k \in \gamma_k$ such that $|y_k| = |y_k - x_k|$. Then clearly $|y_k| \geq |x_k|/2 = Ck$, $2|y_k| \leq \mathrm{length}(\gamma_k) \leq 2C^2k$ and further

$$\operatorname{dist}\bigl(y_k, D^c\bigr) \geq (1/C)(Ck) = k.$$

Accordingly D contains a sequence of open balls $\{B(y_k, k), k \geq 1\}$ with $Ck \leq |y_k| \leq C^2k$.

Let $X = (X_t, \mathbf{P}_x)$ be the Brownian motion in $\mathbb{R}^d$, which is transient as $d \geq 3$, and $e_1 := (1, 0, \dots, 0) \in \mathbb{R}^d$. Observe that by the Brownian scaling and the rotation invariance of the Brownian motion X,

$$\begin{aligned}
&\mathbf{P}_0(X_t \in D \text{ for some } t \geq Ck) \\
&\quad \geq \mathbf{P}_0\bigl(X_t \in B(y_k, k) \text{ for some } t \geq Ck\bigr) \\
&\quad \geq \mathbf{P}_0\bigl(X_t \in B(y_k, k) \text{ for some } t \geq |y_k|\bigr) \\
&\quad = \mathbf{P}_0\bigl(X_t \in B\bigl(e_1, k/|y_k|\bigr) \text{ for some } t \geq 1/|y_k|\bigr) \\
&\quad \geq \mathbf{P}_0\bigl(X_t \in B\bigl(e_1, C^{-2}\bigr) \text{ for some } t \geq C^{-1}\bigr) =: p_0 > 0.
\end{aligned}$$

Consequently,

$$\mathbf{P}_0\Biggl(\bigcap_{n\geq 1}\bigcup_{k\geq n}\{X_t \in D \text{ for some } t \geq Ck\}\Biggr) \geq p_0. \tag{2.18}$$

Let ∂ be the one-point compactification of $\mathbb{R}^d$. As is well known,

$$\mathbf{P}_x\Big(\lim_{t\to\infty} X_t = \partial\Big) = 1, \quad x \in \mathbb{R}^d. \tag{2.19}$$

Any $f \in W_e^{1,2}(\mathbb{R}^d)$ admits a version $\widetilde{f}$ which is quasi-continuous in the restricted sense with respect to the transient extended Dirichlet space $(W_e^{1,2}(\mathbb{R}^d), \frac{1}{2}\mathbf{D})$, namely, there exists a decreasing sequence of open subsets $\{G_n\}$ of $\mathbb{R}^d$ such that $\mathrm{Cap}_{(0)}(G_n) \to 0, n \to \infty$, and the restriction of $\widetilde{f}$ to each set $(\mathbb{R}^d \setminus G_n) \cup \{\partial\}$ is continuous there if we set $\widetilde{f}(\partial) = 0$. By (2.9) of [3], it then holds that $\mathbf{P}_x(\sigma_{G_n} = \infty \text{ for some } n \geq 1) = 1$, q.e. $x \in \mathbb{R}^d$, which combined with (2.19) yields that $\varphi(x) = 1$ for q.e. $x \in \mathbb{R}^d$, where $\varphi(x) = \mathbf{P}_x(\lim_{t\to\infty} \widetilde{f}(X_t) = 0)$. Hence, by the Markov property,

$$\mathbf{P}_0\Big(\lim_{t\to\infty} \widetilde{f}(X_t) = 0\Big) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-|x|^2/2}\varphi(x)\,\mathrm{d}x = 1. \tag{2.20}$$

Suppose $f \in W_e^{1,2}(\mathbb{R}^d)$ equals a constant c a.e. on D, then $\widetilde{f} = c$ q.e. on D, and we can deduce from (2.18) and (2.20) that $c = 0$. □

Remark 2.9. *We can make the statement* (2.12) *more specific as follows*: *When $d \geq 3$, $W_e^{1,2}(\mathbb{R}^d)$ is obtained from the quotient space $\dot{\mathrm{BL}}(\mathbb{R}^d)$ by choosing from each equivalence class a special function whose quasi-continuous version has zero limit at ∂ in the sense of* (2.20).

Theorem 2.10. *Assume that $d \geq 3$ and D is a domain in $\mathcal{D}$ that contains an unbounded uniform domain. Then RBM on $\overline{D}$ is transient*; *in other words*, *D satisfies condition* (A.2).

Proof. In view of the comparison result Lemma 2.2(ii), we may assume, without loss of generality, that D is an unbounded uniform domain.

