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M D

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**Abstract.** A w H -
. W w w G L U . W
L . T -
. W w
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**Résumé.** N
. C ,
L . N
L . P - ,
. N , , .

*MSC:* P 15A52; 17B10

*Keywords:* R ; D ; I ; G T ; C ; M ; R

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## 1. Introduction

O (RMT) . I
1928, w W 61
G . I 1951, W 60 H , w

***Definition 1.1.*** *A random matrix, or an ensemble, $M$ with values in $\mathcal{P}_n(\mathbb{F})$ is called invariant if its distribution is invariant under conjugation by $U_n(\mathbb{F})$.*

. I GUE , J N 34 , O R 47 .
A -
. I S 7, H
,
. F S 9. T
$\mathbb{F}=\mathbb{C}$, . A
. T . T
K . A S 9,
- , , . F S 10,
RMT .
O H . F , C S
15 , . T
- . W
16 , F N 24 GUE($\mathbb{R}$) .

***Notation 1.2.*** *In this paper, for an integer n we will write*

$$\tilde{n}=\begin{cases} n, & \text{when } \mathbb{F}=\mathbb{C} \text{ and } \mathbb{H},\\ \lfloor n/2\rfloor, & \text{when } \mathbb{F}=\mathbb{R}.\end{cases}$$

*We let $c=1$ if $\mathbb{F}=\mathbb{C},\mathbb{R}$, $c=2$ if $\mathbb{F}=\mathbb{H}$ and $\epsilon=1$ if $n$ is odd and 0 otherwise.*

## *Part* 1. RANDOM MATRICES

## 2. Ensembles of Hermitian matrices

### 2.1. *Some invariant set of Hermitian matrices*

T $\mathcal{P}_n(\mathbb{C})$ $n\times n$ H $M$ $M^*=M$,
$M^*$ $M$. M $\mathcal{P}_n(\mathbb{C})$. L
,
- , $\beta=2$ . T
. O .
T $\mathcal{P}_n(\mathbb{C})$ . T $\mathcal{P}_n(\mathbb{R})$ H $M$
$M=X$ $X$ . I $X$ ( . . $X+X^t=0$). T $\mathcal{P}_n(\mathbb{R})$
, M 45 .
I , $C$- A 1 . I
H $\mathcal{H}$

$$\mathcal{H}=\begin{pmatrix} H & S\\ S^* & -\bar{H}\end{pmatrix}, \tag{1}$$

$H$ $S$ $n\times n$ , $H$ H $S$ . I
H L . O B
G H ( ). A
. F , $\mathbb{H}$ $2\times 2$ $Z$

$$Z=\begin{pmatrix} a & b\\ -\bar{b} & \bar{a}\end{pmatrix},$$

$a, b \in \mathbb{C}$. I $Z^*$ $Z$. W $\mathcal{P}_n(\mathbb{H})$ $2n \times 2n$
H $M$ $M =$ $X$ $X$ $n \times n$ .
L $W$ $\mathbb{C}^{2n}$:

$$(x_1, x_2, \ldots) \mapsto (x_1, x_{n+1}, x_2, x_{n+2}, x_3, \ldots).$$

T $\mathcal{H}$ H (1)

$$\tilde{\mathcal{H}} = W\mathcal{H}W^{-1} \tag{2}$$

$\mathcal{P}_n(\mathbb{H})$. T $\mathcal{P}_n(\mathbb{H})$ $C$ A . N
GSE $\mathbb{H}$. W :

***Definition 2.1.*** *For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $\mathcal{P}_n(\mathbb{F})$ is the set of $n \times n$ Hermitian matrices with entries in $\mathbb{F}$.*

O $\mathcal{P}_n(\mathbb{F})$ C ( )
. I , $U_n(\mathbb{F})$ $\mathbb{F}$. I L
$\mathfrak{U}_n(\mathbb{F})$ $M$ $\mathbb{F}$ $M + M^* = 0$. T $\mathcal{P}_n(\mathbb{F}) = \mathfrak{U}_n(\mathbb{F})$, C
$U_n(\mathbb{F})$

$$G = U_n(\mathbb{F}) \times_\sigma \mathcal{P}_n(\mathbb{F}),$$

$U_n(\mathbb{F})$ $\mathcal{P}_n(\mathbb{F})$ $\sigma$ ( . . ). I ,
$\mathcal{P}_n(\mathbb{C})$ A $U_n(\mathbb{C})$ . W $n = 2r$, $\mathcal{P}_n(\mathbb{R})$ D $n = 2r + 1$,
$\mathcal{P}_n(\mathbb{R})$ B, $U_n(\mathbb{R})$ $SO(n)$ . A , $\mathcal{P}_n(\mathbb{H})$ C
$U_n(\mathbb{H})$ $Sp(n)$ .

## 2.2. *Eigenvalues and radial part*

C $M$ $\mathcal{P}_n(\mathbb{F})$. S $M$ H , $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$
$\mathbb{F} = \mathbb{R}$ $\mathbb{C}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n}$ $\mathbb{F} = \mathbb{H}$. W $\mathbb{F} = \mathbb{C}$ ,
$\mathbb{F} = \mathbb{R}$, $\lambda_{n-k+1} = -\lambda_k$, $k = 1, \ldots, \tilde{n} + 1$, $\lambda_{\tilde{n}+1} = 0$ $n$ (R $\tilde{n} = [n/2]$
$\mathbb{F} = \mathbb{R}$). W $\mathbb{F} = \mathbb{H}$ $\lambda_{2n-k+1} = -\lambda_k$, $k = 1, \ldots, n$W W $\mathcal{C}_n$
: $\mathbb{F} = \mathbb{C}$,

$$\mathcal{C}_n = \{\lambda \in \mathbb{R}^n; \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\},$$

$\mathbb{F} = \mathbb{R}$, $n$ ,

$$\mathcal{C}_n = \{\lambda \in \mathbb{R}^{\tilde{n}}; \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\tilde{n}} \geq 0\},$$

$\mathbb{F} = \mathbb{R}$, $n$ ( R 2.3),

$$\mathcal{C}_n = \{\lambda \in \mathbb{R}^{\tilde{n}}; \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\tilde{n}-1} \geq |\lambda_{\tilde{n}}| \geq 0\},$$

$\mathbb{F} = \mathbb{H}$,

$$\mathcal{C}_n = \{\lambda \in \mathbb{R}^n; \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}.$$

T W $U_n(\mathbb{F})$ $\mathcal{P}_n(\mathbb{F})$. M ,
. F $\mathbb{F} = \mathbb{C}$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ $\mathbb{R}^n$, $\Omega_n(\lambda)$ $n \times n$

$$\Omega_n(\lambda) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

W $\mathbb{F} = \mathbb{R}$, w $\omega(\alpha) = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$ w $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{R}^{\tilde{n}}$, w w $\Omega_n(\lambda)$ $n \times n$ -
, w $n$ ,

$$\Omega_n(\lambda) = \begin{pmatrix} \omega(\lambda_1) & & \\ & \ddots & \\ & & \omega(\lambda_{\tilde{n}}) \end{pmatrix}$$

w $n$ ,

$$\Omega_n(\lambda) = \left( \begin{matrix} \omega(\lambda_1 \end{matrix} \right.$$

H $\lambda$ $\mathcal{C}_n$. T . A ,
$\nu_n$ $\mathcal{P}_n(\mathbb{F})$ $n$ ,
$\nu$ $\mathcal{P}_\infty(\mathbb{F})$ H . M , $\mathbb{F}=\mathbb{R},\mathbb{C}$
$\mathbb{H}$, $\mathcal{P}_\infty(\mathbb{F})$ $\{M_{k,l}, 1\le k,l <+\infty\}$, $\mathbb{F}$ $M_{l,k}=M^*_{k,l}$. F
$n\in\mathbb{N}$, $U_n(\mathbb{F})$ $\mathcal{P}_\infty(\mathbb{F})$. A $\mathcal{P}_\infty(\mathbb{F})$
$U_n(\mathbb{F})$.

I , P 52 O V 51 ,
. A F ' , ,
: . . . ( ). I
$\mathcal{P}_\infty(\mathbb{F})$.

L $\mathcal{M}_{n,m}(\mathbb{F})$ $n\times m$ $\mathbb{F}$. I . W
E ,

$$\langle M,N\rangle = a\,\mathrm{R}\ \big(MN^*\big),\quad M,N\in\mathcal{M}_{n,m}(\mathbb{F}),$$

$a=1$ $\mathbb{F}=\mathbb{R}$, $a=2$ $\mathbb{F}=\mathbb{C},\mathbb{H}$. R G E
$d$

$$x\mapsto(2\pi)^{-d/2}\ {}^{-\langle x,x\rangle/2}.$$

O E G $\mathcal{M}_{n,m}(\mathbb{F})$. T
$m=n=1$ G $\mathbb{F}$ . W $\mathcal{P}_n(\mathbb{F})$

$$\langle M,N\rangle = b\ (MN),\quad M,N\in\mathcal{P}_n(\mathbb{F}),$$

$b=1$ $\mathbb{F}=\mathbb{C}$ $b=1/2$ $\mathbb{F}=\mathbb{R},\mathbb{H}$, G $\mathcal{P}_n(\mathbb{F})$.
W , $\mathbb{F}$, $\Omega_n(\lambda)$ $\lambda\in\mathcal{C}_n$. F $k\le\tilde{n}$,

$$\Omega_n^k=\Omega_n(1,\ldots,1,0,\ldots,0),\tag{3}$$

1 $k$ , , 1 $\tilde{n}$ ,

$$\Omega_n=\Omega_n^{\tilde{n}}=\Omega_n(1,\ldots,1).\tag{4}$$

***Definition 2.5.*** *For $\mathbb{F}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$, and $k,n\in\mathbb{N}$, we define*:

1. *The ensemble* $\mathrm{GUE}_n(\mathbb{F})$ *as the set of matrices in $\mathcal{P}_n(\mathbb{F})$ with the standard Gaussian distribution.*
2. *The ensemble* $\mathrm{LUE}_{n,k}(\mathbb{F})$ *as the set of matrices $M\Omega_k M^*$ when $M$ is a standard Gaussian random variable in $\mathcal{M}_{n,k}(\mathbb{F})$.*

N $\mathrm{LUE}_{n,k}(\mathbb{F})$ , F ( $\tilde{k}=k$
$\mathbb{F}=\mathbb{C},\mathbb{H}$ $\tilde{k}=[k/2]$ $\mathbb{F}=\mathbb{R}$, $c=1$ $\mathbb{F}=\mathbb{C},\mathbb{R}$ $c=2$ $\mathbb{F}=\mathbb{H}$):

**Lemma 2.6.** *Let $M$ be a standard Gaussian random variable in $\mathcal{M}_{n,k}(\mathbb{F})$. Then the Fourier transform of $M\Omega_k M^*$ is given by*

$$\mathbb{E}\big({}^{-\langle N,M\Omega_k M^*\rangle}\big)=\ \Big(I+\frac{}{c}N\Big)^{-\tilde{k}},\quad N\in\mathcal{P}_n(\mathbb{F}).$$

**Proof.** A $M\Omega_k M^*$ , $N=\Omega_n(\lambda)$, $\lambda\in\mathcal{C}_n$. W $\mathbb{F}=\mathbb{C}$,

$$\big\langle N,M\Omega_k M^*\big\rangle=\sum_{i=1}^n\sum_{j=1}^k\lambda_i|M_{i,j}|^2,$$

$M_{i,j}$ G . . W , $\alpha \in \mathbb{R}$,

$$\mathbb{E}\big( {}^{-\alpha|M_{1,1}|^2}\big) = \frac{1}{1+\ \alpha},$$

. W $\mathbb{F} = \mathbb{H}$,

$$\big\langle N M\Omega \ \ _k M^*\big\rangle = \sum_{i=1}^{n}\sum_{j=1}^{j}\lambda_i\big(|a_{i,j}|^2 - |b_{i,j}|^2\big),$$

$\begin{pmatrix} a_{i,j} & -\bar{b}_{i,j} \\ b_{i,j} & \bar{a}_{i,j}\end{pmatrix}$ G $\mathbb{H}$. W

$$\mathbb{E}\big( {}^{-\alpha(|a_{1,1}|^2 - |b_{1,1}|^2)}\big) = \frac{1}{1+(\alpha/2)^2},$$

. W $\mathbb{F} = \mathbb{R}$,

$$\big\langle N M\Omega \ \ _k M^*\big\rangle = \sum_{i=1}^{\tilde{n}}\sum_{j=1}^{\tilde{k}}\lambda_i\,(M_{2i,2j-1}M_{2i-1,2j} - M_{2i-1,2j-1}M_{2i,2j}),$$

$M_{i,j}$' G . W

$$\mathbb{E}\big( {}^{-\alpha(M_{2,1}M_{1,2} - M_{1,1}M_{2,2})}\big) = \frac{1}{1+\alpha^2},$$

. □

W $\mathbb{F} = \mathbb{C}$ LUE, L U C W , . T $\mathbb{R}$ $\mathbb{H}$:
F

$$\left(I + \frac{}{c} N\right)^{-\tilde{k}} = \quad \left(I + \frac{N^2}{c^2}\right)^{-\tilde{k}/2}$$

, LUE($\mathbb{F}$) . A $M\Omega_n M^*$ $\mathcal{P}_n(\mathbb{F})$. O $\mathbb{F} = \mathbb{H}$ $\mathbb{F} = \mathbb{R}$ $n$ ,
$\mathcal{P}_n(\mathbb{F})$ .
L . W $\mathrm{LUE}^1_\infty(\mathbb{F})$ $M\Omega^{\,1}_{\infty} M^*$ $M \in \mathcal{M}_\infty(\mathbb{F})$ $\{M_{i,j}, i, j = 1, \dots, n\}$ G $\mathcal{M}_n(\mathbb{F})$ $\mathrm{GUE}_\infty(\mathbb{F})$ $M \in \mathcal{P}_\infty(\mathbb{F})$ $\{M_{i,j}, i, j = 1, \dots, n\}$ G $\mathcal{P}_n(\mathbb{F})$. A $\mathcal{P}_\infty(\mathbb{F})$ $U_n(\mathbb{F})$. A ,
P 52 . I
, B ' T 5.4

**Theorem 2.7.** *Each ergodic invariant random matrix $M$ in $\mathcal{P}_\infty(\mathbb{F})$ is sum of elements of* $\mathrm{GUE}_\infty(\mathbb{F})$ *and* $\mathrm{LUE}^1_\infty(\mathbb{F})$*: it can be written as*

$$M = aI + bG + \sum_{k=1}^{+\infty} d_k L_k,$$

*where $G$ belongs to* $\mathrm{GUE}_\infty(\mathbb{F})$, *$L_k$ belongs to* $\mathrm{LUE}^1_\infty(\mathbb{F})$, *the random variables $G, L_1, L_2, \dots$ are independent, and $a, b, d_k$ are constants such that $\sum d_k^2 < +\infty$, $I$ is the identity matrix. Moreover, $a = 0$ when $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{H}$.*

**Proof.** T O , O 48,49 F 22 . G $G$ $K$, $(G, K)$ O - $\pi$ $G$ H $H$, $\{u \in H; \pi(k)u = u,$ $k \in K\}$ - - . F , G O .

