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Superdiffusivity for Brownian motion in a Poissonian potential with long range correlation II: Upper bound on the volume exponent

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Abstract. This paper continues a study on trajectories of Brownian Motion in a field of soft trap whose radius distribution is unbounded. We show here that for both point-to-point and point-to-plane model the volume exponent (the exponent associated to transversal fluctuation of the trajectories) ξ is strictly less than 1 and give an explicit upper bound that depends on the parameters of the problem. In some specific cases, this upper bound matches the lower bound proved in the first part of this work and we get the exact value of the volume exponent.

Résumé. Cet article est la seconde partie d'une étude sur les trajectoires Brownienne dans un champs de pièges mous dont le rayon est aléatoire et a une distribution non-bornée. Nous montrons que l'exposant de volume (qui est l'exposant associé aux fluctuations transversales des trajectoires) ξ est strictement inférieur à 1 et nous donnons une borne supérieure explicite qui dépend des paramètres du problème, et ceci aussi bien pour le modèle dans la configuration point-à-point que pour celui dans la configuration point à plan. Dans certains cas particulier, cette borne supérieure coïncide avec la borne inférieure démontrée dans la première partie de cette étude, ce qui nous permets d'identifier la valeur de l'exposant de volume.

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1. Introduction

In this paper we investigate properties of the trajectories of Browian Motion in a disordered medium: given a random function V defined on $\mathbb{R}^d$ and $\lambda > 0$, we study trajectories of a Brownian motion $(B_t)_{t\geq 0}$ killed at (space-dependent) rate $\lambda + V(B_t)$ conditioned to survive up to the hitting time either of a distant hyperplane or a distant ball (we refer to these two cases respectively as point-to-plane and point-to-point). We focus more specifically on transversal fluctuation, i.e. fluctuation of the trajectories along the directions that are normal to the line that links the two points.

In an homogeneous medium these fluctuation are of order $\sqrt{L}$ where L is the distance to the hyperplane or point. It is commonly believed that disorder should make these fluctuation larger, e.g. of order L^{ξ} where $\xi > 1/2$ is called volume exponent. This phenomenon is called superdiffusivity and should hold for low dimension ($d \leq 3$) or when amplitude of the variations of V are large enough. We study it in a model where the random potential V is generated by a field of soft trap of random IID radii. The tail distribution of the radius of a trap is heavy-tailed so that our potential presents long range correlation. This model is a variant of a more studied model of Brownian motion among soft obstacles extensively studied by Sznitman (see the monograph [10] and reference therein) and for which superdiffusivity was shown to hold in dimension 2 by Wüthrich ($\xi \geq 3/5$, [11]) who also proved a universal bound $\xi \leq 3/4$, valid for any dimension [12].

For our model with correlated potential, we proved in [5] that superdiffusivity holds when $d=2$ and in larger dimension when correlations in the environment are strong enough (see (2.12)). The lower-bound that we get for ξ depends on the parameter of the model and in certain cases it is larger than $3/4$ (which is an upper bound for the volume exponent in any dimension in a large variety of model in the same universality class see e.g. [6] for directed polymer, and [4] for directed Brownian Polymer in an environment with long-range transversal correlation).

In this paper, our aim is to find an upper-bound for the volume exponent ξ. It turns out that for some particular choices of the parameter, the upper bound one finds for ξ matches the lower bound found in [5] and therefore allows us to derive the existence and exact value of the volume exponent (Corollary 2.2).

It is quite rare to be able to derive volume exponent for disordered model, even at the level of physicists prediction. For the two dimensional model studied in [11] and a whole class of related random growth model (e.g. two dimensional first-passage percolation, oriented first-passage percolation and directed polymer in random environment in $1+1$ dimension) it is predicted that $\xi=2/3$ and it has been proved in very particular cases ([1,2,7] and some more). These works have in common that they rely on exact calculation and therefore cannot be exported to general cases yet.

Here, lower-bound and upper-bound are both derived using energy v.s. entropy comparisons, and the reason why we are able to get the exact exponent is somehow different. When the tail distribution of radiuses of traps gets heavy, most of the fluctuation are caused by very large traps and this makes the system almost "one-dimensional" in a sense, and therefore easier to handle.

2. Model and result

Let $V^\omega(x)$, $x\in\mathbb{R}^d$, be a random potential defined as follows: we consider first a Poisson Point Process, in $\mathbb{R}^d\times\mathbb{R}^+$, viewed as a set of points

$$\omega:=\{(\omega_i,r_i)\in\mathbb{R}^d\times\mathbb{R}_+\mid i\in\mathbb{N}\}\tag{2.1}$$

(the ordering of the points (ω_i,r_i) being made in some arbitrary deterministic way, e.g. such that $|\omega_i|$ is an increasing sequence), whose intensity is given by $\mathcal{L}\times\nu$ where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^d$ and ν is a probability measure on $\mathbb{R}^+$. For the sake of simplicity we restrict to the case of ν satisfying

$$\forall r\geq 1,\quad \nu([r,\infty])=r^{-\alpha}\tag{2.2}$$

for some $\alpha>0$ (but the result would hold with more generality, e.g. assuming only that ν has power-law decay at infinity). Denote by $\mathbb{P}$ and $\mathbb{E}$ the associated probability law and expectation.

This process represents a field random traps centered at ω_i of and radius r_i. From ω we construct the potential $V^\omega:\mathbb{R}^d\to\mathbb{R}_+\cup\infty$ defined by

$$V^\omega(x):=\sum_{i=1}^{\infty}r_i^{-\gamma}\mathbf{1}_{\{|x-\omega_i|\leq r_i\}}\tag{2.3}$$

for some $\gamma>0$. Note that $V^\omega(x)<\infty$ for every $x\in\mathbb{R}^d$, for almost every realization of ω if and only if the condition $\alpha+\gamma-d>0$ holds. We suppose in what follows that we have the stronger condition $\alpha-d>0$ (mainly not too have to treat too many different cases in the proof, but we could have results without this condition) which means that a given point lies almost surely in finitely many traps.

Given $L>0$, we consider the hyperplane $\mathcal{H}_L$ at distance L from the origin.

$$\mathcal{H}_L:=\{L\}\times\mathbb{R}^{d-1}.\tag{2.4}$$

Denote by $\mathbf{P}$, $\mathbf{E}$ (resp. $\mathbf{P}_x$, $\mathbf{E}_x$) the law and expectation associated to standard d-dimensional Brownian motion $(B_t)_{t\geq 0}$ started from the origin (resp. from x).

Given $\lambda > 0$ we study the trajectories of a Brownian Motion started from the origin killed with rate $(V^\omega(\cdot) + \lambda)$ conditioned to survive till it hits $\mathcal{H}_L$. The survival probability is equal to

$$Z_L^\omega := \mathbf{E}\left[\exp\left(-\int_0^{T_{\mathcal{H}_L}} \left(V^\omega(B_t) + \lambda\right) \mathrm{d}t\right)\right]. \tag{2.5}$$

(For any set A, T_A denotes the hitting time of A.) The law of the trajectories conditioned to survival μ_L^ω is absolutely continuous with respect to $\mathbf{P}$, and its density is given by

$$\frac{\mathrm{d}\mu_L^\omega}{\mathrm{d}\mathbf{P}}(B) := \frac{1}{Z_L^\omega} \exp\left(-\int_0^{T_{\mathcal{H}_L}} \left(V^\omega(B_t) + \lambda\right) \mathrm{d}t\right). \tag{2.6}$$

To study transversal fluctuation of the trajectory around the axis $\mathbb{R}\mathbf{e}_1$ ($\mathbf{e}_1 = (1, 0, \dots, 0)$ being the first coordinate vector), one has to give a true definition to the notion of volume exponent discussed in the introduction. In that aim, define

$$\mathcal{C}_L^\xi := \left\{z \in \mathbb{R}^d \mid \exists \alpha \in [0, L], |z - \alpha \mathbf{e}_1| \le L^\xi\right\} = \bigcup_{\alpha \in [0,L]} B\left(\alpha \mathbf{e}_1, L^\xi\right) \tag{2.7}$$

and

$$\mathcal{A}_L^\xi := \left\{(B_t)_{t \ge 0} \mid \forall s \in [0, T_{\mathcal{H}_L}], B_s \in \mathcal{C}_L^\xi\right\} \tag{2.8}$$

the event "the trajectories stays in the tube $\mathcal{C}_L^\xi$ till the hitting time of $\mathcal{H}_L$." We define the upper and lower volume exponent ξ_0 and ξ_1 as follows:

$$\begin{aligned} \xi_1 &:= \inf\left\{\xi \mid \lim_{L\to\infty} \mathbb{E}\left[\mu_L^\omega\left(\mathcal{A}_L^\xi\right)\right] = 1\right\}, \\ \xi_0 &:= \sup\left\{\xi \mid \lim_{L\to\infty} \mathbb{E}\left[\mu_L^\omega\left(\mathcal{A}_L^\xi\right)\right] = 0\right\}. \end{aligned} \tag{2.9}$$

From the definition, $\xi_1 \ge \xi_0$ but one expects that $\xi_1 = \xi_0$ and their common value is referred to as the volume exponent. The main result of this paper is to get an upper-bound on ξ_1. Set

$$\widetilde{\xi}(\alpha, \gamma, d) := \max\left(\frac{3}{4}, \frac{1}{1 + \alpha - d}, \min\left(\frac{1}{1+\gamma}, \frac{2+d}{2\alpha}\right)\right) < 1. \tag{2.10}$$

Theorem 2.1. *For all $\xi > \widetilde{\xi}(\alpha, \gamma, d)$, one has*

$$\lim_{L\to\infty} \mathbb{E}\left[\mu_L^\omega\left(\mathcal{A}_L^\xi\right)\right] = 1. \tag{2.11}$$

Or equivalently $\xi_1 \le \widetilde{\xi}$.

In some special case, when $(\alpha - d) = \gamma \le 1/3$, then the upper bound above coincides with the lower-bound proved in a first study on this model [5], Theorem 2.1,

$$\xi_0 \ge \min\left(\frac{1}{2}, \frac{1}{1 + \alpha - d}, \frac{3}{3 + 2\gamma + \alpha - d}\right) \tag{2.12}$$

(to be more precise the definition of ξ_0 in [5] is a bit different because the set $\mathcal{C}_L^\xi$ there is not exactly the same, but Theorem 2.1 there implies (2.12)). And therefore

Corollary 2.2. *For any value of α, d and γ that satisfies $(\alpha - d) = \gamma \le 1/3$, one has*

$$\xi_1 = \xi_0 = \frac{1}{1+\gamma}. \tag{2.13}$$

A much related problem is the study of trajectories conditioned to survive up to the hitting time of a distant ball. We introduce this model now for two reason:

- We use it as a tool for the proof of the result above.
- An analogous result can be proved using the same method for this model.