We first show that $W_e^{1,2}(D) = W_e^{1,2}(\mathbb{R}^d)|_D$. By [8], Theorem 4.13, there is a linear bounded extension operator $S: \mathrm{BL}(D) \to \mathrm{BL}(\mathbb{R}^d)$ in the following sense: for every $u \in \mathrm{BL}(D)$, $Su \in \mathrm{BL}(\mathbb{R}^d)$ with $Su = u$ a.e. on D, moreover,

$$\big\|\nabla(Su)\big\|_{L^2(\mathbb{R}^d)} \leq M\|\nabla u\|_{L^2(D)} \quad \text{for every } u \in \mathrm{BL}(D). \tag{2.21}$$

For $u \in W^{1,2}(D)$, $Su \in \mathrm{BL}(\mathbb{R}^d)$ and so by Lemma 2.5, $Su = v_0 + c$ for some $v_0 \in W_e^{1,2}(\mathbb{R}^d)$ and constant c. On the other hand, by (2.9), there is $Tu \in W^{1,2}(\mathbb{R}^d)$ so that $Tu = u$ a.e. on D. It follows that $Tu - v_0 \in W_e^{1,2}(\mathbb{R}^d)$ and $Tu - v_0 = c$ on D. Proposition 2.8 now implies that $c = 0$. This proves that the extension operator S maps $W^{1,2}(D)$ into $W_e^{1,2}(\mathbb{R}^d)$. Now for $u \in W_e^{1,2}(D)$, there is a $\mathbf{D}$-Cauchy sequence $\{u_k, k \geq 1\} \subset W^{1,2}(D)$ so that $u_k \to u$ a.e. on D. By (2.21) and what we just established, $\{Su_k, k \geq 1\}$ is a $\mathbf{D}$-Cauchy sequence in $W_e^{1,2}(\mathbb{R}^d)$. Since $(W_e^{1,2}(\mathbb{R}^d), \mathbf{D})$ is a Hilbert space, Su_k is $\mathbf{D}$-convergent to some $f \in W_e^{1,2}(\mathbb{R}^d)$. Taking a subsequence if necessary, Su_k converges to f a.e. on $\mathbb{R}^d$. It follows then $u = f$ a.e. on D. This proves that $W_e^{1,2}(D) = W_e^{1,2}(\mathbb{R}^d)|_D$, as clearly $W_e^{1,2}(\mathbb{R}^d)|_D \subset W_e^{1,2}(D)$. (The above also implies that S maps $W_e^{1,2}(D)$ into $W_e^{1,2}(\mathbb{R}^d)$. This is because for $u \in W_e^{1,2}(D)$, $Su \in \mathrm{BL}(\mathbb{R}^d)$ and so $Su = v_0 + c$ for some $v_0 \in W_e^{1,2}(\mathbb{R}^d)$ and constant c. On the other hand, there is some $f \in W_e^{1,2}(\mathbb{R}^d)$ so that $f = u$ a.e. on D. Thus we have $f - v_0 = c$ on D. Since $f - v_0 \in W_e^{1,2}(\mathbb{R}^d)$, we have by Proposition 2.8 that $c = 0$. This proves that $Su = v_0 \in W_e^{1,2}(\mathbb{R}^d)$.)

To show that RBM on $\overline{D}$ is transient, by (2.2) and Lemma 2.3, it suffices to show that if $u \in W_e^{1,2}(D)$ has $\mathbf{D}(u,u) = 0$, then $u = 0$ on D. Suppose $u \in W_e^{1,2}(D)$ and $\mathbf{D}(u,u) = 0$. Clearly, $u = c$ a.e. on D for some constant c. We know from the above that there is $u_0 \in W_e^{1,2}(\mathbb{R}^d)$ so that $u_0 = u = c$ a.e. on D. It follows from Proposition 2.8 that $c = 0$ and so $u = 0$. □

3. Unique symmetric extension and the space of BL-functions

We start this section with a domain $D \in \mathcal{D}$, together with a measure $m(\mathrm{d}x) = m(x)\,\mathrm{d}x$ on D whose density $m(x)$ is strictly positive for every $x \in \overline{D}$ and satisfies $m \in C_b(\overline{D})$. Consider the following symmetric form

$$\big(\mathcal{E}^{(0)}, \mathcal{F}^{(0)}\big) = \left(\frac{1}{2}\mathbf{D}, W_e^{1,2}(D) \cap L^2(D; m)\right), \tag{3.1}$$

obtained by just replacing $\mathrm{BL}(D)$ with $W_e^{1,2}(D)$ in (2.13). As for (2.13), the above form can be readily checked to be a Dirichlet form on $L^2(D; m)$. Furthermore, it is a regular and strongly local Dirichlet form on $L^2(\overline{D}; m)$. The associated diffusion process $Y = (Y_t, \mathbf{P}_x)$ on $\overline{D}$ is obtained from the reflecting Brownian motion X by a time change with respect to the additive functional $A_t = \int_0^t m(X_s)\,\mathrm{d}s$ in view of [6], Section 6.2.

