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Upper bounds for the density of solutions to stochastic differential equations driven by fractional Brownian motions

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Abstract. In this paper we study upper bounds for the density of solution to stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H > 1/3$. We show that under some geometric conditions, in the regular case $H > 1/2$, the density of the solution satisfies the log-Sobolev inequality, the Gaussian concentration inequality and admits an upper Gaussian bound. In the rough case $H > 1/3$ and under the same geometric conditions, we show that the density of the solution is smooth and admits an upper sub-Gaussian bound.

Résumé. Dans ce papier nous étudions des bornes supérieures pour la densité d'une solution déquation différentielle conduite par un mouvement brownien fractionnaire d'indice de Hurst $H > 1/3$. Nous montrons, que sous certaines conditions géomètriques, dans le cas régulier $H > 1/2$, la densité de la solution satisfait l'inégalité de log-Sobolev, l'inégalité de concentration gaussienne et admet une borne supérieure gaullienne. Dans le cas $H > 1/3$ et sous la même condition géomètrique, nous montrons que la densité est infiniment différentiable et admet une borne supérieure sous-gaussienne.

1. Introduction

Let $B = (B^1, \dots, B^d)$ be a d dimensional fractional Brownian motion (fBm in the sequel) defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with Hurst parameter $H \in (0,1)$. Recall that it means that B is a centered Gaussian process indexed by $\mathbb{R}_+$, whose coordinates are independent and satisfy

$$\mathbb{E}\big[\big(B_t^j - B_s^j\big)^2\big] = |t-s|^{2H} \quad \text{for } s, t \in \mathbb{R}_+. \tag{1}$$

In particular, by considering the family $\{B^H; H \in (0,1)\}$, one obtains some Gaussian processes with any prescribed Hölder regularity, while fulfilling some intuitive scaling properties. This converts fBm into the most natural generalization of Brownian motion to this day.

We are concerned here with the following class of equations driven by B:

$$X_t^x = x + \int_0^t V_0\big(X_s^x\big)\,\mathrm{d}s + \sum_{i=1}^d \int_0^t V_i\big(X_s^x\big)\,\mathrm{d}B_s^i, \tag{2}$$

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where x is a generic initial condition and $\{V_i; 0 \le i \le d\}$ is a collection of smooth vector fields of $\mathbb{R}^d$. Owing to the fact that fBm is a natural generalization of Brownian motion, this kind of model is often used by practitioners in different contexts, among which we would like to highlight recent sophisticated models in Biophysics [24,33,34].

As far as mathematical results are concerned, equation (2) is now a fairly well understood object: existence and uniqueness results are obtained for $H > \frac{1}{2}$ thanks to Young integral type tools [31,36], while rough paths methods [16,26] are required for $\frac{1}{4} < H < \frac{1}{2}$. Numerical schemes can be implemented for this kind of systems [14,16], and a notion of ergodicity is also available [19,20]. Finally, the law of X_t^x has been analyzed by means of semi-group type methods [1,28] and its density has also been investigated in [2,8,23,32].

In spite of these advances, concentrations results and Gaussian bounds for the solution to (2) are scarce: we are only aware of the large deviation results [27] in this line of investigation. The current article is thus an attempt to make a step in this direction, by analyzing a special but nontrivial situation.

Indeed, we consider here equation (2) driven by a fBm with Hurst parameter $H \in (\frac{1}{3}, 1)$, and we suppose that our vector fields $V_0, \dots, V_d$ fulfill either of the following non-degeneracy and antisymmetric hypothesis:

Hypothesis 1.1. *The vector fields $V_0, \dots, V_d$ are C^∞-bounded, and $V_1, \dots, V_d$ satisfy*

(i) *For every $x \in \mathbb{R}^d$, the vectors $V_1(x), \dots, V_d(x)$ form a basis of $\mathbb{R}^d$.*
(ii) *There exist smooth and bounded functions ω_{ij}^k such that:*

$$[V_i, V_j] = \sum_{k=1}^d \omega_{ij}^k V_k \quad \textit{and} \quad \omega_{ij}^k = -\omega_{ik}^j. \tag{3}$$

The second assumption (ii) is of geometric nature and actually means that the Levi–Civita connection associated with the Riemannian structure given by the vector fields V_i's is

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

In a Lie group structure, this is equivalent to the fact that the Lie algebra is of compact type, or in other words that the adjoint representation is unitary. Such geometric assumption already appeared in the work [3] where it was used to prove a small-time asymptotics of the density.