For a Brownian Motion started at x and killed with rate $\lambda + V(\cdot)$, we denote by

$$Z^\omega(x, y) := \mathbf{E}_x\big[\mathrm{e}^{-\int_0^{T_{B(y)}}(\lambda+V(B_t))\,\mathrm{d}t}\mathbf{1}_{\{T_{B(y)}<\infty\}}\big], \tag{2.14}$$

the probability of survival up to the hitting time of $T_{B(y)}$ of $B(y) = B(y, 1)$ the Euclidean ball of radius one and center y, $|x - y| \geq 1$ (we keep this notation for what follows and denote by $B(z, r)$ the Euclidean ball of center z radius $r \in \mathbb{R}_+$), and by $\mu^\omega_{x,y}$ the law of the trajectory $(B_t)_{t\in[0,T_{B(y)}]}$ conditioned to survival, its derivative with respect to $\mathbf{P}_x$ is equal to

$$\frac{\mathrm{d}\mu^\omega_{x,y}}{\mathrm{d}\mathbf{P}_x} := \frac{1}{Z^\omega(x, y)}\mathrm{e}^{-\int_0^{T_{B(y)}}(\lambda+V(B_t))\,\mathrm{d}t}\mathbf{1}_{\{T_{B(y)}<\infty\}}. \tag{2.15}$$

For this reason, for a given $y \in \mathbb{R}^d$ one defines in analogy with $\mathcal{A}^\xi_L$ and $\mathcal{C}^\xi_L$.

$$\mathcal{C}^\xi_y := \big\{z \in \mathbb{R}^d \mid \exists\alpha \in [0, 1], |z - \alpha y| \leq |y|^\xi\big\} = \bigcup_{\alpha\in[0,1]} B\big(\alpha y, |y|^\xi\big) \tag{2.16}$$

and

$$\mathcal{A}^\xi_y := \big\{(B_t)_{t\geq 0} \mid \forall s \in [0, T_{B(y)}], B_s \in \mathcal{C}^\xi_y\big\}. \tag{2.17}$$

The following analogous of Theorem 2.1

Theorem 2.3. *For all $\xi > \widetilde{\xi}(\alpha, \gamma, d)$, one has*

$$\lim_{|y|\to\infty} \mathbb{E}\big[\mu^\omega_{0,y}\big(\mathcal{A}^\xi_y\big)\big] = 1. \tag{2.18}$$

Remark 2.4. *For the point-to-point model, one does not have an equivalent of Corollary 2.2, the reason being that the lower-bound that we have on ξ_0 in [5] was slightly suboptimal. However we strongly believe that the analogous results hold.*

The ideas of this proof are inspired by [9] and [12] where an upper bound on the volume exponent is proved for model with traps of bounded range ($\xi_1 \leq 3/4$). In [9], Sznitman uses martingale techniques to prove concentration of $Z^\omega(x, y)$ around its mean, and in [12] Wüthrich uses these concentration results to prove the bound on the volume exponent.

These techniques cannot directly apply to our model, and in fact both bounds proved in [9] and [12] do not hold when there are too strong correlations in the environment. This is not surprising as in [5] it was shown that the upper-bound $\xi_1 \leq 3/4$ proved in [12] does not always hold.

Our strategy is to study the model with a slightly modified potential:

- First in Section 3 we present our modification of the potential and show that it does not modify much that probabilities of $\mathcal{A}^\xi_y$, $\mathcal{A}^\xi_L$ (Proposition 3.1).
- In Section 4, we show that the partition function associated to the modified potential concentrates around its mean, using a multiscale analysis (Proposition 4.1).
- In Section 5, we use Proposition 4.1 to prove Theorems 2.1 and 2.3.

Remark 2.5. *Some of the refinement of the techniques (in particular, the multiscale analysis) used here are not needed if one simply wants to prove that $\mathbb{E}[\mu^\omega_{0,y}(\mathcal{A}^\xi_y)]$ tends to zero for some $\xi < 1$. The reason we use them is that they allow us to get a slightly better bound, and that they are absolutely necessary to get Corollary 2.2.*

3. Modification of the potential V

We slightly modify V in order to have a potential with nicer properties. In particular we want to

- Make it bounded (by a constant depending on L).
- Suppress traps whose radius is too large to have only finite range correlation (what "too large" depends also on L) in order to treat potential for far away region independently.

In this section we define this modified potential and show that with our choice for modifications of the potential does not significantly change the probability of $\mathcal{A}_L^\xi$ (or if it does, that it does it in the right direction). Given $\xi > \widetilde{\xi}(\alpha, t, d)$ we define

$$\bar{\xi} := \min(\xi, d/\alpha). \tag{3.1}$$

The modified potential $\bar{V}_L^\omega$ by

$$\bar{V}_L^\omega(x) := \sum_{n=0}^{\bar{\xi}\log_2 L} \min\left(\sum_{i=1}^{\infty} \mathbf{1}_{\{r_i\in[2^n,2^{n+1})\}} r_i^{-\gamma} \mathbf{1}_{\{|x-\omega_i|\le r_i\}}, 2^{-n\gamma}\log L\right) \tag{3.2}$$

(it is the same as V except that it ignores traps whose radius is larger than $2L^{\bar{\xi}}$, and that it cuts the contribution of traps of diameter $[2^n, 2^{n+1})$ at the level $2^{-n\gamma}\log L$). In analogy with (2.5), (2.6), (2.14), (2.15) one defines $\bar{Z}_L^\omega$, $\bar{\mu}_L^\omega$, $\bar{Z}^\omega(x,y)$, $\bar{\mu}_L^\omega(x,y)$, by replacing V^ω by $\bar{V}_L^\omega$.

This is not a very drastic modification and it should not change the probability of $\mathcal{A}_L^\xi$ (and that of $\mathcal{A}_y^\xi$ for $|y| = L$) and for two reasons:

- With $\mathbb{P}$-probability going to one, there is no trap of radius more than $2L^{\bar{\xi}}$ that intersects $\mathcal{C}_L^\xi$.
- With $\mathbb{P}$-probability going to one,

$$\max\left(\sum_{i=1}^{\infty} \mathbf{1}_{\{r_i\in[2^n,2^{n+1})\}} r_i^{-\gamma} \mathbf{1}_{\{|x-\omega_i|\le r_i\}}, 2^{-n\gamma}\log L\right)$$

  is equal to $\sum_{i=1}^{\infty} \mathbf{1}_{\{r_i\in[2^n,2^{n+1})\}} r_i^{-\gamma} \mathbf{1}_{\{|x-\omega_i|\le r_i\}}$ for all x in $B(0, L^2)$ the Euclidean ball of radius L^2 centered at zero.

And indeed one has

Proposition 3.1. *There exists c such that, for all $\xi \ge \widetilde{\xi}$, for any y such that $|y| = L$, with probability going to one when L tends to infinity,*

$$\begin{aligned} \mu_{0,y}^\omega\big(\mathcal{A}_y^\xi\big) &\ge \bar{\mu}_{0,y}^\omega\big(\mathcal{A}_y^\xi\big) - \mathrm{e}^{-cL^2}, \\ \mu_L^\omega\big(\mathcal{A}_L^\xi\big) &\ge \bar{\mu}_L^\omega\big(\mathcal{A}_L^\xi\big) - \mathrm{e}^{-cL^2}. \end{aligned} \tag{3.3}$$

Proof. We only prove the first line in (3.3) which is the result concerning the point-to-point model. The other one is proved analogously. Set

$$\widetilde{V}_L^\omega(x) := \sum_{i=1}^{\infty} \mathbf{1}_{\{r_i\le 2L^{\bar{\xi}}\}} r_i^{-\gamma} \mathbf{1}_{\{|x-\omega_i|\le r_i\}} \tag{3.4}$$

(the only difference with V is that traps with radius larger than $2L^{\bar{\xi}}$ are not taken into account) and define $\widetilde{\mu}_{0,y}^\omega$ and $\widetilde{Z}^\omega(0,y)$ as in (2.14) and (2.15).

Our first job is to show that $\bar{\mu}^{\omega}_{0,y}$ and $\widetilde{\mu}^{\omega}_{0,y}$ are close in total variation, then we compare $\widetilde{\mu}^{\omega}_{0,y}(\mathcal{A}^{\xi}_y)$ with $\bar{\mu}^{\omega}_{0,y}(\mathcal{A}^{\xi}_y)$. We notice that $\bar{V}^{\omega}$ and $\widetilde{V}^{\omega}$ coincide with probability tending to one on $B(0,L^2)$, indeed a consequence of Lemma A.1 (proved in the Appendix) is that

$$\mathbb{P}\big[\exists x\in B\big(0,L^2\big),\bar{V}^{\omega}(x)\neq\widetilde{V}^{\omega}(x)\big]\leq\frac{1}{L}. \tag{3.5}$$

When the event $\{\forall x\in B(0,L^2),\bar{V}^{\omega}(x)=\widetilde{V}^{\omega}(x)\}$ holds then $\bar{\mu}^{\omega}_{0,y}(\cdot|\mathcal{S}_L)$ and $\widetilde{\mu}_{0,y}(\cdot|\mathcal{S}_L)$, the measures conditioned on the event

$$\mathcal{S}_L=\big\{\forall t\in[0,T_{B(y)}],|B_t|\leq L^2\big\}, \tag{3.6}$$

are equal, and therefore it remains only to show that with large probability $\bar{\mu}_{0,y}((\mathcal{S}_L)^c)$ and $\widetilde{\mu}_{0,y}((\mathcal{S}_L)^c)$ are small. Set $\tau_{L^2}:=\inf\{t,|B_t|\geq L^2\}$, then

$$\bar{\mu}_{0,y}\big((\mathcal{S}_L)^c\big)\leq\frac{\mathbf{E}[\mathrm{e}^{-\lambda T_{B(y)}}\mathbf{1}_{\{\tau_{L^2}\leq T_{B(y)}<\infty\}}]}{\bar{Z}(0,y)}\leq\frac{\mathbf{E}[\mathrm{e}^{-\lambda\tau_{L^2}}]}{\bar{Z}(0,y)}. \tag{3.7}$$