For a general regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, the notions of its reflected Dirichlet space $(\mathcal{F}^{\mathrm{ref}}, \mathcal{E}^{\mathrm{ref}})$ and its active reflected Dirichlet form $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_a^{\mathrm{ref}})$ were originally introduced by Silverstein (see [14]) and have been further studied in [2] when $(\mathcal{E}, \mathcal{F})$ is transient. These notions are well defined for the present Dirichlet form (3.1).

Assume that $m \in C_b(\overline{D})$ with $m(x) > 0$ for every $x \in \overline{D}$ and let $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ be the Dirichlet form on $L^2(D; m)$ defined by (3.1). Following [2], let

$$\begin{aligned}
&\mathcal{F}^{\mathrm{ref}} := \left\{ f\colon\ f_k := \varphi^k \circ f \in \mathcal{F}^{(0)}_{\mathrm{loc}} \text{ and } \sup_{k \ge 1} \mathbf{D}(f_k, f_k) < \infty \right\}, \\
&\mathcal{F}_a^{\mathrm{ref}} := \mathcal{F}^{\mathrm{ref}} \cap L^2(D; m), \\
&\mathcal{E}^{\mathrm{ref}}(f, f) := \frac{1}{2} \int_D \big|\nabla f(x)\big|^2 \,\mathrm{d}x = \frac{1}{2}\mathbf{D}(f, f) \quad \text{for } f \in \mathcal{F}^{\mathrm{ref}}.
\end{aligned}$$

We call $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}^{\mathrm{ref}})$ and $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_a^{\mathrm{ref}})$ the reflected Dirichlet form and the active reflected Dirichlet form, respectively, of $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ on $L^2(D; m)$. In [2], the above notions are defined for transient $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ and it is proved there that $(\mathcal{E}^{\mathrm{ref}}, \mathcal{F}_a^{\mathrm{ref}})$ is a Dirichlet form on $L^2(D; m)$. But these notions can also be defined when $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ is recurrent.

Theorem 3.1. *Let $D \in \mathcal{D}$ be a domain in $\mathbb{R}^d$ with $d \ge 1$. Assume that $m \in C_b(\overline{D})$ with $m(x) > 0$ for every $x \in \overline{D}$. Then the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ defined by* (2.13) *with $m(\mathrm{d}x) = m(x)\,\mathrm{d}x$ is the active reflected Dirichlet form of the Dirichlet form $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ given by* (3.1) *and* $\mathrm{BL}(D)$ *is the reflected Dirichlet space of $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$.*

Proof. For $f \in \mathcal{F}^{\mathrm{ref}} \cap L^\infty(D)$, $f \in \mathcal{F}^{(0)}_{\mathrm{loc}}$ and so $f \in \mathrm{BL}(D)$. For general $f \in \mathcal{F}^{\mathrm{ref}}$ and $k \ge 1$, define $f_k = \varphi^k \circ f$ for $k \ge 1$, where φ^k is defined in (2.15). From above we have $f_k \in \mathrm{BL}(D)$ with $\mathcal{E}^{\mathrm{ref}}(f_k, f_k) = \frac{1}{2}\mathbf{D}(f_k, f_k)$. Since $\sup_{k \ge 1} \mathbf{D}(f_k, f_k) < \infty$, there is a Cesàro mean $\{g_k, k \ge 1\}$ of $\{f_k, k \ge 1\}$ that is $\mathbf{D}$-Cauchy. Observe that $g_k \in \mathcal{F}^{\mathrm{ref}} \cap L^\infty(D) \subset \mathrm{BL}(D)$. By (BL.1), there is $u \in \mathrm{BL}(D)$ and a sequence of constants $\{c_k, k \ge 1\}$ so that g_k is $\mathbf{D}$-convergent to u and $g_k + c_k$ converges to u locally in $L^2(D)$ as $k \to \infty$. But $\lim_{k\to\infty} g_k = f$ on D. This implies that $\lim_{k\to\infty} c_k = c$ and $f + c = u$. Therefore $f \in \mathrm{BL}(D)$ and so $\mathcal{F}^{\mathrm{ref}} \subset \mathrm{BL}(D)$.