T $\mathcal{P}_\infty(\mathbb{F})$ $\mathcal{P}_n(\mathbb{F})$' , $\mathcal{P}^{(\infty)}(\mathbb{F})$ $M$ $\mathcal{P}_\infty(\mathbb{F})$ $M_{i,j}=0$ $i+j$ . E $U_n(\mathbb{F})$ $\mathcal{P}^{(\infty)}(\mathbb{F})$ $\mathcal{P}_\infty(\mathbb{F})$. L $U^{(\infty)}(\mathbb{F})$ $U_n(\mathbb{F})$' . R $\mathcal{P}_n(\mathbb{F}) = \{U\Omega_n(\lambda)U^*, U \in U_n(\mathbb{F}), \lambda \in \mathcal{C}_n\}$. L $\lambda = (\lambda_1, \lambda_2, \dots)$ , $\lambda_k = 0$ $k$ . I , $\Omega(\lambda)$ $\Omega_\infty(\lambda)$. N $\mathcal{P}^{(\infty)}(\mathbb{F})$ $U\Omega(\lambda)U^*$ $U \in U^{(\infty)}(\mathbb{F})$ $\lambda$. A G $(U^{(\infty)}(\mathbb{F}) \times_\sigma \mathcal{P}^{(\infty)}(\mathbb{F}), U^{(\infty)}(\mathbb{F}))$ O . T , - V O ( O 49 , P 52 ) $\nu$ $\mathcal{P}_\infty(\mathbb{F})$ F $\psi$ $\mathcal{P}^{(\infty)}(\mathbb{F})$ , $\phi : \mathbb{R} \to \mathbb{C}$,

$$\psi\big(\Omega(\lambda_1, \lambda_2, \dots)\big) = \phi(\lambda_1)\phi(\lambda_2)\cdots$$

$\lambda$ . W $\mathbb{F} = \mathbb{C}$ P 52 ( O V 51 ) $a, b \geq 0$ $d_k, k \geq 1,$ $t \in \mathbb{R}$,

$$\phi(t) = \quad at \quad -bt^2 \prod_{k=1}^{\infty} \big[(1 + \ d_k t\ ) \quad d_k t\big]^{-1}. \tag{5}$$

T $\mathbb{F} = \mathbb{C}$. W $\mathbb{F} = \mathbb{R}$, P 52 . T $M \in \mathcal{P}^{(\infty)}(\mathbb{C})$ $f(M) \in \mathcal{P}^{(\infty)}(\mathbb{R})$ $m = x + \ y$, $x, y \in \mathbb{R}$ $M$ $2 \times 2$

$$\tilde{m} = \begin{pmatrix} y & x \\ -x & y \end{pmatrix}.$$

F $\lambda = (\lambda_1, \lambda_2, \dots)$, $f(\Omega(\lambda)) = \Omega(\lambda)$ $\Omega$ , - , $\mathbb{F} = \mathbb{C}$ - $\mathbb{F} = \mathbb{R}$.

C $\mathcal{P}_\infty(\mathbb{R})$ $\psi$ F $\mathcal{P}^{(\infty)}(\mathbb{R})$. T $\psi$ , , - $\phi : \mathbb{R} \to \mathbb{C}$

$$\psi\big(\Omega(\lambda_1, \lambda_2, \dots)\big) = \phi(\lambda_1)\phi(\lambda_2)\cdots.$$

T $\psi \circ f$ $\mathcal{P}^{(\infty)}(\mathbb{C})$ . M , $f \circ \Omega = \Omega$

$$(\psi \circ f)\big(\Omega(\lambda_1, \lambda_2, \dots)\big) = \phi(\lambda_1)\phi(\lambda_2)\cdots.$$

T , $\psi \circ f$ F $\mathcal{P}_\infty(\mathbb{C})$. T $\phi$ (5) . M , $\psi$ $U_n(\mathbb{R}) = SO(n)$ . U $SO(3)$,

$$\psi\big(\Omega(\lambda_1, \lambda_2, \dots)\big) = \psi\big(\Omega(-\lambda_1, \lambda_2, \dots)\big).$$

T , $t \in \mathbb{R}$, $\phi(t) = \phi(-t)$ ,

$$\phi(t) = \quad -bt^2 \prod_{k=1}^{\infty} \big[\big(1 + (d_k t\ )^2\big)\big]^{-1}.$$

U F L 2.6, $\mathbb{F} = \mathbb{R}$. W $\mathbb{F} = \mathbb{H}$ : $\tilde{f} : \mathcal{P}^{(\infty)}(\mathbb{C}) \to \mathcal{P}^{(\infty)}(\mathbb{H})$ . F , $\tilde{f}_n : \mathcal{P}_n(\mathbb{C}) \to \mathcal{P}_n(\mathbb{H})$, $M \in \mathcal{P}_n(\mathbb{C})$ $W$ (2),

$$\tilde{f}_n(M) = W \begin{pmatrix} M & 0 \\ 0 & -\bar{M} \end{pmatrix} W^* \in \mathcal{P}_n(\mathbb{H}).$$

F $M \in \mathcal{P}^{(\infty)}(\mathbb{H})$, $\pi_n(M)$ $n$. T $\pi_n(\tilde{f}_{n+1}(\pi_{n+1}(M))) = \tilde{f}_n(\pi_n(M))$ $\tilde{f} : M \in \mathcal{P}^{(\infty)}(\mathbb{C}) \to \tilde{f}(M) \in \mathcal{P}^{(\infty)}(\mathbb{H})$ $\pi_n(\tilde{f}(M)) = \tilde{f}_n(\pi_n(M))$ . W $\tilde{f} \circ \Omega = \Omega$. T - $\lambda$

, K . 43 , K . 37,38 , G 28 , F N 24 . T LUE($\mathbb{C}$) , LUE($\mathbb{R}$) LUE($\mathbb{H}$) , .

## 3. Minors and Gelfand–Tsetlin polytopes

I , $\mathcal{P}_n(\mathbb{F})$. F $M = \{M_{i,j}, 1 \le i, j \le n\}$ $\mathcal{P}_n(\mathbb{F})$ $k \le n$, $k$ $M$,

$$\pi_k(M) = \{M_{ij}, 1 \le i, j \le k\}$$

( : $M$, ). T $k$ $M$ $\mathcal{P}_k(\mathbb{F})$, $X^{(k)}(M)$. C $\mathcal{P}_n(\mathbb{F})$, ,

$$X(M) = \big(X^{(1)}(M), \dots, X^{(n)}(M)\big),$$

, $\mathbb{F}$ $\mathbb{C}$, $M \in \mathcal{P}_n(\mathbb{C})$ GUE, B 2 , J N 34 . T T 3.4. I $\mathbb{F}$, , $\mathbb{F} = \mathbb{H}$, G T .

O S 7 10, L . I , S 10.

W $M$ H , R ' $\lambda \in \mathbb{R}^n$ $M$ $\beta \in \mathbb{R}^{n-1}$ $\pi_{n-1}(M)$ $n-1$, $\lambda$ $\beta$ $\lambda_i \ge \beta_i \ge \lambda_{i+1}$, $i = 1, \dots, n-1$. O $M$ $\mathcal{P}_n(\mathbb{R})$ $\mathcal{P}_n(\mathbb{H})$, H . T $\mathbb{F} = \mathbb{C}, \mathbb{R}$, $X(M)$ G T , W $\mathbb{F} = \mathbb{H}$. F $x, y \in \mathbb{R}^n$ $x \succeq y$ $x$ $y$ , . .,

$$x_1 \ge y_1 \ge x_2 \ge \dots \ge x_n \ge y_n$$

$x \succ y$

$$x_1 > y_1 > x_2 > \dots > x_n > y_n.$$

W $x \in \mathbb{R}^{n+1}$ $y \in \mathbb{R}^n$ $y_n \ge x_{n+1}$ ( . $y_n > x_{n+1}$). W $|x|$ $x$.

***Definition 3.1.*** *Let $\lambda$ be in the Weyl chamber $\mathcal{C}_n$. The Gelfand–Tsetlin polytope $GT_n(\lambda)$ is defined by*:

- *when $\mathbb{F} = \mathbb{C}$*,

$$GT_n(\lambda) = \big\{\big(x^{(1)}, \dots, x^{(n)}\big) \colon x^{(n)} = \lambda, x^{(k)} \in \mathbb{R}^k, x^{(k)} \succeq x^{(k-1)}, 1 \le k \le n\big\},$$

- *when $\mathbb{F} = \mathbb{H}$*,

$$GT_n(\lambda) = \big\{\big(x^{(1/2)}, x^{(1)}, x^{(3/2)}, \dots, x^{(n-1/2)}, x^{(n)}\big) \colon x^{(n)} = \lambda,$$
$$x^{(k)}, x^{(k-1/2)} \in \mathbb{R}^k_+, x^{(k)} \succeq x^{(k-1/2)} \succeq x^{(k-1)}, 1 \le k \le n\big\},$$

- *when* $\mathbb{F}=\mathbb{R}$,

$$GT_n(\lambda)=\big\{\big(x^{(1)},\dots,x^{(n)}\big):\ x^{(n)}=\lambda,\ x^{(k)}\in\mathbb{R}_+^{i-1}\times\mathbb{R}\ \textit{when}\ k=2i,$$
$$x^{(1)}=0,\ x^{(k)}\in\mathbb{R}_+^{i}\ \textit{when}\ k=2\ \ +1,\ \big|x^{(k)}\big|\succeq\big|x^{(k-1)}\big|,\ 1\le k\le n\big\}.$$

I $M$ $\mathcal{P}_n(\mathbb{H})$ $X^{(n)}(M)=\lambda$, $X(M)$ $GT_n(\lambda)$ $(x^{(1/2)},x^{(1)},\dots,x^{(n)})\in GT_n(\lambda)\mapsto(x^{(1)},x^{(2)},\dots,x^{(n)})$. T , , $r=1,\dots,n$ $X^{(r-1/2)}(M)\in\mathbb{R}^r$ , $r$ $2r-1$ $M$ $\mathcal{P}_{2n}(\mathbb{C})$. T R ,

$$\big(X^{(1/2)}(M),X^{(1)}(M),\dots,X^{(n-1/2)}(M),X^{(n)}(M)\big)$$

G T $GT_n(\lambda)$ $\mathbb{H}$, . U , $x$ G T , , G T , F 1 4. L G T . I . A , $C$ L

$$\begin{array}{ccccccccc}
 & & & & x_1^{(1)} & & & & \\
 & & & x_1^{(2)} & & x_2^{(2)} & & & \\
 & & x_1^{(3)} & & x_2^{(3)} & & x_3^{(3)} & & \\
 & & & & \dots & & & & \\
 & x_1^{(n-1)} & x_2^{(n-1)} & \dots & & \dots & x_{n-2}^{(n-1)} & x_{n-1}^{(n-1)} & \\
x_1^{(n)} & x_2^{(n)} & \dots & & & \dots & & x_{n-1}^{(n)} & x_n^{(n)}
\end{array}$$

F . 1. A G T $\mathbb{F}=\mathbb{C}$.

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & x_1^{(\frac12)} & & -x_1^{(\frac12)} & & & & \\
 & & & x_1^{(1)} & & 0 & & -x_1^{(1)} & & & \\
 & & x_1^{(\frac32)} & & x_2^{(\frac32)} & & -x_2^{(\frac32)} & & -x_1^{(\frac32)} & & \\
 & x_1^{(2)} & & x_2^{(2)} & & 0 & & -x_2^{(2)} & & -x_1^{(2)} & \\
 & & & & \dots & & \dots & & & & \\
 & x_1^{(n-\frac12)} & \dots & & x_n^{(n-\frac12)} & & -x_n^{(n-\frac12)} & & \dots & -x_1^{(n-\frac12)} & \\
x_1^{(n)} & & \dots & & x_n^{(n)} & 0 & -x_n^{(n)} & & \dots & & -x_1^{(n)}
\end{array}$$

F . 2. A G T $\mathbb{F}=\mathbb{H}$.

$$\begin{array}{ccccccccc}
 & & & & x_1^{(2)} & & & & \\
 & & & x_1^{(3)} & & -x_1^{(3)} & & & \\
 & & x_1^{(4)} & & x_2^{(4)} & & -x_1^{(4)} & & \\
 & x_1^{(5)} & & x_2^{(5)} & & -x_2^{(5)} & & -x_1^{(5)} & \\
 & & & \dots & & \dots & & & \\
 & x_1^{(n-1)}\ \dots & & x_{\tilde n-1}^{(n-1)} & x_{\tilde n}^{(n-1)} & -x_{\tilde n-1}^{(n-1)}\ \dots & & -x_1^{(n-1)} & \\
x_1^{(n)} & \dots & & x_{\tilde n}^{(n)} & & -x_{\tilde n}^{(n)} & \dots & & -x_1^{(n)}
\end{array}$$

F . 3. A G T $\mathbb{F}=\mathbb{R}$, $n$ .

$$\begin{array}{ccccccccccccc}
 & & & & & & x_1^{(2)} & & & & & & \\
 & & & & & x_1^{(3)} & & -x_1^{(3)} & & & & & \\
 & & & & x_1^{(4)} & & x_2^{(4)} & & -x_1^{(4)} & & & & \\
 & & & x_1^{(5)} & & x_2^{(5)} & & -x_2^{(5)} & & -x_1^{(5)} & & & \\
 & & & & & \dots & & \dots & & & & & \\
 & x_1^{(n-1)} & & \dots & & x_{\tilde n-1}^{(n-1)} & & -x_{\tilde n-1}^{(n-1)} & & \dots & & -x_1^{(n-1)} & \\
x_1^{(n)} & & \dots & & x_{\tilde n-1}^{(n)} & & x_{\tilde n}^{(n)} & & -x_{\tilde n-1}^{(n)} & & \dots & & -x_1^{(n)}
\end{array}$$

F . 4. A G T $\mathbb{F}=\mathbb{R}$, $n$ .

. W ($C$ ). W L $C$ L $C$ $C$ L $C$.

L $M \in \mathcal{P}_n(\mathbb{F})$ . T $X(M)$ $GT_n = \bigcup_{\lambda\in\mathcal{C}_n} GT_n(\lambda)$. W $X(M)$ G T - .

***Definition 3.2.*** *For $\lambda$ in the Weyl chamber $\mathcal{C}_n$, we let $\mu_\lambda$ be the image of the uniform probability measure on $GT_n(\lambda)$ by the map $p_{n-1} : x \in GT_n(\lambda) \mapsto x^{(n-1)} \in \mathcal{C}_{n-1}$.*

W F 1 4 G T . T $GT_n(\lambda)$, $m_{GT_n(\lambda)}$,

$$m_{GT_n(\lambda)} = \int m_{GT_{n-1}(\beta)} \mu_\lambda(\ \beta), \tag{6}$$

$\mu_\lambda$, $\lambda \in \mathcal{C}_n$. T P 10.1. T $\Omega_n(\lambda)$ S 2.2.

**Lemma 3.3.** *Let $\lambda$ be in the Weyl chamber $\mathcal{C}_n$ and $U \in U_n(\mathbb{F})$ be a Haar distributed random variable. Then the distribution of the radial part of the main minor of order $n-1$ of $U\Omega_n(\lambda)U^*$ is $\mu_\lambda$.*

W $X(M)$ $M$ $\mathcal{P}_n(\mathbb{F})$. I L 2.4 $M = U\Omega_n(\lambda)U^*$, $U$ H $U_n(\mathbb{F})$ $\lambda$ $\mathcal{C}_n$.

**Theorem 3.4.** *Let $M = U\Omega_n(\lambda)U^*$, with $U$ Haar distributed in $U_n(\mathbb{F})$ and $\lambda \in \mathcal{C}_n$. Then $X(M)$ is uniformly distributed on $GT_n(\lambda)$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and is distributed according to the image of the uniform measure on $GT_n(\lambda)$ by the map $(x^{(1/2)}, \dots, x^{(n-1/2)}, x^{(n)}) \in GT_n(\lambda) \mapsto (x^{(1)}, x^{(2)}, \dots, x^{(n)})$ for $\mathbb{F} = \mathbb{H}$.*

**Proof.** I (6) $k \in \{1, \dots, n-1\}$ $f : \mathcal{C}_k \to \mathbb{R}$,

$$\mathbb{E}\big[f\big(X^{(k)}(M)\big)\big|\sigma\big\{X^{(k+1)}(M), \dots, X^{(n)}(M)\big\}\big] = \mathbb{E}\big[f\big(X^{(k)}(M)\big)\big|\sigma\big\{X^{(k+1)}(M)\big\}\big].$$

F $V \in U_{k+1}(\mathbb{F})$ $V\ MV^*$

$$\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix} M \begin{pmatrix} V^* & 0 \\ 0 & I \end{pmatrix},$$

w I w . L w $\pi_{k+1}(M) =$
$V\,\Omega_{k+1}(X^{(k+1)}(M))V^*$, w $V \in U_{k+1}(\mathbb{F})$. L $W$ $M$, H
$U_{k+1}(\mathbb{F})$. W $W\pi_{k+1}(M)W^* = \pi_{k+1}(WMW^*)$ $X^{(r)}(WMW^*) = X^{(r)}(M)$, $r = k+1, \ldots, n$

W λ W , $d_n(\lambda)$ , , $U_n(\mathbb{F})$. W $c_n = c_n(\lambda)$ . W .