As $\bar{V}(x)\leq\log L$ for all x, thanks to standard tubular estimate for Brownian motion (see e.g. (1.11) of [8]) for C large enough

$$\log\bar{Z}^{\omega}(0,y)\geq -CL\log L. \tag{3.8}$$

Other standard estimates give that there exists c such that

$$\mathbf{E}\big[\mathrm{e}^{-\lambda\tau_{L^2}}\big]\leq\mathrm{e}^{-cL^2} \tag{3.9}$$

so that for L large

$$\bar{\mu}^{\omega}_{0,y}\big((\mathcal{S}_L)^c\big)\leq\mathrm{e}^{-cL^2} \tag{3.10}$$

tends to zero when L tends to infinity. Working on the event "$\bar{V}$ and $\widetilde{V}$ coincide on $B(0,L^2)$" holds we get the same conclusion for $\widetilde{\mu}^{\omega}_{0,y}$ so that with probability going to one

$$\big\|\widetilde{\mu}^{\omega}_{0,y}-\bar{\mu}^{\omega}_{0,y}\big\|_{\mathrm{TV}}\leq\mathrm{e}^{-cL^2}. \tag{3.11}$$

Now we remark that with probability going to one $\widetilde{V}$ and V coincide on $\mathcal{C}^{\xi}_y$ i.e. that

$$\lim_{|y|=L\to\infty}\mathbb{P}\big[\exists x\in\mathcal{C}^{\xi}_y,\bar{V}^{\omega}(x)\neq V^{\omega}(x)\big]=\lim_{|y|=L\to\infty}\mathbb{P}\big[\exists i,r_i\geq 2L^{\bar{\xi}},B(\omega,r_i)\cap\mathcal{C}^{\xi}_y\neq\varnothing\big]=0. \tag{3.12}$$

Indeed the number of traps of radius larger that $2L^{\bar{\xi}}$ that intersects $\mathcal{C}^{\xi}_y$ is a Poisson variable and its mean is

$$\int_{2L^{\bar{\xi}}}^{\infty}\big(\sigma_d\big(r+L^{\xi}\big)^d+L\sigma_{d-1}\big(r+L^{\xi}\big)^{d-1}\big)\alpha r^{-\alpha-1}\,\mathrm{d}r\leq CL^{1+\xi(d-1)-\alpha\bar{\xi}}. \tag{3.13}$$

We let the reader check that with our choice of ξ and $\bar{\xi}$,

$$1+\xi(d-1)-\alpha\bar{\xi}<0 \tag{3.14}$$

so that the r.h.s. of (3.13) tends to zero. For any x in $(\mathcal{C}^{\xi}_y)^c$ we necessarily have $\widetilde{V}^{\omega}(x)\leq V^{\omega}(x)$ from the definitions, so that on the event "$\widetilde{V}$ and V coincide on $\mathcal{C}^{\xi}_y$,"

$$\mu^{\omega}_{0,y}\big(\mathcal{A}^{\xi}_y\big)\geq\widetilde{\mu}^{\omega}_{0,y}\big(\mathcal{A}^{\xi}_y\big). \tag{3.15}$$

A combination of the above and (3.11) allows us to conclude. □

4. Concentration inequalities

In this section, one derives some concentration inequalities similar to the one obtained in [9] for the log partition function with the modified potential $\log \bar{Z}^\omega(u,v)$. It could be shown that for some choice of parameters, these concentration results do not hold for the original potential. We suppose that L is fixed, and set

$$\bar{Z}^\omega(u,v) := \mathbf{E}_x\left[\mathrm{e}^{\int_0^{T_y}(\lambda+\bar{V}^\omega(B_t))\,\mathrm{d}t}\right], \tag{4.1}$$

$$\chi = \chi(\xi) := \max\left(\frac{1}{2}, (1-\gamma)\bar{\xi}, \frac{1}{2}\left(1+\bar{\xi}(1+d-2\gamma-\alpha)\right)\right). \tag{4.2}$$

Proposition 4.1. *Suppose that $\xi \geq \widetilde{\xi}(\alpha,\gamma,d)$. For any $\varepsilon > 0$ one can find δ such that for any $(u,v) \in \mathbb{R}^d$, $|u-v| \leq 2L$*

$$\mathbb{P}\left(\left|\log \bar{Z}^\omega(u,v) - \mathbb{E}\log \bar{Z}^\omega(u,v)\right| \geq L^{\chi+\varepsilon}\right) \leq \exp\left(-L^\delta\right). \tag{4.3}$$

As the environment is translation invariant, we need only to prove the result the case $|v| \leq 2L$, $u = 0$.

The proof of this proposition requires a multi-scale analysis, to treat traps of different scale in separate steps. One could get a result by doing a rougher analysis, but this would never get us something optimal. On the contrary, the multi-scale analysis allows us to get sharper results that are optimal for some special choice of the parameters (i.e. they allow to get an upper bound on the volume exponent that matches the lower bound).

For all n define $\mathcal{F}_n$ to be the sigma-algebra generated by the traps of radius smaller than 2^n

$$\mathcal{F}_n := \sigma\left(\omega(A), A \in \mathcal{B}\left(\mathbb{R}^d \times \mathbb{R}_+\right), A \subset \mathbb{R}^d \times \left[1, 2^n\right]\right) \tag{4.4}$$

($\omega(A)$ above stands for the number of point in A and $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}_+)$ stands for the sigma fields of Borel-sets). We define for $n \geq 0$,

$$M_n := \mathbb{E}\left[\log \bar{Z}^\omega(0,v) | \mathcal{F}_n\right]. \tag{4.5}$$

(Note that $M_{\bar{\xi}\log_2 L} = \log \bar{Z}^\omega(0,v)$.) The sequence $(M_n)_{n\geq 0}$ is a martingale for the filtration $(\mathcal{F}_n)_{n\geq 0}$. We prove Proposition 4.1 by proving concentration for every increment of $(M_n)_{n\geq 0}$ (there are only $\mathrm{O}(\log L)$ increments so that this is sufficient to get the result).

Lemma 4.2. *For any ε there exists δ such that for all $n \in [1, \bar{\xi}\log_2 L]$,*

$$\mathbf{P}\left[|M_n - M_{n-1}| \geq L^{\chi+\varepsilon}\right] \leq \mathrm{e}^{-L^\delta}. \tag{4.6}$$

To prove the above lemma we can adapt and use the technique developed in [9]: given n we partition R^d in disjoint cubes of side length 2^n,

$$\left(2^n x + \left[0, 2^n\right]^d\right)_x \in \mathbb{Z}^d, \tag{4.7}$$

and index them by $\mathbb{N}$ in an arbitrary way and call that sequence $(C_{n,k})_{k\geq 1}$. Then one sets

$$\mathcal{F}_{n,k} := \sigma\left(\omega(A), A \in \mathcal{B}\left(\mathbb{R}^d \times \mathbb{R}_+\right), A \subset \left[\left(\mathbb{R}^d \times \left[1, 2^{n-1}\right]\right) \cup \left(\bigcup_{i=1}^k C_{n,i} \times \left[2^{n-1}, 2^n\right]\right)\right]\right), \tag{4.8}$$

which is the sigma algebra generated by traps of radius smaller than 2^{n-1} and traps of radius in $[2^{n-1}, 2^n]$ whose centers are located in the set of cube $\bigcup_{i=1}^k C_{n,i}$ ($\omega(A)$ above stands for the number of point in A and $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}_+)$ stands for the sigma field of Borel-sets).

One defines for $k \geq 0$

$$M_{n,k} := \mathbb{E}\left[\log \bar{Z}^\omega(0,v) | \mathcal{F}_{n,k}\right]. \tag{4.9}$$

One remarks that for fixed n, $(M_{n,k})_{k\geq 0}$ is a martingale for the filtration $(\mathcal{F}_{n,k})_{k\geq 0}$. It is an interpolation between $M_{n-1}=M_{n,0}$ and $M_n=M_{n,\infty}$. This allows us to use a Martingale concentration result by Kesten to prove Lemma 4.2.

Proposition 4.3 (From [3], Theorem 3). *Let $(X_n)_{n\geq 0}$ be a martingale with respect to the filtration $\mathcal{G}_n$, (law P expectation E) that satisfies*

$$|X_{n+1}-X_n|\leq c_1,\quad \forall n\geq \mathbb{N} \tag{4.10}$$

and

$$E\big[(X_{n+1}-X_n)^2|\mathcal{G}_n\big]\leq E[V_n|\mathcal{G}_n] \tag{4.11}$$

for some sequence of random variable $(V_n)_{n\geq 0}$ satisfying:

$$P\Big(\sum_{n\geq 0}V_n\geq x\Big)\leq \mathrm{e}^{-c_2x} \tag{4.12}$$

for all $x\geq c_3$.

Set $x_0:=\max(\sqrt{c_3},c_1)$. Then $X_\infty=\lim_{n\to\infty}X_n$ exists and for all $x\leq c_2x_0^3$

$$P\big(|X_\infty-X_0|\geq x\big)\leq C\Big(1+\frac{1}{c_2x_0}\Big)\mathrm{e}^{-x/(Cx_0)}, \tag{4.13}$$

where C is a universal constant not depending on the $(c_i)_{i=1}^3$.

Our proof of Lemma 4.2 consists simply in checking, for each value of n, the assumptions of Proposition 4.3 for the martingale $(M_{n,k})_{k\geq 0}$. For any cube $C_{n,k}$ one defines

$$\widetilde{C}_{n,k}:=\bigcup_{x\in C_{n,k}}B\big(x,2^n\big) \tag{4.14}$$

(this is the zone where the V can be modified when one adds traps of radius smaller than 2^n with center in $C_{n,k}$) and T_k to be the hitting time of $\widetilde{C}_{n,k}$.

