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Ballistic regime for random walks in random environment with unbounded jumps and Knudsen billiards

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Abstract. We consider a random walk in a stationary ergodic environment in $\mathbb{Z}$, with unbounded jumps. In addition to uniform ellipticity and a bound on the tails of the possible jumps, we assume a condition of strong transience to the right which implies that there are no "traps." We prove the law of large numbers with positive speed, as well as the ergodicity of the environment seen from the particle. Then, we consider Knudsen stochastic billiard with a drift in a random tube in $\mathbb{R}^d$, $d \geq 3$, which serves as environment. The tube is infinite in the first direction, and is a stationary and ergodic process indexed by the first coordinate. A particle is moving in straight line inside the tube, and has random bounces upon hitting the boundary, according to the following modification of the cosine reflection law: the jumps in the positive direction are always accepted while the jumps in the negative direction may be rejected. Using the results for the random walk in random environment together with an appropriate coupling, we deduce the law of large numbers for the stochastic billiard with a drift.

Résumé. Nous considérons une marche aléatoire dans un milieu stationnaire ergodique sur $\mathbb{Z}$, avec des sauts non bornés. En plus de l'uniforme ellipticité et d'une borne uniforme sur la queue de la loi des sauts, nous supposons une condition de transience forte qui garantit l'absence de "pièges." Nous montrons la loi des grands nombres avec vitesse strictement positive, ainsi que l'ergodicité de l'environnement vu de la particule. Par ailleurs, nous étudions aussi le billard stochastique de Knudsen avec dérive dans un tube aléatoire dans $\mathbb{R}^d$, $d \geq 3$, qui constitue l'environnement. Le tube est infini dans la première direction, et, vu comme un processus indéxé par la première coordonnée, il est supposé stationnaire ergodique. Une particule se déplace en ligne droite à l'intérieur du tube, avec des rebonds aléatoires sur le bord, selon la modification suivante de la loi de reflexion en cosinus: les sauts dans la direction positive sont toujours acceptés, tandis que ceux dans l'autre direction peuvent être rejetés. En utilisant les résultats pour la marche aléatoire en milieu aléatoire et un couplage approprié, nous obtenons la loi des grands nombres pour le billard stochastique avec dérive.

MSC: Primary 60K37; secondary 37D50; 60J25

Keywords: Cosine law; Stochastic billiard; Knudsen random walk; Random medium; Random walk in random environment; Unbounded jumps; Stationary ergodic environment; Regenerative structure; Point of view of the particle

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1. Introduction

Stochastic billiards deal with the motion of a particle inside a connected domain in the Euclidean space, travelling in straight lines inside the domain and subject to random bouncing when hitting the boundary. They are motivated by

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problems of transport and diffusion inside nanotubes, where the complex microscopic structure of the tube boundary allows for a stochastic description of the collisions: they can be viewed as limits of deterministic billiards on tables with rough boundary as the ratio of macro to micro scales diverges [8]. See also [5] and [7] for a detailed perspective from physics and chemistry, and [5,8] for basic results. A natural reflection law is when the outgoing direction has a density proportional to the cosine of its angle with the inner normal vector, independently from the past. This model, originally introduced by Martin Knudsen, is called the Knudsen stochastic billiard. It has two important features: the uniform measure on the phase space is invariant for the dynamics, and moreover it is reversible. To understand large time behavior of Knudsen billiards, one needs to consider infinite domains. Recurrence and transience is studied in [13], for billiards in a planar tube extending to infinity in the horizontal direction, under assumptions of regularity and growth on the tube. For the physically relevant case of an infinite tube which is irregular but has some homogeneity properties at large scale, the description of the tube as a random environment has been introduced in [6]: the domain is the realization of a stationary ergodic process indexed by the horizontal coordinate. Diffusivity of the particle is studied in this paper in dimension $d = 1 + (d - 1) \geq 2$: generically, when the tube does not have arbitrarily long cavities, the billiard is diffusive in dimension $d \geq 3$, and also for $d = 2$ when the billiard has "finite horizon." Reversibility allows to find the limit of the environment seen from the particle, and to use the appropriate techniques which have been extended to Random Walks in Random Environments (RWRE). We briefly mention [7] for the nonequilibrium dynamics aspects of the billiard and some features as a microscopic model for diffusion.

In the model of [6] the large-scale picture of the motion of the particle is purely diffusive; in particular, the limiting velocity of the particle equals zero. In this paper, we consider a stochastic billiard in a random tube as in [6] traversed by a flow with constant current to the right; our goal is to prove the law of large numbers (with positive limiting velocity). This current is modelled in the following way: the jumps in the positive direction are always accepted, but the jumps in the negative direction are accepted with probability $e^{-\lambda u}$, where u is the horizontal size of the attempted jump. This method of giving a drift to the particle has the following advantage: the reversibility of the stochastic billiard is preserved (although, of course, the reversible measure is no longer the same), which simplifies considerably the analysis of the model. In view of the above, the large scale picture is expected to be similar to the one-dimensional RWRE with a drift when the environment is given by a "resistor network" with a similar acceptance/rejection mechanism; the environment is not i.i.d. but stationary ergodic, the jumps are not nearest neighbor but *unbounded*.

We review known results on the law of large numbers for transient RWREs on $\mathbb{Z}$. For nearest neighbor jumps, the sub-ballistic and ballistic regimes – meaning that the speed is zero, resp. nonzero – are fully understood (e.g., [18] and Section 1 in [16]) with the explicit formula of Solomon for the speed [15] in the case of i.i.d. environment; the extension to stationary ergodic environment is given in [1]. When the jumps are bounded but not nearest neighbor, Key [10] shows that transience of the walk to the right amounts to positivity of some middle Lyapunov exponent of a product of random matrices. The regime where the law of large numbers holds with a positive speed is characterized in Brémont's [4] by the positivity of this exponent and existence of an invariant law for the environment absolutely continuous to the static law, but no explicit formula is anymore available. Goldsheid [9] gives sufficient conditions (which are also necessary in the case of i.i.d. environment), and also for the quenched central limit theorem. For completeness, we mention a result of Bolthausen and Goldsheid [3] for recurrent RWREs with bounded jumps: if the quenched drift is not a.s. zero, the typical displacement at time n is of order $\ln^2 n$, i.e., the RWRE has a similar lingering behavior as Sinaï's walk.

The case of unbounded jumps has been very seldom considered; in fact, we can only mention that Andjel [2] proves a 0–1 law when the jumps have uniform exponential tails.

In this paper, we prove the law of large numbers with a positive speed for RWRE on $\mathbb{Z}$ with unbounded jumps, under the following assumptions: stationary ergodic environment, (E) uniform ellipticity; (C) uniform (and integrable, but not necessarily exponential) tails for the jumps; (D) strong uniform transience to the right. We do not assume reversibility of the RWRE. The strategy is to consider an auxiliary RWRE with truncated jumps, to prove the existence of limits for the speed and the environment seen from the walker, then let the truncation parameter tend to infinity, and find a limit point for the environment measure. We mention also that assumption (D) precludes the existence of arbitrarily long traps – i.e., pieces of the environment where the random walk can spend an unusually large time –, and it is rather strong. We emphasize that we do not assume any mixing – hence, no independence – on the environment. As we see below, this set of conditions is adapted to our purpose. In our opinion, it is a challenging problem to find weaker conditions that still permit to obtain the law of large numbers for RWREs with unbounded jumps with only polynomial tails. In particular, it would be especially interesting to substitute the current condition (D) by a weaker one; however, at the moment we do not have any concrete results and/or plausible conjectures which go in that direction.

To apply this result to the billiard in random tube, we need a discretization procedure to compare the billiard to a random walk. This can be performed by coupling the billiard with an independent coin tossing; the integer part of the horizontal coordinate of hitting points on the boundary, sampled at success times of the coin tossing, is an embedded RWRE in some environment determined by the random tube. We check condition (D) for the RWRE by making use of the reversibility of the billiard and spectral estimates (as in [14] for a reversible RWRE on $\mathbb{Z}^d$). This coupling allows to transfer results from the RWRE – a simplified model – to the stochastic billiard – a much more involved one. Under fairly reasonable assumptions on the random tube, we obtain for the billiard the law of large numbers with positive speed.

The paper is organized as follows: we define the two models and state the results in the next section. Section 3 contains the proofs for RWRE, and Section 4 those for the stochastic billiard, including the construction of the coupling with the RWRE.

2. Formal definitions and results

Now, we formally define the random billiard with drift in a random tube and the one-dimensional random walk in stationary ergodic random environment with unbounded jumps.

Already at this point we warn the reader that the (continuous) random environment for the billiard processes and the (discrete) random environment for the random walk are denoted by the same letter ω. Hopefully, this creates no confusion since at all times we tried to make it clear which model is under consideration. Also, we use the same notation $\mathbb{P}$ (the law of the random environment) for both models.

2.1. *Random billiards with drift*

We define the model of random billiard in a random tube, basically keeping the notations of [6].

In this paper, $\mathbb{R}^{d-1}$ will always stand for the linear subspace of $\mathbb{R}^d$ which is perpendicular to the first coordinate vector $\mathbf{e}$, we use the notation $\|\cdot\|$ for the Euclidean norm in $\mathbb{R}^d$. Let $\mathcal{B}(x,\varepsilon)=\{y\in\mathbb{R}^d\colon\ \|x-y\|<\varepsilon\}$ be the open ε-neighborhood of $x\in\mathbb{R}^d$. Define $\mathbb{S}^{d-1}=\{y\in\mathbb{R}^d\colon\ \|y\|=1\}$ to be the unit sphere in $\mathbb{R}^d$. We write $|A|$ for the d-dimensional Lebesgue measure in case $A\subset\mathbb{R}^d$, and $(d-1)$-dimensional Hausdorff measure in case $A\subset\mathbb{S}^{d-1}$. Let

$$\mathbb{S}_h=\bigl\{w\in\mathbb{S}^{d-1}\colon\ h\cdot w>0\bigr\}$$

be the half-sphere looking in the direction h. For $x\in\mathbb{R}^d$, it will frequently be convenient to write $x=(\alpha,u)$, being α the first coordinate of x and $u\in\mathbb{R}^{d-1}$; then, $\alpha=x\cdot\mathbf{e}$, and we write $u=\mathcal{U}x$, where $\mathcal{U}$ is the projector on $\mathbb{R}^{d-1}$. Fix some positive constant $\widehat{M}$, and define

$$\Lambda=\bigl\{u\in\mathbb{R}^{d-1}\colon\ \|u\|\le\widehat{M}\bigr\}. \tag{1}$$

We denote by ∂A the boundary of $A\subset\mathbb{R}^d$, by $\bar{A}=A\cup\partial A$ the closure of A and by A° the interior of A (i.e., the largest open set contained in A).

Definition 2.1. *Let $k\in\{d-1,d\}$, and A a subset of $\mathbb{R}^k$. We say that ∂A is $(\hat{\varepsilon},\hat{L})$-Lipschitz, if for any $x\in\partial A$ there exist an affine isometry $\mathfrak{I}_x:\mathbb{R}^k\to\mathbb{R}^k$ and a function $f_x:\mathbb{R}^{k-1}\to\mathbb{R}$ such that*

- *f_x satisfies Lipschitz condition with constant $\hat{L}$, i.e., $|f_x(z)-f_x(z')|\le\hat{L}\|z-z'\|$ for all z,z';*
- *$\mathfrak{I}_x x=0$, $f_x(0)=0$, and*

$$\mathfrak{I}_x\bigl(A^\circ\cap\mathcal{B}(x,\hat{\varepsilon})\bigr)=\bigl\{z\in\mathcal{B}(0,\hat{\varepsilon})\colon\ z^{(k)}>f_x\bigl(z^{(1)},\dots,z^{(k-1)}\bigr)\bigr\}.$$

Now, fix $\widehat{M}$, and define $\mathcal{E}$ to be the set of all open domains A such that $A \subset \Lambda$ and ∂A is $(\hat{\varepsilon}, \hat{L})$-Lipschitz for some $(\hat{\varepsilon}, \hat{L})$ (which may depend on A). We turn $\mathcal{E}$ into a metric space by defining the distance between A and B to be equal to $|(A \setminus B) \cup (B \setminus A)|$. Let Ω be the space of all càdlàg functions $\mathbb{R} \to \mathcal{E}$, let $\mathcal{A}$ be the sigma-algebra generated by the cylinder sets with respect to the Borel sigma-algebra on $\mathcal{E}$, and let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{A})$. This defines a $\mathcal{E}$-valued process $\omega = (\omega_\alpha, \alpha \in \mathbb{R})$. Write θ_α for the spatial shift: $\theta_\alpha \omega_\cdot = \omega_{\cdot+\alpha}$. We suppose that the process ω is stationary and ergodic with respect to the family of shifts $(\theta_\alpha, \alpha \in \mathbb{R})$. With a slight abuse of notation, we denote also by

$$\omega = \{(\alpha, u) \in \mathbb{R}^d \colon u \in \omega_\alpha\}$$

the random domain ("tube") where the billiard lives. Intuitively, ω_α is the "slice" obtained by crossing ω with the hyperplane $\{\alpha\} \times \mathbb{R}^{d-1}$.

We will assume that the domain ω is connected. A trivial sufficient condition is that ω_α is connected for all α; a typical example is when $\partial\omega$ is generated by rotating around the horizontal axis the graph of a one-dimensional (stationary ergodic) process with values in $[1, \widehat{M}]$. In this paper, we will work under the more general Condition P below, which implies that ω is arc-connected.