**Lemma 3.6.** *Let* $\lambda$ *be in the interior of* $\mathcal{C}_n$. *Then*

- *when* $\mathbb{F} = \mathbb{C}$,

$$d_n(\lambda) = c_n \quad \left(\lambda_i^{j-1}\right)_{n\times n},$$

- *when* $\mathbb{F} = \mathbb{H}$,

$$d_n(\lambda) = c_n \quad \left(\lambda_i^{2j-1}\right)_{n\times n},$$

- *when* $\mathbb{F} = \mathbb{R}$,

$$d_n(\lambda) = c_n \quad \left(\lambda_i^{2j-2+\epsilon}\right)_{\tilde{n}\times\tilde{n}}.$$

I , $\mathbb{F} = \mathbb{C}$ λ W , $d_n(\lambda)$ V . T .

**Lemma 3.7.** *For any* $\lambda$ *in the Weyl chamber* $\mathcal{C}_n$, *the volume of* $GT_n(\lambda)$ *is* $d_n(\lambda)$.

**Proof.** I L 10.1. □

F λ W , $l_\lambda$ L $p_{n-1}(GT_n(\lambda))$, $p_{n-1}$ D 3.2.

**Lemma 3.8.** *Let* $\lambda$ *be in the Weyl chamber. Then,*

- *when* $\mathbb{F} = \mathbb{R}, \mathbb{C}$,

$$\mu_\lambda(\ \beta) = \frac{d_{n-1}(\beta)}{\ _n(\lambda)} l_\lambda(\ \beta),$$

- *when* $\mathbb{F} = \mathbb{H}$,

$$\mu_\lambda(\ \beta) = \frac{d_{n-1}(\beta)}{d_n(\lambda)} \quad \left(\left\{z \in \mathbb{R}^n :\ \lambda \succeq z \succeq \beta\right\}\right) l_\lambda(\ \beta).$$

**Proof.** F $x \in GT_n(\lambda)$, $(x^{(1)}, \ldots, x^{(n-1)})$ $\mathbb{F} = \mathbb{C}, \mathbb{R}$ $(x^{(1/2)}, x^{(1)}, x^{(3/2)}, \ldots, x^{(n-1)})$ $\mathbb{F} = \mathbb{H}$, G T $GT_{n-1}(x^{(n-1)})$, $d_{n-1}(x^{(n-1)})$ L 3.7. T . □

W $\mu_\lambda$ : λ W λ . F , C B ( , , 33 ). L $(E, \mathcal{B}, m)$ , $\phi_i$ $\psi_j$, $1 \le i, j \le n$, $\phi_i \psi_j$' . T C B

$$\left(\int_E \phi_i(x)\psi_j(x) \quad m(x)\right) = \frac{1}{n!}\int_{E^n} \quad \left(\phi_i(x_j)\right) \quad \left(\psi_i(x_j)\right) \prod_{k=1}^n \quad m(x_k). \tag{7}$$

L ( W 59 ). L $x$ $y$ $\mathbb{R}^n$ $x_1 > \cdots > x_n$ $y_1 > \cdots > y_n$. T

$$1_{\{x \succ y\}} = \quad (1_{\{x_i > y_j\}})_{n\times n}. \tag{8}$$

**Proposition 3.9.** *Let $\lambda$ be in the Weyl chamber $\mathcal{C}_n$. If $\lambda$ is in the interior of $\mathcal{C}_n$, then the measure $\mu_\lambda$ has a density $f_\lambda$ with respect to the Lebesgue measure on $\mathcal{C}_{n-1}$ defined by* :

- *when* $\mathbb{F}=\mathbb{C}$, $f_\lambda(\beta)=\frac{d_{n-1}(\beta)}{d_n(\lambda)}1_{\{\lambda\succeq\beta\}}$,
- *when* $\mathbb{F}=\mathbb{R}$, $f_\lambda(\beta)=\frac{d_{n-1}(\beta)}{d_n(\lambda)}1_{\{|\lambda|\succeq|\beta|\}}$,
- *when* $\mathbb{F}=\mathbb{H}$, $f_\lambda(\beta)=\frac{d_{n-1}(\beta)}{d_n(\lambda)}$ $((\lambda_i-\beta_j)1_{\{\lambda_i\geq\beta_j\}})_{n\times n}$, *with the convention* $\beta_n=0$.

*If* $\lambda=(\theta,0,\dots,0)$, $\theta\in\mathbb{R}_+$, *then the measure* $\mu_\lambda$ *is equal to* $\tilde{\mu}_\lambda\otimes\delta_0^{\tilde{n}-1}$, $\tilde{\mu}_\lambda$ *having a density* $g_\theta$ *with respect to the Lebesgue measure on* $\mathbb{R}_+$ *defined by*:

- *when* $\mathbb{F}=\mathbb{C}$, $g_\theta(\beta)=(n-1)\frac{\beta^{n-2}}{\theta^{n-1}}1_{[0,\theta]}(\beta)$,
- *when* $\mathbb{F}=\mathbb{R}$, $g_\theta(\beta)=(n-2)\frac{\beta^{n-3}}{\theta^{n-2}}1_{[0,\theta]}(\beta)$,
- *when* $\mathbb{F}=\mathbb{H}$, $g_\theta(\beta)=(2n-2)(2n-1)\frac{\beta^{2n-3}}{\theta^{2n-1}}(\theta-\beta)1_{[0,\theta]}(\beta)$.

**Proof.** L 3.8 $f_\lambda$ $g_\theta$ $\mathbb{F}=\mathbb{C},\mathbb{R}$. F $\mathbb{F}=\mathbb{H}$ $\lambda$ W , , $\beta\in\mathbb{R}_+^n$,

$$f_\lambda(\beta)=\frac{d_{n-1}(\beta)}{d_n(\lambda)}\int_{\mathbb{R}^n}1_{\{\lambda\succeq z\}}1_{\{z\succeq\beta\}}\quad z.$$

W (8) C B (7). F $\mathbb{F}=\mathbb{H}$ $\lambda=(\theta,0,\dots,0)$ L 3.8 , $\beta\in\mathbb{R}_+$,

$$g_\theta(\beta)=\frac{d_{n-1}(\beta)}{d_n(\lambda)}\int_{\mathbb{R}}1_{\{\theta\geq z\geq\beta\}}\quad z=\frac{d_{n-1}(\beta)}{d_n(\lambda)}(\theta-\beta)1_{[0,\theta]}(\beta).$$

□

## 4. Rank one perturbation on $\mathcal{P}_n(\mathbb{F})$

T LUE D 2.5. W . A $\mathrm{LUE}_{n,k}(\mathbb{F})$ $\sum_{i=1,\dots,k}M_i\Omega_n^1M_i$, $M_i$' G $\mathcal{M}_n(\mathbb{F})$ ( . L 2.6). W $k$. T , L , . . $\mathrm{LUE}_n^1(\mathbb{F})$, , T 4.5 $\Omega_n(\lambda)+M\Omega_n^1M^*$, $\lambda\in\mathcal{C}_n$. F $\lambda$ $\mathcal{C}_n$ $k$ , , $\Omega_n(\lambda_1,\dots,\lambda_k)$ $\Omega_n(\lambda)$. L $\theta\in\mathbb{R}_+$ $U$ H $U_n(\mathbb{F})$. W - $\Omega_n(\lambda)+U\Omega_n(\theta)U^*$. W .

***Definition 4.1.*** *For* $\lambda\in\mathcal{C}_n$, $\theta\in\mathbb{R}_+^*$, *we define the set* $\mathcal{E}(\lambda,\theta)$ *by*:

- *when* $\mathbb{F}=\mathbb{C}$,

$$\mathcal{E}(\lambda,\theta)=\left\{(\beta,x)\in\mathbb{R}^n\times GT_n\colon\ \beta\succeq\lambda,\ \sum_{i=1}^n(\beta_i-\lambda_i)=\theta,\ x\in GT_n(\beta)\right\},$$

- *when* $\mathbb{F}=\mathbb{H}$,

$$\mathcal{E}(\lambda,\theta)=\Bigg\{(\beta,z,x)\in\mathbb{R}^n\times\mathbb{R}^n\times GT_n\colon\ \lambda,\beta\succeq z,\\ \sum_{i=1}^n(\lambda_i+\beta_i-2z_i)=\theta,\ x\in GT_n(\beta)\Bigg\},$$

- *when* $\mathbb{F}=\mathbb{R}$, $n=2r+1$,

$$\mathcal{E}(\lambda,\theta)=\Big\{(\beta,z,x,s)\in\mathbb{R}_+^r\times\mathbb{R}_+^r\times GT_n\times\{0,1\}:\ \lambda,\beta\succeq z,\ \sum_{i=1}^{r}(\lambda_i+\beta_i-2z_i)=\theta,\ x\in GT_n(\beta),\ s=0 \text{ if } \lambda_r=0\Big\},$$

- *when* $\mathbb{F}=\mathbb{R}$, $n=2r$,

$$\mathcal{E}(\lambda,\theta)=\Big\{(\beta,z,x)\in\mathbb{R}^r\times\mathbb{R}_+^{r-1}\times GT_n:\ \lambda,\beta\succeq z,\quad \big(|\lambda_r|,|\beta_r|\big)\le z_{r-1},\ \sum_{k=1}^{r-1}(\lambda_k+\beta_k-2z_k)+|\lambda_r-\beta_r|=\theta,\ x\in GT_n(\beta)\Big\}.$$

E $\mathcal{E}(\lambda,\theta)$ . T L

.

***Definition 4.2.*** *For $\lambda\in\mathcal{C}_n$, $\theta\in\mathbb{R}_+^*$, we define $\nu_{\lambda,\theta}$ as the image of the uniform probability measure on $\mathcal{E}(\lambda,\theta)$ by the projection on the component $\beta$, denoted by $p$.*

T S 10.2.

**Proposition 4.3.** *Let $\theta\in\mathbb{R}_+^*$, $\lambda\in\mathcal{C}_n$ and $U$ be a Haar distributed random variable in $U_n(\mathbb{F})$. Then the radial part of the random matrix $\Omega_n(\lambda)+U\Omega_n(\theta)U^*$ is distributed according to the measure $\nu_{\lambda,\theta}$.*

R $(\alpha,n)\in\mathbb{R}_+^*\times\mathbb{N}$,

$$\frac{\alpha}{(n-1)!}\quad{}^{-\alpha x}(\alpha x)^{n-1}1_{\mathbb{R}_+}(x).$$

R $c=1$ $\mathbb{F}=\mathbb{C},\mathbb{R}$ $c=2$ $\mathbb{F}=\mathbb{H}$.

**Lemma 4.4.** *Let $M$ be a standard Gaussian variable in $\mathcal{M}_n(\mathbb{F})$. Then the radial part of $M\Omega_n^1M^*$ has only one nonzero component $\Theta$. It has a gamma distribution with parameters $(1,n)$ when $\mathbb{F}=\mathbb{C}$, $(1,n-1)$ when $\mathbb{F}=\mathbb{R}$, and $(2,2n)$ when $\mathbb{F}=\mathbb{H}$. Its density $f_\Theta$ can be written, for some $k>0$ as*

$$f_\Theta(\theta)=kd_n(\theta)\quad{}^{-c\theta}.$$

**Proof.** W $M\Omega_n^1M^*$ $M^*M\Omega_n^1$. I $\mathbb{F}=\mathbb{C}$ $\mathbb{H}$, $M^*M\Omega_n^1$ $|V|^2\Omega_n^1$, $V$ G $\mathbb{F}^n$. T $M\Omega_n^1M^*$ , $(1,n)$ $\mathbb{F}=\mathbb{C}$ $(2,2n)$ $\mathbb{H}$. I $\mathbb{F}=\mathbb{R}$, $M\Omega_n^1M^*$ . T $(1,n-1)$ . A , P 5.5 T $d_n(\theta)$

$$\frac{\theta^{n-1}}{(n-1)!}\quad \mathbb{F}=\mathbb{C},\qquad \frac{2\theta^{n-2}}{(n-2)!}\quad \mathbb{F}=\mathbb{R},\qquad \frac{\theta^{2n-1}}{(2n-1)!}\quad \mathbb{F}=\mathbb{H}. \tag{9}$$

□

T , , . A , P 5.5, $d_n$ .

**Theorem 4.5.** *Let $M$ be a standard Gaussian variable in $\mathcal{M}_n(\mathbb{F})$ and $\lambda$ be an element of the Weyl chamber $\mathcal{C}_n$. Then the distribution of the radial part of $\Omega_n(\lambda) + M\Omega_n^1 M^*$, that we denote $\nu_\lambda$, is the probability measure proportional to*

$$\int_{\mathbb{R}_+} \nu_{\lambda,\theta} d_n(\theta) \quad {}^{-c\theta} \quad \theta.$$

**Proof.** T $M\Omega_n^1 M^*$ $\mathcal{P}_n(\mathbb{F})$. T L 2.4 4.4 $M$ $U\Omega_n(\Theta)U^*$, $U$ $\Theta$ H $U_n(\mathbb{F})$ $f_\Theta$. I P 4.3 $\nu_\lambda = \int_{\mathbb{R}} \nu_{\lambda,\theta} f_\Theta(\theta) \quad \theta$. ☐

I $\nu_\lambda$. W $\nu_{\lambda,\theta}$.

**Lemma 4.6.** *For $\lambda \in \mathcal{C}_n$, $\theta \in \mathbb{R}_+$, the volume of $\mathcal{E}(\lambda,\theta)$ is equal to $d_n(\lambda)d_n(\theta)$.*

**Proof.** T L 10.3. ☐

W $l_{\lambda,\theta}$ L $p(\mathcal{E}(\lambda,\theta))$, $p$ D 4.2.

**Lemma 4.7.** *Let $\lambda$ be in the Weyl chamber and $\theta$ in $\mathbb{R}_+$. Then*

- *when $\mathbb{F} = \mathbb{C}$,*

$$\nu_{\lambda,\theta}(\ \beta) = \frac{d_n(\beta)}{d_n(\lambda)d_n(\theta)} l_{\lambda,\theta}(\ \beta),$$

- *when $\mathbb{F} = \mathbb{H}, \mathbb{R}$,*

$$\nu_{\lambda,\theta}(\ \beta) = \frac{d_n(\beta)}{d_n(\lambda)d_n(\theta)} \quad (\mathcal{M}_{\lambda,\theta}(\beta)) l_{\lambda,\theta}(\ \beta),$$

*where $\mathcal{M}_{\lambda,\theta}(\beta)$ is the projection, for $\beta$ fixed, of $\mathcal{E}(\lambda,\theta)$ on the component $z$ when $\mathbb{F} = \mathbb{H}$ or $\mathbb{F} = \mathbb{R}$ with $n$ even, and on the component $(z,s)$ when $\mathbb{F} = \mathbb{R}$ with $n$ odd.*

**Proof.** B $\nu_{\lambda,\theta}$ $\mathcal{E}(\lambda,\theta)$ $p$. T , L 4.6 $d_n(\beta)$ $x \in GT_n(\beta)$. ☐

W $(\mathcal{M}_{\lambda,\theta}(\beta))$ . I $\mathbb{F} = \mathbb{C}$ P 9.2 9.5 - . L $\nu_\lambda$ .