Now for $f \in \mathrm{BL}(D)$, clearly $f_k = \varphi^k \circ f \in \mathrm{BL}(D) \cap L^2_{\mathrm{loc}}(D) \subset \mathcal{F}^{(0)}_{\mathrm{loc}}$ for every $k \ge 1$. Since $\mathbf{D}(f_k, f_k) \le \mathbf{D}(f, f)$, we conclude that $f \in \mathcal{F}_a^{\mathrm{ref}}$. This completes the proof that $\mathrm{BL}(D) = \mathcal{F}^{\mathrm{ref}}$. It then follows immediately that $\mathcal{F}^* = \mathrm{BL}(D) \cap L^2(D; m) = \mathcal{F}^{\mathrm{ref}} \cap L^2(D; m) = \mathcal{F}_a^{\mathrm{ref}}$. □

Taking $m(x) \equiv 1$ on D in Theorem 3.1 yields in particular that $\mathrm{BL}(D)$ is the reflected Dirichlet space for $(\frac{1}{2}\mathbf{D}, W^{1,2}(D))$ for every $D \in \mathcal{D}$ in any dimension.

We shall next consider a domain $D \in \mathcal{D}$ satisfying the transience condition (A.2) and the function m satisfying condition (A.1). Then $d \ge 3$ and D is of infinite Lebesgue measure as is noted in Section 2. Abusing the notation a bit, we also use ∂ to denote the point at infinity of $\overline{D}$, namely, we let $\overline{D}^* = \overline{D} \cup \{\partial\}$ be the one point compactification of $\overline{D}$. Under condition (A.2), the reflecting Brownian motion $X = (X_t, \mathbf{P}_x)$ on $\overline{D}$ satisfies

$$\mathbf{P}_x\Big(\lim_{t\to\infty} X_t = \partial\Big) = 1 \quad \text{for q.e. } x \in \overline{D} \tag{3.2}$$

on account of [3], Theorem 2.4. In this case, the Dirichlet form $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ of (3.1) is transient and its associated diffusion process, time-changed Brownian motion $Y = (Y_t, \mathbf{P}_x)$ on $\overline{D}$ has lifetime ζ satisfying $\mathbf{P}_x(\zeta < \infty) = 1$ for q.e. $x \in \overline{D}$. This is because, since $m \in L^1(D, \mathrm{d}x)$ and the RBM X on $\overline{D}$ is transient, by Lemma 1.5.1 of [6], $\mathbf{E}^Y_x[\zeta] = \mathbf{E}_x[\int_0^\infty m(X_s)\,\mathrm{d}s] < \infty$ for q.e. $x \in \overline{D}$. Moreover, Y_t approaches to ∂ as $t \to \zeta$ in view of (3.2).

Let us further make the following assumption on the domain D:

(A.3) $\mathbb{H}^*(D)$ consists of constant functions on D.

Here $\mathbb{H}^*(D)$ is the space of harmonic functions on D appearing in Lemma 2.3. Thus under the transience condition (A.2), (A.3) amounts to assuming that

$$\mathrm{BL}(D) \text{ is a linear space spanned by } W_e^{1,2}(D) \text{ and constant functions.} \tag{3.3}$$

We will show in Theorem 3.5 below that conditions (A.2) and (A.3) hold for every unbounded uniform domain. Moreover Proposition 3.6 will imply that the condition (A.3) is satisfied by any domain $D \in \mathcal{D}$ that is the complement of a compact set.

Under the stated three conditions, we shall compare the form $(\mathcal{E}^*, \mathcal{F}^*)$ defined by (2.13) and the $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ defined by (3.1). They are Dirichlet forms on $L^2(D; m)$ and related to each other by

$$\begin{cases} \mathcal{F}^* = \{u = u_0 + c\colon u_0 \in \mathcal{F}^{(0)}, c \text{ is constant}\}, \\ \mathcal{E}^*(u, u) = \mathcal{E}^{(0)}(u_0, u_0) = \frac{1}{2}\mathbf{D}(u_0, u_0) \end{cases} \tag{3.4}$$

in view of Proposition 2.4.

Let m^* be the extension of m from D to $\overline{D}^*$ obtained by setting

$$m^*\bigl(\partial D \cup \{\partial\}\bigr) = 0.$$

By identifying $L^2(D; m)$ with $L^2(\overline{D}^*; m^*)$, we can regard $(\mathcal{E}^*, \mathcal{F}^*)$ as a Dirichlet form on $L^2(\overline{D}^*; m^*)$.