We also assume the following condition.

Condition L. *There exist $\tilde{\varepsilon}, \tilde{L}$ such that $\partial\omega$ is $(\tilde{\varepsilon}, \tilde{L})$-Lipschitz (in the sense of Definition 2.1) $\mathbb{P}$-a.s.*

Denote by ν^ω the $(d-1)$-dimensional Hausdorff measure on $\partial\omega$; from Condition L one obtains that ν^ω is locally finite. We keep the usual notation $\mathrm{d}x, \mathrm{d}v, \mathrm{d}h, \dots$ for the $(d-1)$-dimensional Lebesgue measure on Λ (usually restricted to ω_α for some α) or the surface measure on $\mathbb{S}^{d-1}$.

Define the set of regular points

$$\mathcal{R}_\omega = \{x \in \partial\omega \colon \partial\omega \text{ is continuously differentiable in } x\}.$$

For all $x = (\alpha, u) \in \mathcal{R}_\omega$, let us define also the normal vector $\mathbf{n}_\omega(x) = \mathbf{n}_\omega(\alpha, u) \in \mathbb{S}^{d-1}$ pointing inside the domain ω.

We suppose that the following condition holds:

Condition R. *We have $\nu^\omega(\partial\omega \setminus \mathcal{R}_\omega) = 0$, $\mathbb{P}$-a.s.*

We say that $y \in \bar{\omega}$ is *seen from* $x \in \bar{\omega}$ if there exists $h \in \mathbb{S}^{d-1}$ and $t_0 > 0$ such that $x + th \in \omega$ for all $t \in (0, t_0)$ and $x + t_0 h = y$. Clearly, if y is seen from x then x is seen from y, and we write "$x \overset{\omega}{\leftrightarrow} y$" when this occurs.

One of the main objects of study in the paper [6] is the Knudsen random walk (KRW) which is a discrete time Markov process on $\partial\omega$, defined through its transition density K with respect to the surface measure ν^ω: for $x, y \in \partial\omega$

$$K(x, y) = \gamma_d \frac{((y - x) \cdot \mathbf{n}_\omega(x))((x - y) \cdot \mathbf{n}_\omega(y))}{\|x - y\|^{d+1}} \mathbf{1}\{x, y \in \mathcal{R}_\omega, x \overset{\omega}{\leftrightarrow} y\}, \tag{2}$$

where $\gamma_d = (\int_{\mathbb{S}_\mathbf{e}} h \cdot \mathbf{e}\, \mathrm{d}h)^{-1}$ is the normalizing constant. We also refer to the Knudsen random walk as the random walk with cosine reflection law, since it can be easily seen from (2) that the density of the outgoing direction is proportional to the cosine of the angle between this direction and the normal vector (see, e.g., formula (4) in [5]). In this paper, however, we shall consider the walk which "prefers" the positive direction: a jump in the direction $\mathbf{e}$ is always accepted, but if the walk attempts to jump in the negative direction $(-\mathbf{e})$, it is accepted with probability $\mathrm{e}^{-\lambda u}$, where u is the horizontal size of the attempted jump and $\lambda > 0$ is a given parameter. Formally, define

$$\hat{K}(x, y) = \begin{cases} K(x, y) & \text{if } (y - x) \cdot \mathbf{e} \geq 0, \\ \mathrm{e}^{\lambda (y-x)\cdot \mathbf{e}} K(x, y) & \text{if } (y - x) \cdot \mathbf{e} < 0, \end{cases} \tag{3}$$

and let

$$\Theta(x) = 1 - \int_{\partial\omega} \hat{K}(x, y)\, \mathrm{d}\nu^\omega(y). \tag{4}$$

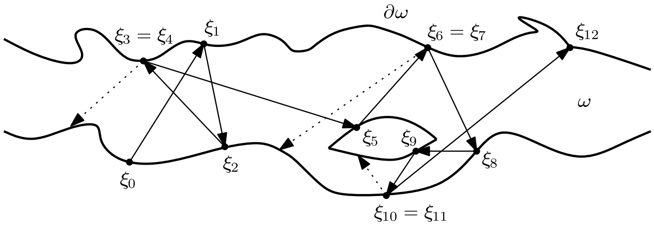

Fig. 1. Knudsen random walk with drift.

For a fixed ω, we define the *Knudsen random walk with drift* (KRWD) $(\xi_n; n \geq 0)$ – and denote by $\mathtt{P}_\omega, \mathtt{E}_\omega$ the corresponding quenched probability and expectation – as the Markov chain on $\partial\omega$ starting from $\xi_0 = 0$ such that, for any $x \in \mathcal{R}_\omega$ and any measurable $B \subset \partial\omega$ such that $x \notin B$,

$$\mathtt{P}_\omega[\xi_{n+1} \in B | \xi_n = x] = \int_B \hat{K}(x, y)\, \mathrm{d}\nu^\omega(y),$$

and

$$\mathtt{P}_\omega[\xi_{n+1} = x | \xi_n = x] = \Theta(x).$$

On Fig. 1 one can see a typical path of the random walk (rejected jumps are shown as dotted lines).

As observed in [5], $K(\cdot, \cdot)$ is symmetric (that is, $K(x, y) = K(y, x)$ for all $x, y \in \mathcal{R}_\omega$), so that the $(d-1)$-dimensional Hausdorff measure ν^ω is reversible for K. Then, with $\pi(x) = \mathrm{e}^{\lambda(x\cdot\mathbf{e})}$, the measure ν_λ^ω on $\partial\omega$ given by

$$\frac{\mathrm{d}\nu_\lambda^\omega}{\mathrm{d}\nu^\omega}(x) = \pi(x) = \mathrm{e}^{\lambda(x\cdot\mathbf{e})} \tag{5}$$

is such that $\pi(x)\hat{K}(x, y) = \pi(y)\hat{K}(y, x)$, showing that ν_λ^ω is reversible for the KRWD ξ. With some abuse of notation, we shall sometimes write $\pi(B) := \nu_\lambda^\omega(B)$ for $B \subset \partial\omega$.

We need also require a last technical assumption:

Condition P. *There exist constants N, ε, δ such that for $\mathbb{P}$-almost every ω, for any $x, y \in \mathcal{R}_\omega$ with $|(x - y)\cdot\mathbf{e}| \leq 2$ there exist $B_1, \ldots, B_n \subset \partial\omega$, $n \leq N - 1$ with $\nu^\omega(B_i) \geq \delta$ for all $i = 1, \ldots, n$, and such that*

- $K(x, z) \geq \varepsilon$ *for all* $z \in B_1$,
- $K(y, z) \geq \varepsilon$ *for all* $z \in B_n$,
- $K(z, z') \geq \varepsilon$ *for all* $z \in B_i$, $z' \in B_{i+1}$, $i = 1, \ldots, n-1$

(if $N = 1$ we only require that $K(x, y) \geq \varepsilon$). In other words, there exists a "thick" path of length at most N joining x and y. This assumption is already used in [6], *it prevents the tube from splitting into separate channels of arbitrary length, which could slow down the homogeneization.*

We prove the existence of the speed of KRWD:

Theorem 2.2. *Assume that $d \geq 3$. There exists a positive deterministic $\hat{v}$ such that for $\mathbb{P}$-almost every ω*

$$\frac{\xi_n\cdot\mathbf{e}}{n} \to \hat{v} \quad \text{as } n \to \infty,\ \mathtt{P}_\omega\text{-a.s.} \tag{6}$$

The assumption $d \geq 3$ crucially enters estimate (61). In dimension 2, depending on the geometry, KRWD can have large jumps, and then may not obey the law of large numbers. Naturally, if one assumes a strong additional condition that the size of the jumps is a.s. uniformly bounded (so-called finite horizon condition in the billiard literature), then our argument works in the case $d = 2$ as well. Still, we feel that Theorem 2.2 can hold in dimension 2 even without the finite horizon condition; for the proof, however, one would need estimates on the size of the jump that are finer than the "uniform" one provided by (59). Note that in the driftless case we can control the average size of the jump using the explicit form of the stationary measure for the environment seen from the particle (cf. Lemma 4.1 in [6]). Unfortunately, in the presence of the drift one does not obtain the stationary measure for this process in such an explicit way, and this is the reason why the situation in dimension 2 is less clear.

2.2. *One-dimensional random walk in random environment*

Let us consider a collection of nonnegative numbers $\omega = (\omega_{xy}; x, y \in \mathbb{Z})$, with the property $\sum_y \omega_{xy} = 1$ for all x. This collection is called the *environment*, and we denote by Ω the space of all environments. Next, we consider a Markov chain $(S_n, n = 0, 1, 2, \ldots)$ with the transition probabilities

$$\mathtt{P}_\omega^{x_0}[S_{n+1} = x + y | S_n = x] = \omega_{xy} \quad \text{for all } n \geq 0, \qquad \mathtt{P}_\omega^{x_0}[S_0 = x_0] = 1,$$

so that $\mathtt{P}_\omega^{x_0}$ is the *quenched* law of the Markov chain starting from x_0 in the environment ω. Let us write $\mathtt{P}_\omega$ for $\mathtt{P}_\omega^0$. The environment is chosen at random from the space Ω according to a law $\mathbb{P}$ before the random walk starts. We denote by $\mathtt{E}_\omega^{x_0}$ and $\mathbb{E}$ the expectations with respect to $\mathtt{P}_\omega^{x_0}$ and $\mathbb{P}$ correspondingly. Also, we assume that the sequence of random vectors $(\omega_{x\cdot}, x \in \mathbb{Z})$ is stationary and ergodic.

We need the following (one-sided) uniform ellipticity condition:

Condition E. *There exists* $\tilde{\varepsilon}$ *such that* $\mathbb{P}[\omega_{01} \geq \tilde{\varepsilon}] = 1$.

For any integer $\varrho > 1$ let us define also the "truncated" environment ω^ϱ by

$$\omega_{xy}^\varrho = \begin{cases} \omega_{xy} & \text{if } 0 < |y| < \varrho, \\ 0 & \text{if } |y| \geq \varrho, \\ \omega_{x0} + \sum_{y:|y|\geq\varrho} \omega_{xy} & \text{if } y = 0, \end{cases}$$

and observe also that formally $\omega = \omega^\infty$. The truncated random walk S^ϱ is then defined by

$$\mathtt{P}_\omega^{x_0}\big[S_{n+1}^\varrho = x + y | S_n^\varrho = x\big] = \omega_{xy}^\varrho \quad \text{for all } n \geq 0, \qquad \mathtt{P}_\omega^{x_0}\big[S_0^\varrho = x_0\big] = 1.$$

In words, the random walk S^ϱ in the truncated environment ω^ϱ is the modification of the original random walk where jumps of lengths less than ρ are kept, but larger jumps are rejected and the particle does not move. We shall sometimes also write e.g. $\mathtt{P}_\omega[S^{\varrho_1} \in \cdot, S^{\varrho_2} \in \cdot]$ meaning here the natural coupling of two versions of the random walk with different truncation but in the same environment. This coupling is defined in the following way:

- if $S_n^{\varrho_1} \neq S_n^{\varrho_2}$, then $S_{n+1}^{\varrho_1}$ and $S_{n+1}^{\varrho_2}$ are independent given $S_i^{\varrho_1}, S_i^{\varrho_2}, i \leq n$;
- if $S_n^{\varrho_1} = S_n^{\varrho_2} = x$, and Y_n is a random variable with $\mathtt{P}_\omega[Y_n = y] = \omega_{xy}$ and independent of $S_i^{\varrho_1}, S_i^{\varrho_2}, i \leq n$, then

$$S_{n+1}^{\varrho_i} = \begin{cases} x + Y_n & \text{if } |Y_n| < \varrho_i, \\ x & \text{if } |Y_n| \geq \varrho_i, \end{cases}$$

for $i = 1, 2$.

Let us assume the following condition on the tails of the possible jumps of the random walks:

Condition C. *There exist* $\gamma_1 > 0$ *and* $\alpha > 1$ *such that for all* $s \geq 1$ *we have*

$$\sum_{y:|y|\geq s} \omega_{0y} \leq \gamma_1 s^{-\alpha} \quad \mathbb{P}\text{-}a.s. \tag{7}$$

For $I \subset \mathbb{Z}_+$ and $A \subset \mathbb{Z}$ we denote by $N_I^\varrho(A)$ the number of visits to A of the random walk S^ϱ during the time set I, i.e.,

$$N_I^\varrho(A) = \sum_{k \in I} \mathbf{1}\{S_k^\varrho \in A\}.$$

We use the shorter notations $N_I^\varrho(x)$ for $N_I^\varrho(\{x\})$, $N_k^\varrho(A) := N_{[0,k]}^\varrho(A)$ for the number of visits to A during the time interval $[0,k]$, and $N_k^\varrho(x) := N_k^\varrho(\{x\})$.

Next, we make another assumption that says, essentially, that the random walk is "uniformly" transient to the right (i.e., there are no "traps").