**Proposition 4.8.** *Let $k$ be an integer smaller that $\tilde{n}$ and $\lambda \in \mathbb{R}^{\tilde{n}}$ be equal to $(\lambda_1, \ldots, \lambda_k, 0, \ldots 0)$. When $\mathbb{F} = \mathbb{R}$, if $n$ is even and $k = \tilde{n}$, we suppose that $\lambda_1 > \cdots > \lambda_{k-1} > |\lambda_k|$. For every other case we suppose that $\lambda_1 > \cdots > \lambda_k > 0$.*

*Then the measure $\nu_\lambda$ is equal to $\tilde{\nu}_\lambda \otimes \delta_0^{\otimes(n-(k+1)\wedge n)}$, where $\tilde{\nu}_\lambda$ has a density $L_\lambda$ with respect to the Lebesgue measure on $\mathbb{R}^{(k+1)\wedge\tilde{n}}$ defined by*

- *when $\mathbb{F} = \mathbb{C}$,*

$$L_\lambda(\beta) = \frac{d_n(\beta)}{d_n(\lambda)} 1_{\{\beta \succeq \lambda\}} \quad {}^{-\sum_{i=1}^{(k+1)\wedge n}(\beta_i - \lambda_i)},$$

- *when $\mathbb{F} = \mathbb{H}$,*

$$L_\lambda(\beta) = 2^n \frac{d_n(\beta)}{d_n(\lambda)} \left[ \int_{\mathbb{R}_+^k} 1_{\{\lambda,\beta \succeq z\}} \quad {}^{-2\sum_{i=1}^k(\lambda_i + \beta_i - 2z_i) - \beta_{k+1} 1_{\{k<n\}}} \quad z \right] 1_{\{\beta_{(k+1)\wedge n} \geq 0\}}.$$

- *when* $\mathbb{F}=\mathbb{R}$, $n=2r$, $k=r,r-1$,

$$L_\lambda(\beta)=\frac{1}{2}\frac{d_n(\beta)}{d_n(\lambda)}\left[\int_{\mathbb{R}_+^{r-1}} 1_{\{\lambda,\beta\succeq z,\,|\lambda_r|,|\beta_r|\le z_{r-1}\}} \quad -\sum_{i=1}^{r-1}(\lambda_i+\beta_i-2z_i)-|\lambda_r-\beta_r| \quad z\right],$$

- *when* $\mathbb{F}=\mathbb{R}$, $n=2r$, $k\le r-2$,

$$L_\lambda(\beta)=\frac{1}{2}\frac{d_n(\beta)}{d_n(\lambda)}\left[\int_{\mathbb{R}_+^{k}} 1_{\{\lambda,\beta\succeq z\}} \quad -\sum_{i=1}^{k}(\lambda_i+\beta_i-2z_i)-\lambda_{k+1} \quad z\right]1_{\{\beta_{k+1}\ge 0\}}.$$

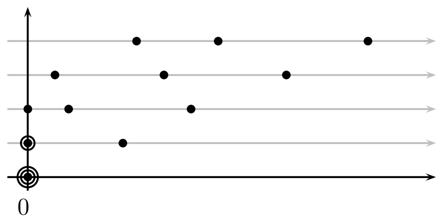
**Proof.** U T 4.5 $\nu_\lambda=\int_{\mathbb{R}_+}\nu_{\lambda,\theta}f_\Theta(\theta)\ \ \theta$. T - L 4.7 $f_\Theta(\theta)=d_n(\theta)\ {}^{-\theta}$ $\mathbb{F}=\mathbb{C}$, $f_\Theta(\theta)=2^n d_n(\theta)\ {}^{-2\theta}$ $\mathbb{F}=\mathbb{H}$, $f_\Theta(\theta)=\frac{1}{2}d_n(\theta)\ {}^{-\theta}$ $\mathbb{F}=\mathbb{R}$. □

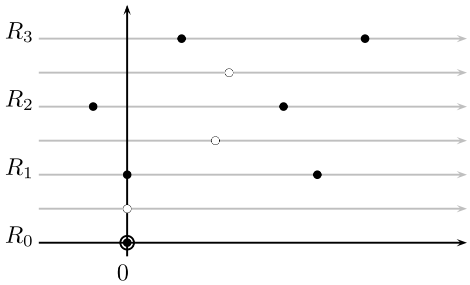
***Remark 4.9.*** *We observe in Lemma* 4.7 *that the measures* $\nu_{\lambda,\theta}$ *are the same, in the cases* $\mathbb{F}=\mathbb{R}$, $n=2r+1$ *and* $\mathbb{F}=\mathbb{H}$, $n=r$ *(see Section* 8.2 *for explanations). Moreover, for that two cases, the functions* $d_n$ *are the same, up to a constant. Thus the measures* $\nu_\lambda$ *defined at Theorem* 4.5 *are the same, up to the constant* $c$. *That is why we did not write both cases in the previous proposition.*

## 5. Generalised Laguerre ensembles

I LUE$_{n,k}(\mathbb{F})$ - , . . $(S_k)_{k\ge0}$ $\mathcal{P}_n(\mathbb{F})$ $S_k=\sum_{i=1}^k M_i\Omega_n^1 M_i$, $M_i$' G $\mathcal{M}_n(\mathbb{F})$. W $R_k$ $S_k$ . T $(R_k)_{k\ge0}$. I F 5 7 $\mathbb{F}=\mathbb{C},\mathbb{H}$ $\mathbb{R}$. T .

**Proposition 5.1.** *The process* $(R_k)_{k\ge0}$ *is a Markov chain whose transition probability* $P(\lambda,\cdot)$ *is equal to* $\nu_\lambda$. *When* $R_0=0$, $R_k$ *has* $k\wedge\tilde{n}$ *nonzero components.*






F . 7. R $\mathcal{P}_4(\mathbb{R})$.

**Proof.** A L 2.4, $(S_k, R_k, \ldots, R)$

$(U\Omega_n(R_k)U^*, R_k, \ldots, R)$,

$U \in U_n(\mathbb{F})$ H $(R_k, \ldots, R)$. T -
$f : \mathcal{P}_n(\mathbb{F}) \to \mathbb{R}$,

$$\mathbb{E}\big(f(S_{k+1})|\sigma\{R_k, \ldots, R\}\big) = \mathbb{E}\big(f\big(U\Omega_n(R_k)U^* + M_{k+1}\Omega_n^1 M_{k+1}^*\big)|\sigma\{R_k, \ldots, R\}\big)$$
$$= \mathbb{E}\big(f\big(U\Omega_n(R_k)U^* + M_{k+1}\Omega_n^1 M_{k+1}^*\big)|\sigma\{R_k\}\big).$$

T $(R_k)_{k\geq 0}$ M . T T 4.5 ,
L 4.7, . □

W
$\mathrm{LUE}_{n,k}(\mathbb{F})$. F $\lambda \in \mathbb{R}^n$, V

$$\Delta_n(\lambda) = \prod_{1\leq i<j\leq n} (\lambda_i - \lambda_j).$$

R $\tilde{n} = n$ $\mathbb{F} = \mathbb{C}, \mathbb{H}$, $\tilde{n} = [n/2]$ $\mathbb{F} = \mathbb{R}$, $c = 1$ $\mathbb{F} = \mathbb{R}, \mathbb{C}$, $c = 2$ $\mathbb{F} = \mathbb{H}$ $\Omega_k$
(4).

**Theorem 5.2.** *Let $M$ be a standard Gaussian variable in $\mathcal{M}_{n,k}(\mathbb{F})$. Then the positive eigenvalues of $M\Omega_k M^*$ have a density $f_{n,k}$ with respect to the Lebesgue measure on $\mathbb{R}^{\tilde{n}\wedge\tilde{k}}$ and there exists a constant $C > 0$ such that for $\lambda \in \mathbb{R}^{\tilde{n}\wedge\tilde{k}}$,*

$$f_{n,k}(\lambda) = C\, d_n(\lambda)\Delta_{\tilde{n}\wedge\tilde{k}}(\lambda)\prod_{i=1}^{\tilde{n}\wedge\tilde{k}} \lambda_i^{(\tilde{k}-\tilde{n})\vee 0} \quad {}^{-c\lambda_i}. \tag{10}$$

**Proof.** L $k$ $\mathbb{F} = \mathbb{C}, \mathbb{H}$. T $M\Omega_k M^*$
$S_k$ . T $k = 1$ L 4.4.
S $k \geq 1$. L $\gamma = (\gamma_1, \ldots, \gamma_{k\wedge n}, 0, \ldots 0)$ $\mathbb{R}^n$ $\gamma_1 > \cdots > \gamma_{k\wedge n} > 0$
$M$ G $\mathcal{M}_{n,1}(\mathbb{F})$. P 4.8
$\Omega_n(\gamma) + M\Omega_n^1 M^*$ $L_\gamma$ L $\mathbb{R}^{n\wedge(k+1)}$,
, $\lambda \in \mathbb{R}^{n\wedge(k+1)}$,

$$f_{n,k+1}(\lambda) = \int_{\mathbb{R}_+^{n\wedge k}} f_{n,k}(\gamma)L_\gamma(\lambda) \quad \gamma. \tag{11}$$

L ▾ . W $\mathbb{F}=\mathbb{C}$, (11)
$C_1$

$$f_{n,k+1}(\lambda)=C_1 d_n(\lambda) \quad {}^{-\sum_i \lambda_i} \int_{\mathbb{R}_+^{k\wedge n}} \Delta_{k\wedge n}(\gamma)\prod_{i=1}^{n\wedge k}\gamma_i^{(k-n)\vee 0} 1_{\{\lambda\succeq\gamma\}} \quad \gamma. \tag{12}$$

W $k < n$ , $\frac{1}{2}k(k+1)$, ▾ $\lambda_i=\lambda_j$, $i\neq j$, $\Delta_{k+1}(\lambda)$. T $k+1\leq n$. T $MM^*$ $M^*M$, ▾ $k\geq n$ ▾ . I $c_k>0$,

$$\int_{\mathbb{R}_+^{k\wedge n}} \Delta_{k\wedge n}(\gamma)\prod_{i=1}^{n\wedge k}\gamma_i^{(k-n)\vee 0} 1_{\{\lambda\succeq\gamma\}} \quad \gamma = c_k\Delta_{(k+1)\wedge n}(\lambda)\prod_{i=1}^{n\wedge(k+1)}\lambda_i^{(k+1-n)\vee 0}. \tag{13}$$

W $\mathbb{F}=\mathbb{H}$, ▾ $f_{n,k+1}(\lambda)$

$$d_n(\lambda) \quad {}^{-2\sum_i\lambda_i}\int_{\mathbb{R}_+^{k\wedge n}} 1_{\{\lambda\succeq z\}}\left[\int_{\mathbb{R}_+^{k\wedge n}} 1_{\{\gamma\succeq z\}} \quad {}^{-4\sum_i(\gamma_i-z_i)}\Delta_{k\wedge n}(\gamma)\prod_{i=1}^{k\wedge n}\gamma_i^{(k-n)\vee 0} \quad \gamma\right] \quad z.$$

T C B

$$\begin{aligned}
&\int_{\mathbb{R}_+^{k\wedge n}} \Delta_{k\wedge n}(\gamma) 1_{\gamma\succeq z} \quad {}^{-4\sum_{i=1}^{k\wedge n}(\gamma_i-z_i)}\prod_{i=1}^{k\wedge n}\gamma_i^{(k-n)\vee 0} \quad \gamma_i\\
&=\frac{1}{(k\wedge n)!}\int_{\mathbb{R}_+^{k\wedge n}} \quad \left(\gamma_i^{j-1+(k-n)\vee 0}\right) \quad \left(1_{\{\gamma_i\succeq \ j\}} \quad {}^{-4(\gamma_i-z_j)}\right) \quad \gamma\\
&= \quad \left(\int_{\mathbb{R}_+}\gamma^{j-1+(k-n)\vee 0}1_{\{\gamma\succeq \ i\}} \quad {}^{-4(\gamma-z_i)} \quad \gamma\right)\\
&=C_2 \quad \left(z_i^{j-1+(k-n)\vee 0}\right)=C_2\Delta_{k\wedge n}(z)\prod_{i=1}^{k\wedge n} z_i^{(k-n)\vee 0},
\end{aligned}$$

▾ $C_2$ . U (13), $k+1$.
L ▾ ▾ $\mathbb{F}=\mathbb{R}$. B R 4.9
$n$, $k$ $c=2$ $\tilde{n}$, $\tilde{k}$ $c=1$. T , . I $n$ ,
▾ ▾ ▾ P 4.8 . L

$$N=\begin{pmatrix} M\\ X\end{pmatrix},$$

$X$ G $\mathcal{M}_{1,k}(\mathbb{R})$, $M$. T , -
$N\Omega_k N^*$ $f_{n+1,k}$. T ▾ $U_{n+1}(\mathbb{R})$
$n$ $M\Omega_k M^*$. T , L 3.8, ▾ $\lambda\in\mathbb{R}_+^{\tilde{n}\wedge\tilde{k}}$, $f_{n,k}(\lambda)$

$$\int_{\mathbb{R}_+^{\tilde{n}\wedge\tilde{k}}}\frac{d_n(\lambda)}{d_{n+1}(\gamma)}f_{n+1,k}(\gamma)1_{\{\gamma\succeq\lambda\}} \quad \gamma.$$

T $n+1$ , ▾ $f_{n+1,k}$ (10). A
. □

L ▼ LUE($\mathbb{F}$)
L B 8 . M , ▼
. L $\epsilon$ 1 $n \notin 2\mathbb{N}$ 0 ▼ .

**Theorem 5.3.** *When $k \geq n$ the distribution of a matrix of the* $\mathrm{LUE}_{n,k}(\mathbb{F})$ *has a density $l(H)$ with respect to the Lebesgue measure $H$ on $\mathcal{P}_n(\mathbb{F})$ proportional to*

$$\prod_{i=1}^{n} \lambda_i^{k-n} \; {}^{-\lambda_i} 1_{\mathbb{R}_+}(\lambda_i) \quad \text{for } \mathbb{F} = \mathbb{C},$$

$$\frac{1}{\prod_{1 \leq i < j \leq n}(\lambda_i + \lambda_j)} \prod_{i=1}^{n} \lambda_i^{k-n-1} \; {}^{-2\lambda_i} \quad \text{for } \mathbb{F} = \mathbb{H},$$

$$\frac{1}{\prod_{1 \leq i < j \leq \tilde{n}}(\lambda_i + \lambda_j)} \prod_{i=1}^{\tilde{n}} \lambda_i^{\tilde{k}-\tilde{n}-\epsilon} \; {}^{-\lambda_i} \quad \text{for } \mathbb{F} = \mathbb{R},$$

*where $\lambda$ is the vector of eigenvalues of $H$ when $\mathbb{F} = \mathbb{C}$, of positive eigenvalues of $H$ when $\mathbb{F} = \mathbb{R}, \mathbb{H}$.*

**Proof.** T $d_n$ W , W '
( H 32 , T I.5.17) $C > 0$
$f : \mathcal{P}_n(\mathbb{F}) \to \mathbb{R}^+$, ▼

$$\int_{\mathcal{P}_n(\mathbb{F})} f(H) \quad H = C \int_{C_n} d_n(\lambda)^2 f(\lambda) \quad \lambda,$$

▼ $H$ $\lambda$ L $\mathcal{P}_n(\mathbb{F})$ $\mathcal{C}_n$. T T 5.2
$\mathrm{LUE}_{n,k}(\mathbb{F})$

$$H \in \mathcal{P}_n(\mathbb{F}) \mapsto \frac{\Delta_{\tilde{n}}(\lambda)}{d_n(\lambda)} \prod_{i=1}^{\tilde{n}} \lambda_i^{\tilde{k}-\tilde{n}} \; {}^{-c\lambda_i} 1_{\mathbb{R}_+}(\lambda_i).$$

W $d_n(\lambda)$ . □

F $\lambda$ $\mathcal{C}_n$, $\Omega_n(\lambda) + \sum_{i=1,\ldots,\tilde{k}} M_i \Omega_n(\alpha_i) M_i^*$, ▼ $M_i$'
G $\mathcal{M}_n(\mathbb{F})$ $\alpha_i$' , $\Omega_n(\lambda) +$
$M\Omega_k(\alpha)M^*$, ▼ $M$ G $\mathcal{M}_{n,k}(\mathbb{F})$. W $\lambda = 0$ $\mathbb{F} = \mathbb{C}$,
W . O ▼ ▼ H C (
W 58 ). B ▼ . O , ▼
▼ , ▼ ▼ $\Omega_n(\lambda) + M\Omega_k(\alpha)M^*$
$\mathbb{F}$. N , ▼ . A , ▼
$\lambda = 0$ $k \leq n$, L 3.3 . T , ▼
▼ ▼ B 5 ( O 49 , P 52 ).