Theorem 3.2. *Assume that conditions* (A.1), (A.2) *and* (A.3) *hold.*

(i) *$(\mathcal{E}^*, \mathcal{F}^*)$ is a recurrent, strongly local and regular Dirichlet form on $L^2(\overline{D}^*, m^*)$.*
(ii) *The associated diffusion process $Y^* = (Y^*_t, \mathbf{P}^*_x)$ on $\overline{D}^*$ is a conservative extension of the time changed transient reflected Brownian motion Y on $\overline{D}$ and satisfies*

$$\mathbf{P}^*_x(\sigma_\partial < \infty) = 1 \quad \textit{for q.e. } x \in \overline{D}^*. \tag{3.5}$$

Proof. (i) The recurrence condition (2.3) is trivially satisfied by $(\mathcal{E}^*, \mathcal{F}^*)$. If we let $\mathcal{C} = \{u + c\colon u \in C_c^\infty(\overline{D}), c \text{ is constant}\}$, then $\mathcal{C} \subset C(\overline{D}^*)$ and $\mathcal{C}$ is readily seen to be a core of the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$. The strong locality of $(\mathcal{E}^*, \mathcal{F}^*)$ can be proved in the same way as in the proof of Theorem 3.2 of [7], where a Dirichlet form quite similar to (3.4) was studied in a rather general context.

(ii) Since Y is associated with $(\mathcal{E}^0, \mathcal{F}^0)$, Y^* is an extension of Y from $\overline{D}$ to $\overline{D}^* = \overline{D} \cup \{\partial\}$. The point ∂ is not m^*-polar for X^* because $\mathbf{P}^*_{m^*}(\sigma_\partial < \infty)$ dominates the quantity $\mathbf{P}_m(\zeta < \infty) > 0$ for Y. On the other hand, the irreducibility of $(\mathcal{E}^*, \mathcal{F}^*)$ can be verified in a similar manner to [7]. Hence (3.5) holds in view of [6], Theorem 4.6.6. □

Remark 3.3. *The extended diffusion Y^* of Y in the above theorem can be also constructed stochastically by a darning of Y at ∂ as in* [7] *and* [3], *namely, by piecing together the excursions of Y around ∂ according to an excursion-valued Poisson point process $\{\mathbf{p}_t\}$. The characteristic measure $\mathbf{n}$ of $\{\mathbf{p}_t\}$ is described as* [7], (4.4) *in terms of the transition function $\{q_t\}$ of Y and the $\{q_t\}$-entrance law $\{\mu_t\}$ on E uniquely determined by the equation*

$$m = \int_0^\infty \mu_t\,\mathrm{d}t.$$

Using Theorem 3.1, we can further show that Y^* is the unique m-symmetric continuous genuine extension of Y in the following sense.

Theorem 3.4. *Suppose that there is a Luzin space E so that $\overline{D}$ is embedded continuously in E as a dense open subset and there is an m-symmetric diffusion Z on E that admits no killings inside and that Y is the proper subprocess of Z killed upon leaving $\overline{D}$. Here the measure m is extended to E by setting $m(E \setminus \overline{D}) = 0$. Then the symmetric Dirichlet form of Z on $L^2(E; m)$ coincides with that of X^* on $L^2(\overline{D}; m)$; in other words, Z under $\mathbf{P}_m$ has the same finite-dimensional distributions as that of Y^* under $\mathbf{P}^*_m$.*

Note that although Z and X^* live on different state spaces, since $m(E \setminus \overline{D}) = 0$ and $m(\{\partial\}) = 0$,

$$\mathbf{P}_m(Z_t \in E \setminus \overline{D}) = \mathbf{P}^*_m\big(Y^*_t = \partial\big) = 0 \quad \text{for every } t \geq 0$$

and so we can compare the finite-dimensional distributions of Z under $\mathbf{P}_m$ with those of Y^* under $\mathbf{P}^*_m$.

Proof of Theorem 3.4. Let $(\mathcal{E}^Z, \mathcal{F}^Z)$ denote the symmetric Dirichlet form of Z on $L^2(E; m)$. It is known (cf. [11]) that $(\mathcal{E}^Z, \mathcal{F}^Z)$ is quasi-regular on E and so, by [4], it is quasi-homeomorphic to a regular Dirichlet form on a locally compact metric space. Consequently, the results established under the regular Dirichlet form framework in [6] and [14] apply to Z and $(\mathcal{E}^Z, \mathcal{F}^Z)$.