Lemma 4.4. *For every n and k set $\Delta M_{n,k}=M_{n,k}-M_{n,k-1}$. One can find a constant C such that for every n and k,*

$$|\Delta M_{n,k}|\leq C(\log L)^2 2^{n(1-\gamma)} \tag{4.15}$$

and

$$\mathbb{E}\big[|\Delta M_{n,k}|^2|\mathcal{F}_{n,k-1}\big]\leq \mathbb{E}[U_{n,k}|\mathcal{F}_{n,k-1}], \tag{4.16}$$

where

$$U_{n,k}=C\bar{\mu}^{\omega}_{0,y}(T_k\leq T_{B(y)})2^{n[2(1-\gamma)+d-\alpha]}(\log L)^2. \tag{4.17}$$

Moreover

$$\mathbb{P}\Bigg[\sum_{k=1}^{\infty}U_{n,k}\geq x\Bigg]\leq \mathrm{e}^{-C^{-2}2^{n(-1+2\gamma+\alpha-d)}(\log L)^{-2}x} \tag{4.18}$$

for all $x\geq C^2L2^{n(1-2\gamma+d-\alpha)}(\log L)^3$.

Proof of Lemma 4.2. According to Lemma 4.4 the assumptions of Proposition 4.3 are satisfied with

$$\begin{aligned} c_1 &= c_1(L,n) := C(\log L)^2 2^{n(1-\gamma)} \le C(\log L)^2 L^{\bar\xi(1-\gamma)_+}, \\ c_2 &= c_2(L,n) := C^{-2} 2^{n(-1+2\gamma+\alpha-d)}(\log L)^{-2} \ge C^{-2} L^{-\bar\xi(1-2\gamma+d-\alpha)_+}(\log L)^{-2}, \\ c_3 &= c_3(L,n) := C^2 L 2^{n(1-2\gamma+d-\alpha)}(\log L)^3 \le C^2(\log L)^3 L^{1+\bar\xi(1-2\gamma+d-\alpha)_+}. \end{aligned} \tag{4.19}$$

And therefore we get that for $x_0(L) = C(\log L)^2 L^{\max(\bar\xi(1-\gamma)_+,(1+\bar\xi(1-2\gamma+d-\alpha)_+)/2)}$, for all $t \le c_2 x_0^2/C$ (and note that $c_2 x_0^2 \ge L$)

$$\mathbb{P}\big[|M_{n-1} - M_n| \ge C x_0 t\big] \le \big(1 + L^d\big)\mathrm{e}^{-t} \tag{4.20}$$

provided the constance C has been chosen large enough. This is enough to conclude. □

At this point of the proof, we can explain a bit better our choice for the multi-scale analysis, and for the modification of the potential. Both are aimed to optimize the constant c_1, c_2, c_3 above.

Proof of Lemma 4.4. One defines $\widetilde\omega$, to be an independent copy of the environment ω (let its law be denoted by $\widetilde{\mathbb{E}}$). Let $\omega_{n,k}$ be an interpolation between ω and $\widetilde\omega$ defined by

$$\begin{aligned} \omega_{n,k} := &\left\{(\omega_i, r_i) \,\middle|\, (\omega_i, r_i) \in \big(\mathbb{R}^d \times \big[1, 2^{n-1}\big]\big) \cup \left(\bigcup_{j=1}^{k} C_{n,j} \times \big[2^{n-1}, 2^n\big]\right)\right\} \\ &\cup \left\{(\widetilde\omega_i, r_i) \,\middle|\, (\omega_i, r_i) \in \left(\bigcup_{j=k+1}^{\infty} C_{n,j} \times \big[2^{n-1}, 2^n\big]\right) \cup \big(\mathbb{R}^d \times \big[2^n, \infty)\big)\right\}. \end{aligned} \tag{4.21}$$

And set

$$\begin{aligned} V_{n,k} &:= \bar V^{\omega_{n,k}}, \\ Z_{n,k}(u,v) &:= \bar Z^{\omega_{n,k}}(u,v), \\ \mu_{u,v}^{n,k} &:= \bar\mu_{u,v}^{\omega_{n,k}}. \end{aligned} \tag{4.22}$$

With this notation, note that $\omega_{n,k}$ has the same distribution as ω and that

$$M_{n,k} = \widetilde{\mathbb{E}}\big[\log Z_{n,k}(0,v)\big]. \tag{4.23}$$

Furthermore

$$|\Delta M_{n,k}| \le \widetilde{\mathbb{E}}\left[\log\max\left(\frac{Z_{n,k-1}}{Z_{n,k}}(0,y), \frac{Z_{n,k}}{Z_{n,k-1}}(0,y)\right)\right]. \tag{4.24}$$

The first step of our proof is to bound $\frac{Z_{n,k-1}}{Z_{n,k}}(0,y)$ and $\frac{Z_{n,k}}{Z_{n,k-1}}(0,y)$ by simpler functional depending only on $V_{n,k}$, $V_{n,k-1}$ in $\widetilde C_k$. We use the following (abuse of) notation

$$(V_{n,k} - V_{n,k-1})_+ := \max_{x\in\mathbb{R}^d}\big(V_{n,k}(x) - V_{n,k-1}(x)\big)_+. \tag{4.25}$$

Lemma 4.5. *There exists a constant C such that for all n and k, for all v in $\mathbb{R}^d$*

$$\begin{aligned} \frac{Z_{n,k-1}}{Z_{n,k}}(0,v) &\le 1 + \mu_{0,v}^{n,k}[T_k \le T_{B(v)}]\big(\mathrm{e}^{C(V_{n,k}-V_{n,k-1})_+ 2^n \log L} - 1\big), \\ \frac{Z_{n,k}}{Z_{n,k-1}}(0,v) &\le 1 + \mu_{0,v}^{n,k-1}[T_k \le T_{B(v)}]\big(\mathrm{e}^{C(V_{n,k-1}-V_{n,k})_+ 2^n \log L} - 1\big). \end{aligned} \tag{4.26}$$

Proof. By symmetry of the problem it is sufficient to show that

$$\frac{Z_{n,k-1}}{Z_{n,k}}(0,v)-1\le \mu^{n,k}_{0,v}[T_k\le T_{B(v)}]\big(\mathrm{e}^{C(V_{n,k}-V_{n,k-1})_+2^n\log L}-1\big). \tag{4.27}$$

Using the Markov property at T_k one gets

$$\frac{Z_{n,k-1}}{Z_{n,k}}(0,v)=\mu^{n,k}_{0,v}[T_k>T_{B(v)}]+\mu^{n,k}_{0,v}\biggl[T_k\le T_{B(v)};\ \frac{Z_{n,k-1}}{Z_{n,k}}(B_{T_k},v)\biggr], \tag{4.28}$$

and hence

$$\begin{aligned}\frac{Z_{n,k-1}}{Z_{n,k}}(0,v)-1&=\mu^{n,k}_{0,v}\biggl[T_k\le T_{B(v)};\ \biggl(\frac{Z_{n,k-1}}{Z_{n,k}}(B_{T_k},v)-1\biggr)\biggr]\\&\le \mu^{n,k}_{0,v}[T_k\le T_{B(v)}]\max_{z\in\partial\widetilde{C}_k}\biggl(\frac{Z_{n,k-1}}{Z_{n,k}}(z,v)-1\biggr).\end{aligned} \tag{4.29}$$

We are left with showing that for all $z\in\partial\widetilde{C}_k$

$$\frac{Z_{n,k-1}}{Z_{n,k}}(z,v)\le \mathrm{e}^{C(V_{n,k}-V_{n,k-1})_+2^n\log L}. \tag{4.30}$$

One has

$$\begin{aligned}\frac{Z_{n,k-1}}{Z_{n,k}}(z,v)&=\mu^{n,k}_{z,v}\big[\mathrm{e}^{\int_0^{T_{B(v)}}(V_{n,k}-V_{n,k-1})(B_t)\,\mathrm{d}t}\big]\\&\le \mu^{n,k}_{z,v}\big[\mathrm{e}^{\int_0^{T_{B(v)}}(V_{n,k}-V_{n,k-1})_+(B_t)\,\mathrm{d}t}\big].\end{aligned} \tag{4.31}$$

We study the tail distribution of the variable $\int_0^{T_{B(v)}}(V_{n,k}-V_{n,k-1})_+(B_t)\,\mathrm{d}t$ under $\mu^{n,k}_{z,v}$. On the event $\int_0^{T_{B(v)}}(V_{n,k}-V_{n,k-1})_+(B_t)\,\mathrm{d}t>a$ one can define

$$\tau_a:=\min\biggl\{t>0\ \Big|\ \int_0^t(V_{n,k}-V_{n,k-1})_+(B_s)\,\mathrm{d}s=a\biggr\}. \tag{4.32}$$

Necessarily, (as $V_{n,k}-V_{n,k-1}\equiv 0$ outside of $\widetilde{C}_{n,k}$)

$$\tau_a\ge\frac{a}{(V_{n,k}-V_{n,k-1})_+}\quad\text{and}\quad B_{\tau_a}\in\widetilde{C}_{n,k}. \tag{4.33}$$

Using the Markov property and the above one gets that

$$\begin{aligned}&\mu^{n,k}_{z,v}\biggl(\int_0^{T_{B(v)}}(V_{n,k}-V_{n,k-1})_+(B_t)\,\mathrm{d}t\ge a\biggr)\\&\quad=\frac{1}{Z_{n,k}(z,v)}\mathbf{E}_z\big[\mathrm{e}^{-\int_0^{\tau_a}(\lambda+V_k(B_t))\,\mathrm{d}t}Z_{n,k}(B_{\tau_a},v)\big]\\&\quad\le\frac{1}{Z_{n,k}(z,v)}\mathbf{E}_z\big[\mathrm{e}^{-\lambda\tau_a-a}Z_{n,k}(B_{\tau_T},v)\big]\\&\quad\le \mathrm{e}^{-(\lambda/(V_{n,k}-V_{n,k-1})_++1)a}\max_{x\in\widetilde{C}_k}\frac{Z_{n,k}(x,v)}{Z_{n,k}(z,v)}\le \mathrm{e}^{-(\lambda/(V_{n,k}-V_{n,k-1})_++1)a+c2^n\log L},\end{aligned} \tag{4.34}$$

where in the last inequality one used an Harnack-type inequality (it is proved in (2.22), p. 225, in [10] for x and z such that $|x-z|\le 1$ so that we can get the result below by iterating it) there exists a constant c such that:

$$\forall x\,\forall z\in\mathbb{R}^d,\qquad \left|\log\frac{Z_{n,k-1}(x,v)}{Z_{n,k-1}(z,v)}\right|\le c\bigl(1+|x-z|\bigr)\|V_{n,k-1}\|_\infty. \tag{4.35}$$