**Theorem 5.4.** *Let $(U_N)_{N \geq 1}$ be a sequence of random variable such that $U_N$ is Haar distributed in $U_N(\mathbb{F})$. Then the main minor of order $n$ of $\sqrt{N}U_N$ converges in distribution to a standard Gaussian variable in $\mathcal{M}_n(\mathbb{F})$, when $N$ goes to $+\infty$.*

**Proposition 5.5.** *Let $k$ be an integer smaller than $n$, $M$ be a standard Gaussian variable in $\mathcal{M}_{n,k}(\mathbb{F})$ and $\alpha \in \mathbb{R}^{\tilde{k}}$ such that $\alpha_1 > \cdots > \alpha_{\tilde{k}} > 0$. Then there is a constant $C$ such that the positive eigenvalues of $M\Omega_k(\alpha)M^*$ have a density $g_{n,k}$ with respect to the Lebesgue measure on $\mathbb{R}_+^{\tilde{k}}$ defined by*

$$g_{n,k}(\lambda) = C \frac{d_n(\lambda)}{d_n(\alpha) \prod_{i=1}^{\tilde{k}} \alpha_i} \quad \left( {}^{-c\lambda_i/\alpha_j} \right)_{1 \leq i,j \leq \tilde{k}}.$$

*In particular, when $\tilde{k} = 1$ and $\alpha_1 = 1$, this density is proportional to the function $\theta \in \mathbb{R} \mapsto d_n(\theta) \; {}^{-c\theta} 1_{\mathbb{R}_+}(\theta)$.*

**Proof.** L $N \in \mathbb{N}$ $n$. W H $U_N \in U_N(\mathbb{F})$, $M_N = U_N \Omega_N(\alpha) U_N^*$ $n$ $\pi_n(M_N)$. U L 3.8 $\tilde{k}$ $\pi_{N-1}(M_N)$

$$\frac{d_{N-1}(\lambda)}{d_N(\alpha)} \quad \left( \frac{(\alpha_i - \lambda_j)^{c-1}}{(c-1)!} 1_{\{\alpha_i \geq \lambda_j\}} \right)_{1 \le i,j \le \tilde{k}}.$$

I C B $\pi_n(M_N)$

$$\frac{d_n(\lambda)}{d_N(\alpha)} \quad \left( \frac{(\alpha_i - \lambda_j)^{c(N-n)-1}}{(c(N-n)-1)!} 1_{\alpha_i \geq \lambda_j} \right)_{1 \le i,j \le \tilde{k}}.$$

S $N\pi_n(M_N)$ -

$$\frac{d_n(\lambda)}{d_n(\alpha) \prod_{i=1}^{\tilde{k}} \alpha_i} \quad \left( {}^{-c\lambda_i/\alpha_j} \right)_{1 \le i,j \le \tilde{k}}.$$

T 5.4 $N\pi_n(M_N)$ $M\Omega_n(\alpha)M^*$, $N$ , . □

T L LUE($\mathbb{C}$) 29 . F LUE($\mathbb{F}$) $\mathbb{F} = \mathbb{H}$ $\mathbb{F} = \mathbb{R}$, G S 2.4. W : LUE$_{n,k}$($\mathbb{F}$) $\mathbb{C}$ $\tilde{k}$ $\mathcal{P}_n(\mathbb{F})$ , $k$ H . F , $M = (M_t)_{t \ge 0}$ B $\mathcal{M}_n(\mathbb{C})$, $M_t M_t^*$ M . T ( B 11 ), B $U(n,k)/U(n) \times U(k)$ ( F 23 , R 56 ). T : $(M_t)$ B $\mathcal{M}_2(\mathbb{R})$, $M_t \Omega_2 M_t^*$ ( !) M . I .

## 6. Interlaced determinantal processes

L $E$ B $\mathbb{R}^r$. A $\xi$ $E$ $\xi(B)$ B $B$ $E$. L $(T_k)_{k \ge 1}$ $E$ $\Xi = \sum_{k \in \mathbb{N}} \delta_{T_k}$. I $\Xi$ $E$, $\Xi$ $E$. L $m$ $E$. A $\rho_n$ $E^n$

$$\mathbb{E}\left[ \prod_{i=1}^n \Xi(B_i) \right] = \int_{B_1 \times \cdots \times B_n} \rho_n(x_1, \ldots, x_n) m(x_1) \cdots m(x_n)$$

B $B_1, \ldots, B_n$ $E$, $n$ . T $m$ .

***Definition 6.1.*** *If there exists a function $K : E \times E \to \mathbb{C}$ such that for all $n \ge 1$,*

$$\rho_n(x_1, \ldots, x_n) = \quad \left( K(x_i, x_j) \right)_{n \times n}$$

*for* $x_1, \ldots, x_n \in E$, *then one says that the point process is determinantal and* $K$ *is called the correlation kernel of the process.*

L . F ( B 8 ). S $\mu_n$

$E^n$ $u_n$ $m^{\otimes n}$ $E^n$

$$u_n(\lambda_1, \ldots, \lambda_n) = C \quad \big(\psi_i(\lambda_j)\big)_{n\times n} \quad \big(\phi_i(\lambda_j)\big)_{n\times n}, \tag{14}$$

$C$ $\psi_i$' $\phi_i$' $\psi_i \phi_j$

$i, j$. W $A = (A_{ij})_{1\le i, j\le n}$

$$A_{ij} = \int_E \psi_i(x)\phi_j(x)m( \quad x).$$

T $A$ P 2.2 8 $\mu_n$

$(\lambda_1, \ldots, \lambda_n) \mapsto \sum_{i=1}^n \delta_{\lambda_i}$

$$K(x, y) = \sum_{i,j=1}^{n} \quad )_{ij} \quad j(y), \quad x, y \in E \tag{15}$$

S $E = \mathbb{R}, m$ L

$$u \quad n\lambda) = \Delta \quad n\lambda) \; {}^2 \prod_{i=1}^{n} u(\lambda \quad i \tag{16}$$

$w$ $\mathbb{R}$ $\int x$ $ku(x) \quad x < +\infty$ $k$. I $(p$ $i)_{i\ge0}$

$i$' $p_i$

$$\int_E p_i(x)p_j(x)u(x) \quad x = \delta_{ij}, \quad i, j \in \mathbb{N},$$

$\Delta_n$

A . I $\lambda_i' = \lambda_i^2$ (18) (16) $u(x) = x^{\alpha} \; ^{-x/2}$, $\alpha = \frac{1}{2}$ $\mathbb{F} = \mathbb{H}$ $\mathbb{F} = \mathbb{R}, n$ $\alpha = -\frac{1}{2}$ $\mathbb{F} = \mathbb{R}$, $n$ . T L .
F $\mathbb{F} = \mathbb{H}$ $\mathbb{R}$, T 5.2
$\mathrm{LUE}_{n,k}(\mathbb{F})$, $k \geq n$, (14) $\tilde{n}$ $n$ $\psi_i(x) = x^{2i-2+\epsilon+\tilde{k}-\tilde{n}}$, $\phi_i(x) = x^{i-1} \; ^{-cx}$.
T , (15). N , .

I , . U , J 33 B . 7 K .

### 6.1. *“Triangular” interlaced processes*

T $\mathcal{P}_n(\mathbb{F})$. I $E = \{1, \ldots, n\} \times \mathbb{R}$ $m$ $\{1, \ldots, n\}$ L $\mathbb{R}$ $\mathbb{F} = \mathbb{C}$, $\mathbb{R}_+$ $\mathbb{F} = \mathbb{H}$ $\mathbb{F} = \mathbb{R}$.

***Definition 6.2.*** *We say that an invariant random matrix $M$ in $\mathcal{P}_n(\mathbb{F})$ belongs to the class $\mathcal{K}$ if the eigenvalues of $M$ for $\mathbb{F} = \mathbb{C}$, and the positive eigenvalues of $M$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{H}$, have a joint density with respect to the Lebesgue measure on $\mathbb{R}^{\tilde{n}}$ proportional to*

$$d_n(\lambda) \quad \left(\psi_j(\lambda_i)\right)_{\tilde{n}\times\tilde{n}},$$

*where the $\psi_i$'s are real continuous functions on $\mathbb{R}$, equal to zero on $\mathbb{R}_-$ for $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{H}$, and such that for all $k \in \mathbb{N}$, the function $x \mapsto x^k \psi_i(x)$ is integrable on $\mathbb{R}$.*

M $\mathcal{K}$, GUE($\mathbb{F}$) LUE($\mathbb{F}$).

**Theorem 6.3.** *Let $M$ be an invariant random matrix in $\mathcal{P}_n(\mathbb{F})$, which belongs to the class $\mathcal{K}$. Let us consider the random vector $X = X(M)$ and the associated point process $\Xi$ on $E$ defined by*

$$\Xi = \sum_{k=1}^{n} \sum_{i=1}^{k} \delta_{(k, X_i^{(k)})} \quad \text{when } \mathbb{F} = \mathbb{C}, \mathbb{H}, \quad \text{and} \quad \Xi = \sum_{k=1}^{n} \sum_{i=1}^{\tilde{k}} \delta_{(k, |X_i^{(k)}|)} \quad \text{when } \mathbb{F} = \mathbb{R}.$$

*Then*

( ) *The point process $\Xi$ is determinantal.*
( ) *The correlation kernel of $\Xi$ is, for $(r, x), (s, y) \in E$,*

$$K((r, x), (s, y)) = -\frac{(y-x)^{c(s-r)-1}}{(c(s-r)-1)!} 1_{\{s>r, y \geq x\}} + \alpha \sum_{k=1}^{\tilde{n}} \psi_{r-k}^r(x) \int \frac{\partial^{c(n-s)} d_n}{\partial z_k^{c(n-s)}}(z_1, \ldots, z_{k-1}, y, z_{k+1}, \ldots, z_{\tilde{n}}) \prod_{\substack{i=1 \\ i\neq k}}^{\tilde{n}} \psi_i(z_i) \; z_i,$$

*where $\psi_{r-k}^r(x) = \int_x^{+\infty} \frac{1}{(c(n-r)-1)!}(z-x)^{c(n-r)-1}\psi_k(z) \; z$, if $r < n$, $\psi_{n-k}^n(x) = \psi_k(x)$ and $\alpha^{-1} = \int d_n(z) \times \prod_{i=1}^{\tilde{n}} \psi_i(z_i) \; z_i$.*

W $(X^{(1)}(M), \ldots, X^{(n)}(M))$ $M$ $\mathcal{P}_n(\mathbb{H})$, $(X^{(3)}(N), X^{(5)}(N), \ldots, X^{(2n+1)}(N))$ $N$ $\mathcal{P}_{2n+1}(\mathbb{R})$, $X^{(n)}(M)$

$X^{(2n+1)}(N)$ . S (
S 8.2).

**Corollary 6.4.** *Under the hypothesis of the previous theorem, suppose that we can write* $d_n(\lambda) = \quad (\chi_i(\lambda_j))_{\tilde{n}\times\tilde{n}}$, *where* $(\chi_k)_{k\geq 1}$ *is a sequence of real functions on* $\mathbb{R}$ *such that the* $\chi_i\psi_j$*'s are integrable on* $\mathbb{R}$ *and* $\int_{\mathbb{R}} \chi_i(x)\psi_j(x) \quad x = \delta_{ij}$. *Then*

$$K\big((r,x),(s,y)\big) = -\frac{(y-x)^{c(s-r)-1}}{(c(s-r)-1)!}1_{\{s>r,\, y\ \geq x\}} + \sum_{k=1}^{\tilde{n}} \psi^r_{r-k}(x)\frac{d^{c(n-s)}\chi_k}{dx^{c(n-s)}}(y).$$

I M λ W ,
, $\psi_i(z)$ $z$ D $\delta_{\lambda_i}(\ z)$ $\mathbb{F}=\mathbb{C}$
$\delta_{|\lambda_i|}(\ z)$ $\mathbb{F}=\mathbb{R}$, $\{1,\ldots,n\}$
$\{1,\ldots,n-1\}$. L .
R $\epsilon$ 1 $n \notin 2\mathbb{N}$ 0 .

*The Gaussian case:* GUE($\mathbb{F}$). A G $M$ $\mathcal{P}_n(\mathbb{F})$
$\psi_i(x) = x^{i-1}\ {}^{-x^2/2}$ $\mathbb{F}=\mathbb{C}$ $\psi_i(x) = x^{2i-2+\epsilon}\ {}^{-x^2/2}1_{\{x>0\}}$ . B , -
$\chi_i = h_{i-1}$ $\mathbb{F}=\mathbb{C}$, $\chi_i = h_{2i-2+\epsilon}$ $\mathbb{F}=\mathbb{R}$ $\chi_i = h_{2i-1}$
$\mathbb{F}=\mathbb{H}$, $(h_i)_{i\geq 0}$ H ${}^{-x^2/2}$, $h_i$
$i$.

I GUE($\mathbb{C}$), J N 34 , O
R 47 . T ,
GUE$_\infty(\mathbb{R})$, 16 . F N
24 .

**Proposition 6.5.** *Let* $M$ *be a standard Gaussian variable in* $\mathcal{P}_\infty(\mathbb{R})$. *We consider the radial part* $X^{(k)} \in \mathbb{R}^{\tilde{k}}$ *of the main minor of order* $k$ *of* $M$. *Then the point process* $\sum_{k=1}^{+\infty}\sum_{i=1}^{\tilde{k}} \delta_{(k,\,|X_i^{(k)}|)}$ *is determinantal on* $\mathbb{N}^*\times\mathbb{R}_+$ *with correlation kernel*

$$\begin{aligned} R\big((r,x),(s,y)\big) = &-\frac{1_{\{r<s\}}}{(s-r-1)!}(y-x)^{s-r-1}1_{\{y\geq x\}} \\ &+ \sum_{i=1}^{\tilde{r}\wedge\tilde{s}} \frac{((r-2i)!)^{1/2}}{((s-2i)!)^{1/2}} h_{s-2i}(y)h_{r-2i}(x)\ {}^{-x^2/2} \\ &+ \sum_{i=\tilde{r}+1}^{\tilde{s}} \frac{h_{s-2i}(y)}{((s-2i)!\sqrt{\pi})^{1/2}} \int_x^{+\infty} \frac{(z-x)^{2i-r-1}}{(2i-r-1)!}\ {}^{-z^2/2}\ z. \end{aligned}$$

**Proof.** L $n$ $M$ G $\mathcal{P}_n(\mathbb{R})$. T $M$
$\mathcal{K}$ $\psi_i = h_{2i-1}$, $i = 1,\ldots\tilde{n}$. T $\chi_i = h_{2i-1}$, $i=1,\ldots\tilde{n}$, C 6.4. S
$\sum_{k=1}^{n}\sum_{i=1}^{\tilde{k}} \delta_{(k,\,|X_i^{(k)}|)}$ $\mathbb{N}^*\times\mathbb{R}_+$ $K$

$$K\big((r,x),(s,y)\big) = -\frac{1_{\{s<r\}}}{(r-s-1)!}(y-x)^{r-s-1}1_{\{y\geq x\}} + \textstyle\sum_{k=1}^{\tilde{n}} h_{2k-1}^{(n-s)}(y)\xi_k(r,x),$$

$\xi_k(r,x) = \int_x^\infty \frac{(z-x)^{n-r-1}}{(n-r-1)!}h_{2k-1}(z)\ {}^{-z^2/2}\ z$. L

$$H_n(x) = (-1)^n\ {}^{x^2/2}\frac{{}^n}{x^n}\ {}^{-x^2/2}.$$

W $h_n = \frac{1}{(n!\sqrt{\pi})^{1/2}} H_n$ $h'_n = \sqrt{n} h_{n-1}$. L $i = \tilde{n} - k + 1$,

$$\sum_{k=1}^{\tilde{n}} h_{2k-1}^{(n-s)}(y)\xi_k(r,x) = \sum_{i=1}^{\tilde{n}} \left[\frac{(n-2i)!}{(s-2i)!}\right]^{1/2} h_{s-2i}(y)\xi_k(r,x).$$

I $i \geq \tilde{r} + 1$,

$$\xi_k(r,x) = \frac{1}{((n-2i)!\sqrt{\pi})^{1/2}} \int_x^{\infty} \frac{(z-x)^{2i-r-1}}{(2i-r-1)!} \quad {}^{-z^2/2} \quad z,$$

$i \leq \tilde{r}$,

$$\xi_k(r,x) = \left[\frac{(r-2i)!}{(n-2i)!}\right]^{1/2} h_{r-2i}(x) \quad {}^{-x^2/2},$$

. □

*The Laguerre case:* LUE($\mathbb{F}$). A LUE$_{n,k}$ ($\mathbb{F}$) $k \geq n$ T - 6.3 $\psi_i(x) = x^{i-1+\tilde{k}-\tilde{n}} \quad {}^{-cx} 1_{\mathbb{R}_+}(x)$. T C 6.4 $\mathbb{F} = \mathbb{C}$ $\psi_i = \chi_i = L_i$, $(L_i)_{i\geq 0}$ L $x^{k-n} \quad {}^{-x}$, $L_i$ $i$.

L T 6.3. T 7 .