Let $\mathcal{L}$ and $\mathcal{L}^0$ be the L^2-generators of the Dirichlet forms $(\mathcal{E}^Z, \mathcal{F}^Z)$ on $L^2(E; m)$ and $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ on $L^2(\overline{D}; m)$, respectively. Since Y is the part process of Z killed upon leaving $\overline{D}$, for every $f \in \mathrm{Dom}(\mathcal{L})$, there is a unique $\mathcal{E}^Z$-orthogonal decomposition $f = f_0 + h$, where $f_0 \in \mathcal{F}^{(0)}$ and $h \in \mathcal{F}^Z$ so that $\mathcal{E}^Z(g, h) = 0$ for every $g \in \mathcal{F}^{(0)}$. Note that h is harmonic with respect to the diffusion Y (see [6]). Since $f \in \mathrm{Dom}(\mathcal{L})$, we have in particular

$$(-\mathcal{L} f, g)_{L^2(\overline{D}; m)} = \mathcal{E}^Z(f, g) = \mathcal{E}^{(0)}(f_0, g) \quad \text{for every } g \in \mathcal{F}^{(0)} \subset \mathcal{F}^Z.$$

The above implies that $f_0 \in \mathrm{Dom}(\mathcal{L}^0)$ and $\mathcal{L}^{(0)} f_0 = \mathcal{L} f$. In the terminology of Silverstein ([14], Definition 15.1 on p. 152), $\mathcal{L}$ is contained in the *local generator* of $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$. So by [14], Theorem 15.2, and its proof as well as Theorem 3.1 above, we have

$$\mathcal{F}^Z \subset \mathcal{F}^* \quad \text{and} \quad \mathcal{E}^Z(f, f) \geq \mathcal{E}^*(f, f) \quad \text{for } f \in \mathcal{F}^Z.$$

Since Z is a genuine extension of Y, $\mathcal{F}^Z$ must be the same as $\mathcal{F}^*$ in view of (3.4). For any $f = f_0 + c_1 \in \mathcal{F}^*$ with $f_0 \in \mathcal{F}^{(0)}$, since $(\mathcal{E}^Z, \mathcal{F}^Z)$ is local and admits no killing, we have

$$\mathcal{E}^Z(f, f) = \mathcal{E}^Z(f_0, f_0) = \mathcal{E}^{(0)}(f_0, f_0) = \mathcal{E}^*(f, f).$$

This proves that $(\mathcal{E}^Z, \mathcal{F}^Z) = (\mathcal{E}^*, \mathcal{F}^*)$. □

Finally we give sufficient conditions for (A.3) to hold.

Theorem 3.5. *Let $D \subset \mathbb{R}^d$ be a uniform domain. Then $\mathbb{H}^*(D)$ consists of all constant functions on D. In particular, if $d \geq 3$ and $D \subset \mathbb{R}^d$ is an unbounded uniform domain, then D satisfies conditions* (A.2) *and* (A.3).

Proof. Again we use [8], Theorem 4.13, which states that there is a linear bounded extension operator $S : \mathrm{BL}(D) \to \mathrm{BL}(\mathbb{R}^d)$ in the following sense: for every $u \in \mathrm{BL}(D)$, $Su \in \mathrm{BL}(\mathbb{R}^d)$ with $Su = u$ a.e. on D and with bound (2.21).

For $u \in \mathbb{H}^*(D) \subset \mathrm{BL}(D)$, $Su \in \mathrm{BL}(\mathbb{R}^d)$. Therefore, by Lemma 2.5, $Su = f_0 + c$ for some $f_0 \in W^{1,2}_e(\mathbb{R}^d)$ and some constant c. Note that $f_0|_D \in W^{1,2}_e(D)$. Since $f_0|_D = u - c \in \mathbb{H}^*(D)$, we obtain

$$\mathbf{D}(f_0, f_0) = 0, \tag{3.6}$$

which implies that $f_0 = c_1$ for some constant. Hence $u \in \mathbb{H}^*(D)$ equals a constant $c + c_1$ on D.

The second statement of the theorem follows from the above and Theorem 2.10. □

We can readily deduce the next proposition from the above theorem. In particular, any unbounded domain $D \subset \mathbb{R}^d$, $d \geq 3$, in $\mathcal{D}$ with compact boundary satisfies (A.3).