Condition D. *There is a nonincreasing function $g_1 \geq 0$ with the property $\sum_{k=1}^\infty k g_1(k) < \infty$ and a finite ϱ_0, such that for all $x \leq 0$ and all $\varrho \geq \varrho_0$, $\mathbb{P}$-almost surely it holds that $\mathrm{E}_\omega^0 N_\infty^\varrho(x) \leq g_1(|x|)$.*

With these assumptions, we can prove that the speed of the random walk is well defined and positive:

Theorem 2.3. *For all $\varrho \in [\varrho_0, \infty]$ there exists $v_\varrho > 0$ such that for $\mathbb{P}$-a.a. ω we have*

$$\frac{S_n^\varrho}{n} \to v_\varrho \quad \text{as } n \to \infty, \ \mathrm{P}_\omega\text{-a.s.} \tag{8}$$

Next, we are interested in the environment seen from the particle. Let θ_z be the shift to z acting on ω in the following way: $(\theta_z\omega)_{xy} = \omega_{x+z,y}$. The process of the environment viewed from the particle (with respect to S^ϱ) is defined by $\omega(n) = \theta_{S_n^\varrho}\omega$.

Theorem 2.4. *For all $\varrho \in [\varrho_0, +\infty]$ there exists an unique invariant measure $\mathbb{Q}^\varrho$ for the process of the environment viewed from the particle with $\mathbb{Q}^\varrho \ll \mathbb{P}$. Then, we have*

$$v_\varrho = \int_\Omega \mathrm{E}_\omega^0 S_1^\varrho \, \mathrm{d}\mathbb{Q}^\varrho. \tag{9}$$

Moreover, for all $\varrho \in [\varrho_0, +\infty]$ the measure $\mathbb{Q}^\varrho$ is ergodic and $\mathbb{Q}^\varrho$ weakly converges to $\mathbb{Q}^\infty$ as $\varrho \to \infty$. Finally, it holds that $v_\varrho \to v_\infty$ as $\varrho \to \infty$.

Remark 2.5. *In the case $\varrho < \infty$ the invariant measure $\mathbb{Q}^\varrho$ is given by the formula (27) in Section 3.*

3. Proofs for RWRE

Denote by $T_z^\varrho = \min\{k \geq 0:\ S_k^\varrho \geq z\}$. We use the simplified notation $T^\varrho := T_0^\varrho$. Let

$$r_x^\varrho(z) = \mathrm{P}_\omega^x\big[S_{T_z^\varrho}^\varrho = z\big]$$

be the probability that, at moment T_z^ϱ, the (truncated) random walk is located *exactly* at z. We also use the shorter notation $r_x^\varrho := r_x^\varrho(0)$. Of course, the quantity $r_x^\varrho(z)$ depends also on ω, but, for the sake of simplicity, we keep it this way.

The key fact needed in the course of the proof of our results is the following lemma:

Lemma 3.1. *Assume Conditions E, C, D. Then, there exists $\varepsilon_1 > 0$ such that, $\mathbb{P}$-a.s.,*

$$r_x^\varrho \geq 2\varepsilon_1$$

for all $x \leq 0$ and for all $\varrho \in [\varrho_0, \infty]$.

Proof. Let us denote $\mathbb{Z}_-^* := \mathbb{Z} \cap (-\infty, -1]$, $\mathbb{Z}_- := \mathbb{Z} \cap (-\infty, 0]$. For $\varrho \geq \varrho_0$ and $x \leq -1$,

$$\begin{aligned}
\mathrm{E}_\omega^x N_\infty^\varrho(\mathbb{Z}_-) &= \sum_{z \leq x} \mathrm{E}_\omega^x N_\infty^\varrho(z) + \sum_{x<z\leq 0} \mathrm{E}_\omega^x N_\infty^\varrho(z) \\
&= \sum_{z \leq x} \mathrm{E}_\omega^x N_\infty^\varrho(z) + \sum_{x<z\leq 0} \mathrm{E}_\omega^z N_\infty^\varrho(z) \quad \text{(Markov property)} \\
&\leq \sum_k g_1(k) + |x| g_1(0) \quad \text{(Condition D)} \\
&\leq C_1 |x| \qquad (10)
\end{aligned}$$

for $\mathbb{P}$-almost all ω, with some finite constant C_1. Since

$$\{T^\varrho > k\} = \{N_k^\varrho(\mathbb{Z}_-^*) \geq k+1\} \subset \{N_k^\varrho(\mathbb{Z}_-) \geq k\}$$

for such an x, using Chebyshev's inequality we obtain, for $\mathbb{P}$-almost all ω and for all $\varrho \geq \varrho_0$,

$$\begin{aligned}
\mathrm{P}_\omega^x[T^\varrho > k] &\leq \mathrm{P}_\omega^x[N_k^\varrho(\mathbb{Z}_-) \geq k] \leq \mathrm{P}_\omega^x[N_\infty^\varrho(\mathbb{Z}_-) \geq k] \\
&\leq \frac{\mathrm{E}_\omega^x N_\infty^\varrho(\mathbb{Z}_-)}{k} \leq \frac{C_1|x|}{k}. \qquad (11)
\end{aligned}$$

Let us fix $\delta_1 > 0$ such that $1 + \delta_1 < \alpha$ with α from Condition C, and fix some $\beta \in (1, \frac{\alpha}{1+\delta_1})$. Observe that (11) implies that for any $x \in [-n^\beta, 0]$

$$\mathrm{P}_\omega^x[T^\varrho > n^{\beta(1+\delta_1)}] \leq C_1 n^{-\beta\delta_1} \qquad (12)$$

for $\mathbb{P}$-almost all ω and for all $\varrho \geq \varrho_0$.

Fix a real number $s \geq 1$ and denote

$$\sigma_s = \min\{k \geq 0: S_k^\varrho \in [-s, 0]\}.$$

Let G_s be the event defined as

$$G_s = \{|S_k^\varrho - S_{k-1}^\varrho| \leq s \text{ for all } k \leq s^{\beta(1+\delta_1)}\}.$$

By Condition C, it is straightforward to obtain that, for some $C_2 > 0$

$$\mathrm{P}_\omega[G_s] \geq 1 - C_2 s^{-(\alpha-(1+\delta_1)\beta)} \qquad (13)$$

for all $\varrho \geq \varrho_0$ (observe that, by definition, $\alpha > (1+\delta_1)\beta$). Also, note that, by the Markov property, when $x \leq y$ and $-s \leq y \leq 0$,

$$r_y^\varrho = \mathrm{P}_\omega^x[S_{T^\varrho}^\varrho = 0 | \sigma_s < T^\varrho, S_{\sigma_s}^\varrho = y]. \qquad (14)$$

On the event $G_s \cap \{T^\varrho \leq s^{\beta(1+\delta_1)}\}$ we have $\sigma_s < T^\varrho$ a.s. for $x < 0$, and using also (12), (13), (14), we obtain for any $x \in [-s^\beta, 0)$

$$\begin{aligned}
r_x^\varrho &= \mathrm{P}_\omega^x[S_{T^\varrho}^\varrho = 0] \\
&\geq \mathrm{P}_\omega^x[S_{T^\varrho}^\varrho = 0 | \sigma_s < T^\varrho] \mathrm{P}_\omega^x[\sigma_s < T^\varrho] \\
&\geq \Bigl(\min_{y\in[-s,0]} r_y^\varrho\Bigr) \mathrm{P}_\omega^x[G_s, T^\varrho \leq s^{\beta(1+\delta_1)}] \\
&\geq \Bigl(\min_{y\in[-s,0]} r_y^\varrho\Bigr)\bigl(1 - C_1 s^{-\beta\delta_1} - C_2 s^{-(\alpha-(1+\delta_1)\beta)}\bigr). \qquad (15)
\end{aligned}$$

For any real number $k \geq 1$, define

$$u_k = \operatorname*{ess\,inf}_{\mathbb{P}} \min_{y \in [-k,0]} r_y^{\varrho},$$

which depends also on ω, and let $\varphi := \min\{\beta\delta_1, \alpha - (1+\delta_1)\beta\}$. Now, (15) implies that for some $C_3 > 0$

$$u_{s^\beta} \geq \left(1 - C_3 s^{-\varphi}\right) u_s. \tag{16}$$

By the ellipticity Condition E, we have $u_2 > 0$. Iterating (16), we obtain that $u_m \geq 2\varepsilon_1 > 0$ for all $m \geq 2$, where

$$\varepsilon_1 = \frac{1}{2} u_2 \left(1 - C_3 2^{-\varphi}\right)\left(1 - C_3 2^{-\beta\varphi}\right)\left(1 - C_3 2^{-\beta^2\varphi}\right)\left(1 - C_3 2^{-\beta^3\varphi}\right)\cdots$$

is indeed positive since it holds that $\sum_j 2^{-\beta^j \varphi} < \infty$. This concludes the proof of Lemma 3.1. □

Now, fix some integer $\varrho \in [\varrho_0, \infty)$, and consider a sequence of i.i.d. random variables $\zeta_1, \zeta_2, \zeta_3, \ldots$ with $P[\zeta_j = 1] = 1 - P[\zeta_j = 0] = \varepsilon_1$ (the parameter ε_1 is from Lemma 3.1, and P stands for the law of this sequence; in the sequel we shall use also E for the expectation corresponding to P). Then, our strategy can be described in words in the following way. For all $j \geq 1$, Lemma 3.1 implies that $r_x^{\varrho}(j\varrho) \geq 2\varepsilon_1$ for all $x \in [(j-1)\varrho, j\varrho - 1]$. We couple the sequence $\zeta = (\zeta_1, \zeta_2, \zeta_3, \ldots)$ with the random walk S^{ϱ} in such a way that $\zeta_j = 1$ implies that $S^{\varrho}_{T^{\varrho}_{j\varrho}} = j\varrho$. Denote

$$\ell_1 = \min\{j \colon \zeta_j = 1\}. \tag{17}$$

Then, since ζ (and therefore ℓ_1) is independent of ω, $\theta_{\ell_1\varrho}\omega$ has the same law $\mathbb{P}$ as ω. This allows us to break the trajectory of the random walk into stationary ergodic (after suitable shift) sequence of pieces, and then apply the ergodic theorem to obtain the law of large numbers. The stationary measure of the environment seen from the particle (for the truncated random walk) can also be obtained from this construction by averaging along the cycle. Then, we pass to the limit as $\rho \to \infty$.

So, let us now construct the quenched law $\mathtt{P}_{\omega,\zeta}$, i.e., the law of the random walk S^{ϱ} when *both* the environment ω and the sequence ζ are fixed. This is done inductively: first, the law of $(S^{\varrho}_k, k \leq T^{\varrho}_{\varrho})$ is defined by

$$\mathbf{1}\{\zeta_1 = 1\}\mathtt{P}_\omega\left[\cdot \mid S^{\varrho}_{T^{\varrho}_{\varrho}} = \varrho\right] + \mathbf{1}\{\zeta_1 = 0\}\left(\frac{r_0^{\varrho}(\varrho) - \varepsilon_1}{1 - \varepsilon_1}\mathtt{P}_\omega\left[\cdot \mid S^{\varrho}_{T^{\varrho}_{\varrho}} = \varrho\right] + \frac{1 - r_0^{\varrho}(\varrho)}{1 - \varepsilon_1}\mathtt{P}_\omega\left[\cdot \mid S^{\varrho}_{T^{\varrho}_{\varrho}} > \varrho\right]\right).$$

Then, given $S^{\varrho}_{T^{\varrho}_{j\varrho}} = y \in [j\varrho, (j+1)\varrho - 1]$, the law of $(S^{\varrho}_k, T^{\varrho}_{j\varrho} + 1 \leq k \leq T^{\varrho}_{(j+1)\varrho})$ is

$$\begin{aligned}
&\mathbf{1}\{\zeta_{j+1} = 1\}\mathtt{P}^y_\omega\left[\cdot \mid S^{\varrho}_{T^{\varrho}_{(j+1)\varrho}} = (j+1)\varrho\right] \\
&\quad + \mathbf{1}\{\zeta_{j+1} = 0\}\left(\frac{r_0^{\varrho}((j+1)\varrho) - \varepsilon_1}{1 - \varepsilon_1}\mathtt{P}^y_\omega\left[\cdot \mid S^{\varrho}_{T^{\varrho}_{(j+1)\varrho}} = (j+1)\varrho\right]\right. \\
&\qquad\qquad \left. + \frac{1 - r_0^{\varrho}((j+1)\varrho)}{1 - \varepsilon_1}\mathtt{P}_\omega\left[\cdot \mid S^{\varrho}_{T^{\varrho}_{(j+1)\varrho}} > (j+1)\varrho\right]\right).
\end{aligned}$$

Let $\mathbb{P}' := \mathbb{P} \otimes P$ (where P is the law of ζ), and $\mathbb{E}'$ the expectation corresponding to $\mathbb{P}'$. With $\ell_0 := 0$, let us define consistently with (17)

$$\ell_{k+1} = \min\{j > \ell_k \colon \zeta_j = 1\}, \qquad k \geq 0. \tag{18}$$

Note that, by construction,

$$S^{\varrho}_{T^{\varrho}_{\ell_k\varrho}} = \ell_k \varrho$$

for all $k \geq 1$. We now define a regeneration structure, which is fundamental in our construction. Following Chapter 8 of [17], we recall the definition of cycle-stationarity of a stochastic process together with a sequence of points. Consider, on some probability space, (i) a sequence $Z = (Z_n)_n$ of random variables with values in some measurable space $(F, \mathcal{F})$, (ii) a sequence of random times Σ (called "time points"), $0 < \Sigma_1 < \Sigma_2 < \cdots$. Define the kth cycle $\mathcal{C}_k = (Z_n; \Sigma_k \leq n \leq \Sigma_{k+1} - 1) \in \bigcup_{m \geq 1} F^m$. The sequence Z is cycle-stationary with points Σ if $(\mathcal{C}_k; k \geq 1)$ has the same law as $(\mathcal{C}_{k+1}; k \geq 1)$. It is cycle-stationary and ergodic if $(\mathcal{C}_k; k \geq 1)$ is stationary and ergodic.

