Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2010, Vol. 46, No. 4, 1025–1041
DOI: 10.1214/09-AIHP338

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Continuous differentiability of renormalized intersection local times in R^1

Jay S. Rosen[1]

Department of Mathematics, College of Staten Island, CUNY, Staten Island, NY 10314. E-mail: jrosen30@optimum.net

Received 15 July 2009; revised 19 August 2009; accepted 1 September 2009

Abstract. We study $\gamma_k(x_2,\dots,x_k;t)$, the k-fold renormalized self-intersection local time for Brownian motion in R^1. Our main result says that $\gamma_k(x_2,\dots,x_k;t)$ is continuously differentiable in the spatial variables, with probability 1.

Résumé. Nous étudions $\gamma_k(x_2,\dots,x_k;t)$, le temps local renormalisé d'auto-intersection d'ordre k du mouvement brownien dans R^1. Notre résultat principal montre que $\gamma_k(x_2,\dots,x_k;t)$ est presque sûrement continûment différentiable dans les variables spatiales.

MSC: 60J55; 60J65

Keywords: Continuous differentiability; Intersection local time; Brownian motion

1. Introduction

The object of this paper is to establish the almost sure continuous differentiability of renormalized intersection local time for the multiple intersections of Brownian motion in R^1.

Intersection local times were originally envisioned as a means of "measuring" the amount of self-intersections of Brownian motion $W_t \in R^m$. Formally, the k-fold intersection local time is

$$\alpha_k(x_2,x_3,\dots,x_k;t)=\int\cdots\int_{\{0\le t_1\le\cdots\le t_k\le t\}}\prod_{j=2}^k \delta(W_{t_j}-W_{t_{j-1}}-x_j)\,\mathrm{d}t_1\cdots\mathrm{d}t_k,$$

where $\delta(x)$ denotes the δ-function. Intuitively, $\alpha_k(0,0,\dots,0;t)$ measures the "amount" of k-fold intersections.

More precisely, we can set

$$\begin{aligned}&\alpha_{k,\varepsilon}(x_2,x_3,\dots,x_k;t)\\&\quad=\int\cdots\int_{\{0\le t_1\le\cdots\le t_k\le t\}}\prod_{j=2}^k f_\varepsilon(W_{t_j}-W_{t_{j-1}}-x_j)\,\mathrm{d}t_1\cdots\mathrm{d}t_k,\end{aligned}\tag{1.1}$$

where f_ε is an approximate δ-function, and try to take the $\varepsilon\to 0$ limit.

In two dimensions, $\lim_{\varepsilon\to 0}\alpha_{k,\varepsilon}(x_2,x_3,\dots,x_k;t)$ will not exist unless all $x_i\neq 0$! This gave rise to the problem of renormalization: to subtract from $\alpha_{k,\varepsilon}(x_2,x_3,\dots,x_k;t)$ terms involving lower order intersection local times,

[1] Supported in part by grants from PSC-CUNY and the NSF.

$\alpha_{j,\varepsilon}(x_2, x_3, \ldots, x_k; t)$ for $j < k$, in such a way that a finite and continuous, $\varepsilon \to 0$ limit results. This was originally done for double intersections of Brownian motion by Varadhan [14], and gave rise to a large literature, see Bass and Khoshnevisan [1], Dynkin [2], Le Gall [3–5] and Rosen [7,8,10–12].

In this paper we are concerned with one dimensional Brownian motion. In this case, as we show below, the limit

$$\alpha_k(x_2, x_3, \ldots, x_k; t) = \lim_{\varepsilon \to 0} \alpha_{k,\varepsilon}(x_2, x_3, \ldots, x_k; t) \tag{1.2}$$

exists a.s. Although, as we will see, $\alpha_k(x_2, x_3, \ldots, x_k; t)$ is almost surely continuous, it is not C^1 in the spatial variable. It is here that renormalization enters in the one dimensional case.

Let

$$g(x) = \int_0^\infty \mathrm{e}^{-t/2} p_t(x)\,\mathrm{d}t = \mathrm{e}^{-|x|}, \tag{1.3}$$

where $p_t(x)$ is the Brownian density function. We define the renormalized k-fold intersection local time for $x = (x_2, \ldots, x_k) \in R^{k-1}$ by

$$\gamma_k(x; t) = \sum_{A \subseteq \{2,\ldots,k\}} (-1)^{|A|} \left(\prod_{j \in A} g(x_j) \right) \alpha_{k-|A|}(x_{A^c}; t), \tag{1.4}$$

where for any $B = \{i_1 < \cdots < i_{|B|}\} \subseteq \{2, \ldots, k\}$

$$x_B = (x_{i_1}, x_{i_2}, \ldots, x_{i_{|B|}}). \tag{1.5}$$

Here, we use the convention $\alpha_1(t) = t$. Simple combinatorics then show that

$$\alpha_k(x; t) = \sum_{A \subseteq \{2,\ldots,k\}} \left(\prod_{j \in A} g(x_j) \right) \gamma_{k-|A|}(x_{A^c}; t). \tag{1.6}$$

The renormalization (1.4) used here is similar to that used in [11] and [1] for two dimensional Brownian motion, but in that case $g(x) = \int_0^\infty \mathrm{e}^{-t/2} p_t(x)\,\mathrm{d}t$ is infinite when $x = 0$, compare (1.3). One key result of those papers is that $\gamma_k(x; t)$ has a continuous extension from $(R^2 - \{0\})^{k-1} \times R_+$ to $(R^2)^{k-1} \times R_+$.

Here is our main result.

Theorem 1. *For Brownian motion in* R^1

$$\alpha_k(x; t) = \lim_{\varepsilon \to 0} \alpha_{k,\varepsilon}(x; t) \tag{1.7}$$

exists and is jointly continuous a.s. Furthermore, $\gamma_k(x; t)$ *is differentiable in* x *and* $\nabla_x \gamma_k(x; t)$ *is jointly continuous with probability* 1.

For $k = 2$, this was established in [9] by very different techniques. It seems impossible to use those techniques for $k > 3$.

In [13] we use Theorem 1 to give a simple proof of the CLT for the L^2 modulus of continuity of local time.

Note that $g(x)$ is continuously differentiable for $x \neq 0$. Equation (1.6) then exhibits precisely the non-differentiability of $\alpha_k(x; t)$. This justifies our choice of renormalization (1.4). Simple combinatorics show that we obtain a similar result if we add any C^1 function to $g(x)$. We have chosen $g(x)$ as a potential density to simplify our proofs.

Our paper is organized along the lines of [11]. That paper was concerned with the continuity of $\gamma_k(x_2, x_3, \ldots, x_k; t)$ in two dimensions, for Brownian motion and stable processes. Our challenge here is to study differentiability, and for ease of exposition we consider only Brownian motion. After laying the groundwork in Section 2, we establish the existence and almost sure continuity of $\alpha_k(x_2, x_3, \ldots, x_k; \zeta)$ in Section 3, where ζ is an independent mean-2 exponential random variable. In Section 4 we show that the renormalized k-fold intersection local time $\gamma_k(x_2, x_3, \ldots, x_k; \zeta)$ is almost surely differentiable with a continuous derivative, again at an independent exponential time. In Section 5 we use martingale techniques to obtain a.s. joint continuty.

2. Intersection local times: moments

Let W_t denote Brownian motion in R^1 with transition densites $p_t(x)$. In this section we introduce approximate intersection local times for W_t and present formulas for the expectations of their moments.

Let f denote a smooth positive function supported on $[-1, 1]$, such that $\int f(x)\,\mathrm{d}x = 1$, and for any $\varepsilon > 0$ let

$$f_\varepsilon(y) = \frac{1}{\varepsilon} f\left(\frac{y}{\varepsilon}\right)$$

and $f_{\varepsilon,x}(y) = f_\varepsilon(y - x)$. We define the approximate intersection local time of order k as

$$\begin{aligned} &\alpha_{k,\varepsilon}(x_2, x_3, \ldots, x_k; t) \\ &\quad = \int \cdots \int_{\{0\le t_1\le\cdots\le t_k\le t\}} \prod_{j=2}^{k} f_{\varepsilon,x_j}(W_{t_j} - W_{t_{j-1}})\,\mathrm{d}t_1 \cdots \mathrm{d}t_k. \end{aligned} \tag{2.1}$$

We often abbreviate this as $\alpha_{k,\varepsilon}(x;t)$ where $x = (x_2, x_3, \ldots, x_k) \in R^{k-1}$. Let $g(x) = \int_0^\infty \mathrm{e}^{-t/2} p_t(x)\,\mathrm{d}t = \mathrm{e}^{-|x|}$ denote the Green's function for W_t, and let ζ denote a mean-2 exponential random variable independent of W_t. The following follows appear in [11], Section 2, and are reproduced here for the convenience of the reader. The first formula follows easily from the Markov property for W_t.

Lemma 1.

$$\begin{aligned} &E\left(\prod_{i=1}^{n} \alpha_{k_i,\varepsilon_i}(x^i;\zeta)\right) \\ &\quad = \sum_{v\in\mathcal{V}} \int \prod_{i=1}^{n}\prod_{j=2}^{k_i} f_{\varepsilon_i,x^i_j}(y^i_j - y^i_{j-1}) \prod_{p=1}^{k} g(w_{v(p)} - w_{v(p-1)})\,\mathrm{d}w_1 \cdots \mathrm{d}w_k, \end{aligned} \tag{2.2}$$

where $x^i = (x^i_2, x^i_3, \ldots, x^i_{k_i})$, $k = \sum_{i=1}^n k_i$, $(w_1, \ldots, w_k) = (y^1, \ldots, y^n) \in (\mathbf{R}^1)^k$ *and* $\mathcal{V}$ *is the set of permutations* v *of* $\{1, 2, \ldots, k\}$ *such that whenever* $w_{v(p)} = y^i_j$, $w_{v(\tilde p)} = y^i_{\tilde j}$ *we have* $p > \tilde p \Longleftrightarrow j > \tilde j$.