Proposition 3.6. *If $D \subset \mathbb{R}^d$, $d \geq 3$ is an unbounded domain in $\mathcal{D}$ such that $D \setminus \overline{B(0,r)}$ is a uniform domain for some $r > 0$, then condition* (A.3) *holds.*

Proof. By normal contraction property, it suffices to show that for every bounded nonnegative $u \in \mathrm{BL}(D)$, $u = u_0 + c$ for some $u_0 \in W_e^{1,2}(D)$ and a constant c.

We put $U = D \setminus B(0,r)$ and take $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi = 1$ on $B(0, r+1)$. As $u \in \mathrm{BL}(D) \subset \mathrm{BL}(U)$ and U is an unbounded uniform domain, we see by the above theorem that there is a constant c with $u_0 := u - c \in W_e^{1,2}(U)$. We claim that $(1-\phi)u_0 \in W_e^{1,2}(U)$. This is because there is a sequence $f_n \in W^{1,2}(U)$ such that $\sup_{n\geq 1}\int_U |\nabla f_n|^2\,\mathrm{d}x = 0$ and $f_n \to u_0, n \to \infty$, a.e. on U. Since u_0 is bounded, by the normal contraction property, we may assume that $\{f_n, n \geq 1\}$ are uniformly bounded. Clearly $(1-\phi)f_n \in W^{1,2}(D)$, $(1-\phi)f_n$ converges a.e. on D to $(1-\phi)u_0$ and

$$
\begin{aligned}
&\sup_{n\geq 1}\int_D \bigl|\nabla\bigl((1-\phi)f_n\bigr)\bigr|^2\,\mathrm{d}x \\
&\quad\leq 2\sup_{n\geq 1}\|f_n\|^2_{L^\infty(U)}\int_{\mathbb{R}^d}|\nabla\phi|^2\,\mathrm{d}x + 2\bigl\|(1-\phi)\bigr\|^2_\infty \sup_{n\geq 1}\int_U |\nabla f_n|^2\,\mathrm{d}x < \infty.
\end{aligned}
$$

We conclude that $(1-\phi)u_0 \in W_e^{1,2}(D)$. On the other hand, since $\phi u_0 \in W^{1,2}(D)$, we have $u - c = u_0 = \phi u_0 + (1-\phi)u_0 \in W_e^{1,2}(D)$. This proves the theorem. □

Remark 3.7. *The second statement of Theorem* 3.5 *is "sharp" for* (A.3) *in view of the following example.*

If D is an unbounded domain in $\mathbb{R}^d$ with two or more infinite branches, then condition (A.3) *may fail. Consider*

$$
D = B(0,1) \cup \left\{ x = (x_1, x_2, \ldots, x_d):\ x_d^2 > \sum_{k=1}^{d-1} x_k^2 \right\}
$$

in $\mathbb{R}^d$ for $d \geq 3$. Here $B(x,r)$ denote the ball centered at x with radius $r > 0$. Clearly D is a Lipschitz domain but it is not a uniform domain as it has a bottleneck B_1. However D contains an unbounded uniform domain

$$
C_1 := B(0,1)^c \cap \bigl\{x_d > \sqrt{x_1^2 + \cdots + x_{d-1}^2}\bigr\}.
$$

So by Theorem 2.10, *condition* (A.2) *holds for D.*

We claim that condition (A.3) *does not hold for D. Define*

$$
C_2 = B(0,1)^c \cap \bigl\{x_d < -\sqrt{x_1^2 + \cdots + x_{d-1}^2}\bigr\}.
$$

Let $f \in C_b^2(\overline{D})$ be such that $f = 1$ on C_1 and $f = 2$ on C_2. Clearly $f \in \mathrm{BL}(D)$ and so we can write it as $f = f_0 + h$ with $f_0 \in W_e^{1,2}(D)$ and $h \in \mathbb{H}^*(D)$. Let X be RBM on $\overline{D}$, which is transient as is noted in the above. By (3.2)

$$
\mathbf{P}_x\Bigl(\lim_{t\to\infty} f(X_t) = 1\Bigr)\cdot\mathbf{P}_x\Bigl(\lim_{t\to\infty} f(X_t) = 2\Bigr) > 0 \quad \text{for q.e. } x \in D.
$$