Lemma 3.2. *Let $\rho < \infty$. The pair $(\theta_{S^{\varrho}_{\cdot}}\omega, T^{\varrho}_{\ell_{\cdot}\varrho})$ is cycle-stationary and ergodic. In particular, $\theta_{\ell_k \varrho}\omega$ has the same law as ω for all $k = 1, 2, 3, \ldots$.*

In short, the kth cycle $\mathcal{C}_k$ is the sequence of environments seen from the truncated walk $S^{\varrho}_{\cdot}$ from time $T^{\varrho}_{\ell_{k-1}\varrho}$ to time $T^{\varrho}_{\ell_k \varrho} - 1$ $(k = 1, 2, \ldots)$. The first statement in the lemma is that the sequence $(\mathcal{C}_k; k \geq 1)$ is stationary under the measure $P \otimes \mathbb{P} \otimes \mathrm{P}_{\omega,\zeta}$.

Proof of Lemma 3.2. Let us denote by $\boldsymbol{\mathcal{C}}$ the above sequence, and by ϑ the shift $(\vartheta \boldsymbol{\mathcal{C}})_k = \mathcal{C}_{k+1}$. With $f \geq 0$ a measurable function on the appropriate space, we write

$$
\begin{aligned}
E\mathbb{E}\mathrm{E}_{\omega,\zeta} f(\vartheta \boldsymbol{\mathcal{C}}) &= \sum_{m \geq 1} E\mathbb{E}\mathbf{1}\{\ell_1 = m\}\mathrm{E}_{\omega,\zeta} f(\vartheta \boldsymbol{\mathcal{C}}) \\
&= \sum_{m \geq 1} E\mathbb{E}\mathbf{1}\{\ell_1 = m\}\mathrm{E}^{m\varrho}_{\omega,\zeta} f(\boldsymbol{\mathcal{C}}) \quad \text{(Markov property)} \\
&= \sum_{m \geq 1} \varepsilon_1 (1 - \varepsilon_1)^{m-1} \mathbb{E}\mathrm{E}^{m\varrho}_{\omega,\zeta} f(\boldsymbol{\mathcal{C}}) \quad \text{(independence)} \\
&= \sum_{m \geq 1} \varepsilon_1 (1 - \varepsilon_1)^{m-1} \mathbb{E}\mathrm{E}_{\omega,\zeta} f(\boldsymbol{\mathcal{C}}) \quad \text{($\mathbb{P}$-stationarity)} \\
&= E\mathbb{E}\mathrm{E}_{\omega,\zeta} f(\boldsymbol{\mathcal{C}}),
\end{aligned}
$$

which shows the cycle-stationarity. The ergodicity then follows from the ergodicity of $\mathbb{P}$ and independence of ω and ζ, see Section 7 of Chapter 8 of [17]. $\square$

Now, we are able to prove the existence of the speed v_{ϱ} for the truncated random walk. First, we prove the following lemma.

Lemma 3.3. *There exist $C_4, C_5 > 0$ such that for $\mathbb{P}$-almost all ω and for all $\varrho \in [\varrho_0, \infty)$ we have*

$$
C_4 \varrho \leq E\mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1 \varrho} \leq C_5 \varrho. \tag{19}
$$

Proof. We begin by proving the second inequality in (19). Write

$$
\mathrm{P}^x_{\omega}\big[T^{\varrho} > k\big] \leq \mathrm{P}^x_{\omega}\big[N^{\varrho}_k\big(\mathbb{Z}^*_-\big) \geq k + 1\big] \leq \mathrm{P}^x_{\omega}\big[N^{\varrho}_{\infty}(\mathbb{Z}_-) \geq k + 1\big],
$$

so, using (10) we obtain

$$
\begin{aligned}
\mathrm{E}^x_{\omega} T^{\varrho} &= \sum_{k \geq 0} \mathrm{P}^x_{\omega}\big[T^{\varrho} > k\big] \\
&\leq \sum_{k \geq 0} \mathrm{P}^x_{\omega}\big[N^{\varrho}_{\infty}(\mathbb{Z}_-) \geq k + 1\big] \\
&= \mathrm{E}^x_{\omega} N^{\varrho}_{\infty}(\mathbb{Z}_-) \\
&\leq C_1 |x|.
\end{aligned} \tag{20}
$$

Using the elementary inequality $\mathtt{E}(Y|A) \le \frac{\mathtt{E}Y}{\mathtt{P}[A]}$ for $Y \ge 0$ together with Lemma 3.1, we obtain that on $\{\zeta_1 = 1\}$

$$\mathtt{E}_{\omega,\zeta} T_\varrho^\varrho = \mathtt{E}_\omega\big(T_\varrho^\varrho | S^\varrho_{T_\varrho^\varrho} = \varrho\big) \le \frac{\mathtt{E}_\omega T_\varrho^\varrho}{r_0^\varrho(\varrho)} \le \frac{1}{\varepsilon_1}\mathtt{E}_\omega T_\varrho^\varrho,$$

and that on $\{\zeta_1 = 0\}$ (observe that $\frac{r_0^\varrho(\varrho)-\varepsilon_1}{r_0^\varrho(\varrho)} \le \frac{1}{\varepsilon_1}$ by Lemma 3.1)

$$\begin{aligned}
\mathtt{E}_{\omega,\zeta} T_\varrho^\varrho &= \frac{r_0^\varrho(\varrho)-\varepsilon_1}{1-\varepsilon_1}\mathtt{E}_\omega\big(T_\varrho^\varrho | S^\varrho_{T_\varrho^\varrho} = \varrho\big) + \frac{1-r_0^\varrho(\varrho)}{1-\varepsilon_1}\mathtt{E}_\omega\big(T_\varrho^\varrho | S^\varrho_{T_\varrho^\varrho} > \varrho\big) \\
&= \frac{r_0^\varrho(\varrho)-\varepsilon_1}{(1-\varepsilon_1)r_0^\varrho(\varrho)} r_0^\varrho(\varrho)\mathtt{E}_\omega\big(T_\varrho^\varrho | S^\varrho_{T_\varrho^\varrho} = \varrho\big) \\
&\quad + \frac{1}{1-\varepsilon_1}\big(1-r_0^\varrho(\varrho)\big)\mathtt{E}_\omega\big(T_\varrho^\varrho | S^\varrho_{T_\varrho^\varrho} > \varrho\big) \\
&\le \frac{1}{\varepsilon_1(1-\varepsilon_1)}\mathtt{E}_\omega\big(T_\varrho^\varrho\big),
\end{aligned}$$

so for any ζ we obtain

$$\mathtt{E}_{\omega,\zeta} T_\varrho^\varrho \le \frac{1}{\varepsilon_1(1-\varepsilon_1)}\mathtt{E}_\omega T_\varrho^\varrho. \tag{21}$$

In the same way, we show that for any ζ and for all $j \ge 1$

$$\mathtt{E}^y_{\omega,\zeta} T^\varrho_{(j+1)\varrho} \le \frac{1}{\varepsilon_1(1-\varepsilon_1)}\mathtt{E}^y_\omega T^\varrho_{(j+1)\varrho} \tag{22}$$

for all $y \in [j\varrho, (j+1)\varrho - 1]$.

Writing

$$T^\varrho_{k\varrho} = T^\varrho_\varrho + \big(T^\varrho_{2\varrho} - T^\varrho_\varrho\big) + \cdots + \big(T^\varrho_{k\varrho} - T^\varrho_{(k-1)\varrho}\big),$$

and using (20), (21), (22), we obtain on $\{\ell_1 = k\}$ that

$$\mathtt{E}_{\omega,\zeta} T^\varrho_{k\varrho} \le \frac{1}{\varepsilon_1(1-\varepsilon_1)} kC_1\varrho.$$

Since $P[\ell_1 = k] = \varepsilon_1(1-\varepsilon_1)^{k-1}$, we see that, for $\mathbb{P}$-a.a. ω

$$\begin{aligned}
E\mathtt{E}_{\omega,\zeta} T^\varrho_{\ell_1\varrho} &\le \sum_{k=1}^\infty (1-\varepsilon_1)^{k-2} kC_1\varrho \\
&= C_5\varrho
\end{aligned}$$

with $C_5 = C_1\varepsilon_1^{-2}(1-\varepsilon_1)^{-1}$, and the proof of the second inequality in (19) is finished.

Let us prove the first inequality in (19). Consider a sequence of i.i.d. positive integer-valued random variables $Y_1, Y_2, Y_3, \ldots$ with the law

$$\mathtt{P}[Y_1 \ge s] = \big(\gamma_1 s^{-\alpha}\big) \wedge 1,$$

$s \ge 1$, with α, γ_1 from Condition C. Then, on $\{S_0^\varrho = 0\}$, it holds that for any ϱ and for $\mathbb{P}$-almost all ω, S_n^ϱ is dominated by $Y_1 + \cdots + Y_n$. From this, we easily obtain

$$\mathtt{E}_\omega T_\varrho^\varrho \ge C_7\varrho. \tag{23}$$

Next, note that $r_0^\varrho(\varrho) - \varepsilon_1 \ge \varepsilon_1$, so on the event $\{\zeta_1 = 0\}$ we have

$$\begin{aligned}
\mathrm{E}_{\omega,\zeta} T_\varrho^\varrho &= \frac{r_0^\varrho(\varrho) - \varepsilon_1}{(1-\varepsilon_1) r_0^\varrho(\varrho)} r_0^\varrho(\varrho) \mathrm{E}_\omega\big(T_\varrho^\varrho \mid S^\varrho_{T_\varrho^\varrho} = \varrho\big) \\
&\quad + \frac{1}{1-\varepsilon_1}\big(1 - r_0^\varrho(\varrho)\big) \mathrm{E}_\omega\big(T_\varrho^\varrho \mid S^\varrho_{T_\varrho^\varrho} > \varrho\big) \\
&\ge \frac{\varepsilon_1}{1-\varepsilon_1} \mathrm{E}_\omega T_\varrho^\varrho.
\end{aligned} \tag{24}$$

So, using (23) we obtain that on $\{\zeta_1 = 0\}$

$$\mathrm{E}_{\omega,\zeta} T^\varrho_{\ell_1 \varrho} \ge \mathrm{E}_{\omega,\zeta} T_\varrho^\varrho \ge \frac{\varepsilon_1}{1-\varepsilon_1} \mathrm{E}_\omega T_\varrho^\varrho \ge \frac{C_7 \varepsilon_1}{1-\varepsilon_1} \varrho,$$

and, finally,

$$\mathbb{E}' \mathrm{E}_{\omega,\zeta} T^\varrho_{\ell_1 \varrho} \ge \frac{C_7 \varepsilon_1}{1-\varepsilon_1} \varrho P[\zeta_1 = 0] = C_7 \varepsilon_1 \varrho.$$

Proof of Lemma 3.3 is finished. □

Now, we show that

$$v_\varrho = \frac{\varrho E \ell_1}{\mathbb{E}' \mathrm{E}_{\omega,\zeta} T^\varrho_{\ell_1 \varrho}} = \frac{\varrho}{\varepsilon_1 \mathbb{E}' \mathrm{E}_{\omega,\zeta} T^\varrho_{\ell_1 \varrho}}, \tag{25}$$

which implies that $v_\varrho > 0$ by Lemma 3.3. Indeed, suppose that n is such that $T^\varrho_{\ell_k \varrho} \le n < T^\varrho_{\ell_{k+1} \varrho}$. Then, we have $\ell_{k+1}\varrho - (T^\varrho_{\ell_{k+1}\varrho} - T^\varrho_{\ell_k \varrho})\varrho < S_n^\varrho < \ell_{k+1}\varrho$, so

$$\frac{\ell_{k+1}\varrho - (T^\varrho_{\ell_{k+1}\varrho} - T^\varrho_{\ell_k \varrho})\varrho}{T^\varrho_{\ell_{k+1}\varrho}} \le \frac{S_n^\varrho}{n} \le \frac{\ell_{k+1}\varrho}{T^\varrho_{\ell_k \varrho}}. \tag{26}$$

Now, we divide the numerator and the denominator by k in (26), we use Lemmas 3.2, 3.3 and the ergodic theorem to get

$$\lim_{n\to\infty} \frac{S_n^\varrho}{n} = v_\varrho,$$

with v_ϱ given by the second member of (25). Since $(\ell_{k+1} - \ell_k)$ has a geometric distribution with parameter ε_1 we get the last expression in (25). This ends the proof of Theorem 2.3 for finite ϱ.

With the help of Lemma 3.2, we derive that there exists an invariant measure for the environment seen from the particle in the truncated case. By formula (4.14°) of Chapter 8 of [17], for $\varrho < \infty$ we can characterize this measure $\mathbb{Q}^\varrho$ by its expectation $\mathbb{E}_\varrho$,

$$\mathbb{E}_\varrho f(\omega) = \frac{1}{\mathbb{E}' \mathrm{E}_{\omega,\zeta} T^\varrho_{\ell_1 \varrho}} \mathbb{E}' \mathrm{E}_{\omega,\zeta} \sum_{k=1}^{T^\varrho_{\ell_1 \varrho}} f(\theta_{S_k^\varrho} \omega). \tag{27}$$

Next, we need to pass to the limit as $\varrho \to \infty$. This requires a fine analysis of the Radon–Nikodym derivative $\frac{\mathrm{d}\mathbb{Q}^\varrho}{\mathrm{d}\mathbb{P}}$, and this is what we are going to do now.