A change of variables leads to the following more useful formula.

Lemma 2.

$$\begin{aligned} &E\left(\prod_{i=1}^{n} \alpha_{k_i,\varepsilon_i}(x^i;\zeta)\right) \\ &\quad = \sum_{s\in\mathcal{S}} \int \prod_{i=1}^{n}\prod_{j=2}^{k_i} f(y^i_j) \prod_{p=1}^{k} g\left(z_{s(p)} + \sum_{j=2}^{c(p)}\left(\varepsilon_{s(p)} y^{s(p)}_j + x^{s(p)}_j\right)\right. \\ &\qquad \left. - \left(z_{s(p-1)} + \sum_{j=2}^{c(p-1)}\left(\varepsilon_{s(p-1)} y^{s(p-1)}_j + x^{s(p-1)}_j\right)\right)\right)\mathrm{d}y^i_j\,\mathrm{d}z_1 \cdots \mathrm{d}z_n, \end{aligned} \tag{2.3}$$

where $x^i = (x^i_2, x^i_3, \ldots, x^i_{k_i})$, $k = \sum_{i=1}^n k_i$, $\mathcal{S}$ *is the set of mappings* $s : \{1, 2, \ldots, k\} \mapsto \{1, \ldots, n\}$ *such that* $|s^{-1}(i)| = k_i, \forall 1 \le i \le n$, *and* $c(p) = |\{u \le p \mid s(u) = s(p)\}|$.

3. Intersection local times: existence and continuity at exponential times

We first consider the intersection local time α_k at an independent mean-2 exponential time ζ.

A function $Z_\varepsilon(x)$ indexed by $\varepsilon \in (0,1]$ and x in a topological space $\mathcal{S}$ will be said to converge locally uniformly in x as $\varepsilon \to 0$ if for any compact $K \in \mathcal{S}$, $Z_\varepsilon(x)$ converges uniformly in $x \in K$ as $\varepsilon \to 0$.

Theorem 2. *Almost surely, $\alpha_{k,\varepsilon}(x;\zeta)$ converges locally uniformly in x as $\varepsilon \to 0$. Hence*

$$\alpha_k(x;\zeta) := \lim_{\varepsilon\to 0} \alpha_{k,\varepsilon}(x;\zeta) \tag{3.1}$$

is continuous.

Furthermore, the occupation density formula holds:

$$\int \Phi(x_2,\dots,x_k)\alpha_k(x_2,\dots,x_k;\zeta)\,\mathrm{d}x_2\cdots\mathrm{d}x_k = \int\cdots\int_{\{0\le t_1\le\cdots\le t_k\le\zeta\}} \Phi(W_{t_2}-W_{t_1},\dots,W_{t_k}-W_{t_{k-1}})\,\mathrm{d}t_1\cdots\mathrm{d}t_k \tag{3.2}$$

for all bounded Borel measurable functions Φ on R^{k-1}.

Remark 1. *The occupation density formula (3.2) shows that $\alpha_k(x;\zeta)$ is independent of the particular f used to define $\alpha_{k,\varepsilon}(x;\zeta)$.*

Proof of Theorem 2. We will show that for n even and $\gamma > 0$ we can find $\delta > 0$ such that

$$E\big(\{\alpha_{k,\varepsilon}(x;\zeta) - \alpha_{k,\varepsilon'}(x';\zeta)\}^n\big) \le c_{n,\gamma}\big|(\varepsilon,x)-(\varepsilon',x')\big|^{\delta n/2} \tag{3.3}$$

for all $0 < \varepsilon, \varepsilon' \le \gamma/2$ and all $x, x' \in R^{k-1}$. The multidimensional version of Kolmogorov's lemma, [6], Chapter 1, Theorem 2.1, then gives us that for any $\delta' < \delta$ and any $M < \infty$ we have

$$\big|\alpha_{k,\varepsilon}(x;\zeta) - \alpha_{k,\varepsilon'}(x';\zeta)\big| \le c_{n,\gamma}(\omega)\big|(\varepsilon,x)-(\varepsilon',x')\big|^{\delta'/2} \tag{3.4}$$

for all rational $0 < \varepsilon, \varepsilon' \le \gamma/2$ and all rational $x, x' \in R^{k-1}$, $|x|, |x'| \le M$. Since $\alpha_{k,\varepsilon}(x;\zeta)$ is clearly continuous as long as $\varepsilon > 0$, this will establish the statements concerning (3.1).

To establish (3.3) we first handle the variation in ε. If h is a function of ε, let

$$\Delta_{\varepsilon,\varepsilon'}h = h(\varepsilon) - h(\varepsilon').$$

From Lemma 2 we have

$$\begin{aligned} &E\left(\prod_{i=1}^n \{\alpha_{k,\varepsilon_i}(x^i;\zeta) - \alpha_{k,\varepsilon_i'}(x^i;\zeta)\}^n\right) \\ &\quad= \prod_{i=1}^n \Delta_{\varepsilon_i,\varepsilon_i'} E\left(\prod_{i=1}^n \alpha_{k,\varepsilon_i}(x^i;\zeta)\right) \\ &\quad= \prod_{i=1}^n \Delta_{\varepsilon_i,\varepsilon_i'} \sum_{s\in\mathcal{S}} \int \prod_{i=1}^n\prod_{j=2}^k f(y_j^i) \prod_{p=1}^{nk} g\left(z_{s(p)} + \sum_{j=2}^{c(p)}(\varepsilon_{s(p)}y_j^{s(p)} + x_j^{s(p)})\right. \\ &\qquad \left. - \left(z_{s(p-1)} + \sum_{j=2}^{c(p-1)} (\varepsilon_{s(p-1)}y_j^{s(p-1)} + x_j^{s(p-1)})\right)\right)\mathrm{d}y_j^i\,\mathrm{d}z_1\cdots\mathrm{d}z_n, \end{aligned} \tag{3.5}$$

where we eventually set all $(\varepsilon_i, \varepsilon_i') = (\varepsilon, \varepsilon')$. We expand this as a sum of many terms using

$$\Delta_{\varepsilon,\varepsilon'}(uv) = (\Delta_{\varepsilon,\varepsilon'}u)v(\varepsilon) + u\big(\varepsilon'\big)(\Delta_{\varepsilon,\varepsilon'}v) \tag{3.6}$$

so that each term contains for each $1 \le i \le n$ a single difference of the form $\Delta_{\varepsilon_i,\varepsilon_i'}g$. Since each g factor in (3.5) involves at most two i's, whenever our procedure gives two differences involving the same g factor we write one of the differences as two terms. The upshot is that after setting all $(\varepsilon_i, \varepsilon_i') = (\varepsilon, \varepsilon')$, the expectation $E(\{\alpha_{k,\varepsilon}(x;\zeta) - \alpha_{k,\varepsilon'}(x;\zeta)\}^n)$ can be written as a sum of many terms of the form appearing in (2.3) except that at least $n/2$ of the g factors have been replaced by factors of the form

$$\Delta_{\varepsilon,\varepsilon',j}\, g\left(z_{s(p)} + \sum_{j=2}^{c(p)}\big(\tilde{\varepsilon}_{s(p)}y_j^{s(p)} + x_j^{s(p)}\big) - \left(z_{s(p-1)} + \sum_{j=2}^{c(p-1)}\big(\tilde{\varepsilon}_{s(p-1)}y_j^{s(p-1)} + x_j^{s(p-1)}\big)\right)\right), \tag{3.7}$$

where $\tilde{\varepsilon}$ can be variously $\varepsilon, \varepsilon'$ and the notation $\Delta_{\varepsilon,\varepsilon',j}$ denotes a difference between two g factors of the above form in which one of the ε's, multiplying $y_j^{s(p)}$ or $y_j^{s(p-1)}$ has been replaced by ε'.

Our result then follows using

$$\big|g(x) - g(y)\big| = \big|\mathrm{e}^{-|x|} - \mathrm{e}^{-|y|}\big| \le |x - y|\big(g(x) + g(y)\big) \tag{3.8}$$

for the variation in ε, and the variation in x is handled similarly. This completes the proof of (3.1).

To prove the occupation density formula (3.2) we note that

$$\begin{aligned}&\int \Phi(x_2,\ldots,x_k)\alpha_{k,\varepsilon}(x_2,\ldots,x_k;\zeta)\,\mathrm{d}x_2\cdots\mathrm{d}x_k\\ &\quad= \int\cdots\int_{\{0\le t_1\le\cdots\le t_k\le\zeta\}} \Phi * F_\varepsilon(W_{t_2} - W_{t_1},\ldots,W_{t_k} - W_{t_{k-1}})\,\mathrm{d}t_1\cdots\mathrm{d}t_k,\end{aligned} \tag{3.9}$$

where $F_\varepsilon(x_2,\ldots,x_k) = \prod_{j=2}^k f_\varepsilon(x_j)$. Hence, by what we have established above, we can take the $\varepsilon \to 0$ limit in (3.9) to yield (3.2) whenever Φ is a bounded continuous function. The monotone convergence theorem then allows us to obtain (3.2) for all bounded Borel measurable Φ. This completes the proof of Theorem 2. □