Hence

$$\begin{aligned}
&\mu_{z,v}^{n,k}\bigl[\mathrm{e}^{\int_0^{T_{B(v)}}(V_{n,k}-V_{n,k-1})_+(B_t)\,\mathrm{d}t}\bigr]\\
&\quad\le 1+\int_0^\infty \mathrm{e}^a\min\bigl(1,\mathrm{e}^{-(\lambda/(V_{n,k}-V_{n,k-1})_++1)a+c2^n\log L}\bigr)\,\mathrm{d}a\\
&\quad=\frac{\lambda+(V_{n,k}-V_{n,k-1})_+}{\lambda}\mathrm{e}^{c2^n\log L(V_{n,k}-V_{n,k-1})_+/(\lambda+(V_{n,k}-V_{n,k-1})_+)}.
\end{aligned}\tag{4.36}$$

□

Let us introduce the notation

$$\begin{aligned}
\mathcal{N}_{n,k,+}&:=\bigl|\{\text{points that are in }\omega_{n,k}\text{ and not in }\omega_{n,k-1}\}\bigr|,\\
\mathcal{N}_{n,k,-}&:=\bigl|\{\text{points that are in }\omega_{n,k-1}\text{ and not in }\omega_{n,k}\}\bigr|.
\end{aligned}\tag{4.37}$$

These two quantities are independent Poisson variable of mean $2^{n(d-\alpha)}(2^\alpha-1)$. According to the definition of $\bar V^\omega$ and $V_{n,k}$ one has

$$\begin{aligned}
(V_{n,k}-V_{n,k-1})_+&\le 2^{-(n-1)\gamma}(\mathcal{N}_{n,k,+}\vee\log L),\\
(V_{n,k-1}-V_{n,k})_+&\le 2^{-(n-1)\gamma}(\mathcal{N}_{n,k,-}\vee\log L).
\end{aligned}\tag{4.38}$$

Combining Lemma 4.5, equations (4.24) and (4.38), one gets (4.15). In order to get (4.16) we use equation (4.24) to get that

$$|\Delta M_k|^2\le\widetilde{\mathbb{E}}\left[\max\left(\left(\log\frac{Z_{n,k-1}}{Z_{n,k}}(0,v)\right)^2,\left(\log\frac{Z_{n,k}}{Z_{n,k-1}}(0,v)\right)^2\right)\right].\tag{4.39}$$

And from Lemma 4.5,

$$\begin{aligned}
\log\frac{Z_{n,k-1}}{Z_{n,k}}(0,v)\quad&\le\quad\log\bigl(1+\mu_{0,v}^{n,k}(T_k\le T_{B(v)})\mathrm{e}^{C2^n(\log L)(V_{n,k-1}-V_{n,k-1})_+}\bigr)\\
&\overset{\text{(Jensen)}}{\le}\quad C2^n(\log L)\mu_{0,v}^{n,k}(T_k\le T_{B(v)})(V_{n,k-1}-V_{n,k-1})_+\\
&\overset{(4.38)}{\le}\quad C2^{n(1-\gamma)}(\log L)\mu_{0,v}^{n,k}(T_k\le T_{B(v)})\mathcal{N}_{n,k,+}.
\end{aligned}\tag{4.40}$$

One can get an analogous bound for $\log\frac{Z_{n,k}}{Z_{n,k-1}}(0,v)$ and get that

$$\begin{aligned}
|\Delta M_k|^2&\le C^24^{n(1-\gamma)}(\log L)^2\\
&\qquad\times\widetilde{\mathbb{E}}\bigl[\max\bigl(\mathcal{N}_{n,k,+}^2\mu_{0,v}^{n,k}(T_k\le T_{B(v)})^2,\mathcal{N}_{n,k,-}^2\mu_{0,v}^{n,k-1}(T_k\le T_{B(v)})^2\bigr)\bigr].
\end{aligned}\tag{4.41}$$

Replacing max by a sum and conditioning to $\mathcal{F}_{n,k-1}$ one gets that to bound $\mathbb{E}[|\Delta M_k|^2|\mathcal{F}_{n,k-1}]$ it is sufficient to bound

$$\begin{aligned}
&\mathbb{E}\bigl[\widetilde{\mathbb{E}}\bigl[\mathcal{N}_{n,k,+}^2\mu_{0,v}^{n,k}(T_k\le T_{B(v)})^2\bigr]|\mathcal{F}_{n,k-1}\bigr],\\
&\mathbb{E}\bigl[\widetilde{\mathbb{E}}\bigl[\mathcal{N}_{n,k,-}^2\mu_{0,v}^{n,k-1}(T_k\le T_{B(v)})^2\bigr]|\mathcal{F}_{n,k-1}\bigr].
\end{aligned}\tag{4.42}$$

The reader can check that

$$\widetilde{\mathbb{E}}\big[\mathcal{N}^2_{n,k,+}\mu^{n,k}_{0,v}(T_k \le T_{B(v)})^2\big] = \mathbb{E}\big[\mathcal{N}^2_{n,k,+}\bar{\mu}^{\omega}_{0,v}(T_k \le T_{B(v)})^2 | \mathcal{F}_{n,k}\big],$$
$$\widetilde{\mathbb{E}}\big[\mathcal{N}^2_{n,k,-}\mu^{n,k-1}_{0,v}(T_k \le T_{B(v)})^2\big] = \mathbb{E}\big[\mathcal{N}^2_{n,k,+}\bar{\mu}^{\omega}_{0,v}(T_k \le T_{B(v)})^2 | \mathcal{F}_{n,k-1}\big]. \tag{4.43}$$

And thus we have just to bound from above control the r.h.s. of the second line. We rewrite it as follows

$$\mathbb{E}\big[\bar{\mu}^{\omega}_{0,v}(T_k \le T_{B(v)})^2\mathcal{N}^2_{n,k,+} | \mathcal{F}_{n,k-1}\big]$$
$$= \mathbb{E}\big[\mathbb{E}\big[\bar{\mu}^{\omega}_{0,v}(T_k \le T_{B(v)})^2 | \mathcal{F}_{n,k-1} \vee \sigma(\mathcal{N}_{n,k,+})\big]\mathcal{N}^2_{n,k,+} | \mathcal{F}_{n,k-1}\big]. \tag{4.44}$$

Then one can remark that

$$\mathbb{E}\big[\bar{\mu}^{\omega}_{0,v}(T_k \le T_{B(v)})^2 | \mathcal{F}_{n,k-1} \vee \sigma(\mathcal{N}_{n,k,+})\big]$$

is a non-increasing function of $\mathcal{N}_{n,k,+}$. If f is a non-increasing function of $\mathcal{N}$, g a non-decreasing function of $\mathcal{N}$ then

$$\mathbb{E}\big[f(\mathcal{N})g(\mathcal{N})\big] \le \mathbb{E}\big[f(\mathcal{N})\big]\mathbb{E}\big[g(\mathcal{N})\big]. \tag{4.45}$$

Therefore the right-hand side of (4.44) is less than

$$\mathbb{E}\big[\mathbb{E}\big[\mathcal{N}^2_{n,k,+}\big]\bar{\mu}^{\omega}_{0,v}(T_k \le T_{B(v)})^2 | \mathcal{F}_{n,k-1}\big]$$
$$= \mathbb{E}\big[\mathcal{N}^2_{n,k,+}\big]^2\mathbb{E}\big[\bar{\mu}^{\omega}_{0,v}(T_k \le T_{B(v)})^2 | \mathcal{F}_{n,k-1}\big]$$
$$\le C2^{n(d-\alpha)}\mathbb{E}\big[\bar{\mu}^{\omega}_{0,v}(T_k \le T_{B(v)}) | \mathcal{F}_{n,k-1}\big], \tag{4.46}$$

which combined with (4.41), (4.42) and (4.44) ends the proof of (4.16)–(4.17).

As for (4.18), notice that

$$\sum_{k=1}^{\infty}\bar{\mu}^{\omega}_{0,v}(T_k \le T_{B(v)}) = \bar{\mu}^{\omega}_{0,v}(A_{T_{B(y)}}), \tag{4.47}$$

where

$$A_T := \big|\big\{x \in \mathbb{Z}^d \mid \widetilde{C}_x \cap \big\{B_t, t \in [0,T]\big\} \neq \varnothing\big\}\big| \tag{4.48}$$

denotes the number of different $\widetilde{C}_x$ visited before T. Large deviation estimates for the upper-tail distribution of $A_{T_{B(y)}}$ under $\bar{\mu}^{\omega}_{0,v}$ are computed in the Appendix (Lemma A.3) and they allow us to obtain (4.18). □

5. Volume exponent from fluctuation

5.1. *Preliminary result*

Before going in to the proof of Theorems 2.1 and 2.3, we need a result that controls the growth of the expected value of $\log\bar{Z}^{\omega}(0,y)$ as a function of $|y|$.

Set $y_r := (r, 0, \dots, 0)$ and define

$$\alpha(r) := -\mathbb{E}\big[\log\bar{Z}^{\omega}(0,y_r)\big]. \tag{5.1}$$

It is natural to think that $r \mapsto \alpha(r)$ is increasing function of r and that its growth is linear, but we cannot prove it. Instead we prove a weaker result that will be sufficient to our purpose.