Proposition 3.4. *Let ϱ be finite. Then*,

$$\frac{\mathrm{d}\mathbb{Q}^{\varrho}}{\mathrm{d}\mathbb{P}}(\omega) = \frac{1}{\mathbb{E}' \mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1 \varrho}} \sum_{x\in\mathbb{Z}} E\,\mathrm{E}_{\theta_{-x}\omega^{\varrho},\zeta} N^{\varrho}_{T^{\varrho}_{\ell_1\varrho}}(x). \tag{28}$$

Moreover, there exist $\hat{\gamma}_1, \hat{\gamma}_2 \in (0,\infty)$ (not depending on ϱ) such that for $\mathbb{P}$-almost all ω

$$\hat{\gamma}_1 \le \frac{\mathrm{d}\mathbb{Q}^{\varrho}}{\mathrm{d}\mathbb{P}}(\omega) \le \hat{\gamma}_2. \tag{29}$$

Proof. Using translation invariance of $\mathbb{P}$, we write expression (27) as

$$\begin{aligned}
\mathbb{E}_{\varrho} f(\omega) &= \frac{1}{\mathbb{E}' \mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1 \varrho}} \int \mathrm{d}\mathbb{P}' \mathrm{E}_{\omega,\zeta} \sum_{k=1}^{T^{\varrho}_{\ell_1\varrho}} f(\theta_{S^{\varrho}_k}\omega) \\
&= \frac{1}{\mathbb{E}' \mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1 \varrho}} \int \mathrm{d}\mathbb{P}' \sum_{k=1}^{\infty} \mathrm{E}_{\omega,\zeta}\big(f(\theta_{S^{\varrho}_k}\omega); k \le T^{\varrho}_{\ell_1\varrho}\big) \\
&= \frac{1}{\mathbb{E}' \mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1 \varrho}} \sum_{k=1}^{\infty}\sum_{x\in\mathbb{Z}} \int \mathrm{d}\mathbb{P}' f(\theta_x\omega) \mathrm{P}_{\omega,\zeta}\big[S^{\varrho}_k = x, k \le T^{\varrho}_{\ell_1\varrho}\big] \\
&= \frac{1}{\mathbb{E}' \mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1 \varrho}} \sum_{k=1}^{\infty}\sum_{x\in\mathbb{Z}} \int \mathrm{d}\mathbb{P}' f(\omega) \mathrm{P}_{\theta_{-x}\omega^{\varrho},\zeta}\big[S^{\varrho}_k = x, k \le T^{\varrho}_{\ell_1\varrho}\big] \\
&= \int \mathrm{d}\mathbb{P}' f(\omega) \frac{1}{\mathbb{E}' \mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1 \varrho}} \sum_{x\in\mathbb{Z}} \mathrm{E}_{\theta_{-x}\omega^{\varrho},\zeta} N^{\varrho}_{T^{\varrho}_{\ell_1\varrho}}(x) \\
&= \int \mathrm{d}\mathbb{P} f(\omega) \frac{1}{\mathbb{E}' \mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1 \varrho}} \sum_{x\in\mathbb{Z}} E\,\mathrm{E}_{\theta_{-x}\omega^{\varrho},\zeta} N^{\varrho}_{T^{\varrho}_{\ell_1\varrho}}(x),
\end{aligned}$$

which proves (28).

Let us prove (29). Write

$$\begin{aligned}
\frac{\mathrm{d}\mathbb{Q}^{\varrho}}{\mathrm{d}\mathbb{P}}(\omega) &= \frac{1}{\mathbb{E}' \mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1 \varrho}} E\Bigg(\sum_{x<0} \mathrm{E}_{\theta_{-x}\omega^{\varrho},\zeta} N^{\varrho}_{T^{\varrho}_{\ell_1\varrho}}(x) + \sum_{x\ge 0} \mathrm{E}_{\theta_{-x}\omega^{\varrho},\zeta} N^{\varrho}_{T^{\varrho}_{\ell_1\varrho}}(x)\Bigg) \\
&\le \frac{1}{\mathbb{E}' \mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1 \varrho}} E \sum_{x<0} \mathrm{E}_{\theta_{-x}\omega^{\varrho},\zeta} N^{\varrho}_{\infty}(x) \\
&\quad + \frac{1}{\mathbb{E}' \mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1 \varrho}} E \sum_{0<x\le(\ell_1+1)\varrho} \mathrm{E}_{\theta_{-x}\omega^{\varrho},\zeta} N^{\varrho}_{\infty}(x) \\
&=: \mathbf{A}_1 + \mathbf{A}_2.
\end{aligned} \tag{30}$$

Next, we need to obtain upper bounds on the terms $\mathbf{A}_1, \mathbf{A}_2$; for that, let us write first

$$N^{\varrho}_{\infty}(x) = N^{\varrho}_{[0,T^{\varrho}_{\varrho})}(x) + N^{\varrho}_{[T^{\varrho}_{\varrho},T^{\varrho}_{2\varrho})}(x) + N^{\varrho}_{[T^{\varrho}_{2\varrho},T^{\varrho}_{3\varrho})}(x) + \cdots. \tag{31}$$

Analogously to (21) and (22), for any ζ we obtain

$$\mathrm{E}_{\omega,\zeta} N^{\varrho}_{[0,T^{\varrho}_{\varrho})}(x) \le \frac{1}{\varepsilon_1(1-\varepsilon_1)} \mathrm{E}_{\omega} N^{\varrho}_{[0,T^{\varrho}_{\varrho})}(x), \tag{32}$$

and for all $j \geq 1$ and all $y \in [j\varrho, (j+1)\varrho - 1]$

$$\mathrm{E}^y_{\omega,\zeta} N^{\varrho}_{[T^{\varrho}_{j\varrho}, T^{\varrho}_{(j+1)\varrho})}(x) \leq \frac{1}{\varepsilon_1(1-\varepsilon_1)} \mathrm{E}^y_{\omega} N^{\varrho}_{[T^{\varrho}_{j\varrho}, T^{\varrho}_{(j+1)\varrho})}(x). \tag{33}$$

Consider the term $\mathbf{A}_1$ of (30). Applying (32) and (33) to (31) and using Condition D, we obtain for $\mathbb{P}$-almost every ω (and so the following holds also with $\theta_{-x}\omega^{\varrho}$ on the place of ω^{ϱ}) and all $x < 0$ that

$$\begin{aligned}
\mathrm{E}_{\omega,\zeta} N^{\varrho}_{\infty}(x) &\leq \frac{1}{\varepsilon_1(1-\varepsilon_1)} \bigl(g_1\bigl(|x|\bigr) + g_1\bigl(\varrho + |x|\bigr) + g_1\bigl(2\varrho + |x|\bigr) + \cdots\bigr) \\
&\leq \frac{1}{\varepsilon_1(1-\varepsilon_1)} \sum_{j=0}^{\infty} g_1\bigl(|x| + j\bigr).
\end{aligned} \tag{34}$$

So, we write

$$\begin{aligned}
\mathbf{A}_1 &\leq \frac{1}{\varepsilon_1(1-\varepsilon_1)\mathbb{E}'\mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1\varrho}} \sum_{x<0} \sum_{j=0}^{\infty} g_1\bigl(|x| + j\bigr) \\
&\leq \frac{1}{\varepsilon_1(1-\varepsilon_1)\mathbb{E}'\mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1\varrho}} \sum_{k=1}^{\infty} k g_1(k) \\
&\leq C_5
\end{aligned} \tag{35}$$

for some $C_5 > 0$.

Let us deal now with the term $\mathbf{A}_2$. Suppose that $x \geq 0$ is such that $x \in [k\varrho, (k+1)\varrho)$. Then, we have $N^{\varrho}_{[T^{\varrho}_{j\varrho}, T^{\varrho}_{(j+1)\varrho})}(x) = 0$ for all $j < k$. So, by (33), we obtain for $\mathbb{P}$-almost every ω (again, this means that it holds also with $\theta_{-x}\omega^{\varrho}$ on the place of ω^{ϱ}) that

$$\mathrm{E}_{\omega,\zeta} N^{\varrho}_{[T^{\varrho}_{j\varrho}, T^{\varrho}_{(j+1)\varrho})}(x) \leq \frac{g_1(0)}{\varepsilon_1(1-\varepsilon_1)} \tag{36}$$

for $j \in \{k, k+1\}$, and

$$\mathrm{E}_{\omega,\zeta} N^{\varrho}_{[T^{\varrho}_{j\varrho}, T^{\varrho}_{(j+1)\varrho})}(x) \leq \frac{1}{\varepsilon_1(1-\varepsilon_1)} g_1\bigl((j-k-1)\varrho\bigr) \tag{37}$$

for $j > k+1$. Using (31), we obtain

$$\mathrm{E}_{\omega,\zeta} N^{\varrho}_{\infty}(x) \leq \frac{2g_1(0)}{\varepsilon_1(1-\varepsilon_1)} + \frac{1}{\varepsilon_1(1-\varepsilon_1)} \sum_{i \geq 1} g_1(i) \leq C_6$$

for some $C_6 > 0$. Then, using also Lemma 3.3, we have

$$\begin{aligned}
\mathbf{A}_2 &\leq \frac{1}{\mathbb{E}'\mathrm{E}_{\omega,\zeta} T^{\varrho}_{\ell_1\varrho}} C_6 \varrho E(\ell_1 + 1) \\
&\leq \frac{C_6}{C_4}\bigl(\varepsilon_1^{-1} + 1\bigr).
\end{aligned} \tag{38}$$

Using (35) and (38), we obtain the second inequality in (29).

As for the first inequality in (29), let us note that, analogously to (24), on the event $\{\zeta_1 = 0\}$ we have for any nonnegative random variable Y which is measurable with respect to the sigma-algebra generated by $(S^{\varrho}_1, \ldots, S^{\varrho}_{T^{\varrho}_{\varrho}})$

$$\mathrm{E}_{\omega,\zeta} Y \geq \frac{\varepsilon_1}{1-\varepsilon_1} \mathrm{E}_{\omega} Y. \tag{39}$$

By Lemma 3.1, we obtain

$$
\begin{aligned}
E\sum_{x\in\mathbb{Z}} \mathrm{E}_{\theta_{-x}\omega^\varrho,\zeta} N^\varrho_{T^\varrho_{\ell_1\varrho}}(x) &\geq (1-\varepsilon_1) E\sum_{x\in[1,\varrho]} \mathrm{E}_{\theta_{-x}\omega^\varrho,\zeta} N^\varrho_{T^\varrho_{\ell_1\varrho}}(x) \\
&\geq (1-\varepsilon_1)\times\frac{\varepsilon_1}{1-\varepsilon_1} E\sum_{x\in[1,\varrho]} \mathrm{E}_{\theta_{-x}\omega^\varrho} N^\varrho_{T^\varrho_{\ell_1\varrho}}(x) \\
&\geq 2\varepsilon_1^2\varrho,
\end{aligned}
$$

and this concludes the proof of Proposition 3.4. □

Now, we pass to the limit as $\varrho\to\infty$, proving the existence of v^∞, $\mathbb{Q}:=\mathbb{Q}^\infty$, and the fact that $v^\varrho\to v^\infty$.

From Proposition 3.4 we obtain that the family of measures $(\mathbb{Q}^\varrho, \varrho\in[\varrho_0,+\infty))$ is tight. By Prohorov's theorem, there exists a sequence $\varrho_k\to\infty$ and a probability measure $\mathbb{Q}^\infty$ such that $\mathbb{Q}^{\varrho_k}\to\mathbb{Q}^\infty$ weakly as $k\to\infty$. Let us prove that $\mathbb{Q}^\infty$ is in fact a stationary measure for the environment of the random walk without truncation S^∞. For this, take a bounded continuous function f and, recalling that $\mathbb{E}_\varrho$ is the expectation corresponding to $\mathbb{Q}^\varrho$, write by stationarity

$$
\begin{aligned}
\bigl|\mathbb{E}_\infty f(\omega)-\mathbb{E}_\infty f(\theta_{S_1^\infty}\omega)\bigr| \leq{}& \bigl|\mathbb{E}_\infty f(\omega)-\mathbb{E}_{\varrho_k} f(\omega)\bigr| \\
&+\bigl|\mathbb{E}_{\varrho_k} f(\theta_{S_1^{\varrho_k}}\omega)-\mathbb{E}_\infty f(\theta_{S_1^{\varrho_k}}\omega)\bigr| \\
&+\mathbb{E}_\infty\bigl|f(\theta_{S_1^{\varrho_k}}\omega)-f(\theta_{S_1^\infty}\omega)\bigr| \\
=:{}& \mathbf{B}_1+\mathbf{B}_2+\mathbf{B}_3.
\end{aligned}\tag{40}
$$

First, we deal with terms $\mathbf{B}_2$ and $\mathbf{B}_1$. By Condition C, for any $\varepsilon>0$ there exists h_0 such that for any ϱ we have

$$
\mathrm{P}_\omega\bigl[\bigl|S_1^\varrho\bigr|>h_0\bigr]<\varepsilon\quad\mathbb{P}\text{-a.s.}
$$

Then, supposing without restriction of generality that $|f|\leq 1$, we write

$$
\begin{aligned}
\mathbf{B}_2 &\leq \biggl|\mathbb{E}_{\varrho_k}\sum_{|m|\leq h_0}\omega_{0m}^{\varrho_k} f(\theta_m\omega)-\mathbb{E}_\infty\sum_{|m|\leq h_0}\omega_{0m} f(\theta_m\omega)\biggr|+2\varepsilon \\
&\leq \biggl|\mathbb{E}_{\varrho_k}\sum_{|m|\leq h_0}\omega_{0m} f(\theta_m\omega)-\mathbb{E}_\infty\sum_{|m|\leq h_0}\omega_{0m} f(\theta_m\omega)\biggr|+\mathbb{E}_{\varrho_k}\sum_{|m|>\varrho_k}\omega_{0m}+2\varepsilon.
\end{aligned}
$$

Since $\mathbb{Q}^{\varrho_k}$ converges to $\mathbb{Q}^\infty$ as $k\to\infty$ and using also Condition C, we can choose k large enough in such a way that $\mathbf{B}_1+\mathbf{B}_2\leq 5\varepsilon$. As for the term $\mathbf{B}_3$, again using Condition C we note that, for the natural coupling of $S_1^{\varrho_k}$ and S_1^∞ (described in Section 2.2), we have

$$
\mathrm{P}_\omega\bigl[S_1^{\varrho_k}\neq S_1^\infty\bigr]<\varepsilon\quad\mathbb{P}\text{-a.s.}
$$

for large enough k. Then, for such k's, we have $\mathbf{B}_3<\varepsilon$ (recall that we assumed that $|f|\leq 1$). Then, since ε is arbitrary, (40) implies that $\mathbb{Q}^\infty$ is stationary for S^∞.