4. Renormalized intersection local times: continuous differentiability at exponential times

We have defined the renormalized k-fold intersection local time for $x = (x_2,\ldots,x_k) \in R^{k-1}$ by

$$\gamma_k(x;t) = \sum_{A\subseteq\{2,\ldots,k\}}(-1)^{|A|}\left(\prod_{j\in A} g(x_j)\right)\alpha_{k-|A|}(x_{A^c};t), \tag{4.1}$$

where for any $B = \{i_1 < \cdots < i_{|B|}\} \subseteq \{2,\ldots,k\}$

$$x_B = (x_{i_1}, x_{i_2},\ldots,x_{i_{|B|}}). \tag{4.2}$$

Here, we use the convention $\alpha_1(t) = t$. Define the approximate renormalized k-fold intersection local time for $x = (x_2,\ldots,x_k) \in R^{k-1}$ by

$$\gamma_{k,\varepsilon}(x;t) = \sum_{A\subseteq\{2,\ldots,k\}}(-1)^{|A|}\left(\prod_{j\in A} g_\varepsilon(x_j)\right)\alpha_{k-|A|,\varepsilon}(x_{A^c};t), \tag{4.3}$$

where $g_\varepsilon(x) = f_\varepsilon * g(x)$. Clearly $\gamma_{k,\varepsilon}(x;\zeta) \in C^1$ for fixed $\varepsilon > 0$ so that for any y_l, z_l

$$\begin{aligned}&\gamma_{k,\varepsilon}(x_2,\ldots,x_{l-1},z_l,x_{l+1},\ldots,x_k;\zeta)\\ &\quad - \gamma_{k,\varepsilon}(x_2,\ldots,x_{l-1},y_l,x_{l+1},\ldots,x_k;\zeta) = \int_{y_l}^{z_l}\frac{\partial}{\partial x_l}\gamma_{k,\varepsilon}(x;\zeta)\,\mathrm{d}x_l.\end{aligned} \tag{4.4}$$

By Theorem 2 and the continuity of g it follows that almost surely $\gamma_{k,\varepsilon}(x;\zeta) \to \gamma_k(x;\zeta)$ as $\varepsilon \to 0$, locally uniformly in x. It follows from the next theorem that $\frac{\partial}{\partial x_l}\gamma_{k,\varepsilon}(x;\zeta) \to \psi_k(x;\zeta)$ as $\varepsilon \to 0$, locally uniformly in x, for some continuous $\psi_k(x)$. Hence it follows from (4.4) that for any y_l, z_l

$$\begin{aligned}&\gamma_k(x_2,\dots,x_{l-1},z_l,x_{l+1},\dots,x_k;\zeta)\\&\quad-\gamma_k(x_2,\dots,x_{l-1},y_l,x_{l+1},\dots,x_k;\zeta)=\int_{y_l}^{z_l}\psi_k(x;\zeta)\,\mathrm{d}x_l.\end{aligned} \tag{4.5}$$

This implies that $\gamma_k(x;\zeta)$ is differentiable with respect to x_l, and $\frac{\partial}{\partial x_l}\gamma_k(x;\zeta) = \psi_k(x;\zeta)$, hence is continuous in x.

Theorem 3. *Almost surely, for each* $2 \le l \le k$

$$\frac{\partial}{\partial x_l}\gamma_{k,\varepsilon}(x;\zeta) \tag{4.6}$$

converges locally uniformly as $\varepsilon \to 0$. *Hence the limit is continuous in* x.

Proof. As in the proof of Theorem 2 it suffices to show that for n even and $\gamma > 0$ we can find $\delta > 0$ such that

$$E\left(\left\{\frac{\partial}{\partial x_l}\gamma_{k,\varepsilon}(x;\zeta)-\frac{\partial}{\partial x_l}\gamma_{k,\varepsilon'}(x';\zeta)\right\}^n\right)\le c_{n,\gamma}\left|(\varepsilon,x)-\left(\varepsilon',x'\right)\right|^{\delta n/2} \tag{4.7}$$

for all $0 < \varepsilon, \varepsilon' \le \gamma/2$ and all $x, x' \in R^{k-1}$.

Note that

$$\begin{aligned}&\frac{\partial}{\partial x_l}\alpha_{k,\varepsilon}(x_2,x_3,\dots,x_k;t)\\&\quad=-\int\cdots\int_{\{0\le t_1\le\cdots\le t_k\le t\}}\prod_{j=2}^{l-1}f_\varepsilon(W_{t_j}-W_{t_{j-1}}-x_j)\\&\qquad\times(f_\varepsilon)'(W_{t_l}-W_{t_{l-1}}-x_l)\prod_{j=l+1}^{k}f_\varepsilon(W_{t_j}-W_{t_{j-1}}-x_j)\,\mathrm{d}t_1\cdots\mathrm{d}t_k.\end{aligned} \tag{4.8}$$

Note also that since $g(x)$ is differentiable for all $x \neq 0$, with $g'(x) = -g(x)$ for $x > 0$ and $g'(x) = g(x)$ for $x < 0$, it follows that for any compactly supported $f \in C^1(R^1)$

$$\begin{aligned}&\int f'(x)g(x)\,\mathrm{d}x\\&\quad=\lim_{\varepsilon\to0}\left(\int_{-\infty}^{-\varepsilon}f'(x)g(x)\,\mathrm{d}x+\int_{-\varepsilon}^{\varepsilon}f'(x)g(x)\,\mathrm{d}x+\int_{\varepsilon}^{\infty}f'(x)g(x)\,\mathrm{d}x\right)\\&\quad=\lim_{\varepsilon\to0}\left(f(-\varepsilon)g(-\varepsilon)-\int_{-\infty}^{-\varepsilon}f(x)g(x)\,\mathrm{d}x\right)+\lim_{\varepsilon\to0}\int_{-\varepsilon}^{\varepsilon}f'(x)g(x)\,\mathrm{d}x\\&\qquad+\lim_{\varepsilon\to0}\left(-f(\varepsilon)g(\varepsilon)+\int_{\varepsilon}^{\infty}f(x)g(x)\,\mathrm{d}x\right)\\&\quad=-\int_{-\infty}^{0}f(x)g(x)\,\mathrm{d}x+\int_{0}^{\infty}f(x)g(x)\,\mathrm{d}x\\&\quad=-\int_{-\infty}^{\infty}f(x)g'(x)\,\mathrm{d}x,\end{aligned} \tag{4.9}$$

where for definiteness we set $g'(0)=0$. Hence as in (2.3), and using the product rule for differentiation

$$
\begin{aligned}
&E\left(\prod_{i=1}^{n}\frac{\partial}{\partial x_l^i}\alpha_{k_i,\varepsilon_i}\left(x^i;\zeta\right)\right)\\
&\quad=(-1)^n\sum_{s\in\mathcal{S}}\int\prod_{i=1}^{n}\left(\prod_{j=2,j\neq l}^{k_i}f_{\varepsilon_i}\left(y_j^i\right)\right)(f_{\varepsilon_l})'\left(y_l^i\right)\\
&\qquad\times\prod_{p=1}^{k}g\left(z_{s(p)}+\sum_{j=2}^{c(p)}\left(y_j^{s(p)}+x_j^{s(p)}\right)-\left(z_{s(p-1)}+\sum_{j=2}^{c(p-1)}\left(y_j^{s(p-1)}+x_j^{s(p-1)}\right)\right)\right)\mathrm{d}y_j^i\,\mathrm{d}z_1\cdots\mathrm{d}z_n\\
&\quad=\sum_{s\in\mathcal{S},a\in\mathcal{A}}(-1)^{\bar{a}_2}\int\prod_{i=1}^{n}\prod_{j=2}^{k_i}f_{\varepsilon_i}(y_j^i)\\
&\qquad\times\prod_{p=1}^{nk}g^{(a_1(p)+a_2(p))}\left(z_{s(p)}+\sum_{j=2}^{c(p)}\left(y_j^{s(p)}+x_j^{s(p)}\right)\right.\\
&\qquad\left.-\left(z_{s(p-1)}+\sum_{j=2}^{c(p-1)}\left(y_j^{s(p-1)}+x_j^{s(p-1)}\right)\right)\right)\mathrm{d}y_j^i\,\mathrm{d}z_1\ldots\mathrm{d}z_n,
\end{aligned}\tag{4.10}
$$

where $g^{(0)}=g$, $g^{(1)}=g'$, $g^{(2)}=g''$, and $\mathcal{A}$ is the set of maps $a=(a_1,a_2):[1,\ldots,kn]\mapsto\{0,1\}\times\{0,1\}$ such that:

- $\sum_{p=1}^{nk}a_1(p)+a_2(p)=n$,
- for each $1\le i\le n$

$$\sum_{p=1}^{nk}a_1(p)1_{\{s(p)=i\}}+a_2(p)1_{\{s(p-1)=i\}}=1,$$

- if $a_1(p)=1$, then $c(p)\ge l$,
- if $a_2(p)=1$, then $c(p-1)\ge l$.

In other words, if $s(p)=i$ then $a_1(p)=1$ if and only if, after using the product rule for differentiation, the pth g factor is the only g factor to which $\frac{\partial}{\partial y_l^i}$ has been applied. Similarly, if $s(p-1)=i$ then $a_2(p)=1$ if and only if, after using the product rule for differentiation, the pth g factor is the only g factor to which $\frac{\partial}{\partial y_l^i}$ has been applied. In (4.10), $\bar{a}_2=\sum_{p=1}^{nk}a_2(p)$.

Then by scaling

$$
\begin{aligned}
&E\left(\prod_{i=1}^{n}\frac{\partial}{\partial x_l^i}\alpha_{k_i,\varepsilon_i}\left(x^i;\zeta\right)\right)\\
&\quad=\sum_{s\in\mathcal{S},a\in\mathcal{A}}(-1)^{\bar{a}_2}\int\prod_{i=1}^{n}\prod_{j=2}^{k_i}f\left(y_j^i\right)\\
&\qquad\times\prod_{p=1}^{k}g^{(a_1(p)+a_2(p))}\left(z_{s(p)}+\sum_{j=2}^{c(p)}\left(\varepsilon_{s(p)}y_j^{s(p)}+x_j^{s(p)}\right)\right.\\
&\qquad\left.-\left(z_{s(p-1)}+\sum_{j=2}^{c(p-1)}\left(\varepsilon_{s(p-1)}y_j^{s(p-1)}+x_j^{s(p-1)}\right)\right)\right)\mathrm{d}y_j^i\,\mathrm{d}z_1\cdots\mathrm{d}z_n,
\end{aligned}\tag{4.11}
$$

where we eventually set all $\varepsilon_i = \varepsilon$.