On the other hand, f_0 is quasi-continuous in the restricted sense with respect to the transient extended Dirichlet space $(W_e^{1,2}(D), \frac{1}{2}\mathbf{D})$: there exists a decreasing sequence of open subsets $\{G_n\}$ of $\overline{D}$ such that $\mathrm{Cap}_{(0)}(G_n) \to 0$ *as $n \to \infty$, and the restriction of f_0 to each set $(\overline{D} \setminus G_n) \cup \{\partial\}$ is continuous there if we set $f_0(\partial) = 0$. By* (2.9) *of* [3], *it then holds that $\mathbf{P}_x(\sigma_{G_n} = \infty$ for some $n \geq 1) = 1$, q.e. $x \in D$, which combined with* (3.2) *leads us to* $\lim_{t\to\infty} f_0(X_t) = 0$ $\mathbf{P}_x$*-a.s. for q.e. $x \in D$. Thus we have* $\lim_{t\to\infty} h(X_t) = \lim_{t\to\infty} f(X_t)$ $\mathbf{P}_x$*-a.s. for q.e. $x \in D$. This yields that h cannot be a constant.*

Remark 3.8. *In the case where the reflecting Brownian motion on $\overline{D}$ is recurrent, we have the identification*

$$
\left(W_e^{1,2}(D), \frac{1}{2}\mathbf{D}\right) = \left(\mathrm{BL}(D), \frac{1}{2}\mathbf{D}\right). \tag{3.7}
$$

In Corollary 2.7, this has been verified for any domain $D \subset \mathbb{R}^d$, $d \geq 1$, with finite Lebesgue measure. When $d \leq 2$, (3.7) holds for any domain $D \subset \mathbb{R}^d$. Here is a proof when $d = 2$.

Take any function $u \in \mathrm{BL}(D)$ such that $|u| \leq \ell$ for some constant. Let $\psi_n \in C_c^1(\mathbb{R}_+)$ be functions satisfying

$$\begin{cases} \psi_n(x) = 1, & \text{for } 0 \leq x < n, \\ \psi_n(x) = 0, & \text{for } x > 2n+1, \\ \left|\psi_n'(x)\right| \leq \frac{1}{n}, & \text{for } n \leq x \leq 2n+1, \\ 0 \leq \psi_n(x) \leq 1, & \text{for } x \in \mathbb{R}_+. \end{cases}$$

and put $u_n(x) = u(x)\psi_n(|x|)$, $x \in D$. Then $u_n \in \mathrm{BL}(D) \cap L^2(D) = W^{1,2}(D)$ and

$$\begin{aligned} \mathbf{D}(u_n, u_n) &\leq 2\int_D |\nabla u|^2(x)\psi_n\big(|x|\big)^2\,\mathrm{d}x + 2\int_D u^2(x)\psi_n'\big(|x|\big)^2\,\mathrm{d}x \\ &\leq 2\mathbf{D}(u,u) + 2\ell^2\int_0^{2\pi}\int_0^{2n+1}\psi_n'(r)^2 r\,\mathrm{d}r\,\mathrm{d}\theta \\ &\leq 2\mathbf{D}(u,u) + \frac{2\ell^2\pi(2n+1)^2}{n^2} \leq 2\mathbf{D}(u,u) + 18\ell^2\pi. \end{aligned}$$

Hence a Cesàro mean of a subsequence of $\{u_n\}$ is $\mathbf{D}$-convergent. Since u_n converges to u pointwise, we conclude that $u \in W_e^{1,2}(D)$ and $\mathcal{E}(u,u) = \frac{1}{2}\mathbf{D}(u,u)$.

Next take any $u \in \mathrm{BL}(D)$ and put $u_\ell = \varphi^\ell \circ u$, $\ell \in \mathbb{N}$, for the normal contraction φ^ℓ of (2.15). By (BL.2), $u_\ell \in \mathrm{BL}(D)$ and we have (2.16), which particularly means that $\mathbf{D}(u_\ell, u_\ell)$ is bounded. We have just shown that $u_\ell \in W_e^{1,2}(D)$ with $\mathcal{E}(u_\ell, u_\ell) = \frac{1}{2}\mathbf{D}(u_\ell, u_\ell)$.

Under the general setting in the beginning of Chapter 2, let $(\mathcal{F}_e, \mathcal{E})$ be the extended Dirichlet space of a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;m)$. Assume that an m-measurable function f on E is finite m-a.e. on E, $f_\ell = \varphi^\ell \circ f \in \mathcal{F}_e$ for each ℓ, and $\sup_\ell \mathcal{E}(f_\ell, f_\ell) < \infty$. Then it can be readily shown that $f \in \mathcal{F}_e$. In the present case, we have therefore $u \in W_e^{1,2}(D)$ and $\mathcal{E}(u,u) = \frac{1}{2}\mathbf{D}(u,u)$.

The general case follows from Theorem 3.1 above and a general result for reflected Dirichlet space of a recurrent Dirichlet form, see Theorem 16.2 in [14].

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