Now, let us prove that $\mathbb{Q}^\varrho$ is ergodic for all $\varrho\in[\varrho_0,+\infty]$. First, we note that (29) holds for $\varrho=\infty$ as well. Then, we argue by contradiction: suppose that $\mathbb{Q}^\varrho$ is not ergodic and let $A\subset\Omega$ be a nontrivial invariant event. Then, (29) implies that $0<\mathbb{P}[A]<1$. Since $\mathbb{P}$ is stationary ergodic under space shift, the random set

$$
\mathfrak{G}(\omega)=\{k>0:\ \theta_k\omega\in A\}
$$

is such that

$$
|\mathfrak{G}|=|\mathfrak{G}^c|=\infty\quad\mathbb{P}\text{-a.s.},
$$

which means by absolute continuity that

$$|\mathfrak{G}| = |\mathfrak{G}^c| = \infty \quad \mathbb{Q}^\varrho\text{-a.s.}$$

Since they are infinite, it follows from Lemma 3.1 that both sets $\mathfrak{G}$ and $\mathfrak{G}^c$ are visited by S^ϱ infinitely many times almost surely. This contradicts the invariance of A for the environment dynamics, which asserts that $\theta_{S_k}\omega \in A$ holds $\mathbb{Q}^\varrho \otimes \mathrm{P}_\omega$-a.s. on $\{\omega \in A\}$ and that $\theta_{S_k}\omega \in A^c$ holds $\mathbb{Q}^\varrho \otimes \mathrm{P}_\omega$-a.s. on $\{\omega \in A^c\}$.

Now, we claim that the ergodicity of $\mathbb{Q}^\infty$ implies that $\mathbb{Q}^\varrho \to \mathbb{Q}^\infty$ weakly as $\varrho \to \infty$: indeed, any possible limit $\tilde{\mathbb{Q}}$ has also to be ergodic by the above argument, and thus singular with respect to $\mathbb{Q}^\infty$ or equal to it; in view of (29), the first case cannot hold, so the second one does hold.

Finally, let us deal with the proof of (9) and the fact that $v_\varrho \to v_\infty$ as $\varrho \to \infty$. The existence of v_ϱ (which has been previously established for finite ϱ) and formula (9) follow from the ergodic theorem. To prove the convergence, note that, by (9),

$$v_\varrho = \mathbb{E}_\varrho \sum_{m\in\mathbb{Z}} m\omega_{0m},$$

and, by Condition C, for any $\varepsilon > 0$ there exists h_1 such that

$$\sum_{|m|>h_1} |m|\omega_{0m} < \varepsilon \quad \mathbb{P}\text{-a.s.} \tag{41}$$

Since ε is arbitrary, the fact that $v_\varrho \to v_\infty$ follows from (41) and from

$$\mathbb{E}_\varrho \sum_{|m|\le h_1} m\omega_{0m} \to \mathbb{E}_\infty \sum_{|m|\le h_1} m\omega_{0m} \quad \text{as } \varrho \to \infty.$$

This concludes the proof of Theorems 2.3 and 2.4.

4. Proofs for the random billiard

Let us define $h_0 := (\varepsilon^{-1}\gamma_d)^{1/(d-1)}$, with ε from Condition P and γ_d from (2). From (2) it directly follows that

$$K(x,y) \le \frac{\gamma_d}{\|x-y\|^{d-1}}. \tag{42}$$

So, if Condition P holds, for all pairs of points z, z' involved in Condition P it holds that $\|z-z'\|^{d-1} \le \varepsilon^{-1}\gamma_d$, so that

$$\left|\left(z-z'\right)\cdot \mathbf{e}\right| \le \left\|z-z'\right\| \le h_0.$$

Thus, we obtain that Condition P holds for $\hat{K}$ on the place of K with $\varepsilon' = \varepsilon \mathrm{e}^{-\lambda h_0}$ on the place of ε.

Let us consider a sequence of i.i.d. random variables $\eta_1, \eta_2, \eta_3, \dots$ with uniform distribution on $\{1, 2, \dots, N\}$ (where N is from Condition P), independent of everything else. Also, define $J(0) = 0$, $J(n) = \eta_1 + \cdots + \eta_n$. Then, analogously to the proof of Lemma 3.6 in [6], using Condition P one can obtain that, for any $x \in \partial\omega$ and $B \subset \{y \in \partial\omega\colon -1 \le (y-x)\cdot\mathbf{e} \le 1\}$, we have

$$\mathrm{P}^x_\omega[\xi_{\eta_1} \in B] \ge \frac{1}{N}\delta^{N-1}\left(\varepsilon \mathrm{e}^{-\lambda h_0}\right)^N \nu^\omega(B). \tag{43}$$

Here, and below, we still use, for simplicity, the notations $\mathrm{P}^x_\omega, \mathrm{E}^x_\omega$ for the enlarged model, meaning that ω is fixed but $\xi_\cdot$ and $\eta_\cdot$ are integrated out.

Let us define for an arbitrary $a, b \in \mathbb{R}$, $a < b$,

$$\tilde{F}^\omega(a,b) = \left\{x \in \partial\omega\colon\ x\cdot\mathbf{e} \in [a,b]\right\}.$$

Then, consider the moment when the particle steps out from $\tilde{F}^\omega(a,b)$:

$$\tau(a,b)=\min\{n\ge 0:\ |\xi_n\cdot\mathbf{e}|\notin[a,b]\}=\min\{n\ge 0:\ \xi_n\notin\tilde{F}^\omega(a,b)\}.$$

Next, we prove the following fact:

Lemma 4.1. *There exist $\tilde{\gamma}_1,\tilde{\gamma}_1'>0$ such that for $\mathbb{P}$-almost all ω, for any $a\le b-1$ we have*

$$\mathrm{P}_\omega^x\left[\tau(a,b)>(b-a)^3t\right]\le\tilde{\gamma}_1\mathrm{e}^{-\tilde{\gamma}_1't} \tag{44}$$

for all $x\in\tilde{F}^\omega(a,b)$ and all $t\ge 1$.

Proof. Observe that, from Condition L we obtain that for some positive constants $\tilde{\gamma}_2,\tilde{\gamma}_3$ we have

$$\tilde{\gamma}_2\le\nu^\omega\big(x:\ x\cdot\mathbf{e}\in[s,s+1]\big)\le\tilde{\gamma}_3\quad\mathbb{P}\text{-a.s.} \tag{45}$$

for all $s\in\mathbb{R}$ (without restriction of generality one may assume that $\tilde{\gamma}_2\le 1$, $\tilde{\gamma}_3\ge 1$).

Now, suppose without restriction of generality that $a=0$, and b is a positive integer. Denote

$$\tilde{\tau}=\min\{n\ge 0:\ \xi_{J(n)}\notin\tilde{F}^\omega(a,b)\}.$$

Clearly, by definition of the random variables (η_i), we have $\{\tau(a,b)>b^3t\}\subset\{\tilde{\tau}>N^{-1}b^3t\}$. Now, let us consider a sub-Markov kernel $Q_\omega:=Q_\omega^1$, which acts on functions $f:\partial\omega\to\mathbb{R}$ in the following way:

$$\big(Q_\omega^n f\big)(x)=\mathrm{E}_\omega^x\big(f(\xi_{J(n)})\mathbf{1}\{\tilde{\tau}>n\}\big). \tag{46}$$

Let

$$\tilde{K}(x,y)=\frac{1}{N}\sum_{j=1}^{N}\hat{K}^j(x,y)$$

be the transition density of the process $(\xi_{J(n)},n\ge 0)$. Observe that this process is reversible with the reversible measure ν_λ^ω, so that $\pi(x)\tilde{K}(x,y)=\pi(y)\tilde{K}(y,x)$ for all $x,y\in\partial\omega$. Let

$$\mathcal{E}(f,g)=\int_{(\partial\omega)^2}\pi(x)\tilde{K}(x,y)\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)\,\mathrm{d}\nu^\omega(x)\,\mathrm{d}\nu^\omega(y),$$

and define

$$\tilde{\Lambda}=\inf\left\{\frac{\mathcal{E}(f,f)}{2\int_{\partial\omega}f^2(x)\,\mathrm{d}\nu_\lambda^\omega(x)}:\ f\not\equiv 0,\ f|_{(\tilde{F}^\omega(0,b))^c}=0,\ f\in L^2\big(\nu_\lambda^\omega\big)\right\}. \tag{47}$$

From the variational formula for the top of the spectrum of symmetric operators, $\|Q_\omega\|_{L^2(\nu_\lambda^\omega)}=1-\tilde{\Lambda}$, so we look for a lower bound for $\tilde{\Lambda}$. Denote

$$U_j=\{x\in\partial\omega:\ x\cdot\mathbf{e}\in(j,j+1]\},\qquad j\in\mathbb{Z},$$

so that $\tilde{F}^\omega(j,j+1)=\overline{U_j}$. Observe that, by (43), for any $j\in\mathbb{Z}$ we have

$$\tilde{K}(x,y)\ge\frac{1}{N}\delta^{N-1}\big(\varepsilon\mathrm{e}^{-\lambda h_0}\big)^N$$

for all $x\in U_j$, $y\in U_{j+1}$. Also, it is clear that

$$\nu^\omega\big(\{x\in\partial\omega:\ x\cdot\mathbf{e}=0\}\big)=0\quad\mathbb{P}\text{-a.s.},$$

and

$$e^{\lambda n} \le \pi(x) \le e^{\lambda(n+1)} \quad \text{for all } x \in U_n. \tag{48}$$

So, using also (45), the Cauchy–Schwarz inequality, and the fact that $f(y) = 0$ for all $y \in U_b$, we can write

$$\begin{aligned}
&\int_{\partial\omega} f^2(x)\,d\nu_\lambda^\omega(x) \\
&\quad = \int_{\partial\omega} \pi(x) f^2(x)\,d\nu^\omega(x) \\
&\quad = \sum_{i=0}^{b-1} \int_{U_i} \pi(x) f^2(x)\,d\nu^\omega(x) \\
&\quad = \sum_{i=0}^{b-1} \frac{1}{\nu^\omega(U_{i+1})\cdots\nu^\omega(U_b)} \int_{U_i} \pi(x_i)\,d\nu^\omega(x_i) \int_{U_{i+1}} d\nu^\omega(x_{i+1}) \\
&\qquad \times\cdots\times \int_{U_b} d\nu^\omega(x_b) \Biggl(\sum_{j=i}^{b-1} \bigl(f(x_j) - f(x_{j+1})\bigr)\Biggr)^2 \\
&\quad \le b \sum_{i=0}^{b-1} \frac{1}{\nu^\omega(U_{i+1})\cdots\nu^\omega(U_b)} \int_{U_i} \pi(x_i)\,d\nu^\omega(x_i) \int_{U_{i+1}} d\nu^\omega(x_{i+1}) \\
&\qquad \times\cdots\times \int_{U_b} d\nu^\omega(x_b) \Biggl(\sum_{j=i}^{b-1} \bigl(f(x_j) - f(x_{j+1})\bigr)^2\Biggr) \\
&\quad \le b \sum_{i=0}^{b-1} e^{\lambda(i+1)} \Biggl(\frac{1}{\nu^\omega(U_{i+1})} \int_{U_i} d\nu^\omega(x_i) \int_{U_{i+1}} d\nu^\omega(x_{i+1}) \bigl(f(x_i) - f(x_{i+1})\bigr)^2 \\
&\qquad + \sum_{j=i+1}^{b-1} \frac{\nu^\omega(U_i)}{\nu^\omega(U_j)\nu^\omega(U_{j+1})} \int_{U_j} d\nu^\omega(x_j) \int_{U_{j+1}} d\nu^\omega(x_{j+1}) \bigl(f(x_j) - f(x_{j+1})\bigr)^2\Biggr) \\
&\quad \le \frac{b\tilde\gamma_3}{\tilde\gamma_2^2} \sum_{i=0}^{b-1} e^{\lambda(i+1)} \sum_{j=i}^{b-1} \int_{U_j} d\nu^\omega(x_j) \int_{U_{j+1}} d\nu^\omega(x_{j+1}) \bigl(f(x_j) - f(x_{j+1})\bigr)^2 \quad \bigl(\text{by (45)}\bigr) \\
&\quad \le \frac{b\tilde\gamma_3}{\tilde\gamma_2^2} \biggl(\frac{1}{N}\delta^{N-1}\bigl(\varepsilon e^{-\lambda h_0}\bigr)^N\biggr)^{-1} \sum_{j=0}^{b-1} \Biggl(\sum_{i=0}^{j} e^{\lambda(i+1)}\Biggr) e^{-\lambda j} \\
&\qquad \times \int_{U_j} d\nu^\omega(x_j) \int_{U_{j+1}} d\nu^\omega(x_{j+1}) \pi(x_j) \tilde K(x_j, x_{j+1}) \bigl(f(x_j) - f(x_{j+1})\bigr)^2 \\
&\quad \le \frac{b\tilde\gamma_3 N}{\tilde\gamma_2^2 \delta^{N-1}(\varepsilon e^{-\lambda h_0})^N} \Biggl(\sum_{i=-\infty}^{0} e^{\lambda(i+1)}\Biggr) \mathcal{E}(f, f),
\end{aligned}$$

and so, for some positive constant C_1 it holds that

$$\tilde\Lambda \ge \frac{C_1}{b}. \tag{49}$$

Then, since $\tilde{\Lambda} = 1 - \|Q_\omega\|_{L^2(\nu_\lambda^\omega)}$ from the spectral variational formula we have

$$\|Q_\omega^n\|_{L^2(\nu_\lambda^\omega)} \le \left(1 - \frac{C_1}{b}\right)^n. \tag{50}$$