We note in particular that if p is a "bad integer," i.e. $s(p) = s(p-1)$, the $g^{(m)}$ factor in the above product has the form

$$g^{(m)}\big(\varepsilon_{s(p)} y^{s(p)}_{c(p)} + x^{s(p)}_{c(p)}\big) \tag{4.12}$$

and in this case $m = 0$ or 1.

Let us now analyze the changes which occur in (4.11) when we replace the factor $\frac{\partial}{\partial x^r_l}\alpha_{k_r,\varepsilon_r}(x^r;\zeta)$ by $\frac{\partial}{\partial x^r_l}\{(\prod_{j\in B} g_{\varepsilon_r}(x^r_j))\alpha_{k_r-|B|,\varepsilon_r}(x^r_{B^c};\zeta)\}$. Keeping in mind (4.12) we see that now s runs over those $s \in \mathcal{S}$ such that $s(p) = r, c(p) \in B \Rightarrow s(p-1) = r$, i.e. such p's are bad, and in the integrand on the right hand side of (4.11), aside from the factor $\prod_{j\in B} g^{(\cdot)}(\varepsilon_r y^r_j + x^r_j)$, all other occurences of $\varepsilon_r y^r_i + x^r_i$, $i \in B$ are deleted.

If $h(x)$ is any function of the variable x we use the notation

$$\mathcal{D}_x h = h(x) - h(0)$$

for the difference between the value of h at x and it's value at $x = 0$. If $s \in \mathcal{S}$ we set $B_s = \{p | s(p) = s(p-1)\}$. The upshot is that we have

$$\begin{aligned}
&E\left(\prod_{i=1}^{n} \frac{\partial}{\partial x^i_l}\gamma_{k_i,\varepsilon}\big(x^i;\zeta\big)\right)\\
&\quad= \sum_{s\in\mathcal{S},a\in\mathcal{A}} (-1)^{\bar{a}_2} \int \prod_{i=1}^{n}\prod_{j=2}^{k_i} f\big(y^i_j\big)\left(\prod_{p\in B_s} g^{(a_1(p)+a_2(p))}\big(\varepsilon_{s(p)} y^{s(p)}_{c(p)} + x^{s(p)}_{c(p)}\big)\right)\\
&\qquad\times\left(\prod_{p\in B_s}\mathcal{D}_{\varepsilon_{s(p)} y^{s(p)}_j + x^{s(p)}_{c(p)}}\right)\prod_{p\in B^c_s} g^{(a_1(p)+a_2(p))}\left(z_{s(p)} + \sum_{j=2}^{c(p)}\big(\varepsilon_{s(p)} y^{s(p)}_j + x^{s(p)}_j\big)\right.\\
&\qquad\left.-\left(z_{s(p-1)} + \sum_{j=2}^{c(p-1)}\big(\varepsilon_{s(p-1)} y^{s(p-1)}_j + x^{s(p-1)}_j\big)\right)\right) \mathrm{d}y^i_j\,\mathrm{d}z_1\cdots\mathrm{d}z_n,
\end{aligned}\tag{4.13}$$

where we eventually set all $(\varepsilon_i, \varepsilon'_i) = (\varepsilon, \varepsilon')$.

To establish (4.7) we first handle the variation in ε. By (4.13)

$$\begin{aligned}
&E\left(\prod_{i=1}^{n}\left\{\frac{\partial}{\partial x^i_l}\gamma_{k_i,\varepsilon}\big(x^i;\zeta\big) - \frac{\partial}{\partial x^i_l}\gamma_{k_i,\varepsilon'}\big(x^i;\zeta\big)\right\}\right)\\
&\quad= \prod_{i=1}^{n}\Delta_{\varepsilon_i,\varepsilon'_i}\sum_{s\in\mathcal{S},a\in\mathcal{A}} (-1)^{\bar{a}_2}\int\prod_{i=1}^{n}\prod_{j=2}^{k_i} f\big(y^i_j\big)\\
&\qquad\times\left(\prod_{p\in B_s} g^{(a_1(p)+a_2(p))}\big(\varepsilon_{s(p)} y^{s(p)}_{c(p)} + x^{s(p)}_{c(p)}\big)\right)\left(\prod_{p\in B_s}\mathcal{D}_{\varepsilon_{s(p)} y^{s(p)}_j + x^{s(p)}_{c(p)}}\right)\\
&\qquad\times\prod_{p\in B^c_s} g^{(a_1(p)+a_2(p))}\left(z_{s(p)} + \sum_{j=2}^{c(p)}\big(\varepsilon_{s(p)} y^{s(p)}_j + x^{s(p)}_j\big)\right.\\
&\qquad\left.-\left(z_{s(p-1)} + \sum_{j=2}^{c(p-1)}\big(\varepsilon_{s(p-1)} y^{s(p-1)}_j + x^{s(p-1)}_j\big)\right)\right) \mathrm{d}y^i_j\,\mathrm{d}z_1\cdots\mathrm{d}z_n,
\end{aligned}\tag{4.14}$$

where we eventually set all $(\varepsilon_i, \varepsilon'_i) = (\varepsilon, \varepsilon')$. We will show that this is bounded in absolute value by $c_n|\varepsilon - \varepsilon'|^{\delta n/2}$ for all $|\varepsilon|, |\varepsilon'| \le \gamma$. (At this stage, and for ease in generalizing to the variation in x, we allow $\varepsilon, \varepsilon'$ to be zero or negative.) Compared to our proof of Theorem 2, the main difficulty here comes from the fact that $g'(x), g''(x)$, although uniformly bounded, are not continuous at $x=0$. It is here that the operators $\prod_{p\in B_s}\mathcal{D}_{\varepsilon_{s(p)}y_j^{s(p)}+x_{c(p)}^{s(p)}}$ will play a critical role.

In the following we let $g^\sharp = g, g'$ or g''. Fix $s\in\mathcal{S}, a\in\mathcal{A}$. As in the proof of Theorem 2, the corresponding term on the left-hand side of (4.14) can be written as a sum of many terms of the form appearing in (2.3) except that at least $n/2$ of the $g^\sharp$ factors have been replaced by factors of the form

$$\Delta_{\varepsilon_i,\varepsilon_i'} g^\sharp\big(z_i+\varepsilon_i y_j^i - z_{i'} + b_i\big) \tag{4.15}$$

or

$$\begin{aligned}\Delta_{\varepsilon_i,\varepsilon_i'}\big(g^\sharp\big(\varepsilon_i y_j^i+b_i\big)\mathcal{D}_{\varepsilon_i y_j^i+b_i}\big) &= \big(g^\sharp\big(\varepsilon_i y_j^i+b_i\big)-g^\sharp\big(\varepsilon_i' y_j^i+b_i\big)\big)\mathcal{D}_{\varepsilon_i y_j^i+b_i}\\ &\quad + g^\sharp\big(\varepsilon_i' y_j^i+b_i\big)(\mathcal{D}_{\varepsilon_i y_j^i+b_i}-\mathcal{D}_{\varepsilon_i' y_j^i+b_i}).\end{aligned} \tag{4.16}$$

Furthermore, we can assume that any i which appears in (4.16) differs from from any i, i' that appears in (4.15). Otherwise, we simply write one of them as a difference of two terms and consider each separately.

Consider first

$$\begin{aligned}&\Delta_{\varepsilon_i,\varepsilon_i'} g^\sharp\big(z_i+\varepsilon_i y_j^i - z_{i'} + b_i\big)\\ &\quad= \Delta_{\varepsilon_i,\varepsilon_i'} g^\sharp\big(z_i+\varepsilon_i y_j^i - z_{i'} + b_i\big)1_{\{|z_i+\varepsilon_i y_j^i - z_{i'}+b_i|\geq 4|\varepsilon_i-\varepsilon_i'||y_j^i|\}}\\ &\qquad+\Delta_{\varepsilon_i,\varepsilon_i'} g^\sharp\big(z_i+\varepsilon_i y_j^i - z_{i'} + b_i\big)1_{\{|z_i+\varepsilon_i y_j^i - z_{i'}+b_i|\leq 4|\varepsilon_i-\varepsilon_i'||y_j^i|\}}.\end{aligned} \tag{4.17}$$

For the first term since we are away from the discontinuity of $g^\sharp$, we use (3.8) to obtain a factor of $C|\varepsilon_i-\varepsilon_i'|$, since $|y_j^i|\leq 1$, while for the second term we use the fact that $g^\sharp$ is bounded and hence the $\mathrm{d}z_i$ integral contributes a factor of $C|\varepsilon_i-\varepsilon_i'|$. In more detail, on the set $\{|z_i+\varepsilon_i y_j^i - z_{i'}+b_i|\leq 4|\varepsilon_i-\varepsilon_i'||y_j^i|\}$, up to a bounded error, we can replace every occurrence of z_i in a $g^\sharp$ factor by $z_{i'}$, and in particular we simply bound $\Delta_{\varepsilon_i,\varepsilon_i'} g^\sharp(z_i+\varepsilon_i y_j^i - z_{i'}+b_i)$ by 2. This eliminates any occurrence of z_i except in $1_{\{|z_i+\varepsilon_i y_j^i - z_{i'}+b_i|\leq 4|\varepsilon_i-\varepsilon_i'||y_j^i|\}}$ which we can write as $1_{\{B(z_{i'}-\varepsilon_i y_j^i-b_i),4|\varepsilon_i-\varepsilon_i'||y_j^i|)\}}(z_i)$. We then do the the $\mathrm{d}z_i$ integral, to obtain a bound of $C|\varepsilon_i-\varepsilon_i'|$.