F $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$, $h : \mathbb{R} \to \mathbb{R}_+$ $x, y \in \mathbb{R}$, , ,

$(f *$

- $\mathbb{F} = \mathbb{C}$, $x_r^{(r-1)} = a$,

$$f\big(x^{(1)}, \ldots, x^{(n)}\big) = \big(\psi_j\big(x_i^{(n)}\big)\big)_{n\times n} \prod_{r=1}^{n} \big(1_{\{x_j^{(r)} \succ x_i^{(r-1)}\}}\big)_{r\times r},$$

- $\mathbb{F} = \mathbb{R}$, $x_r^{(2r-1)} = 0$,

$$f\big(x^{(1)}, \ldots, x^{(n)}\big) = \big(\psi_j\big(x_i^{(n)}\big)\big)_{\tilde n\times \tilde n} \prod_{r=1}^{n} \big(1_{\{x_j^{(r)} \succ x_i^{(r-1)}\}}\big)_{\tilde r\times \tilde r}.$$

W $(\tilde\Lambda_N)_N$ $\tilde\Lambda_N$ $\frac{1}{N} GT_{n,\mathbb{Z}}$ $\mathbb{P}(\tilde\Lambda_N = (x^{(1)}, \ldots, x^{(n)}))$ $f(x^{(1)}, \ldots, x^{(n)})$. T L 3.4 7, $\mathbb{F} = \mathbb{R}$ ( 24 ), $K_N$ $K$ L $\mathbb{R}$ (19) $\frac{1}{N}\mathbb{Z}$. W $N$ . □

**Proof of Theorem 6.3.** W $\mathbb{F} = \mathbb{R}$. W L 6.6 . W , $r \geq 1$,

$$\phi^{(r)}(0, y) = \frac{y^{r-1}}{(r-1)!} 1_{\{y\geq 0\}}.$$

T , $\phi^{(s-2l+1)}(0, y) = \frac{\partial^{n-s}}{\partial y^{n-s}} \phi^{(n-2l+1)}(0, y)$, $l = 1, \ldots \tilde n$,

$$\sum_{l=1}^{s} \big(B^{-1}\big)_{kl}\, \phi^{(s-2l+1)}(0, y) = \frac{\partial^{n-s}}{\partial y^{n-s}} \sum_{l=1}^{n} \big(B^{-1}\big)_{kl}\, \phi^{(n-2l+1)}(0, y).$$

L $s_{l\,k}(B)$ $B$ $l$ $k$ . W

$$\big(B^{-1}\big)_{kl} = \frac{(-1)^{k+l}}{(B)} \big(s_{l\,k}(B)\big)_{\tilde n-1\times \tilde n-1}.$$

T

$$\begin{aligned}
\sum_{l=1}^{\tilde n} \big(B^{-1}\big)_{kl}\, \phi^{(n-2l+1)}(0, y) &= \sum_{l=1}^{\tilde n} \frac{(-1)^{k+l}}{(B)} \big(s_{l\,k}(B)\big)_{\tilde n-1\times \tilde n-1} \phi^{(n-2l+1)}(0, y) \\
&= \sum_{l=1}^{\tilde n} \frac{(-1)^{k+l}}{(B)} \big(\phi^{(n-2i+1)} * \psi_j(0)\big)_{\substack{i\neq l\\ j\neq k}} \phi^{(n-2l+1)}(0, y) \\
&= \sum_{l=1}^{\tilde n} \frac{(-1)^{k+l}}{(B)} \int_{\mathbb{R}^{n-1}} \big(\phi^{(n-2i+1)}(0, z_j)\big)_{\substack{i\neq l\\ j\neq k}} \phi^{(n-2l+1)}(0, y) \prod_{\substack{j=1\\ j\neq k}}^{\tilde n} \psi_j(z_j) \quad z_j \\
&= \frac{1}{(B)} \int_{\mathbb{R}^{n-1}} \big(\phi^{(n-2i+1)}(0, z_j)\big)_{\tilde n\times \tilde n} \prod_{\substack{j=1\\ j\neq k}}^{\tilde n} \psi_j(z_j) \quad z_j, \qquad z_k = y.
\end{aligned}$$

M , $V_n$ D 3.5,

$$\big(\phi^{(n-2i+1)}(0, z_j)\big)_{\tilde n\times \tilde n} = \left(\frac{z_j^{n-2i}}{(n-2i)!} 1_{\{z_i\geq 0\}}\right)_{\tilde n\times \tilde n} = V_n(z) \prod_{i=1}^{\tilde n} \frac{1_{\{z_i\geq 0\}}}{(n-2i)!},$$

T ( ) $\mathcal{O}(\lambda)$ H $m_K$ $K$, . .
A $(U)\lambda$ w $U$ w w$m_K$. C
( ) $\mathcal{O}(\lambda)$ $\mathcal{O}(\mu)$, w $\lambda, \mu \in \mathfrak{k}^*$,
$p$ L $H$. L w K ' ( 39,
40 , . ). I U ' w ( w $V_\lambda$
$\lambda$):

R 3: I w w $Res^K_H V_\lambda$ w
$p(\mathcal{O}(\lambda))$ A $(H)$ .
R 5: I w w $V_\lambda \otimes V_\mu$ w
$\mathcal{O}(\lambda) + \mathcal{O}(\mu)$ A $(K)$ .

O w , : w w w w
. F w H w w w w
w w *branching rules*. T w
.

## 7.2. *Characters*

L $K$ L w L $\mathfrak{k}$ L $\mathfrak{k}_{\mathbb{C}}$. B ,
w , w $K$ ,
A $(k)x = kxk^*, k \in K, x \in \mathfrak{k}$ $\mathfrak{k}^*$. W $T$ $K$ w
$\mathfrak{t}$ L . W $R = \{\alpha \in \mathfrak{t}^*\colon \exists X \in \mathfrak{k}_{\mathbb{C}} \setminus \{0\}, \forall H \in \mathfrak{t}, [H, X] = \alpha(H)X\}$,
$h_\alpha = 2\alpha/\langle\alpha,\alpha\rangle$, $\alpha \in R$. W $\Sigma$ $R$. W
$R^+$ ( ) W $\mathcal{C} = \{\lambda \in \mathfrak{t}^*\colon \langle\lambda,\alpha\rangle \geq 0$ $\alpha \in \Sigma\}$. T w
$P = \{\lambda \in \mathfrak{t}^*\colon \langle h_\alpha, \lambda\rangle \in \mathbb{Z},$ $\alpha \in R\}$ w $P^+ = P \cap \mathcal{C}$. W $W$ W
.

F $\lambda \in P^+$, w $V_\lambda$ $\mathfrak{k}$- w w $\lambda$ $(\lambda)$ $V_\lambda$. I
$\chi_\lambda$ $\mathfrak{t}$ ,

$$\chi_\lambda(\zeta) = \sum_{\mu \in P} m(\mu\lambda) \quad {}^{\langle\mu\zeta\rangle}, \quad \zeta \in \mathfrak{t},$$

w $m(\mu\lambda)$ w $\mu$ $\mathfrak{k}$- $V_\lambda$. N w
L . W $\rho = \frac{1}{2}\sum_{\alpha \in R^+}\alpha$,
. T $V_\lambda$ $\chi_\lambda(0)$. R W ( K 42 ,
T V.5.84):

$$\chi_\lambda(0) = \prod_{\alpha \in R^+} \frac{\langle\lambda + \rho, \alpha\rangle}{\langle\rho, \alpha\rangle} \tag{20}$$

W L L ( K 42 , T V.5.77):

**Proposition 7.1 (Weyl character formula).** *The character $\chi_\lambda$ is equal to*

$$\chi_\lambda(\zeta) = \frac{\sum_{w \in W} \quad (w) \quad {}^{\langle w(\lambda+\rho),\zeta\rangle}}{\sum_{w \in W} \quad (w) \quad {}^{\langle w(\rho),\zeta\rangle}}.$$

I , $\prod_{\alpha \in R^+}( {}^{(\ /2)\langle\alpha,\zeta\rangle} - {}^{-(\ /2)\langle\alpha,\zeta\rangle})$. W $K = U_n(\mathbb{C})$
$\lambda$ , S ( , , 26 ).
L $K$. L ,
$z \in \mathfrak{t} \oplus \mathfrak{t}$, $\lambda \in \mathfrak{t}^*$,

$$h(z) = \prod_{\alpha \in R^+} \langle\alpha, z\rangle, \qquad d(\lambda) = \prod_{\alpha \in R^+} \langle\alpha,\lambda\rangle/\langle\alpha,\rho\rangle.$$

T $d(\lambda)$ L $\mathcal{O}(\lambda)$ .
F $\lambda \in \mathfrak{t}^*$, $\Phi_\lambda$ $\mathfrak{k}$ $\Phi_\lambda(\zeta) = \Phi_\lambda(k\zeta k^*)$ $\zeta \in \mathfrak{k}, k \in K$,
$\zeta \in \mathfrak{t}$,

$$\Phi_\lambda(\zeta) = \frac{\sum_{w \in W} (w) \ \langle w\lambda, \zeta \rangle}{h( \zeta) d(\lambda)}.$$

W H C ( H 32 , T II.5.35). I
K I . R $m_K$ H
$K$.

**Proposition 7.2.** *For* $\lambda \in \mathfrak{t}^*$, $\zeta \in \mathfrak{k}$

$$\int_K \ \langle k\lambda k^*, \zeta \rangle m_K( k) = \Phi_\lambda(\zeta). \tag{21}$$

T $\Phi_\lambda(\zeta)$ $(\lambda, \zeta)$ $\Phi_\lambda(0) = 1$.

7.3. *A version of Heckman's theorem*

W $H$ $K$ L $\mathfrak{h}$. A ,
$S$ $H$ $T$ ( , , K 42 ). W L $\mathfrak{s}$. T
$K$ $H$. I , $H$
. F $\lambda \in P^+, \beta \in P_H^+$ $m_H^\lambda(\beta)$ $\mathfrak{h}$- $\beta$
$\mathfrak{k}$- $V_\lambda$ $\mathfrak{h}$- . R
$m_H^\lambda$ . W

$$V_\lambda = \bigoplus_{\beta \in P_H^+} m_H^\lambda(\beta) V_\beta^H, \tag{22}$$

$V_\lambda$ $\mathfrak{h}$- $V_\beta^H$ $\mathfrak{h}$- $\beta$. T
$m_H^\lambda$ $P_H^+$ $\mathbb{N}$ : $\zeta \in \mathfrak{s}$,

$$\chi_\lambda(\zeta) = \sum_{\beta \in P_H^+} m_H^\lambda(\beta) \chi_\beta^H(\zeta). \tag{23}$$

F $x \in \mathfrak{k}^*$, $\pi_H(x)$ $x$ $\mathfrak{h}^*$. T $x \in \mathfrak{k}^*$
$K$ W $\mathcal{C}$ $x$
$r(x)$. T $H$ $r_H(x)$ $x \in \mathfrak{h}^*$ W $\mathcal{C}_H$
$H$. W $\varepsilon_n > 0$ 0 $n \to \infty$. T
T 6.4 H 30 . W .

**Theorem 7.3.** *Let* $\lambda$ *be in the Weyl chamber* $\mathcal{C}$ *and* $(\lambda_n)_{n\geq 1}$ *be a sequence of elements in* $P^+$ *such that* $\varepsilon_n \lambda_n$ *converges to* $\lambda$ *as* $n$ *tends to* $+\infty$. *Then*

( ) *the sequence* $(\mu_n)_{n\geq 0}$ *of probability measures on* $\mathcal{C}_H$ *defined by*

$$\mu_n = \sum_{\beta \in P_H^+} \frac{{}_H(\beta)}{(\lambda_n)} m_H^{\lambda_n}(\beta) \delta_{\varepsilon_n \beta}$$

*converges to a probability measure* $\mu$ *which satisfies, for* $\zeta \in \mathfrak{h}$,

$$\int_{\mathcal{C}_H} \Phi_\beta^H(\zeta) \mu( \beta) = \Phi_\lambda(\zeta), \tag{24}$$

( ) $\mu$ *is the law of* $r_H(\pi_H(U\lambda U^*))$, *where* $U$ *is distributed according to* $m_K$.

**Proof.** L $\zeta \in \mathfrak{s}$. W

$$\frac{\chi_{\lambda_n}(\varepsilon_n\zeta)}{\chi_{\lambda_n}(0)} = \Phi_{\varepsilon_n\lambda_n+\varepsilon_n\rho}(\zeta)\prod_{\alpha\in R^+}\frac{\langle\alpha,\varepsilon_n\zeta\rangle}{(\ /2)\langle\alpha,\varepsilon_n\zeta\rangle - \ -(\ /2)\langle\alpha,\varepsilon_n\zeta\rangle}.$$

O ,

$$\frac{\chi_{\lambda_n}(\varepsilon_n\zeta)}{\chi_{\lambda_n}(0)} = \sum_{\beta\in P_H^+}\frac{\chi_\beta^H(\varepsilon_n\zeta)}{\chi_\beta^H(0)}\frac{m_H^{\lambda_n}(\beta)\chi_\beta^H(0)}{\chi_{\lambda_n}(0)}$$

$$= \left[\prod_{\alpha\in R_H^+}\frac{\langle\alpha,\varepsilon_n\zeta\rangle}{(\ /2)\langle\alpha,\varepsilon_n\zeta\rangle - \ -(\ /2)\langle\alpha,\varepsilon_n\zeta\rangle}\right]\int_{\mathcal{C}_H}\Phi^H_{\beta+\varepsilon_n\rho_H}(\zeta)\ \ \mu_n(\beta).$$

T ,

$$_{n\to+\infty}\int_{\mathcal{C}_H}\Phi^H_{\beta+\varepsilon_n\rho_H}(\zeta)\mu_n(\ \beta) = \Phi_\lambda(\zeta).$$

T $\mu_n$ $\varepsilon_n\lambda_n$ W . T $\mu_n$ . U $\Phi$

$$_{n\to+\infty}\int_{\mathcal{C}_H}\Phi^H_\beta(\zeta)\mu_n(\ \beta) = \Phi_\lambda(\zeta). \tag{25}$$

L $\gamma_n$ $m_H\otimes\mu_n$ $(u,\beta)\in H\times\mathcal{C}_H\mapsto u\beta u^*\in\mathfrak{h}^*$. T H C ' $H$

$$_{n\to\infty}\int_{\mathfrak{h}^*}\ ^{\langle x,\zeta\rangle}\gamma_n(\ x) = \Phi_\lambda(\zeta).$$

B H $H$ , $\zeta\in\mathfrak{h}$, $(\gamma_n)_{n\geq 0}$ $(\mu_n)_{n\geq 0}$. W $\mu$ . T (25) , $\zeta\in\mathfrak{h}$,

$$\int_{\mathcal{C}_H}\Phi^H_\beta(\zeta)\mu(\ \ \beta) = \Phi_\lambda(\zeta),$$

. A H C $K$ $H$

$$\int_K\ ^{\langle u\lambda u^*,\zeta\rangle}m_K(\ u) = \int_K\ ^{\langle\pi_H(u\lambda u^*),\zeta\rangle}m_K(\ u)$$

$$= \int_H\int_{\mathcal{C}_H}\ ^{\langle u\beta u^*,\zeta\rangle}\mu(\ \ \beta)m_H(\ u),$$

. □

I $H=T$, $\mu$ $d(\lambda)^{-1}D_\lambda$ $D_\lambda$ D H $\lambda$. T , , D . 18 .