Lemma 5.1. *There exists a constant $c = c(\lambda)$ such that for any $l \geq L^{\chi+\varepsilon}$, $r \leq 2L$ one has, for all large enough L,*

$$\alpha(r+l) \geq \alpha(r) + cl. \tag{5.2}$$

Proof. Let us consider a family of ball $(B(x_i,1))_{i\in\{1,\dots,k_r\}}$, $x_i \in \partial B(0,r)$ with $k_r = O(r^{d-1})$ that cover the sphere $\partial B(0,r)$,

$$\partial B(0,r) \subset \bigcup_{i=1}^{k_r} \big(B(x_i,1)\big). \tag{5.3}$$

In order to reach y_{r+l} starting from zero, a Brownian motion has to touch one of the $B(x_i,1)$ first (as it is shown on Fig. 1) and therefore

$$\bar{Z}^\omega(0,y_{r+l}) \leq \sum_{i=1}^{k_r} \mathbf{E}\big[e^{-\int_0^{T_{B(x_i)}}(\lambda+\bar{V}^\omega(B_t))\,dt}\mathbf{1}_{\{T_{B(x_i)}\leq T_{B(y_{l+r})}\}}e^{-\int_{T_{B(x_i)}}^{T_{B(y_{l+r})}}(\lambda+\bar{V}^\omega(B_t))\,dt}\mathbf{1}_{\{T_{B(y_{l+r})}<\infty\}}\big]. \tag{5.4}$$

Moreover

$$\begin{aligned}
&\mathbf{E}\big[e^{-\int_0^{T_{B(x_i)}}(\lambda+\bar{V}^\omega(B_t))\,dt}\mathbf{1}_{\{T_{B(x_i)}\leq T_{B(y_{l+r})}\}}e^{-\int_{T_{B(x_i)}}^{T_{B(y_{l+r})}}(\lambda+\bar{V}^\omega(B_t))\,dt}\mathbf{1}_{\{T_{B(y_{l+r})}<\infty\}}\big]\\
&\quad\leq \mathbf{E}\big[e^{-\int_0^{T_{B(x_i)}}(\lambda+\bar{V}^\omega(B_t))\,dt}\mathbf{1}_{\{T_{B(x_i)}<\infty\}}\mathbf{E}_{T_{B(x_i)}}\big[e^{-\int_0^{T_{B(y_{l+r})}}(\lambda+\bar{V}^\omega(B_t))\,dt}\mathbf{1}_{\{T_{B(y_{l+r})}<\infty\}}\big]\big]\\
&\quad\leq \bar{Z}^\omega(0,x_i)\max_{z\in B(x_i,1)}\bar{Z}^\omega(z,y_{r+l}),
\end{aligned} \tag{5.5}$$

so that

$$\bar{Z}^\omega(0,y_{r+l}) \leq \sum_{i=1}^{k_r}\bar{Z}^\omega(0,x_i)\max_{z\in B(x_i,1)}\bar{Z}^\omega(z,y_{r+l}). \tag{5.6}$$

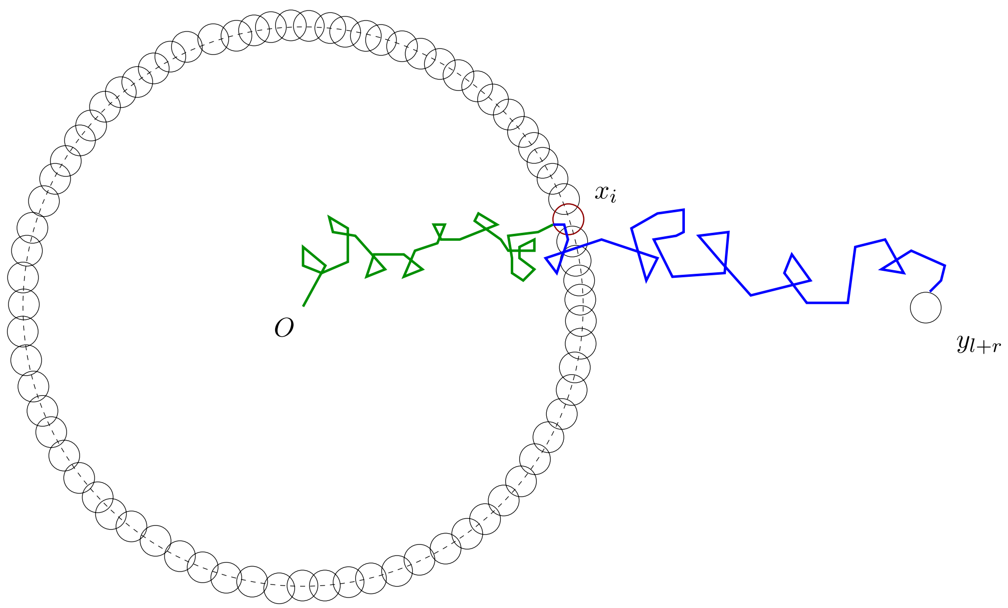

Fig. 1. In order to reach $B(y_{l+r})$, the Brownian motion starting from zero must first hit $\partial B(0,r)$, and thus by (5.3) it must it one of the $B(x_i)$. This observation allows us to get an upper-bound on $\bar{Z}^\omega(0,y_{r+l})$ in terms of $\bar{Z}^\omega(x_i,y_{r+l})$ and $\bar{Z}^\omega(0,x_i)$.

Now recall that

$$\bar{Z}^{\omega}(z, y_{r+l}) \le \mathbf{E}\big[\mathrm{e}^{-\lambda T_{B(y_{r+l}-z)}}\mathbf{1}_{\{T_{B(y_{r+l}-z)}<\infty\}}\big] \le \mathrm{e}^{-C_\lambda(|y_{r+l}-z|-2)} \tag{5.7}$$

for some constant C_λ (it follows from standard estimate for Brownian motion). Then notice that for any choice of z and x_i one has

$$|z-y_{r+l}| \ge |y_{r+l}| - |z| \ge |y_{r+l}| - \big(|z-x_i| + |x_i|\big) \ge l-1, \tag{5.8}$$

so that there exists a constant c such that for all $l \ge L^{\chi+\varepsilon}$, and $z \in B(x_i, 1)$

$$\bar{Z}^{\omega}(z, y_{r+l}) \le \mathrm{e}^{-2cl}. \tag{5.9}$$

As a consequence

$$\bar{Z}^{\omega}(0, y_{r+l}) \le \big(k_r \mathrm{e}^{-2cl}\big) \max_{i\in\{1,\dots,k_r\}} \bar{Z}^{\omega}(0, x_i). \tag{5.10}$$

The different $\bar{Z}^{\omega}(0, x_i)$ are identically distributed. Thanks to Proposition 4.1 one can find a δ such that for all L large enough

$$\mathbb{P}\Big(\log \max_{i\in\{1,\dots,k_r\}} \bar{Z}^{\omega}(0, x_i) - \alpha(r) \ge L^{\chi+(\varepsilon/2)}\Big) \le k_r \mathrm{e}^{-L^{\delta}}. \tag{5.11}$$

As we also have that deterministically

$$\max_{i\in\{1,\dots,k_r\}} \bar{Z}^{\omega}(0, x_i) \le 1. \tag{5.12}$$

This implies

$$\mathbb{E}\Big[\log \max_{i\in\{1,\dots,k_r\}} \bar{Z}^{\omega}(0, x_i)\Big] \le -\alpha(r) + L^{\chi+\varepsilon/2} + \alpha(r)k_r \mathrm{e}^{-L^{\delta}}. \tag{5.13}$$

Altogether by taking the expectation of $-\log$ of (5.10)

$$\alpha(l+r) \ge \alpha(r) + 2cl - \log k_r - \alpha(r)k_r\mathrm{e}^{-L^{\delta}} - L^{\chi+\varepsilon/2} \ge \alpha(r) + cl, \tag{5.14}$$

where the last inequality holds when the assumption given in the lemma for r and l are satisfied and L is large enough. □

5.2. *Proof of Theorem* 2.3

The idea for the proof is the following: Set $|y| = L$, according to Proposition 3.1 it is sufficient to prove that

$$\bar{\mu}^{\omega}_{0,y}\big(\big(\mathcal{A}^{\xi}_y\big)^c\big) \Rightarrow 0, \quad \text{in probability when } |y| \to \infty. \tag{5.15}$$

In order to go out of $\mathcal{C}^{\xi}_y$ before hitting $B(y)$, $(B_t)_{t\ge0}$ has to travel a longer distance that if it went in "straight-line." This extra distance traveled is at least of order $L^{2\xi-1}$. Lemma 5.1 allows to say that the cost of traveling is linear in the distance. However doing this may bring some extra-energy reward by allowing to visit regions that are more favorable energetically. Proposition 4.1 ensures that the energetic gain may not be more than $L^{\chi+\varepsilon}$. As with our choice of parameter

$$2\xi - 1 > \chi(\xi), \tag{5.16}$$

the cost of extra-travel cannot be compensated by this energetic gain and this implies that the probability of $(\mathcal{A}^{\xi}_y)^c$ is small. This is not too complicated to make this heuristic rigorous.

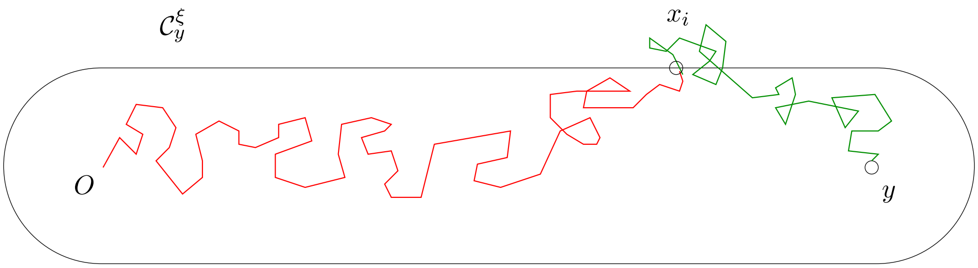

Fig. 2. If a trajectory does not belong to $\mathcal{A}_y^\xi$ then it has to hit $\partial\mathcal{C}_y^\xi$ at some point before $T(B(y))$ and thus, by (5.18) it has to hit one of the $B(x_i)$. This observation allows to get an upper bound on $Y_{0,y}^\omega$.