A similar analysis holds for the last term in (4.16) since $\mathcal{D}_{\varepsilon_i y_j^i+b_i}-\mathcal{D}_{\varepsilon_i' y_j^i+b_i}=\Delta_{\varepsilon_i y_j^i+b_i,\varepsilon_i' y_j^i+b_i}$, and we have not yet "used" the $\mathrm{d}z_i$ integral. Finally, for the term $(g^\sharp(\varepsilon_i y_j^i+b_i)-g^\sharp(\varepsilon_i' y_j^i+b_i))\mathcal{D}_{\varepsilon_i y_j^i+b_i}$ in (4.16), if $|\varepsilon_i y_j^i+b_i|\geq|\varepsilon_i-\varepsilon_i'|$ then $\varepsilon_i y_j^i+b_i$ and $\varepsilon_i' y_j^i+b_i$ have the same sign, so we can use (3.8) to obtain a factor of $C|\varepsilon_i-\varepsilon_i'|$, while if $|\varepsilon_i y_j^i+b_i|\leq|\varepsilon_i-\varepsilon_i'|$ we can use $\mathcal{D}_{\varepsilon_i y_j^i+b_i}$ and the $\mathrm{d}z_i$ integral to obtain a factor of $C|\varepsilon_i y_j^i+b_i|\leq C|\varepsilon_i-\varepsilon_i'|$. □

5. Renormalized intersection local times: joint continuity of the spatial derivative

Recall the approximate kth order renormalized intersection local time.

$$\gamma_{k,\varepsilon}(x;t)=\sum_{A\subseteq\{2,\ldots,k\}}(-1)^{|A|}\Big(\prod_{j\in A}g_\varepsilon(x_j)\Big)\alpha_{k-|A|,\varepsilon}(x_{A^c};t). \tag{5.1}$$

Theorem 4. *Almost surely, $\gamma_{\varepsilon,k}(x;t)$ and $\frac{\partial}{\partial x_l}\gamma_{k,\varepsilon}(x;t)$ converge locally uniformly on $R^{k-1}\times R_+$ as $\varepsilon\to 0$. Hence*

$$\gamma_k(x;t)\stackrel{def}{=}\lim_{\varepsilon\to 0}\gamma_{\varepsilon,k}(x;t) \tag{5.2}$$

is differentiable in x and $\nabla_x\gamma_{\varepsilon,k}(x;t)$ is continuous in $(x,t)\in R^{k-1}\times R_+$.

Proof. Let Y_t denote Brownian motion killed at an independent mean-2 exponential time ζ. From now on, $\gamma_{\varepsilon,k}(x;t)$ will be defined for the process Y_t in place of W_t. Using Fubini's theorem, it suffices as before to show that $\gamma_{\varepsilon,k}(x;t)$ and $\frac{\partial}{\partial x_l}\gamma_{\varepsilon,k}(x;t)$ converge locally uniformly on $R^{k-1}\times[0,\zeta)$ as $\varepsilon\to 0$ with probability 1. We will focus on $\frac{\partial}{\partial x_l}\gamma_{\varepsilon,k}(x;t)$, and leave the easier case of $\gamma_{\varepsilon,k}(x;t)$ to the reader.

If $\mathcal{S}$ is a subset of Euclidean space we will say that $\{Z_\varepsilon(x);(\varepsilon,x)\in(0,1]\times\mathcal{S}\}$ converges rationally locally uniformly on $\mathcal{S}$ as $\varepsilon\to 0$ if for any compact $K\in\mathcal{S}$, $Z_\varepsilon(x)$ converges uniformly in $x\in K$ as $\varepsilon\to 0$ when restricted to dyadic rational x,ε. We note that since $\frac{\partial}{\partial x_l}\gamma_{\varepsilon,k}(x;t)$ for $\varepsilon>0$ is continuous in ε,x,t, saying that $\frac{\partial}{\partial x_l}\gamma_{\varepsilon,k}(x;t)$ converges locally uniformly or converges rationally locally uniformly on $R^{k-1}\times[0,\zeta)$ as $\varepsilon\to 0$ are equivalent.

We know from Theorem 3 that $\frac{\partial}{\partial x_l}\gamma_{\varepsilon,k}(x;\infty)$ converges locally uniformly on R^{k-1} as $\varepsilon\to 0$ with probability 1. Using martingale techniques we will see that the right continuous martingale

$$\Gamma_{k,\varepsilon,l}(x;t)\overset{\text{def}}{=}E\left\{\frac{\partial}{\partial x_l}\gamma_{k,\varepsilon}(x;\infty)\Big|\mathcal{F}_t\right\}$$

converges rationally locally uniformly on $R^{k-1}\times R_+$ as $\varepsilon\to 0$ with probability 1. $\Gamma_{k,\varepsilon,l}(x;t)$ is not the same as $\frac{\partial}{\partial x_l}\gamma_{\varepsilon,k}(x;t)$, but we will see that they differ by terms of "lower order," and we will be able to complete our proof by induction. Given the tools we have developed so far in this paper, the proof is conceptually fairly straightforward, but in order to treat the "lower order" terms systematically we need to introduce some notation. This we now proceed to do.

We first define the approximate kth order generalized intersection local time

$$\alpha_{k,\varepsilon}(x_2,x_3,\ldots,x_k;\phi;t)=\int_{\{0\le t_1\le\cdots\le t_k\le t\}}\prod_{j=2}^{k}f_{\varepsilon,x_j}(Y_{t_j}-Y_{t_{j-1}})\phi(Y_{t_k})\,\mathrm{d}t_1\cdots\,\mathrm{d}t_k \tag{5.3}$$

and set

$$\alpha_{k,\varepsilon}(x_2,x_3,\ldots,x_k;\phi)=\alpha_{k,\varepsilon}(x_2,x_3,\ldots,x_k;\phi;\infty).$$

$\alpha_{k,\varepsilon}(x_2,x_3,\ldots,x_k;\phi)$ is the approximate kth order generalized total intersection local time. For ease of notation in later formulas, we also set

$$\alpha_{1,\varepsilon}(\phi;t)=\int_{\{0\le t_1\le t\}}\phi(Y_{t_1})\,\mathrm{d}t_1 \tag{5.4}$$

and

$$\alpha_{0,\varepsilon}(\phi;t)=\int\phi(z)\,\mathrm{d}z \tag{5.5}$$

although $\alpha_{1,\varepsilon}(\phi;t)$ is independent of ε and $\alpha_{0,\varepsilon}(\phi;t)$ is independent of ε,t.

Observe that

$$\begin{aligned}E\big\{\alpha_{k,\varepsilon}(x_2,x_3,\ldots,x_k;\phi)|\mathcal{F}_t\big\}&=\alpha_{k,\varepsilon}(x_2,x_3,\ldots,x_k;\phi;t)\\&\quad+\sum_{i=0}^{k-1}E\left(\int_{\{0\le t_1\le\cdots\le t_i\le t\le t_{i+1}\cdots\le t_k\}}\prod_{j=2}^{k}f_{\varepsilon,x_j}(Y_{t_j}-Y_{t_{j-1}})\phi(Y_{t_k})\,\mathrm{d}t_1\cdots\,\mathrm{d}t_k\Big|\mathcal{F}_t\right)\end{aligned} \tag{5.6}$$

and for $i \geq 1$

$$E\left(\int_{\{0\leq t_1\leq\cdots\leq t_i\leq t\leq t_{i+1}\cdots\leq t_k\}}\prod_{j=2}^{k} f_{\varepsilon,x_j}(Y_{t_j}-Y_{t_{j-1}})\phi(Y_{t_k})\,\mathrm{d}t_1\cdots\mathrm{d}t_k\Big|\mathcal{F}_t\right)$$

$$=\int_{\{0\leq t_1\cdots\leq t_i\leq t\leq t_{i+1}\cdots\leq t_k\}}\prod_{j=2}^{i} f_{\varepsilon,x_j}(Y_{t_j}-Y_{t_{j-1}})$$

$$\times E\Bigg(f_{\varepsilon,x_{i+1}}\big((Y_{t_{i+1}}-Y_t)+(Y_t-Y_{t_i})\big)\prod_{j=i+2}^{k} f_{\varepsilon,x_j}(Y_{t_j}-Y_{t_{j-1}})$$

$$\times\phi\big((Y_{t_k}-Y_{t_{k-1}})+\cdots+(Y_{t_{i+1}}-Y_t)+Y_t\big)\Big|\mathcal{F}_t\Bigg)\,\mathrm{d}t_1\cdots\mathrm{d}t_k$$

$$=\int_{\{0\leq t_1\cdots\leq t_i\leq t\}}\prod_{j=2}^{i} f_{\varepsilon,x_j}(Y_{t_j}-Y_{t_{j-1}})f_{\varepsilon,x_{i+1}}\big(z_{i+1}+(Y_t-Y_{t_i})\big)$$

$$\times\prod_{j=i+2}^{k} f_{\varepsilon,x_j}(z_j)\phi(z_{i+1}+\cdots+z_k+Y_t)\prod_{j=i+1}^{k} g(z_j)\,\mathrm{d}z_j\,\mathrm{d}t_1\cdots\mathrm{d}t_i$$

$$=\alpha_{i,\varepsilon}\big(x_2,x_3,\ldots,x_i;\lambda_{k-i,\varepsilon}[\phi;x_{i+1},\ldots,x_k;Y_t];t\big),\tag{5.7}$$

where

$$\lambda_{k-i,\varepsilon}[\phi;x_{i+1},\ldots,x_k;u](v)$$

$$=\int f_{\varepsilon,x_{i+1}}(z_{i+1}+u-v)\prod_{j=i+2}^{k} f_{\varepsilon,x_j}(z_j)$$

$$\times\phi(z_{i+1}+\cdots+z_k+u)\prod_{j=i+1}^{k} g(z_j)\,\mathrm{d}z_{i+1}\cdots\mathrm{d}z_k$$

$$=\int g(z_{i+1}+v-u)\prod_{j=i+2}^{k} g(z_j)$$

$$\times\phi(z_{i+1}+\cdots+z_k+v)\prod_{j=i+1}^{k} f_{\varepsilon,x_j}(z_j)\,\mathrm{d}z_{i+1}\cdots\mathrm{d}z_k.\tag{5.8}$$