**Corollary 7.4.** *Let $\lambda$ and $\gamma$ be in $\mathcal{C}$. Let $(\lambda_n)_{n\geq 1}$ and $(\gamma_n)_{n\geq 1}$ be two sequences of elements in $P^+$ such that $\varepsilon_n\lambda_n$ and $\varepsilon_n\gamma_n$ respectively converge to $\lambda$ and $\gamma$, as $n$ tends to $+\infty$. Let us define the sequence $(\nu_n)_{n\geq 0}$ of probability measures on $\mathcal{C}$ by*

$$\nu_n = \sum_{\beta\in P^+} \frac{(\beta)}{(\lambda_n)\qquad(\gamma_n)} M_{\lambda_n,\gamma_n}(\beta)\delta_{\varepsilon_n\beta},$$

*where $M_{\lambda_n,\gamma_n}(\beta)$ is the multiplicity of the highest weight $\beta$ in the decomposition into irreducible components of $V_{\lambda_n}\otimes V_{\gamma_n}$. Then the sequence $(\nu_n)_{n\geq 0}$ converges to the law of the radial part of $\lambda + U\gamma U^*$, where $U$ is distributed according to $m_K$.*

**Proof.** L $V_{\lambda_n}$ $V_{\gamma_n}$ $\mathfrak{k}$- $\lambda_n$ $\gamma_n$. L $K\times K$. T $V_{\lambda_n}\otimes V_{\gamma_n}$ $(\mathfrak{k}\times\mathfrak{k})$- $(\lambda_n,\gamma_n)$. A T 7.3 $K\times K$ $H=\{(k,k),k\in K\}$, $(\nu_n)_{n\geq 1}$ , $n$ $+\infty$, $r_H(\pi_H(\mathrm{A}\ (W)(\lambda,\gamma)))$ , $W$ H $K\times K$, . . $W=(U,V)$, $U$ $V$ $m_K$. T $\pi_H(\mathrm{A}\ (W)(\lambda,\gamma)) = U\lambda U^* + V\gamma V^*$ $r_H(U\lambda U^*+V\gamma V^*) = r_H(\lambda + U^*V\gamma V^*U)$ . □

## 8. Orbit measures and invariant random matrices

### 8.1. *Previous results in the framework random matrices*

I , T 7.3 $\mathcal{P}_n(\mathbb{F})$. F $\mathbb{F}=\mathbb{C},\mathbb{H},\mathbb{R}$ $U_n(\mathbb{F})$ S 2 , , , . I $A_{n-1}$ $\mathbb{F}=\mathbb{C}$, $C_n$ $\mathbb{F}=\mathbb{H}$, $B_r$ $\mathbb{F}=\mathbb{R}$ $n=2r+1$, $D_r$ $\mathbb{F}=\mathbb{R}$ $n=2r$. T L $\mathfrak{U}_n(\mathbb{F})$ $U_n(\mathbb{F})$ $\mathcal{P}_n(\mathbb{F})$. L $\mathfrak{t}_n = \{\ \Omega_n(x)\colon\ x\in\mathbb{R}^{\tilde{n}}\}$. I L $U_n(\mathbb{F})$. W $\epsilon_k:\mathfrak{t}_n\to\mathbb{R}$, $\epsilon_k(\ \Omega_n(x)) = x_k$, $x\in\mathbb{R}^{\tilde{n}}$, $k=1,\ldots\tilde{n}$. W $\mathfrak{U}_n(\mathbb{F})$ $\langle x,y\rangle = Tr(xy^*)$ $\mathbb{F}=\mathbb{C}$ $\langle x,y\rangle = \frac{1}{2}Tr(xy^*)$ $\mathbb{F}=\mathbb{H},\mathbb{R}$. F $U_n(\mathbb{F})$, $\Sigma$ :

- $\mathbb{F}=\mathbb{C}$, $\Sigma=\{\epsilon_i-\epsilon_{i+1}, i=1,\ldots,n-1\}$,
- $\mathbb{F}=\mathbb{H}$, $\Sigma=\{2\epsilon_n, \epsilon_i-\epsilon_{i+1}, i=1,\ldots,n-1\}$,
- $\mathbb{F}=\mathbb{R}$ $n=2r+1$, $\Sigma=\{\epsilon_r, \epsilon_i-\epsilon_{i+1}, i=1,\ldots,r-1\}$,
- $\mathbb{F}=\mathbb{R}$ $n=2r$, $\{\epsilon_{r-1}+\epsilon_r, \epsilon_i-\epsilon_{i+1}, i=1,\ldots,r-1\}$.

I $\mathbb{R}^{\tilde{n}}$ $\mathfrak{t}_n$ $x\in\mathbb{R}^{\tilde{n}}\mapsto\ \Omega_n(x)\in\mathfrak{t}_n$, $\mathfrak{t}_n$ $\mathfrak{t}_n^*$ , $x\in\mathbb{R}^{\tilde{n}}$ $\Omega_n(x)\in\mathfrak{t}_n$ $\sum_{i=1}^{\tilde{n}} x_i\epsilon_i\in\mathfrak{t}_n^*$. U , W $\mathcal{C}_n$ S 2, $U\Omega_n(x)U^*$ $x$, S 2 S 7. A $\mathcal{C}_n$ $\mathbb{Z}$. A , $U_n(\mathbb{F})$. W $K=U_n(\mathbb{F})$, C 7.4 .

**Theorem 8.1.** *Let $\lambda$ and $\beta$ be two elements in the Weyl chamber $\mathcal{C}_n$ and an associated sequence of measures $(\nu_k)_{k\geq 1}$ chosen as in Corollary 7.4. Then $(\nu_k)_{k\geq 1}$ converges to the law of the radial part of $\Omega_n(\lambda)+U\Omega_n(\beta)U^*$ where $U$ is a Haar distributed random variable in $U_n(\mathbb{F})$.*

W $H=\{U\in U_n(\mathbb{F})\colon\ U_{in}=U_{ni}=\delta_{in}, i=1,\ldots,n\}$ L $\{M\in\mathfrak{U}_n(\mathbb{F})\colon\ M_{in}=M_{ni}=0, i=1,\ldots,n\}$. T $U_{n-1}(\mathbb{F})$ $\mathfrak{U}_{n-1}(\mathbb{F})$. T $M$ $\mathfrak{U}_n(\mathbb{F})$ , , $n-1$ $M$. T , $U_n(\mathbb{F})$ $H$, T 7.3 :

**Theorem 8.2.** *Let $\lambda$ be in the Weyl chamber $\mathcal{C}_n$. Let us consider $M = U\Omega_n(\lambda)U^*$, where $U$ is a Haar distributed random variable in $U_n(\mathbb{F})$ and an associated sequence of measures $(\mu_k)_{k\geq 1}$ on $\mathcal{C}_{n-1}$ as in Theorem 7.3. Then $(\mu_k)_{k\geq 1}$ converges to the law of the radial part of the main minor of order $n-1$ of $M$.*

8.2. *Relation between quaternionic and real odd case*

W $\mathbb{F}=\mathbb{R}$
$n=2r+1$ $\mathbb{F}=\mathbb{H}$ $n=r$,
$n-1$ $U\Omega_n(\lambda)U^*$, $U$ H $U_n(\mathbb{H})$,
$2n-1$ $V\Omega_{2n+1}(\lambda)V^*$, $V$ H $U_{2n+1}(\mathbb{R})$. I : (24)
W
. A ,
.

## 9. Tensor product and restriction multiplicities

W . B T 8.1 8.2,
. I
, . R
.
T S 3 $V_\lambda \otimes V_\gamma$, $\lambda$
$\gamma$ , $\gamma$ $\epsilon_1$. U K , ,
S 9.1, . O ,
, ( , , B 3 , N 46 )
: ,
, F K 25 , , .
I S 9.2,
.

9.1. *Tensor product multiplicities and crystal graphs*

L ( , . , K 35 ). A
L $K$ L $\mathfrak{k}$. R $\mathfrak{k}$-
$i \in I$. A $a \xrightarrow{i} b$ $\tilde{f}_i(a) = b$ $\tilde{e}_i(b) = a$ $\tilde{e}_i$
$\tilde{f}_i$ . W $\Lambda_i$, $i = 1, \dots, n$ . F $\mathfrak{k}$- $V$
$B$, $b \in B$ $(b) = \sum_I (\varphi_i(b) - \varepsilon_i(b))\Lambda_i$, $\varphi_i(b) =$
$\{n \geq 0\colon \tilde{f}_i^n(b) \in B\}$ $\varepsilon_i(b) =$ $\{n \geq 0\colon \tilde{e}_i^n(b) \in B\}$, $i \in I$. F $\lambda$ $B(\lambda)$
$\mathfrak{k}$- $V_\lambda$ $\lambda$ $u_\lambda$ . W
P 4.2 35 .

**Proposition 9.1.** *Let $\lambda$ and $\mu$ be two dominant weights and $B(\mu)$ the crystal graph of $V_\mu$. Then*

$$V_\lambda \otimes V_\mu = \bigoplus V_{\lambda + (b)},$$

*where the sum ranges over $b \in B(\mu)$ such that $\varepsilon_i(b) \leq \langle h_i, \lambda\rangle$ (or equivalently $\varepsilon_i(u_\lambda \otimes b) = 0$) for every $i \in I$.*

W $K = U_n(\mathbb{F})$ $V_\lambda \otimes V_{a\epsilon_1}$ . F
L K N 36 . I
, $x$ $(x_1, \dots, x_{\tilde{n}}) \in \mathbb{R}^{\tilde{n}}$ $\sum_{i=1}^{\tilde{n}} x_i \epsilon_i$. N $\epsilon_1$
.

9.1.1. *Tensor product of representations for the type $A_{n-1}$*

T P ' ( F 26 ). B . I $A_{n-1}$, $h_i = \epsilon_i - \epsilon_{i+1}$, $1 \le i \le n-1$. T $V_{\epsilon_1}$ , 35 ,

$$B(\epsilon_1): 1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n.$$

H i $\epsilon_i$, $i = 1, \dots, n$ W $\{1, \dots, n\}$. L m . T 3.4.2 36

$$B(m\epsilon_1) = \{b_m \otimes \cdots \otimes b_1 \in B(\epsilon_1)^{\otimes m}:\ b_{k+1} \ge b_k\}.$$

L λ . L $V_\lambda \otimes V_{m\epsilon_1}$. I P 9.1, $b_m \otimes \cdots \otimes b_1 \in B(m\epsilon_1)$ , $1 \le i \le n$, $\varepsilon_i(u_\lambda \otimes b_m \otimes \cdots \otimes b_1) = 0$, $\varepsilon_i(b_k) \le \langle h_i, \lambda + wt(b_{k+1}) + \cdots + wt(b_m)\rangle$ $1 \le k \le m$. W $b \in B(\epsilon_1)$, $b = i+1$ $\varepsilon_i(b) = 1 = -\langle h_i, wt(b)\rangle$, $\varepsilon_i(b) = 0 \le \langle h_i, wt(b)\rangle$. T

$$\varepsilon_i(b) \le \langle h_i, \lambda\rangle \quad \Leftrightarrow \quad 0 \le \langle h_i, \lambda + wt(b)\rangle. \tag{26}$$

S , , $b_m \otimes \cdots \otimes b_1 \in B(\epsilon_1)^{\otimes m}$ $k \in \{1, \dots, m\}$, $i \in \{1, \dots, n\}$,

$$\begin{cases} b_{k+1} \ge b_k, \\ 0 \le \langle h_i, \lambda + wt(b_m) + \cdots + wt(b_k)\rangle. \end{cases} \tag{27}$$

W F . 8

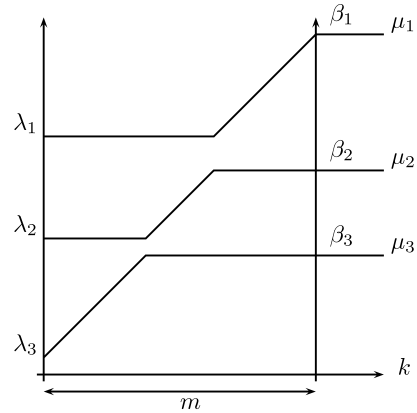
$$k \mapsto \mu_i(k) = \langle \epsilon_i, \lambda + wt(b_m) + \cdots + wt(b_{m-k+1})\rangle.$$

A $k$, $\mu_1, \dots, \mu_n$ . M , $i$ $(i+1)$ $b_m \otimes \cdots \otimes b_1$ $B(m\epsilon_1)$. T $0 \le \langle h_i, \lambda + wt(b_m) + \cdots + wt(b_k)\rangle$. T $b_m \otimes \cdots \otimes b_1 \mapsto \beta \in \mathbb{Z}^n$, $\beta_i = \langle \epsilon_i, \lambda + wt(b_m) + \cdots + wt(b_1)\rangle$, $i = 1, \dots, n$ $\{b \in B(m\epsilon_1): b$ (27)$\}$ $\{\beta \in \mathbb{Z}^n:\ \beta \succeq \lambda, \sum_i(\beta_i - \lambda_i) = m\}$. S P ' ( ):

**Proposition 9.2.** *Let $\lambda, \gamma \in \mathbb{Z}^n$ such that $\lambda_1 \ge \cdots \ge \lambda_n$ and $\gamma = (m, 0, \dots, 0)$, $m \in \mathbb{N}$. Then*

$$V_\lambda \otimes V_\gamma = \bigoplus_\beta V_\beta,$$

*where the sum is over the integral dominant weights such that $\beta \succeq \lambda$, and $m = \sum_{i=1}^n (\beta_i - \lambda_i)$.*




F . 8. I $V_\lambda \otimes V_{m\epsilon_1}$ $A_2$.

9.1.2. *Tensor product of representations for the type $C_n$*

T $h_i = \epsilon_i - \epsilon_{i+1}$, $1 \le i \le n-1$, $h_n = \epsilon_n$, $V_{\epsilon_1}$

$$B(\epsilon_1): 1 \xrightarrow{1} \cdots \xrightarrow{n-1} n \xrightarrow{n} \bar{n} \xrightarrow{n-1} \cdots \xrightarrow{1} \bar{1}.$$

H $i$ $\bar{i}$ $\epsilon_i$ $-\epsilon_i$. W $\le$ $B(\epsilon_1)$ $1 \le \cdots \le n \le \bar{n} \le \cdots \le \bar{1}$. B T 4.5.1 36 , $m \in \mathbb{N}$,

$$B(m\epsilon_1) = \{b_m \otimes \cdots \otimes b_1 \in B(\epsilon_1)^{\otimes m}: \ b_{k+1} \ge b_k\}.$$

L $\lambda$ . A (26) . T , P 9.1, $b_m \otimes \cdots \otimes b_1 \in B(\epsilon_1)^{\otimes m}$ $1 \le k \le m$, $1 \le i \le n$,

$$\begin{cases} b_{k+1} \ge b_k, \\ 0 \le \langle h_i, \lambda + wt(b_m) + \cdots + wt(b_k)\rangle. \end{cases} \tag{28}$$

T $b_m \otimes \cdots \otimes b_1 \mapsto (\beta, c) \in \mathbb{N}^n \times \mathbb{N}^n$, $i = 1, \ldots, n$

$$\beta_i = \langle \epsilon_i, \lambda + wt(b_m) + \cdots + wt(b_1)\rangle$$

$$c_i = \quad \{\langle \epsilon_i, \lambda + wt(b_m) + \cdots + wt(b_k)\rangle, 1 \le k \le m\},$$

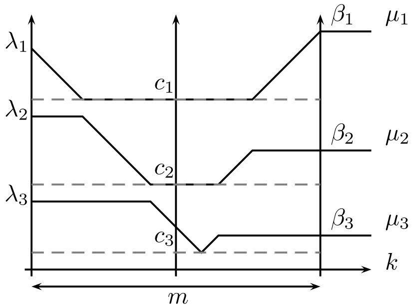
$\{b \in B(m\Lambda_1): \ b$ (28)$\}$ $\{(\beta, c) \in \mathbb{N}^n \times \mathbb{N}^n: \ \lambda \succeq c, \beta \succeq c, \sum_i (\lambda_i - c_i + \beta_i - c_i) = m\}$. L F . 9 . T $i$ ( . ) $(i-1)$ ( . $(i+1)^{th}$) $b_m \otimes \cdots \otimes b_1$ $B(a\epsilon_1)$. M , $0 \le \langle h_i, \lambda + wt(b_k) + \cdots + wt(b_m)\rangle$. S .

**Proposition 9.3.** *Let $\lambda, \gamma \in \mathbb{N}^n$ be such that $\lambda_1 \ge \cdots \ge \lambda_n$, and $\gamma = (m, 0, \ldots, 0)$, $m \in \mathbb{N}$. Then*

$$V_\lambda \otimes V_\gamma = \bigoplus_\beta M_{\lambda,\gamma}(\beta) V_\beta,$$

*where the sum is over all $\beta \in \mathbb{N}^n$ satisfying $\beta_1 \ge \cdots \ge \beta_n$ such that there exists $c = (c_1, \ldots, c_n) \in \mathbb{N}^n$ which verifies $\lambda \succeq c$, $\beta \succeq c$ and $\sum_{i=1}^n (\lambda_i - c_i + \beta_i - c_i) = m$. In addition, the multiplicity $M_{\lambda,\gamma}(\beta)$ of the irreducible module with highest weight $\beta$ is the number of $c \in \mathbb{N}^n$ satisfying these relations.*

W F . 6: $R_i$ $R_{i+1}$ ( ) $\lambda$ $\mu$, $c$.




F . 9. I $V_\lambda \otimes V_{m\epsilon_1}$ $C_3$.