Now, using the notation $\mathbf{1}_B$ for the indicator function of $B \subset \partial\omega$, observe that $\mathtt{P}_\omega^x[\tilde{\tau} > n] = (Q_\omega^n \mathbf{1}_{\tilde{F}^\omega(0,b)})(x)$. For $j \in [0, b-1]$ we can write using (45), (50), and Cauchy–Schwarz inequality

$$\begin{aligned}
\int_{U_j} \mathtt{P}_\omega^x[\tilde{\tau} > n]\, \mathrm{d}\nu_\lambda^\omega(x) &= \big(\mathbf{1}_{U_j}, Q_\omega^n \mathbf{1}_{\tilde{F}^\omega(0,b)}\big)_{L^2(\nu_\lambda^\omega)} \\
&\le \|\mathbf{1}_{U_j}\|_{L^2(\nu_\lambda^\omega)} \|\mathbf{1}_{\tilde{F}^\omega(0,b)}\|_{L^2(\nu_\lambda^\omega)} \|Q_\omega^n\|_{L^2(\nu_\lambda^\omega)} \\
&\le C_2 \mathrm{e}^{\lambda j/2} \mathrm{e}^{\lambda b/2} \left(1 - \frac{C_1}{b}\right)^n
\end{aligned}$$

for some $C_2 > 0$. So, again using (48), we have for $j \le b-1$,

$$\int_{U_j} \mathtt{P}_\omega^x[\tilde{\tau} > n]\, \mathrm{d}\nu^\omega(x) \le C_2 \mathrm{e}^{\lambda(b-j)/2} \left(1 - \frac{C_1}{b}\right)^n. \tag{51}$$

Now, (51) implies that, if b is large enough, then

$$\int_{U_j} \mathtt{P}_\omega^x\big[\tilde{\tau} \le b^3 - 1\big]\, \mathrm{d}\nu^\omega(x) \ge C_3 > 0.$$

So, with $C_4 := C_3 N^{-1} \delta^{N-1} (\varepsilon \mathrm{e}^{-\lambda h_0})^N > 0$, for any $x \in U_j$ we can write (using also (43)) that

$$\begin{aligned}
\mathtt{P}_\omega^x\big[\tilde{\tau} \le b^3\big] &= \int_{\partial\omega} \tilde{K}(x,y) \mathtt{P}_\omega^y\big[\tilde{\tau} \le b^3 - 1\big]\, \mathrm{d}\nu^\omega(y) \\
&\ge \frac{1}{N} \delta^{N-1} \big(\varepsilon \mathrm{e}^{-\lambda h_0}\big)^N \int_{U_j} \mathtt{P}_\omega^y\big[\tilde{\tau} \le b^3 - 1\big]\, \mathrm{d}\nu^\omega(y) \\
&\ge C_4.
\end{aligned}$$

This implies that $\mathtt{P}_\omega^x[\tilde{\tau} > b^3 t] \le \mathrm{e}^{-C_6 t}$ for any $x \in \tilde{F}^\omega(0,b)$, and this (as discussed in the beginning of the proof of this lemma) by its turn implies (44), thus concluding the proof of Lemma 4.1. □

Consider $B \subset \partial\omega$ with positive $(d-1)$-dimensional Hausdorff measure and such that $\sup\{x \cdot \mathbf{e}\colon x \in B\} < +\infty$. By definition, the *stationary distribution* π^B *conditioned on* B is the distribution with the density $\frac{\pi(x)}{\pi(B)} \mathbf{1}_B(x)$ (recall that $\pi(B) := \nu_\lambda^\omega(B)$). We use the notation $\mathtt{P}_\omega^B$, $\mathtt{E}_\omega^B$ for the KRWD starting from the stationary distribution conditioned on B.

Now, we construct the connection with RWRE. Recall that $\eta_1, \eta_2, \eta_3, \ldots$ are i.i.d. random variables with uniform distribution on $\{1, \ldots, N\}$, and $J(0) = 0$, $J(n) = \eta_1 + \cdots + \eta_n = J(n-1) + \eta_n$. We now focus on the process $(\xi_{J(n)}, n \ge 0)$. In view of (43) and (45), we couple this process with a Bernoulli process $\zeta' = (\zeta'_n, n \ge 1)$ (independent of ω) of parameter $r_1 = (N\mathrm{e}^\lambda)^{-1} \tilde{\gamma}_2 \delta^{N-1} (\varepsilon \mathrm{e}^{-\lambda h_0})^N$,

$$P\big[\zeta'_n = 1\big] = 1 - P\big[\zeta'_n = 0\big] = r_1,$$

in such a way that on the event $\{\zeta'_n = 1\}$, $\xi_{J(n)}$ has the stationary distribution on $U_{[\xi_{J(n-1)} \cdot \mathbf{e}]}$. (The choice of the stationary distribution is arbitrary, and the whole construction could be implemented with another distribution, absolutely continuous to the uniform on $U_{[\xi_{J(n-1)} \cdot \mathbf{e}]}$ with density bounded from above and below.) We denote by $E^{\zeta'}$ be the expectation with respect to ζ', and $\mathtt{P}_{\omega,\zeta'}$, $\mathtt{E}_{\omega,\zeta'}$ the probability and expectation with fixed ω and ζ', which is defined as follows: $\mathtt{P}_{\omega,\zeta'}^x(\xi_0 = x) = 1$, and recursively for $n = 1, 2, \ldots$,

$$\mathtt{P}_{\omega,\zeta'}^x[\xi_{J(n)} \in \cdot \mid \xi_{J(k)}, k < n] = \pi^{U_{[\xi_{J(n-1)} \cdot \mathbf{e}]}}(\cdot) \quad \text{if } \zeta'_n = 1,$$

and, for $\zeta'_n = 0$,

$$\mathrm{P}^x_{\omega,\zeta'}[\xi_{J(n)} \in \cdot | \xi_{J(k)}, k < n] = \frac{\mathrm{P}^{\xi_{J(n-1)}}_{\omega}[\xi_{\eta_n} \in \cdot] - r_1 \pi^{U_{[\xi_{J(n-1)}\cdot \mathbf{e}]}}(\cdot)}{1 - r_1}.$$

Set $\kappa_0 = 0$, and define the times of success in the new Bernoulli process,

$$\kappa_{m+1} = \min\{k > \kappa_m \colon \zeta'_k = 1\}$$

for $m \geq 1$. It is easy to see that, under $P^{\zeta'} \otimes \mathrm{P}^x_{\omega,\zeta'}$, the sequence $(\xi_{J(\kappa_m)}, m \geq 0)$ is a Markov chain, and $\xi_{J(\kappa_m)}$ for $m \geq 1$ has "piecewise stationary" law, i.e., of the form $\sum_{i\in\mathbb{Z}} a_i \pi^{U_i}$ with $a_i \geq 0$, $\sum_i a_i = 1$. It follows that, starting ξ_0 from such a law, the Markov chain is weakly lumpable and can be reduced to another Markov chain on a smaller space, see [11], or [12] for a more modern account. More precisely, we prove:

Lemma 4.2. *Under $P^{\zeta'} \otimes \mathrm{P}^{U_0}_{\omega,\zeta'}$, the sequence $([\xi_{J(\kappa_m)} \cdot \mathbf{e}], m \geq 0)$ is a RWRE on $\mathbb{Z}$, with transition probabilities*

$$Q_\omega(i,j) = P^{\zeta'} \otimes \mathrm{P}^{U_i}_{\omega,\zeta'}[\xi_{J(\kappa_1)} \in U_j].$$

Proof. Under $P^{\zeta'} \otimes \mathrm{P}^x_{\omega,\zeta'}$, the transition density from x to y for the Markov chain $\xi_{J(\kappa_\cdot)}$ can be written as $a_j(x,\omega)\pi^{U_j}(\mathrm{d}y)$, with j such that $U_j \ni y$. Hence,

$$\begin{aligned}
&P^{\zeta'} \otimes \mathrm{P}^{U_0}_{\omega,\zeta'}\big[[\xi_{J(\kappa_1)} \cdot \mathbf{e}] = j_1, \ldots, [\xi_{J(\kappa_m)} \cdot \mathbf{e}] = j_m\big] \\
&\quad = P^{\zeta'} \otimes \mathrm{P}^{U_0}_{\omega,\zeta'}[\xi_{J(\kappa_1)} \in U_{j_1}, \ldots, \xi_{J(\kappa_m)} \in U_{j_m}] \\
&\quad = \int \cdots \int \pi^{U_0}(\mathrm{d}x_0) \prod_{k=1}^m a_{j_k}(x_{k-1},\omega)\pi^{U_{j_k}}(\mathrm{d}x_k) \\
&\quad = \prod_{k=1}^m \int a_{j_k}(x_{k-1},\omega)\pi^{U_{j_{k-1}}}(\mathrm{d}x_{k-1}) \times \int \pi^{U_{j_m}}(\mathrm{d}x_m) \\
&\quad = \prod_{k=1}^m Q_\omega(j_{k-1}, j_k) \times 1,
\end{aligned}$$

which ends the proof. □

This result is the bridge between the two main processes considered in this paper: obviously, starting ξ_0 from the origin or distributed in the interval U_0 will not make any difference for the law of large numbers. But it is not quite enough to conclude the proof for the billiard. In the sequel we shall need the following two results about hitting times of sets:

Lemma 4.3. *For any $m \geq 0$ and arbitrary $B, F \subset \partial\omega$ we have for $\mathbb{P}$-almost all ω*

$$\mathrm{P}^B_\omega[\textit{there exists } k \leq m \textit{ such that } \xi_k \in F] \leq m\frac{\pi(F)}{\pi(B)}. \tag{52}$$

Proof. Using reversibility, it is straightforward to obtain that $\pi(B)\mathrm{P}^B_\omega[\xi_k \in F] = \pi(F)\mathrm{P}^F_\omega[\xi_k \in B]$, so $\mathrm{P}^B_\omega[\xi_k \in F] \leq \frac{\pi(F)}{\pi(B)}$. Using the union bound, we obtain (52). □

Lemma 4.4. *There exist $\tilde\gamma_4, \tilde\gamma_5 > 0$ such that for any $m \geq 0$, $H \geq 1$, and $x \in \partial\omega$ we have for $\mathbb{P}$-almost all ω*

$$\mathrm{P}^x_\omega\big[\textit{there exists } k \leq m \textit{ such that } (\xi_k - x)\cdot \mathbf{e} < -H\big] \leq \tilde\gamma_4 m \mathrm{e}^{-\tilde\gamma_5 H^{1/2}}. \tag{53}$$

Proof. This fact would be a trivial consequence of Lemma 4.3 if one starts from the stationary distribution on a set (of not too small measure) instead of starting from a single point. So, the idea is the following: we first wait for the moment $J(\kappa_1)$ (when the particle has the stationary distribution on U_j for some random j), and then note that it is likely that the particle did not go too far to the left until this moment. Then, it is already possible to apply Lemma 4.3.