Similarly, for $i=0$ we have

$$E\left(\int_{\{0\leq t\leq t_1\cdots\leq t_k\}}\prod_{j=2}^{k} f_{\varepsilon,x_j}(Y_{t_j}-Y_{t_{j-1}})\phi(Y_{t_k})\,\mathrm{d}t_1\cdots\mathrm{d}t_k\Big|\mathcal{F}_t\right)$$

$$=\int_{\{0\leq t\leq t_1\cdots\leq t_k\}} E\Bigg(\prod_{j=2}^{k} f_{\varepsilon,x_j}(Y_{t_j}-Y_{t_{j-1}})$$

$$\times\phi\big((Y_{t_k}-Y_{t_{k-1}})+\cdots+(Y_{t_1}-Y_t)+Y_t\big)\Big|\mathcal{F}_t\Bigg)\,\mathrm{d}t_1\cdots\mathrm{d}t_k$$

$$= \int \prod_{j=2}^{k} f_{\varepsilon,x_j}(z_j)\phi(z_1 + \cdots + z_k + Y_t) \prod_{j=1}^{k} g(z_j)\,\mathrm{d}z_j$$

$$= \int \left(\int g(v)\phi(v + z_2 + \cdots + z_k + Y_t) \prod_{j=2}^{k} f_{\varepsilon,x_j}(z_j) g(z_j)\,\mathrm{d}z_j \right) \mathrm{d}v$$

$$= \alpha_{0,\varepsilon}\big(\lambda_{k,\varepsilon}[\phi; x_2, \ldots, x_k; Y_t]; t\big), \tag{5.9}$$

where, recall our convention (5.5),

$$\lambda_{k,\varepsilon}[\phi; x_2, \ldots, x_k; u](v)$$
$$= \int g(v-u)\phi(z_2 + \cdots + z_k + v) \prod_{j=2}^{k} f_{\varepsilon,x_j}(z_j) g(z_j)\,\mathrm{d}z_j. \tag{5.10}$$

By abuse of notation we can introduce a fictitious x_1, and letting $f_{\varepsilon,x_1}(z_1)$ denote $\delta(z_1)$, the δ-function, we can write

$$\lambda_{k,\varepsilon}[\phi; x_2, \ldots, x_k; u](v)$$
$$= \int g(z_1 + v - u) \prod_{j=2}^{k} g(z_j)\phi(z_1 + z_2 + \cdots + z_k + v) \prod_{j=1}^{k} f_{\varepsilon,x_j}(z_j)\,\mathrm{d}z_j. \tag{5.11}$$

Finally setting

$$\lambda_{k,\varepsilon}[\phi; x_1, \ldots, x_k; Y_t] =: \lambda_{k,\varepsilon}[\phi; x_2, \ldots, x_k; Y_t] \tag{5.12}$$

this now takes the same form as (5.8) with $i = 0$. Then we can write

$$E\big\{\alpha_{k,\varepsilon}(x_2, x_3, \ldots, x_k; \phi)|\mathcal{F}_t\big\}$$
$$= \alpha_{k,\varepsilon}(x_2, x_3, \ldots, x_k; \phi; t)$$
$$+ \sum_{i=0}^{k-1} \alpha_{i,\varepsilon}\big(x_2, x_3, \ldots, x_i; \lambda_{k-i,\varepsilon}[\phi; x_{i+1}, \ldots, x_k; Y_t]; t\big). \tag{5.13}$$

Setting

$$\lambda_{k-i}[\phi; z_{i+1}, \ldots, z_k; u](v)$$
$$= g(z_{i+1} + v - u) \prod_{j=i+2}^{k} g(z_j)\phi(z_{i+1} + \cdots + z_k + v) \tag{5.14}$$

we have

$$\lambda_{k-i,\varepsilon}[\phi; x_{i+1}, \ldots, x_k; u](v)$$
$$= \int \lambda_{k-i}[\phi; z_{i+1}, \ldots, z_k; u](v) \prod_{j=i+1}^{k} f_{\varepsilon,x_j}(z_j)\,\mathrm{d}z_{i+1} \cdots \mathrm{d}z_k. \tag{5.15}$$

We next define the approximate kth order generalized renormalized intersection local time

$$\gamma_{k,\varepsilon}(x; \phi; t) = \sum_{A \subseteq \{2,\ldots,k\}} (-1)^{|A|} \left(\prod_{j \in A} g_\varepsilon(x_j) \right) \alpha_{k-|A|,\varepsilon}(x_{A^c}; \phi; t) \tag{5.16}$$

and set

$$\gamma_{k,\varepsilon}(x_2, x_3, \ldots, x_k; \phi) = \gamma_{k,\varepsilon}(x_2, x_3, \ldots, x_k; \phi; \infty).$$

$\gamma_{k,\varepsilon}(x_2, x_3, \ldots, x_k; \phi)$ is the approximate kth order generalized total renormalized intersection local time. As before, for ease of notation in later formulas, we also set

$$\gamma_{1,\varepsilon}(\phi; t) = \int_{\{0 \le t_1 \le t\}} \phi(Y_{t_1})\,\mathrm{d}t_1$$

and

$$\gamma_{0,\varepsilon}(\phi; t) = \int \phi(z)\,\mathrm{d}z.$$

Using (5.13) we find that

$$\begin{aligned} &E\{\gamma_{k,\varepsilon}(x_2, x_3, \ldots, x_k; \phi)|\mathcal{F}_t\} \\ &\quad = \gamma_{k,\varepsilon}(x_2, x_3, \ldots, x_k; \phi; t) + \sum_{A \subseteq \{2,\ldots,k\}} (-1)^{|A|}\left(\prod_{j \in A} g_\varepsilon(x_j)\right) \\ &\qquad \times \sum_{i=0}^{k-|A|-1} \alpha_{i,\varepsilon}\bigl(x_{A^c(1)}, x_{A^c(2)}, \ldots, x_{A^c(i-1)}; \lambda_{k-|A|-i,\varepsilon}[\phi; x_{A^c(i)}, \ldots, x_{A^c(k-|A|)}; Y_t]; t\bigr), \end{aligned} \tag{5.17}$$

where

$$A^c = \bigl\{A^c(1) < A^c(2) < \cdots < A^c\bigl(k - |A|\bigr)\bigr\}.$$

We reorganize this by writing A as the disjoint union of

$$A_i = \bigl\{j \in A | j < A^c(i)\bigr\} \quad \text{and} \quad B_i = \bigl\{j \in A | j > A^c(i)\bigr\}$$

so that

$$\begin{aligned} &(-1)^{|A|}\left(\prod_{j \in A} g_\varepsilon(x_j)\right) \\ &\qquad \times \alpha_{i,\varepsilon}\bigl(x_{A^c(1)}, x_{A^c(2)}, \ldots, x_{A^c(i-1)}; \lambda_{k-|A|-i,\varepsilon}[\phi; x_{A^c(i)}, \ldots, x_{A^c(k-|A|)}; Y_t]; t\bigr) \\ &\quad = (-1)^{|A_i|}\left(\prod_{j \in A_i} g_\varepsilon(x_j)\right) \\ &\qquad \times \alpha_{i,\varepsilon}\left(x_{A^c(1)}, x_{A^c(2)}, \ldots, x_{A^c(i-1)}; (-1)^{|B_i|}\left(\prod_{j \in B_i} g_\varepsilon(x_j)\right)\lambda_{k-|A|-i,\varepsilon}[\phi; x_{A^c(i)}, \ldots, x_{A^c(k-|A|)}; Y_t]; t\right). \end{aligned}$$

It is now easy to see that if we fix $0 \le l \le k-1$ and sum in (5.17) over all $A \subseteq \{2, \ldots, k\}$ with $A^c(i) = l+1$ we will obtain

$$\gamma_{l,\varepsilon}\bigl(x_2, x_3, \ldots, x_l; \Lambda_{k-l,\varepsilon}[\phi; x_{l+1}, \ldots, x_k; Y_t]; t\bigr), \tag{5.18}$$

where

$$\begin{aligned} &\Lambda_{k-l,\varepsilon}[\phi; x_{l+1}, \ldots, x_k; u](v) \\ &\quad = \int \Lambda_{k-l}[\phi; z_{l+1}, \ldots, z_k; u](v) \prod_{j=l+1}^{k} f_{\varepsilon,x_j}(z_j)\,\mathrm{d}z_{i+1} \cdots \mathrm{d}z_k \end{aligned} \tag{5.19}$$

with

$$\begin{aligned}
&\Lambda_{k-l}[\phi; z_{l+1}, \dots, z_k; u](v) \\
&\quad = \sum_{B\subseteq\{l+2,\dots,k\}} (-1)^{|B|} \left(\prod_{j\in B} g(z_j)\right) \lambda_{k-|B|-l,\varepsilon}[\phi; x_{\{l+1,\dots,k\}-B}; u](v) \\
&\quad = g(z_{l+1}+v-u) \sum_{B\subseteq\{l+2,\dots,k\}} (-1)^{|B|} \left(\prod_{j\in B} g(z_j)\right) \prod_{j\in\{l+2,\dots,k\}-B} g(z_j) \\
&\qquad \times \phi\left(z_{l+1} + \sum_{i\in\{l+2,\dots,k\}-B} z_i + v\right) \\
&\quad = g(z_{l+1}+v-u) \prod_{j=l+2}^{k} g(z_j) \sum_{B\subseteq\{l+2,\dots,k\}} (-1)^{|B|} \phi\left(z_{l+1} + \sum_{i\in\{l+2,\dots,k\}-B} z_i + v\right) \\
&\quad = g(z_{l+1}+v-u) \prod_{j=l+2}^{k} g(z_j) \mathcal{D}_{z_j} \phi(z_{l+1}+\dots+z_k+v).
\end{aligned} \tag{5.20}$$