Our aim is to compare $\bar{Z}^\omega(0,y)$ with

$$Y_{0,y} = \bar{Z}^\omega(0,y)\bar{\mu}_{0,y}^\omega\big(\big(\mathcal{A}_y^\xi\big)^c\big) = \mathbf{E}\big[\mathrm{e}^{-\int_0^{T_{B(y)}}(\lambda+\bar{V}(B_t))\,\mathrm{d}t}\mathbf{1}_{\{T_{\partial\mathcal{C}_y^\xi}<T_{B(y)}\}}\big]. \tag{5.17}$$

Let us consider a family of ball $(B(x_i,1))_{i\in\{1,\dots,m_L\}}$, $x_i\in\partial\mathcal{C}_y^\xi$ with $m_L=\mathrm{O}(L^{(d-2)\xi+1})$ that satisfies

$$\partial\mathcal{C}_L^\xi\subset\bigcup_{i=1}^{m_L}B(x_i,1). \tag{5.18}$$

Trajectories in $(\mathcal{A}_y^\xi)^c$ have to hit one of the $B(x_i,1)$ before hitting $B(y)$ and therefore with a computation analogous to the one we made to obtain (5.6) (see Fig. 2), we get that

$$Y_{0,y}\le\sum_{i=1}^{m_L}\bar{Z}^\omega(0,x_i)\max_{z\in B(x_i,1)}\bar{Z}^\omega(z,y). \tag{5.19}$$

Note that one can find a constant C such that for any $z\in B(x,1)$ (cf. (2.22), p. 225, in [10]),

$$\big|\log\bar{Z}^\omega(z,y)-\log\bar{Z}^\omega(x,y)\big|\le C\log L. \tag{5.20}$$

Moreover, according to Proposition 4.1, for any $\varepsilon>0$ one has, for L large enough,

$$\begin{aligned}&\mathbb{P}\big(\exists i\in\{1,\dots,m_L\},\log\bar{Z}^\omega(0,x_i)+\alpha\big(|x_i|\big)\ge L^{\chi+\varepsilon}\big)\le m_L\mathrm{e}^{-L^\delta},\\&\mathbb{P}\big(\exists i\in\{1,\dots,m_L\},\log\bar{Z}^\omega(x_i,y)+\alpha\big(|y-x_i|\big)\ge L^{\chi+\varepsilon}\big)\le m_L\mathrm{e}^{-L^\delta}\end{aligned} \tag{5.21}$$

so that combining (5.20) and (5.21) one gets that with high probability

$$\begin{aligned}Y_{0,y}&\le m_L\mathrm{e}^{2L^{\chi(\xi)+\varepsilon}+C\log L}\max_{i\in\{0,\dots,m_L\}}\mathrm{e}^{-\alpha(|x_i|)-\alpha(|y-x_i|)}\\&\le m_L\mathrm{e}^{L^{\chi(\xi)+\varepsilon}+C\log L}\max_{x\in\partial\mathcal{C}_L^\xi}\mathrm{e}^{-\alpha(|x|)-\alpha(|y-x|)}.\end{aligned} \tag{5.22}$$

One also has that for any choice of $r\in[0,3L/4]$ (recall $L=|y|$), with large probability (using Proposition 4.1 and (5.20))

$$\bar{Z}(0,y)\ge\bar{Z}^\omega\big(0,(r/L)y\big)\min_{z\in B((r/L)y,1)}\bar{Z}^\omega(z,y)\ge\mathrm{e}^{-2L^{\chi(\xi)+\varepsilon}-C\log L}\mathrm{e}^{-\alpha(r)-\alpha(L-r)}. \tag{5.23}$$

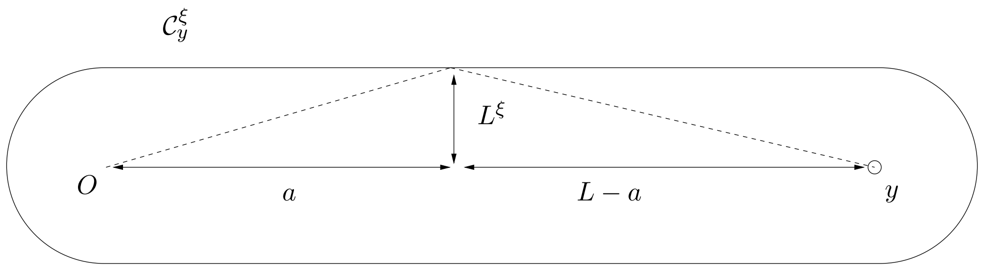

Fig. 3. Suppose that x is on $\partial\mathcal{C}_y^\xi$ for $|y|=L$ then if x is on the "cylindric" part then $|y-x|+|x|=\sqrt{a^2+L^{2\xi}}+\sqrt{(L-a)^2+L^{2\xi}}\geq 2\sqrt{(L/2)^2+L^2\xi}=2L^{2\xi-1}(1+o(1))$. We let the reader check that this also holds when x is on one of the "hemispheres."

Set $x_0\in\operatorname{argmin}_{x\in\partial\mathcal{C}_L^\xi}\alpha(|x|)+\alpha(|y-x|)$. For large values of L, either $|x_0|\leq 3L/4$ or $|y-x_0|\leq 3L/4$ holds, and by symmetry one can assume that $|x_0|\leq 3L/4$. Then taking $r=|x_0|$ in (5.23) one obtains that with probability going to one

$$\log\bar{\mu}^\omega_{0,y}\big(\big(\mathcal{A}_y^\xi\big)^c\big)=\log\frac{Y_{0,y}}{\bar{Z}^\omega(0,y)}\leq\alpha\big(L-|x_0|\big)-\alpha\big(|y-x_0|\big)+4L^{\chi+\varepsilon}+C'\log L. \tag{5.24}$$

Note that necessarily $|y-x_0|-(L-|x_0|)\geq L^{2\xi-1}$ for large L as it is the case for any $x\in\partial C_y^\xi$ (see Fig. 3).

With our choice of ξ

$$2\xi-1>\chi(\xi)>0, \tag{5.25}$$

so that one can use Lemma 5.1 to get that

$$\alpha\big(L-|x|\big)-\alpha\big(|y-x|\big)\geq cL^{2\xi-1} \tag{5.26}$$

and hence

$$\log\bar{\mu}^\omega_{0,y}\big(\mathcal{A}_y^\xi\big)\leq 4L^{\chi+\varepsilon}+C'\log L-cL^{2\xi-1}\leq-\frac{c}{2}L^{2\xi-1}. \tag{5.27}$$

5.3. *Proof of Theorem* 2.1

To treat the point to plane model needs a bit more care but the general idea is the same. Thanks to Proposition 3.1, the result is proved if $\bar{\mu}_L^\omega((\mathcal{A}_L^\xi)^c)$ tends to zero in probability.

Therefore our aim is to compare

$$Y_L^\omega:=\bar{Z}_L^\omega\bar{\mu}_L^\omega\big(\big(\mathcal{A}_L^\xi\big)^c\cap\mathcal{S}_L\big)=\mathbf{E}\big[\mathrm{e}^{-\int_0^{T_{\mathcal{H}_L}}(\lambda+\bar{V}(B_t))\,\mathrm{d}t}\mathbf{1}_{\{T_{\partial\mathcal{C}_L^\xi}<T_{\mathcal{H}_L}\}}\big] \tag{5.28}$$

with $\bar{Z}_L^\omega$. First one can remark that

$$\bar{Z}_L^\omega\geq\bar{Z}^\omega(0,y_{L+1}), \tag{5.29}$$

where $y_L=(L+1,0,\dots,0)$. Then one has to find a good upper bound on Y_L.

Consider a family of ball $(B(x_i,1))_{i\in\{0,\dots,m_L\}}$, $x_i\in\partial\mathcal{C}_L^\xi$ with $m_L=O(L^{1+\xi(d-2)})$ that satisfies

$$\partial\mathcal{C}_L^\xi\subset\bigcup_{i\in\{0,\dots,m_L\}}B(x_i,1). \tag{5.30}$$

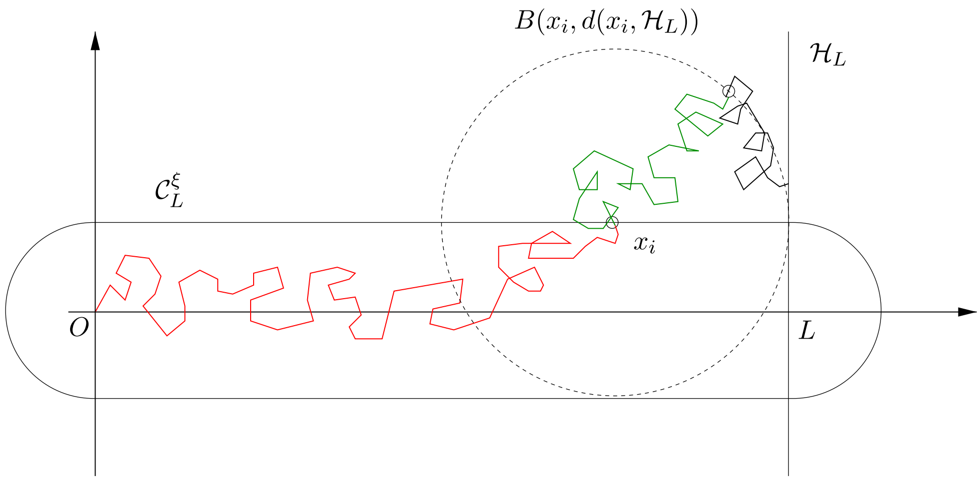

Fig. 4. If a trajectory does not belong to $\mathcal{A}_L^\xi$ then it has to hit $\partial\mathcal{C}_y^\xi$ at some point before $T_{\mathcal{H}_L}$ and thus, by (5.30) it must hit one of the $B(x_i)$. Then before hitting $\mathcal{H}_L$ is has to hit $\partial B(x_i, d(x_i, \mathcal{H}_L))$ (because of distance consideration) and thus, by (5.31) one of the $B(y_{i,j})$. We use this information to get an upper bound on Y_L^ω.

Then for each i set $r_{i,L} := d(x_i, \mathcal{H}_L)$, consider a family of balls $(B(y_{i,j}, 1))_{j\in\{0,\dots,n_{i,L}\}}$, with $y_{i,j} \in B(x_i, r_{i,l})$, $n_{i,L} = O(L^d)$ that cover entirely the boundary of $B(x_i, r_{i,l})$.

$$\partial B(x_i, r_{i,L}) \subset \bigcup_{j\in\{0,\dots,n_{i,L}\}} B(y_{i,j}, 1). \tag{5.31}$$