9.1.3. *Tensor product of representations for type $B_r$*
T $h_i = \epsilon_i - \epsilon_{i+1}, i = 1, \ldots, r-1, h_r = 2\epsilon_r,$ $V_{\epsilon_1}$

$$B(\epsilon_1): 1 \xrightarrow{1} \cdots \xrightarrow{r-1} r \xrightarrow{r} 0 \xrightarrow{r} \overline{r} \xrightarrow{r-1} \cdots \xrightarrow{1} \overline{1},$$

$i, \overline{i}$ 0 $\epsilon_i, -\epsilon_i$ 0 $i = 1, \ldots, r$W $B(\epsilon_1)$ $1 \le \cdots \le r \le 0 \le \overline{r} \le \cdots \le \overline{1}$. B T 5.7.1 36 ,

$$B(m\epsilon_1) = \{b_m \otimes \cdots \otimes b_1 \in B(\epsilon_1)^{\otimes m} : b_{k+1} \ge b_k, b_{k+1} \otimes b_k \ne 0 \otimes 0\}.$$

L $\lambda$ . A $C_n$, $V_\lambda \otimes V_{m\epsilon_1}$ $b_m \otimes \cdots \otimes b_1 \in B(m\epsilon_1)$ $\varepsilon_i(b_k) \le \langle h_i, \lambda + wt(b_{k+1}) + \cdots + wt(b_m)\rangle$ $1 \le k \le m, 1 \le i \le r$. L $b \in B(\epsilon_1)$. F $i \le r-1$, $\langle h_i, wt(b)\rangle = -1$ $b = i+1$ $b = \overline{i}$. M , $\langle h_r, wt(b)\rangle = -2$ $b = \overline{r}$. I $\langle h_i, wt(b)\rangle$ . T

$$\varepsilon_i(b) \le \langle h_i, \lambda\rangle \quad \Leftrightarrow \quad \begin{cases} (b \ne 0 & 0 \le \langle h_i, \lambda + wt(b)\rangle) \\ (b = 0 & \langle h_r, \lambda\rangle \ge 1). \end{cases}$$

S , , $b_m \otimes \cdots \otimes b_1 \in B(\epsilon_1)^{\otimes m}$ $(k, i) \in \{1, \ldots, m\} \times \{1, \ldots, r\}$

$$\begin{cases} b_{k+1} \ge b_k, b_{k+1} \otimes b_k \ne 0 \otimes 0, \\ 0 \le \langle h_i, \lambda + wt(b_m) + \cdots + wt(b_k)\rangle, \\ 1 \le \langle h_r, \lambda + wt(b_m) + \cdots + wt(b_k)\rangle \quad b_k = 0. \end{cases} \tag{29}$$

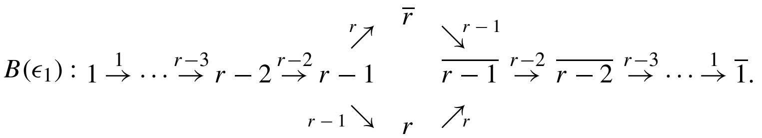
T .

**Proposition 9.4.** *Let $\lambda, \gamma \in \mathbb{N}^r$ be such that $\lambda_1 \ge \cdots \ge \lambda_r$ and $\gamma = (m, 0, \ldots, 0)$, $a \in \mathbb{N}$. Then*

$$V_\lambda \otimes V_\gamma = \bigoplus_\beta M_{\lambda,\gamma}(\beta) V_\beta,$$

*where the sum is over all $\beta \in \mathbb{N}^r$ such that $\beta_1 \ge \cdots \ge \beta_r$ such that there exists an integer $s \in \{0, 1\}$ and $c \in \mathbb{N}^r$ which verifies $\lambda \succeq c$, $\beta \succeq c$ and $\sum_{i=1}^r (\lambda_i - c_i + \beta_i - c_i) + s = m$, $s$ being equal to 0 if $c_r = 0$. In addition, the multiplicity $M_{\lambda,\gamma}(\beta)$ of the irreducible module with highest weight $\beta$ is the number of $(c, s) \in \mathbb{N}^r \times \{0, 1\}$ satisfying these relations.*

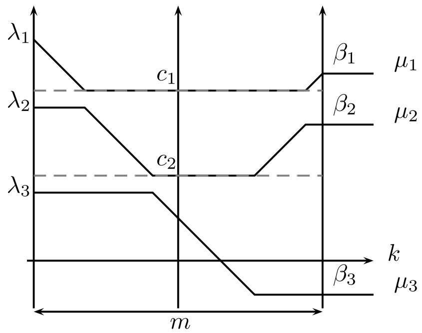
9.1.4. *Tensor product of representations for type $D_r$*
T $h_i = \epsilon_i - \epsilon_{i+1}, i = 1, \ldots, r-1,$ $h_r = \epsilon_r + \epsilon_{r-1},$ $V_{\epsilon_1}$



H $i$ $\overline{i}$ $\epsilon_i$ $-\epsilon_i$ , $i = 1, \ldots, r$W $\le$ $B(\epsilon_1)$ $1 \le \cdots \le r-1 \le {\overline{r} \atop r} \le \overline{r-1} \le \cdots \le \overline{1}$. F $m \in \mathbb{N}$ T 6.7.1 36 ,

$$B(m\epsilon_1) = \{b_m \otimes \cdots \otimes b_1 \in B(\epsilon_1)^{\otimes m} : b_{k+1} \le b_k\}.$$

L $\lambda$ $\langle \epsilon_r, \lambda\rangle \in \mathbb{N}$. F $b \in B(\epsilon_1)$, $A_{n-1}$ $C_n$ (26). S P 9.5, F . 10. W F . 7.




F . 10. I $V_\lambda \otimes V_{m\epsilon_1}$ $D_3$.

**Proposition 9.5.** *Let $\lambda, \gamma \in \mathbb{N}^r$ be such that $\lambda_1 \geq \cdots \geq |\lambda_r|$, and $\gamma = (m, 0, \ldots, 0)$, $m \in \mathbb{N}$. Then*

$$V_\lambda \otimes V_\gamma = \bigoplus_\beta M_{\lambda,\gamma}(\beta) V_\beta,$$

*where the sum is over all $\beta \in \mathbb{N}^r$ satisfying $\beta_1 \geq \cdots \geq \beta_r$ such that there exists $c \in \mathbb{N}^{r-1}$ which verify $\lambda \succeq c$, $\beta \succeq c$, $(|\lambda_r|, |\beta_r|) \leq c_{r-1}$ and $\sum_{k=1}^{r-1}(\lambda_k - c_k + \beta_k - c_k) + |\lambda_r - \mu_r| = m$. In addition, the multiplicity $M_{\lambda,\gamma}(\beta)$ of the irreducible module with highest weight $\beta$ is the number of $c \in \mathbb{N}^{r-1}$ satisfying these relations.*

9.2. *Classical restriction multiplicities*

F $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $K = U_n(\mathbb{F})$ $H = U_{n-1}(\mathbb{F})$, ( , , K 42 ). L . W $\mathbb{Z}$ G T $GT_n(\lambda)$ .

**Proposition 9.6.** *Let $\lambda$ be an integral point of $\mathcal{C}_n$. Let $V_\lambda$ be an irreducible module with highest weight $\lambda$. The irreducible decomposition (22) when $K = U_n(\mathbb{F})$ and $H = U_{n-1}(\mathbb{F})$ is the following one:*

$$V_\lambda = \bigoplus_\beta m^\lambda_{U_{n-1}}(\beta) V^{U_{n-1}}_\beta,$$

*where the sum is over all $\beta$ such that there exists $x \in GT_{n,\mathbb{Z}}(\lambda)$ such that $x^{(n-1)} = \beta$. Moreover, for $\mathbb{F} = \mathbb{C}, \mathbb{R}$, $m^\lambda_{U_{n-1}}(\beta) = 1$ and for $\mathbb{F} = \mathbb{H}$, $m^\lambda_{U_{n-1}}(\beta)$ is the number of $c \in \mathbb{N}^n$ for which there exists $x \in GT_{n,\mathbb{Z}}(\lambda)$ with $x^{(n-1)} = \beta$ and $x^{(n-1/2)} = c$.*

## 10. Asymptotic multiplicities and limit measures

I , L 3.3 P 4.3.

10.1. *Proof of Lemma* 3.3

W P 9.6 $K = U_n(\mathbb{F})$ $H = U_{n-1}(\mathbb{F})$. L $U_n(\mathbb{F}) \supset \cdots \supset U_1(\mathbb{F})$ . I G T $GT_n(\lambda)$ $\lambda$ $\mathcal{C}_n$, $GT_n(\lambda)$ $\mathfrak{U}_n(\mathbb{F})$- $\lambda$. A G T 27 . T (20) . L $\epsilon$ 1 $n \notin 2\mathbb{N}$ 0 .

**Lemma 10.1.** *Let $\lambda$ be an integer point in $\mathcal{C}_n$. The number of points in $GT_{n,\mathbb{Z}}(\lambda)$, denoted* C $GT_{n,\mathbb{Z}}(\lambda)$, *is equal to*:

- *when* $\mathbb{F}=\mathbb{C}$,

$$\prod_{1\le i<j\le n}\frac{\lambda_i-\lambda_j+j-i}{j-i},$$

- *when* $\mathbb{F}=\mathbb{H}$,

$$\prod_{1\le i<j\le n}\frac{(\lambda_i-\lambda_j+j-i)(\lambda_i+\lambda_j+2n+2-j-i)}{(j-i)(2n+2-j-i)}\prod_{i=1}^{n}\frac{\lambda_i+n+1-i}{n+1-i},$$

- *when* $\mathbb{F}=\mathbb{R}$,

$$\prod_{1\le i<j\le \tilde{n}}\frac{(\lambda_i-\lambda_j+j-i)(\lambda_i+\lambda_j+2\tilde{n}+\epsilon-j-i)}{(j-i)(2\tilde{n}+\epsilon-j-i)}\prod_{i=1}^{\tilde{n}}\left[\frac{\lambda_i+\tilde{n}+1/2-i}{\tilde{n}+1/2-i}\right]^{\epsilon}.$$

B L 3.3 . R $\mu_\lambda$ $GT_n(\lambda)$ $x\in GT_n(\lambda)\mapsto x^{(n-1)}$, $(\varepsilon_k)_{k\ge 1}$ 0.

**Lemma 10.2.** *Let $\lambda$ be in the Weyl chamber $\mathcal{C}_n$. Let us consider a sequence $(\lambda_k)_{k\ge1}$ of integer points in $\mathcal{C}_n$ such that $\varepsilon_k\lambda_k$ converges to $\lambda$, as $k$ goes to infinity, and the associated sequence of measures $(\mu_k)_{k\ge0}$ defined as in Theorem 7.3 for $K=U_n(\mathbb{F})$ and $H=U_{n-1}(\mathbb{F})$. Then $(\mu_k)_{k\ge1}$ converges to the measure $\mu_\lambda$.*

**Proof.** W $m^{\lambda_k}_{U_{n-1}}(\beta)$ P 9.6. S $\mathfrak{U}_n(\mathbb{F})$- $\gamma$ $GT_n(\gamma)$, $\mu_k$

$$\sum_\beta\frac{\text{C}\quad GT_{n-1,\mathbb{Z}}(\beta)}{\text{C}\quad GT_{n,\mathbb{Z}}(\lambda_k)}m^{\lambda_k}_{U_{n-1}}(\beta)\delta_{\varepsilon_k\beta}. \tag{30}$$

C G T P 9.6, $\mu_k$ $x\in GT_n(\varepsilon_k\lambda_k)\mapsto x^{(n-1)}$ $\frac{1}{(\lambda_k)}\sum_{x\in GT_{n,\mathbb{Z}}(\lambda_k)}\delta_{\varepsilon_k x}$. T $GT_n(\lambda)$. T $\mu_k$ $\mu_\lambda$. □

**Proof of Lemma 3.3.** L $\lambda\in\mathcal{C}_n$ $U\in U_n(\mathbb{F})$ H . W $(\mu_k)_{k\ge1}$ $\mathcal{C}_n$ L 10.2 $(\mu_k)_{k\ge1}$ $n-1$ $U\Omega_n(\lambda)U^*$. L 10.2 $\mu_\lambda$. □

10.2. *Proof of Proposition 4.3*

T $\mathcal{E}(\lambda,\theta)$ S 4 S 9. I , $\mathcal{E}_{\mathbb{Z}}(\lambda,\theta)$ $\mathcal{E}(\lambda,\theta)$ $\mathbb{Z}$. I $\mathbb{F}=\mathbb{R}$ $n=2r+1$,

$$\mathcal{E}_{\mathbb{Z}}(\lambda,\theta)=\Bigg\{(\beta,z,x,s)\in\mathbb{N}^r\times\mathbb{N}^r\times GT_{n,\mathbb{Z}}\times\{0,1\}:\ \lambda,\beta\succeq z,$$
$$\sum_{i=1}^{r}(\lambda_i+\beta_i-2z_i)+s=\theta,x\in GT_n(\beta),s=0\quad z_r=0\Bigg\}.$$

**Lemma 10.3.** *Let $\lambda$ and $\gamma=(a,0,\dots,0)$ be integer points of $\mathcal{C}_n$. Then the number of points in $\mathcal{E}_{\mathbb{Z}}(\lambda,a)$ is equal to* $(\lambda)$ $(\gamma)$.

**Proof.** W $\beta$ $\mathcal{E}_{\mathbb{Z}}(\lambda, a)$ $V_\lambda \otimes V_\gamma$ P 9.2 9.5. R C $GT_{n,\mathbb{Z}}(\beta)$ $(V_\beta)$ C $\mathcal{E}_{\mathbb{Z}}(\lambda, a) = \sum_\beta$ $(V_\beta)$, $\beta$ ( ) $V_\lambda \otimes V_\gamma$. T C $\mathcal{E}_{\mathbb{Z}}(\lambda, a) =$ $(V_\gamma \otimes V_\gamma) =$ $(\lambda)$ $(\gamma)$. □

**Lemma 10.4.** *Let $\lambda$ be in the Weyl chamber $\mathcal{C}_n$ and $\theta > 0$. Let us consider two sequences $(\lambda_k)_{k\geq 1}$ and $(\gamma_k)_{k\geq 1}$ of integer points in $\mathcal{C}_n$ such that $\gamma_k$ can be written as $(a_k, 0, \ldots, 0)$. We suppose that $\varepsilon_k\lambda_k$ converges to $\lambda$ and $\varepsilon_k a_k$ converges to $\theta$, as $k$ goes to infinity. Then the associated sequence of measures $(\nu_k)_{k\geq 1}$ given in Corollary 7.4 for $K = U_n(\mathbb{F})$ converges to the measure $\nu_{\lambda,\theta}$.*

**Proof.** T $\nu_k$

$$\sum_\beta \frac{(\beta)}{(\lambda_k) \quad (\gamma_k)} M_{\lambda_k,\gamma_k}(\beta)\delta_{\varepsilon_k\beta},$$

$M_{\lambda_k,\gamma_k}(\beta)$ $\beta$ $V_{\lambda_k} \otimes V_{\gamma_k}$. T - S 9, P 9.2 9.5, $(\beta) =$ C $GT_{n,\mathbb{Z}}(\beta)$ $\nu_k$ $\beta$

$$\frac{1}{(\lambda_k) \quad (\gamma_k)} \sum_{x\in\mathcal{E}_{\mathbb{Z}}(\lambda_k,a_k)} \delta_{\varepsilon_k x},$$

. □

**Proof of Proposition 4.3.** L $\lambda \in \mathcal{C}_n$, $\theta \in \mathbb{R}_+^*$ $(\nu_k)_{k\geq 1}$ $\mathcal{C}_n$ L 10.4. T $(\nu_k)_{k\geq 1}$ $\Omega_n(\lambda) + U\Omega_n(\theta)U^*$ $U$ H . L 10.4 $\nu_{\lambda,\theta}$. □

## 11. Concluding remarks

### 11.1. *Random processes with values in $GT_n$*

L $M = (M_t)_{t\geq 0}$ B $\mathcal{P}_n(\mathbb{C})$. T $X(M) = (X(M_t))_{t\geq 0}$ M . F $\mathbb{F} = \mathbb{C}$, $X(M)$ M $n = 1$ $n = 2$. A B $\mathcal{P}_n(\mathbb{C})$ , , ( B 4 $\mathcal{P}_2(\mathbb{C})$). T $X(M)$ M $n \geq 3$, C .67 62 , M .

### 11.2. *Rank one perturbation on classical complex Lie groups*

K 41 $G$ $K$, $K$. H D W 19 . U B $G/K$ .

### 11.3. *Rank one perturbation on $U_n(\mathbb{F})$*

L . F , D S 17 , P 53,54 R 55 , $U_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, , H $U_n(\mathbb{F})$. T 19 .

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T B C R I , U
C C . T w T D , A M , N O'C , J W
P B .

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