The reader can check that this is consistent with our conventions when $l = 0$. Combining the above we obtain

$$\begin{aligned}
&E\{\gamma_{k,\varepsilon}(x_2, x_3, \dots, x_k; \phi)|\mathcal{F}_t\} \\
&\quad = \gamma_{k,\varepsilon}(x_2, x_3, \dots, x_k; \phi; t) \\
&\qquad + \sum_{i=0}^{k-1} \gamma_{i,\varepsilon}\big(x_2, x_3, \dots, x_i; \Lambda_{k-i,\varepsilon}[\phi; x_{i+1}, \dots, x_k; Y_t]; t\big).
\end{aligned} \tag{5.21}$$

We will say that $\Lambda_{k-i}[\phi; z_{i+1}, \dots, z_k; u]$ is obtained from ϕ by adjunction of $z_{i+1}, \dots, z_k; u$.

In order to make the sequel easier to follow, we make some explanatory comments. We will use (5.21) inductively to show that almost surely $\frac{\partial}{\partial x_l}\gamma_{k,\varepsilon}(x_2, x_3, \dots, x_k; t)$ converges locally uniformly in x, t as $\varepsilon \to 0$. The convergence of the conditional expectation will follow easily from martingale inequalities and the techniques we have used to obtain convergence at exponential times. When looking at the last line in (5.21) we encounter something new, the presence of Y_t. Rather than try to deal with this directly we prove that almost surely

$$\frac{\partial}{\partial x_l}\gamma_{i,\varepsilon}\big(x_2, x_3, \dots, x_i; \Lambda_{k-i,\varepsilon}[\phi; x_{i+1}, \dots, x_k; u]; t\big)$$

converges locally uniformly in x, u, t as $\varepsilon \to 0$. Using the fact that Y is almost surely locally bounded, this will show that almost surely

$$\frac{\partial}{\partial x_l}\gamma_{i,\varepsilon}\big(x_2, x_3, \dots, x_i; \Lambda_{k-i,\varepsilon}[\phi; x_{i+1}, \dots, x_k; Y_t]; t\big)$$

converges locally uniformly in x, t as $\varepsilon \to 0$ which will allow us to complete the induction step. This explains the presence of u in (5.19) and (5.20). We will refer to such u as a "new" parameter, while the $\{x_2, \dots, x_k\}$ will be referred to as "old" parameters.

We will say that a function $\varphi_{y_1,\dots,y_n}(v)$ is *an admissable function of* v with auxiliary parameters $y_1, \dots, y_n$ if it can be written in the form

$$\varphi_{y_1,\dots,y_n}(v) = \prod_{j\in B_0} g(y_j) \mathcal{D}_{y_j} \prod_{i=1}^{p} g\left(v + \sum_{v\in B_i} \pm y_v\right), \tag{5.22}$$

where $B_i \subseteq \{1, \ldots, n\}, \forall i = 0, 1, \ldots, p$, and $\{1, \ldots, n\} = \bigcup_{i=1}^{p} B_i$. Here p is an arbitrary positive integer. If $\varphi_{y_1,\ldots,y_n}(v)$ is of the above form we will say that $\varphi_{y_1,\ldots,y_n}(v)$ is of weight $|B_0| + p$. Note that the weight of $\varphi_{y_1,\ldots,y_n}(v)$ is the number of g factors in (5.22). We will also consider the function $\varphi(v) \equiv 1$ to be an admissable function of v (of weight 0 and with no auxiliary parameters).

If $\varphi_{y_1,\ldots,y_n}(v)$ is an admissable function of v with auxiliary parameters $y_1, \ldots, y_n$ we will use the notation $\varphi_{y_1,\ldots,y_n;\varepsilon}(v)$ to denote the function in which some of the auxiliary variables have been smoothed. More precisely, we will say that $\varphi_{y_1,\ldots,y_n;\varepsilon}(v)$ is a *totally ε-smoothed version of* $\varphi_{y_1,\ldots,y_n}(v)$ if

$$\varphi_{y_1,\ldots,y_n;\varepsilon}(v) = \int \varphi_{y_1+\varepsilon z_1,\ldots,y_n+\varepsilon z_n}(v) \prod_{i\in A} f(z_i)\,\mathrm{d}z_i \prod_{i\in A^c} \mathrm{d}\mu_0(z_i) \tag{5.23}$$

for some subset $A \subseteq \{1, \ldots, n\}$ such that (with the notation of (5.22)) $B_0 \subseteq A$ and $B_i \cap A \neq \varnothing$ for all $i = 1, \ldots, p$. In other words, we require that each g factor in (5.22) contain at least one element of the set y_j, $j \in A$. Here μ_0 is the Dirac measure which puts unit mass at the origin. It would be more precise to refer to the function defined in (5.23) as $\varphi_{y_1,\ldots,y_n;\varepsilon,A}(v)$, but in order to avoid further cluttering of the notation, and because the actual nature of the set A will be irrelevant for us, we shall simply drop it from the notation. The reader will note in the sequel that it is precisely the "old" parameters which are integrated against an f. □

The next lemma assembles some facts about adjunction which follow easily from the definitions.

Lemma 3. *Let $\varphi_{y_1,\ldots,y_n}(v)$ be an admissable function of v of weight q and auxiliary parameters $y_1, \ldots, y_n$, and let $\Lambda_{k-i}[\varphi_{y_1,\ldots,y_n}; x_{i+1}, \ldots, x_k; u]$ denote the function in* (5.20) *obtained from $\varphi_{y_1,\ldots,y_n}(v)$ by adjunction of $x_{i+1}, \ldots, x_k; u$. Then*:

1. *$\Lambda_{k-i}[\varphi_{y_1,\ldots,y_n}; x_{i+1}, \ldots, x_k; u](z)$ is an admissable function of z of weight $q + k - i$ and auxiliary parameters $y_1, \ldots, y_n, x_{i+1}, \ldots, x_k, u$.*
2. *If $\varphi_{y_1,\ldots,y_n;\varepsilon}(v)$ is a totally ε-smoothed version of $\varphi_{y_1,\ldots,y_n}(v)$, then the function $\Lambda_{k-i,\varepsilon}[\varphi_{y;\varepsilon}; x_{i+1}, \ldots, x_k; u]$ defined in* (5.19) *is a totally ε-smoothed version of $\Lambda_{k-i}[\varphi_y; x_{i+1}, \ldots, x_k; u]$.*

In the following, the notation $\nabla\gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon})$ will denote the gradient with respect to x and y. In fact, we are not interested in differentiating with respect to "new" parameters, but to avoid excessive notation we consider them also.

The next lemma generalizes Theorem 3.

Lemma 4. *Let $\varphi_y(z)$ be an admissable function of z of weight $k - i$ and auxiliary parameters $y = (y_1, \ldots, y_j)$ and let $\varphi_{y;\varepsilon}(z)$ be a totally ε-smoothed version of $\varphi_y(z)$. Then there exists $\delta > 0$ such that for each n and $M < \infty$ we can find $c_{n,M} < \infty$ such that*

$$E\left(\left\{\sup_{F_M} \frac{|\nabla\gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon}) - \nabla\gamma_{i,\varepsilon'}(x'; \varphi_{y';\varepsilon'})|}{|(\varepsilon, x, y) - (\varepsilon', x', y')|^{\delta}}\right\}^n\right) \leq c_{n,M}, \tag{5.24}$$

where $\sup_{F_M}$ is taken over all dyadic rational pairs $(\varepsilon, x, y) \neq (\varepsilon', x', y')$ such that $0 < \varepsilon, \varepsilon' \leq 1$ and $|x|, |x'|, |y|, |y'| \leq M$.

Proof. According to [6], Chapter 1, Theorem 2.1, it suffices to show that there exists $\delta > 0$ such that for each n and $M < \infty$ we can find $c_{n,M} < \infty$ such that

$$E\left(\left|\nabla\gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon}) - \nabla\gamma_{i,\varepsilon'}\left(x'; \varphi_{y';\varepsilon'}\right)\right|^n\right) \leq c_{n,M}\left|(\varepsilon, x, y) - \left(\varepsilon', x', y'\right)\right|^{\delta n} \tag{5.25}$$

for all $(\varepsilon, x, y), (\varepsilon', x', y')$ such that $0 < \varepsilon, \varepsilon' \leq 1$ and $|x|, |x'|, |y|, |y'| \leq M$. Equation (5.25) follows as in the proof of Theorem 3. □

Proof of Theorem 4 (continued). We will show by induction on $i = 0, 1, \ldots, k$ that $\nabla\gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon}; t)$ converges locally uniformly in $(x, y, t) \in R^{j+i-1} \times [0, \zeta)$ as $\varepsilon \to 0$ for all admissable functions $\varphi_y(z)$ of z of weight $k - i$ and auxiliary parameters $y = (y_1, \ldots, y_j)$.

The case $i = k$ and $\varphi_y(z) \equiv 1$ will prove our theorem.