Then one remarks that trajectories in $(\mathcal{A}_L^\xi)^c$ have to hit, first one of the $B(x_i, 1)$ (they have to hit $\partial\mathcal{C}_L^\xi$ first), then one of the $B(y_{i,j}, 1)$ (starting from x_i one has to hit $\partial B(x_i, r_{i,L})$ before hitting $\mathcal{H}_L$ see Fig. 4), so that with a computation similar to the one made to obtain (5.6), we obtain that

$$Y_L \le \sum_{i\in\{0,\dots,m_L\}} \sum_{j\in\{0,\dots,n_{i,L}\}} \bar{Z}^\omega(0, x_i) \max_{z\in B(x_i,1)} \bar{Z}^\omega(z, y_{i,j}), \tag{5.32}$$

with the convention that $\bar{Z}^\omega(a, b) = 1$ if $|b - a| \le 1$. Then recall (5.20)

$$\max_{z\in B(0,x_i)} \bar{Z}^\omega(z, y_j) \le e^{c\log L} Z^\omega(x_i, y_{i,j}), \tag{5.33}$$

and that concentration inequalities from Proposition 4.1 tells us that with high-probability, all the $\log \bar{Z}^\omega(0, x_i)$ and $\log \bar{Z}^\omega(x_i, y_{i,j})$ are not further than $L^{\chi+\varepsilon}$ away from their respective mean value, or more precisely

$$\mathbb{P}\big(\exists i \in \{1, \dots, m_L\}, \log \bar{Z}^\omega(0, x_i) + \alpha\big(|x_i|\big) \ge L^{\chi+\varepsilon}\big) \le m_L e^{-L^\delta} \tag{5.34}$$

and

$$\begin{aligned}&\mathbb{P}\big(\exists i \in \{1, \dots, m_L\}, \exists j \in \{0, \dots, n_{i,L}\}, \log \bar{Z}^\omega(x_i, y_{i,j}) + \alpha\big(|y_{i,j} - x_i|\big) \ge L^{\chi+\varepsilon}\big)\\ &\quad\le m_L\Big(\max_i n_{i,L}\Big)e^{-L^\delta}.\end{aligned} \tag{5.35}$$

Hence similarly to (5.22) there exists a constant C' such that with high probability

$$\log Y_L^\omega \le C'\log L + 2L^{\chi+\varepsilon} - \min_{x\in\partial\mathcal{C}_L^\xi} \big(\alpha\big(|x|\big) + \alpha\big(d(x, \mathcal{H}_L)\big)\big). \tag{5.36}$$

Consider $x_0 \in \operatorname{argmin}_{x\in\partial\mathcal{C}_L^\xi} \alpha(|x|) + \alpha(d(x,\mathcal{H}_L))$. Note that either $|x_0|$ or $d(x_0,\mathcal{H}_L)$ is smaller than $3L/4$. Suppose $|x_0| \le 3/4L$ (the proof would work the same way in the other case). For any $r \in [0, 3L/4]$ one has that with high probability (cf. (5.23)),

$$\bar{Z}_L^\omega \ge \bar{Z}^\omega(0, y_{L+1}) \ge \mathrm{e}^{-2L^{\chi(\xi)+\varepsilon} - C\log L}\mathrm{e}^{\alpha(|x_0|)+\alpha(L+1-|x_0|)}, \tag{5.37}$$

so that

$$\log \bar{\mu}_{0,L}^\omega\big(\big(\mathcal{A}_L^\xi\big)^c\big) = \log Y_L^\omega/\bar{Z}_L^\omega \le \alpha\big(L+1-|x_0|\big) - \alpha\big(d(x_0,\mathcal{H}_L)\big) + 4L^{\chi+\varepsilon} + C'\log L. \tag{5.38}$$

From geometric consideration as $x_0 \in \partial\mathcal{C}_L^\xi$ one has

$$|x_0| + d(x_0,\mathcal{H}_L) \ge \sqrt{L^2 + L^{2\xi}} \ge L + \frac{1}{4}L^{2\xi-1}. \tag{5.39}$$

So that from Lemma 5.1 (as $2\xi - 1 > \chi(\xi)$)

$$\alpha\big(d(x_0,\mathcal{H}_L)\big) - \alpha\big(L+1-|x_0|\big) \ge c\left(\frac{1}{4}L^{2\xi-1} - 1\right) \tag{5.40}$$

and hence with high probability, provided ε is small enough

$$\log \bar{\mu}_{0,L}^\omega\big(\big(\mathcal{A}_L^\xi\big)^c\big) \le -c\left(\frac{1}{4}L^{2\xi-1} - 1\right) + 4L^{\chi+\varepsilon} + C'\log L \le -\frac{cL^{2\xi-1}}{8}. \tag{5.41}$$

Appendix: Technical estimates

We present here the proof of two technical statement. The first one, Lemma A.1, is the fact that with our setup, each x in $B(0,L^2)$ lies in at most $\log L$ different traps with high probability.

The second statement Lemma A.3 is that under our Gibbs measure, B_t does not visit too many different cubes of side-length l.

Lemma A.1. *One has that for all L large enough*,

$$\mathbb{P}\left[\max_{x\in B(0,L^2)}\left(\sum_{i=1}^\infty \mathbf{1}_{\{|x-\omega_i|\le r_i\}}\right) \ge \log L\right] \le \frac{1}{L}. \tag{A.1}$$

Proof. Note that it is sufficient to show that

$$\mathbb{P}\left[\max_{x\in B(0,1)}\sum_{i=1}^\infty \mathbf{1}_{\{|x-\omega_i|\le r_i\}} \ge \log L\right] \le \frac{1}{L^{2d+2}}. \tag{A.2}$$

Indeed, one can cover up $B(0,L^2)$ with $\mathrm{O}(L^{2d})$ balls of radius one, and use union bound and translation invariance. Then we remark that $\max_{x\in B(0,1)}\dots$ is less than

$$\left(\sum_{i=1}^\infty \mathbf{1}_{\{|\omega_i|\le r_i+1\}}\right) \tag{A.3}$$

which is a Poisson variable whose mean is

$$\int_1^\infty \alpha r^{-\alpha-1}\sigma_d(r+1)^d\,\mathrm{d}r < \infty, \tag{A.4}$$

this is enough to conclude. □

For $l \geq 0$, $x \in \mathbb{Z}^d$ define $C_x := lx + [0,l]^d$ and $\widetilde{C}_x := \bigcup_{y \in C_y} B(x,l)$. For $T \geq 0$ define

$$A_T := \left|\left\{x \in \mathbb{Z}^d \mid \widetilde{C}_x \cap \left\{B_t, t \in [0,T]\right\}\right\}\right|, \tag{A.5}$$

the number of $\widetilde{C}_x$ that are visited by $(B_t)_{t\in[0,T]})$. Scaling properties of the Brownian motion implies that A_T is typically of order $\mathrm{O}(T/l^2)$ (and smaller than this when B is recurrent, i.e. for $d = 1, 2$). We investigate large deviation of A_T above its typical value

Lemma A.2. *There exist a constant C such that if $nl^2/T \geq C$ then*

$$\mathbf{P}[A_T \geq n] \leq \mathrm{e}^{-n^2 l^2/(4CT)}. \tag{A.6}$$

Proof. Set $\mathcal{T}_0 := 0$ and

$$\mathcal{T}_{n+1} := \inf\left\{t \geq \mathcal{T}_n, |B_t - B_{\mathcal{T}_n}| \geq l\right\}. \tag{A.7}$$

Note that in the interval $(\mathcal{T}_n, \mathcal{T}_{n+1})$ the Brownian motion cannot visit more than 5^d different $\widetilde{C}_x$, and therefore

$$\mathbf{P}\left[A_T \geq 5^d n\right] \leq \mathbf{P}[\mathcal{T}_n \leq T]. \tag{A.8}$$

To estimate the second term, one uses Chernov inequality and therefore, the first step is to compute the Laplace transform of $\mathcal{T}_1$

$$\begin{aligned}\mathbf{E}\left[\mathrm{e}^{-u\mathcal{T}_1}\right] &= \int_0^\infty u\mathrm{e}^{-ut}\mathbf{P}[\mathcal{T}_1 \geq t]\,\mathrm{d}t \leq 4d\int_0^\infty u\mathrm{e}^{-ut}\int_{l/\sqrt{t}}^\infty \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C\int_0^\infty \frac{\sqrt{t}}{l}\mathrm{e}^{-ut-l^2/(2t)}\,\mathrm{d}t \leq \mathrm{e}^{-l\sqrt{u}},\end{aligned} \tag{A.9}$$

where the last inequality holds if $l^2 u$ is large enough, say larger than a constant C.

$$\mathbf{P}[\mathcal{T}_n \leq T] \leq \inf_{u \geq 0}\left(\mathbf{P}\left[\mathrm{e}^{-u\mathcal{T}_1 + uT/n}\right]\right)^n \leq \inf_{u \geq C/l^2}\left(\mathbf{P}\left[\mathrm{e}^{-l\sqrt{u}+uT/n}\right]\right)^n = \mathrm{e}^{-n^2l^2/(4T)}, \tag{A.10}$$

where the last equality holds provided nl^2/T is large. □

We use the previous estimate to get a (rather rough) bound on the tail distribution of $A_{T_{B(v)}}$ under $\bar{\mu}^\omega_{0,v}$.

Lemma A.3. *There exists a constant C such that for all L large enough and all $|v| \in \mathbb{R}^d$, for all $l \geq 1$ and for all $n \geq \frac{C|v|\log L}{l}$*

$$\bar{\mu}^\omega_{0,v}[A_{T_{B(v)}} \geq n] \leq \mathrm{e}^{-nl/C}. \tag{A.11}$$

Proof. First recall that via standard tubular estimates for Brownian Motion, one can prove that almost surely, for all v

$$\log \bar{Z}^\omega_v \geq -C|v|\log L. \tag{A.12}$$

And therefore

$$\mu_{0,v}(T_{B(v)} \leq T) \leq \mathrm{e}^{-\lambda T + |v|\log L}. \tag{A.13}$$

On the other hand if $nl^2/T \geq C$

$$\bar{\mu}^{\omega}_{0,v}[A_T \geq n] \leq \frac{1}{\bar{Z}^{\omega}_y}\mathbf{P}[A_T \geq n] \leq \mathrm{e}^{|v|\log L - n^2 l^2/(4T)}. \tag{A.14}$$

Altogether one has that

$$\bar{\mu}^{\omega}_{0,v}[A_{T_{B(v)}} \geq n] \leq \bar{\mu}^{\omega}_{0,v}[A_T \geq n] + \bar{\mu}^{\omega}_{0,v}[T_{B(v)} \geq T] \leq \mathrm{e}^{C|v|\log L}\big(\mathrm{e}^{-\lambda T} + \mathrm{e}^{-n^2 l^2/(4TC)}\big), \tag{A.15}$$

were the last inequality is valid when $T \leq nl^2/C$. Taking $T = nl/C$ one gets that

$$\bar{\mu}^{\omega}_{0,v}[A_{T_{B(v)}} \geq n] \leq \mathrm{e}^{Cv|\log L|}\big(\mathrm{e}^{\lambda nl/C} + \mathrm{e}^{-nl/4}\big) \leq \mathrm{e}^{\lambda nl/(2C)}, \tag{A.16}$$

where the last inequality holds if C is large enough and $n \geq \frac{2C^2|v|\log L}{l}$. □

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