Formally, we write

$$\begin{aligned}
&\mathrm{P}_\omega^x\big[\text{there exists } k \le m \text{ such that } (\xi_k - x)\cdot \mathbf{e} < -H\big] \\
&\quad = E^{\zeta'}\mathrm{P}_{\omega,\zeta'}^x\big[\text{there exists } k \le m \text{ such that } (\xi_k - x)\cdot \mathbf{e} < -H\big] \\
&\quad \le P^{\zeta'}\left[\kappa_1 > \frac{H^{1/2}}{2N}\right] \\
&\qquad + \mathrm{P}_\omega^x\left[\text{there exists } k \le \frac{H^{1/2}}{2N} \text{ such that } (\xi_k - \xi_{k-1})\cdot \mathbf{e} \le -H^{1/2}\right] \\
&\qquad + E^{\zeta'}\mathrm{P}_{\omega,\zeta'}^x\left[\kappa_1 \le \frac{H^{1/2}}{2N}, (\xi_k - \xi_{k-1})\cdot \mathbf{e} > -H^{1/2} \text{ for all } k \le \frac{H^{1/2}}{2N},\right. \\
&\qquad \left. \text{there exists } k \le m \text{ such that } (\xi_{k+J(\kappa_1)} - \xi_{J(\kappa_1)})\cdot \mathbf{e} < -H/2\right].
\end{aligned}$$

Now, the bound on the first term is straightforward (the random variable κ_1 has a geometric distribution with parameter r_1). To estimate the second term, note that for any $h > 0$ one has

$$\mathrm{P}_\omega^x\big[(\xi_1 - x)\cdot \mathbf{e} < -h\big] = \int_{(y-x)\cdot\mathbf{e}<-h} \mathrm{e}^{-\lambda(y-x)\cdot\mathbf{e}} K(x,y)\,\mathrm{d}y \le \mathrm{e}^{-\lambda h}. \tag{54}$$

To deal with the third term, recall that the law of $\{\xi_j, j > J(\kappa_1)\}$ conditional on $\xi_{J(\kappa_1)}$ does not depend on $(\xi_m; m \le J(\kappa_1))$. Then, on the event

$$\left\{\kappa_1 \le \frac{H^{1/2}}{2N}, (\xi_k - \xi_{k-1})\cdot \mathbf{e} > -H^{1/2} \text{ for all } k \le \frac{H^{1/2}}{2N}\right\}$$

we have $[\xi_{J(\kappa_1)}\cdot \mathbf{e}] \ge -H/2$. So, one can estimate the third term using Lemma 4.3, and conclude the proof of Lemma 4.4. □

Now, to prove Theorem 2.2, the idea is to construct an "induced" RWRE, then apply the results of Section 2.2, and then recover the LLN for the original billiard. To apply this approach, we need a few estimates on displacement probabilities that we derive in the following lines. Consider some (large) integer L (to be chosen later) and observe that, since κ_n is a sum of n i.i.d. geometric random variables, we can find some large r_1 such that, for all n,

$$P^{\zeta'}\big[\kappa_n \le 2r_1^{-1}n\big] \ge 1 - C_7\mathrm{e}^{-C_8 n}. \tag{55}$$

So, by Lemma 4.3, (55), and using also the fact that $J(\kappa_{L^4}) \le N\kappa_{L^4}$, we obtain for $\mathbb{P}$-almost all ω that for every $m \ge 1$ it holds that

$$\begin{aligned}
&E^{\zeta'}\mathrm{P}_{\omega,\zeta'}^{U_0}\big[[\xi_{J(\kappa_{L^4})}\cdot \mathbf{e}] \in \big[-(m+1)L, -mL\big]\big] \\
&\quad \le E^{\zeta'}\mathrm{P}_{\omega,\zeta'}^{U_0}\big[\text{there exists } k \le 2mNr_1^{-1}L^4 \text{ such that } \xi_k\cdot \mathbf{e} \in \big[-(m+1)L, -mL\big]\big] \\
&\qquad + P^{\zeta'}\big[\kappa_{L^4} > 2mr_1^{-1}L^4\big] \\
&\quad \le C_9 mL^4\mathrm{e}^{-C_{10}mL} + C_7\mathrm{e}^{-C_8 mL^4}.
\end{aligned} \tag{56}$$

Also, we use Lemma 4.1 (applied to $\tilde{F}^{\omega}(-L,2L)$), (55), Lemma 4.4, and the fact that $J(\kappa_{L^4}) \geq L^4$, to obtain that for $\mathbb{P}$-almost all ω

$$\begin{aligned} E^{\zeta'}\mathrm{P}^{U_0}_{\omega,\zeta'}\bigl[[\xi_{J(\kappa_{L^4})}\cdot\mathbf{e}]\geq L\bigr] &\geq E^{\zeta'}\mathrm{P}^{U_0}_{\omega,\zeta'}\bigl[\kappa_{L^4}\leq 2r_1^{-1}L^4, \tau(-L,2L)<L^4, \xi_k\cdot\mathbf{e}>-L, \\ &\qquad (\xi_{k+\tau(-L,2L)}-\xi_{\tau(-L,2L)})\cdot\mathbf{e}>-L \text{ for all } k\leq 2Nr_1^{-1}L^4\bigr] \\ &\geq 1-C_7\mathrm{e}^{-C_8L^4}-\tilde{\gamma}_1\mathrm{e}^{-\tilde{\gamma}_1'L/27}-2Nr_1^{-1}L^4\mathrm{e}^{-\lambda L}-2\tilde{\gamma}_4Nr_1^{-1}L^4\mathrm{e}^{-\tilde{\gamma}_5L^{1/2}}. \end{aligned} \tag{57}$$

So, from (56) and (57) we obtain

$$\begin{aligned} E^{\zeta'}\mathrm{E}^{U_0}_{\omega,\zeta'}[\xi_{J(\kappa_{L^4})}\cdot\mathbf{e}] &\geq L\bigl(1-C_7\mathrm{e}^{-C_8L^4}-\tilde{\gamma}_1\mathrm{e}^{-\tilde{\gamma}_1'L/27}-2Nr_1^{-1}L^4\mathrm{e}^{-\lambda L}-2\tilde{\gamma}_4Nr_1^{-1}L^4\mathrm{e}^{-\tilde{\gamma}_5L^{1/2}}\bigr) \\ &\quad - L\bigl(C_7\mathrm{e}^{-C_8L^4}+\tilde{\gamma}_1\mathrm{e}^{-\tilde{\gamma}_1'L/27}+2Nr_1^{-1}L^4\mathrm{e}^{-\lambda L}+2\tilde{\gamma}_4Nr_1^{-1}L^4\mathrm{e}^{-\tilde{\gamma}_5L^{1/2}}\bigr) \\ &\quad - \sum_{m=1}^{\infty} mL\bigl(C_7\mathrm{e}^{-C_8mL^4}+C_9mL^4\mathrm{e}^{-C_{10}mL}\bigr) \\ &> 1 \end{aligned} \tag{58}$$

if L is large enough.

Proof of Theorem 2.2. Under the law $E^{\zeta'}\mathrm{P}^{U_0}_{\omega,\zeta'}(\cdot)$, the process S defined by

$$S_n := [\xi_{J(\kappa_{L^4 n})}\cdot\mathbf{e}], \quad n\geq 0,$$

is a RWRE on $\mathbb{Z}$ in a (stationary ergodic) environment given by the tube ω. Let us choose L such that (58) holds. In this case, (56) and (58) show that the process S has uniformly positive drift, and its jumps to the left have uniformly exponential tail.

Now, to apply the results of Section 2.2 to the process S, we need to check that it verifies Conditions E, C, D. First, it is straightforward to obtain that Condition E holds. To check Condition C, first recall that (cf. e.g. formula (54) of [5]) that there exists $\tilde{\gamma}_6>0$ (depending only on $\widehat{M}$) such that for all $x\in\partial\omega$ and all $h\geq 1$

$$\mathrm{P}^x_{\omega}\bigl[\bigl|(\xi_1-x)\cdot\mathbf{e}\bigr|>h\bigr]\leq\tilde{\gamma}_6h^{-(d-1)}. \tag{59}$$

Now, observe that

$$[\xi_{J(j)}\cdot\mathbf{e}]=[\xi_{J(j-1)}\cdot\mathbf{e}] \quad \text{on } \{j=\kappa_n \text{ for some } n\} \tag{60}$$

and, for i such that $i\neq\kappa_n$ for all n,

$$\begin{aligned} &E^{\zeta'}\bigl(\mathrm{P}^{U_0}_{\omega,\zeta'}\bigl[\bigl|[\xi_{J(i)}\cdot\mathbf{e}]-[\xi_{J(i-1)}\cdot\mathbf{e}]\bigr|\geq h\bigr]|\kappa_{L^4}=j\bigr) \\ &\quad= E^{\zeta'}\bigl(\mathrm{P}^{U_0}_{\omega,\zeta'}\bigl[\bigl|[\xi_{J(i)}\cdot\mathbf{e}]-[\xi_{J(i-1)}\cdot\mathbf{e}]\bigr|\geq h\bigr]|\zeta_i'=0\bigr) \\ &\quad\leq \frac{1}{P^{\zeta'}[\zeta_i'=0]}\mathrm{P}^{U_0}_{\omega}\bigl[\bigl|[\xi_{J(i)}\cdot\mathbf{e}]-[\xi_{J(i-1)}\cdot\mathbf{e}]\bigr|\geq h\bigr] \\ &\quad\leq C_{11}h^{-2} \end{aligned} \tag{61}$$

since $d\geq 3$, recall (59). Then, write using (60) and (61)

$$\begin{aligned} &E^{\zeta'}\mathrm{P}^{U_0}_{\omega,\zeta'}\bigl[\bigl|[\xi_{J(\kappa_{L^4})}\cdot\mathbf{e}]\bigr|\geq s\bigr] \\ &\quad=\sum_{j=1}^{\infty}P^{\zeta'}[\kappa_{L^4}=j]E^{\zeta'}\bigl(\mathrm{P}^{U_0}_{\omega,\zeta'}\bigl[\bigl|[\xi_{J(\kappa_{L^4})}\cdot\mathbf{e}]\bigr|\geq s\bigr]|\kappa_{L^4}=j\bigr) \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{\infty} P^{\zeta'}[\kappa_{L^4}=j] E^{\zeta'}\Bigl(\mathrm{P}^{U_0}_{\omega,\zeta'}\bigl[\text{there exists } i \leq j \text{ such that} \\
&\quad \bigl|[\xi_{J(i)}\cdot \mathbf{e}] - [\xi_{J(i-1)}\cdot \mathbf{e}]\bigr| \geq s/j\bigr] | \kappa_{L^4}=j\Bigr) \\
&\leq \sum_{j=1}^{\infty} P^{\zeta'}[\kappa_{L^4}=j] j C_{11}\left(\frac{s}{j}\right)^{-2} \\
&= C_{11} s^{-2} \sum_{j=1}^{\infty} j^3 P^{\zeta'}[\kappa_{L^4}=j] \\
&= C_{12} s^{-2}.
\end{aligned}
\tag{62}
$$

Abbreviating $P^*[\cdot] := E^{\zeta'}\mathrm{P}^{U_0}_{\omega,\zeta'}[\cdot]$, we see that (62) is equivalent to $P^*[|S_1| \geq s] \leq C_{12}s^{-2}$. This means that Condition C holds for the process $(S_n, n \geq 0)$.

Now, we show that Condition D holds for the process $(S_n, n \geq 0)$. First, using (56) and (58) and Condition C, for large enough ϱ_0 one obtains by a straightforward computation that there exist small enough $\tilde{\gamma}_7, \tilde{\gamma}_8 > 0$ such that for all $y \in \mathbb{Z}$ and all $\varrho \geq \varrho_0$

$$
E^*\bigl(\mathrm{e}^{-\tilde{\gamma}_7 S^{\varrho}_{n+1} + \tilde{\gamma}_8(n+1)} - \mathrm{e}^{-\tilde{\gamma}_7 S^{\varrho}_n + \tilde{\gamma}_8 n} \mid S^{\varrho}_n = y\bigr) \leq 0,
$$

so that $(\mathrm{e}^{-\tilde{\gamma}_7 S^{\varrho}_n + \tilde{\gamma}_8 n}, n \geq 0)$ is a positive supermartingale. So, for some positive constants C_{13}, C_{14} it holds that for all $k \geq 1$

$$
P^*\bigl[\text{there exists } n \text{ such that } S^{\varrho}_n \leq -k\bigr] < C_{13}\mathrm{e}^{-\tilde{\gamma}_7 k}
$$

and

$$
E^* \sum_{n=0}^{\infty} 1\bigl\{S^{\varrho}_n = 0\bigr\} < C_{14}.
$$

So, Condition D holds with $g_1(k) = C_{13}C_{14}\mathrm{e}^{-\tilde{\gamma}_7 k}$. This means that we can use Theorem 2.3 for the process S.

Now, it remains to deduce the LLN for the random billiard with drift from the LLN for the random walk in random environment. It is done by a standard argument that we sketch in the following lines. First, observe that, just in the same way as (62) one can prove a slightly more general fact: for any n

$$
E^{\zeta'}\mathrm{P}^{U_0}_{\omega,\zeta'}\Bigl[\max_{m \in [J(\kappa_{L^4 n}), J(\kappa_{L^4(n+1)}))} \bigl|[\xi_m - \xi_{J(\kappa_{L^4 n})}\cdot \mathbf{e}]\bigr| \geq s\Bigr] \leq C_{15}s^{-2}.
\tag{63}
$$

Then, since the limit of $n^{-1}S_n$ exists and is finite, there exists $u \in (0, \infty)$ such that

$$
u = \lim_{n\to\infty} \frac{\xi_{J(\kappa_{L^4 n})}\cdot \mathbf{e}}{J(\kappa_{L^4 n})}.
$$

We then use (63) to obtain that, for $m \in [J(\kappa_{L^4 n}), J(\kappa_{L^4(n+1)}))$

$$
\begin{aligned}
\frac{1}{n}|\xi_{J(\kappa_{L^4 n})} - \xi_m| &\leq \frac{1}{n} \max_{m \in [J(\kappa_{L^4 n}), J(\kappa_{L^4(n+1)}))} \bigl|[\xi_m - \xi_{J(\kappa_{L^4 n})}\cdot \mathbf{e}]\bigr| \\
&\to 0 \quad \text{a.s., as } n \to \infty,
\end{aligned}
$$

and this permits us to conclude the proof of Theorem 2.2. □

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