Consider first the case of $i = 0$. We have to show that if $\varphi_y(z)$ is an admissable function of z of weight k and auxiliary parameters $y = (y_1, \dots, y_j)$ and $\varphi_{y;\varepsilon}(z)$ is a totally ε-smoothed version of $\varphi_y(z)$, then both

$$\gamma_{0,\varepsilon}(x; \varphi_{y;\varepsilon}; t) \equiv \int \varphi_{y;\varepsilon}(z)\,\mathrm{d}z$$

and

$$\frac{\partial}{\partial y_l}\gamma_{0,\varepsilon}(x; \varphi_{y;\varepsilon}; t) \equiv \int \frac{\partial}{\partial y_l}\varphi_{y;\varepsilon}(z)\,\mathrm{d}z$$

converge locally uniformly in $y \in R^j$ as $\varepsilon \to 0$. This follows as in the proof of Theorem 3.

Assume now that for all $p < i$, and for all admissable functions $\Phi_y(z)$ of z of weight $k - p$ and auxiliary parameters $y = (y_1, \dots, y_{j'})$ we have that $\nabla\gamma_{p,\varepsilon}(x; \Phi_{y;\varepsilon}; t)$ converges locally uniformly in $(x, y, t) \in R^{j'+p-1} \times [0, \zeta)$ as $\varepsilon \to 0$ for any totally ε-smoothed version $\Phi_{y;\varepsilon}(z)$ of $\Phi_y(z)$. Let us show that if $\varphi_y(z)$ is an admissable functions of z of weight $k - i$ and auxiliary parameters $y = (y_1, \dots, y_j)$, and $\varphi_{y;\varepsilon}(z)$ is a totally ε-smoothed version of $\varphi_y(z)$, then $\nabla\gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon}; t)$ converges locally uniformly in $(x, y, t) \in R^{j+i-1} \times [0, \zeta)$ as $\varepsilon \to 0$.

With F_M as in Lemma 4, let $F_M^m; m = 1, 2, \dots$ be an exhaustion of F_M by a sequence of finite symmetric subsets. (A set F of pairs (a, b) is symmetric if $(a, b) \in F \Rightarrow (b, a) \in F$). Let us define the right continuous martingale

$$\Gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon}; t) = E\big(\nabla\gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon}) | \mathcal{F}_t\big). \tag{5.26}$$

By [6], Chapter II, Theorem 1.7, applied to the right continuous submartingale

$$A_t^m = \sup_{F_M^m} \frac{|\Gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon}; t) - \Gamma_{i,\varepsilon'}(x'; \varphi_{y';\varepsilon'}; t)|}{|(\varepsilon, x, y) - (\varepsilon', x', y')|^\delta} \tag{5.27}$$

we have that

$$\begin{aligned}
&E\left(\left\{\sup_t \sup_{F_M^m} \frac{|\Gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon}; t) - \Gamma_{i,\varepsilon'}(x'; \varphi_{y';\varepsilon'}; t)|}{|(\varepsilon, x, y) - (\varepsilon', x', y')|^\delta}\right\}^n\right) \\
&\quad \le E\left(\left\{\sup_{F_M^m} \frac{|\nabla\gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon}) - \nabla\gamma_{i,\varepsilon'}(x'; \varphi_{y';\varepsilon'})|}{|(\varepsilon, x, y) - (\varepsilon', x', y')|^\delta}\right\}^n\right) \\
&\quad \le E\left(\left\{\sup_{F_M} \frac{|\nabla\gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon}) - \nabla\gamma_{i,\varepsilon'}(x'; \varphi_{y';\varepsilon'})|}{|(\varepsilon, x, y) - (\varepsilon', x', y')|^\delta}\right\}^n\right) \\
&\quad \le c_{n,M},
\end{aligned} \tag{5.28}$$

where the last line used Lemma 4. Hence

$$E\left(\left\{\sup_t \sup_{F_M} \frac{|\Gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon}; t) - \Gamma_{i,\varepsilon'}(x'; \varphi_{y';\varepsilon'}; t)|}{|(\varepsilon, x, y) - (\varepsilon', x', y')|^\delta}\right\}^n\right) \le c_{n,M}. \tag{5.29}$$

In particular this shows that

$$\sup_t \sup_{F_{1,M}} \big|\Gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon}; t) - \Gamma_{i,\varepsilon'}(x; \varphi_{y';\varepsilon'}; t)\big| \le C(\omega)\big|\varepsilon - \varepsilon'\big|^\delta, \tag{5.30}$$

where $F_{1,M}$ denotes the set of dyadic rational $(x, y) \in R^{j+i-1}$ with $|x|, |y| \le M$. Thus, $\Gamma_{i,\varepsilon}(x; \varphi_{y;\varepsilon}; t)$ converges rationally locally uniformly on $R^{j+i-1} \times R_+$ as $\varepsilon \to 0$ with probability 1.

It is easy to see that $\nabla\gamma_{i,\varepsilon}(x;\varphi_{y;\varepsilon};t)$ is continuous in ε, x, y, t for $\varepsilon > 0$. Thus, as with $\nabla\gamma_{\varepsilon,k}(x;t)$, saying that $\nabla\gamma_{i,\varepsilon}(x;\varphi_{y;\varepsilon};t)$ converges locally uniformly or converges rationally locally uniformly as $\varepsilon \to 0$ are equivalent. By (5.21)

$$\begin{aligned}\Gamma_{i,\varepsilon}(x;\varphi_{y;\varepsilon};t) &= \nabla\gamma_{i,\varepsilon}(x_2,x_3,\ldots,x_i;\varphi_{y;\varepsilon};t)\\ &\quad+\sum_{p=0}^{i-1}\nabla\gamma_{p,\varepsilon}\big(x_2,x_3,\ldots,x_p;\Lambda_{i-p,\varepsilon}[\varphi_{y;\varepsilon};x_{p+1},\ldots,x_i;Y_t];t\big).\end{aligned}\tag{5.31}$$

Hence to show that $\nabla\gamma_{i,\varepsilon}(x;\varphi_{y;\varepsilon};t)$ converges locally uniformly on $R^{j+i-1}\times[0,\zeta)$ as $\varepsilon\to 0$ with probability 1 it suffices to show that for each $p<i$

$$\nabla\gamma_{p,\varepsilon}\big(x_2,x_3,\ldots,x_p;\Lambda_{i-p,\varepsilon}[\varphi_{y;\varepsilon};x_{p+1},\ldots,x_i;Y_t];t\big)$$

converges locally uniformly on $R^{j+i-1}\times[0,\zeta)$ as $\varepsilon\to 0$ with probability 1. However, by Lemma 3, $\Lambda_{i-p,\varepsilon}[\varphi_{y;\varepsilon};x_{p+1},\ldots,x_i;u]$ is a totally ε-smoothed version of $\Lambda_{i-p}[\varphi_y;x_{p+1},\ldots,x_i;u]$, and the latter is an admissable function of weight $k-p$ with auxiliary variables $y, x_{p+1},\ldots,x_i,u$. Therefore, by our induction assumption,

$$\nabla\gamma_{p,\varepsilon}\big(x_2,x_3,\ldots,x_p;\Lambda_{i-p,\varepsilon}[\varphi_{y;\varepsilon};x_{p+1},\ldots,x_i;u];t\big)$$

converges locally uniformly in $(x,y,u,t)\in R^{j+i}\times[0,\zeta)$ as $\varepsilon\to 0$ with probability 1. Since Y_t is locally bounded on $[0,\zeta)$, this completes proof of Theorem 4. □

References

[1] R. Bass and D. Khoshnevisan. Intersection local times and Tanaka formulas. *Ann. Inst. H. Poincaré Probab. Statist.* **29** (1993) 419–452. MR1246641

[2] E. B. Dynkin. Self-intersection gauge for random walks and for Brownian motion. *Ann. Probab.* **16** (1988) 1–57. MR0920254

[3] J.-F. Le Gall. Propriétés d'intersection des marches aléatoires, I. *Comm. Math. Phys.* **104** (1986) 471–507. MR0840748

[4] J.-F. Le Gall. Fluctuation results for the Wiener sausage. *Ann. Probab.* **16** (1988) 991–1018. MR0942751

[5] J.-F. Le Gall. Some properties of planar Brownian motion. In *École d' Été de Probabilités de St. Flour XX, 1990*. 111–235. *Lecture Notes in Math.* **1527** Springer, Berlin, 1992. MR1229519

[6] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Springer, Berlin, 1998.

[7] J. Rosen. Tanaka's formula for multiple intersections of planar Brownian motion. *Stochastic Process. Appl.* **23** (1986) 131–141. MR0866291

[8] J. Rosen. A renormalized local time for the multiple intersections of planar Brownian motion. In *Séminaire de Probabilités XX, 1984/85*. 515–531. *Lecture Notes in Math.* **1204**. Springer, Berlin, 1986. MR0942041

[9] J. Rosen. Derivatives of self-intersection local times. In *Séminaire de Probabilités, XXXVIII* 263–281. *Lecture Notes in Math.* **1857**. Springer, New York, 2005. MR2126979

[10] J. Rosen. Joint continuity and a Doob–Meyer type decomposition for renormalized intersection local times. *Ann. Inst. H. Poincaré Probab. Statist.* **35** (1999) 143–176. MR1678521

[11] J. Rosen. Joint continuity of renormalized intersection local times. *Ann. Inst. H. Poincaré Probab. Statist.* **32** (1996) 671–700. MR1422306

[12] J. Rosen. Dirichlet processes and an intrinsic characterization of renormalized intersection local times. *Ann. Inst. H. Poincaré Probab. Statist.* **37** (2001) 403–420. MR1876838

[13] J. Rosen. A stochastic calculus proof of the CLT for the L^2 modulus of continuity of local time. Preprint.

[14] S. R. S. Varadhan. Appendix to Euclidian quantum field theory by K. Symanzyk. In *Local Quantum Theory*. R. Jost (ed.). Academic Press, New York, 1969.