Hypothesis 1.2. *The Hypothesis 1.1 is satisfied and moreover, the vector fields $V_1, \dots, V_d$ form moreover a uniform elliptic system. That is*

$$\left|v^T V V^T v\right| \ge \lambda |v|^2 \quad \textit{for all } v \in \mathbb{R}^d.$$

Here $V = (V_j^i)_{i,j=1,\dots,d}$ and λ is a positive constant.

When $H > \frac{1}{2}$, under Hypothesis 1.1 our main result can be loosely summarized as follows (see Theorem 3.13 for a precise statement):

Theorem 1.3. *Fix $H > \frac{1}{2}$. Let X^x be the solution to equation (2), and suppose Hypothesis 1.1 is satisfied. Then for any $t \in \mathbb{R}_+^*$, the random variable X_t^x admits a smooth density $p_X(t, \cdot)$. Furthermore, there exist 3 positive constants $c_t^{(1)}, c_t^{(2)}, c_{t,x}^{(3)}$ such that*

$$p_X(t, y) \le c_t^{(1)} \exp\bigl(-c_t^{(3)}\bigl(|y| - c_{t,x}^{(2)}\bigr)^2\bigr),$$

for any $y \in \mathbb{R}^d$.

We do not claim any optimality in the quantities $c_t^{(1)}, c_t^{(2)}$ and $c_{t,x}^{(3)}$ above (whose exact definitions are postponed to Section 3.3). Nevertheless, this is (to the best of our knowledge) the first Gaussian type bound available for solutions of differential equations driven by fBm. Let us observe that such global Gaussian bounds of course does not hold in general and crucially relies on some geometric assumptions. Typically in the Brownian motion case, a non negative lower bound for the Ricci curvature is required. The assumption (ii) here is stronger and it is easily seen that it implies that the Ricci curvature of the Levi–Civita connection associated with the Riemannian structure given by the vector fields V_i's is non negative.

Let us say a few words about the strategy we have followed in order to prove Theorem 1.3. It is mostly based on stochastic analysis tools, and particularly on a general integration by parts formula giving an exact expression for the density $p_X(t,\cdot)$ in terms of Malliavin derivatives in the non-degenerate case we are dealing with. In this context, it is crucial to bound the first Malliavin derivative of X_t^x (called $\mathbf{D}X_t^x$ in the sequel) efficiently. This is where our asymmetry hypothesis on the vector fields $V_1,\dots,V_n$ enter into the picture, and we shall see (at Theorem 3.1) how asymmetry properties yield an easy deterministic bound on $\mathbf{D}X_t^x$. This result enables to get concentration results for the law of X_t^x, and is the key to our density bounds as well.

As another interesting consequences of the deterministic bound on $\mathbf{D}X_t^x$, we also obtain logarithmic Sobolev inequality and Poincaré inequality for the law of X_t.

Once the picture for the smooth case (when $H>\frac{1}{2}$) becomes clear, we are able to extend some of our results described above to the irregular case when $\frac{1}{3}<H<\frac{1}{2}$. In particular, we are able to prove

Theorem 1.4. *Fix $H\in(\frac{1}{3},\frac{1}{2})$. Assume Hypothesis 1.2. Let X^x be the solution to equation (2) and γ_{X_t} the Malliavin matrix of X_t^x, $t>0$. We have $|\det\gamma_{X_t}|^{-1}\in L^\infty(\mathbb{P})$. The random variable X_t^x admits a smooth density $p_X(t,\cdot)$ and for any $\delta<H$ there exist 2 positive constants $c_t^{(1)}$ such that*

$$p_X(t,y)\le c_t^{(1)}\exp\bigl(-|y-x|^\delta\bigr), \tag{4}$$

for all $y\in\mathbb{R}^d$.

The existence of a density for solutions to stochastic differential equations of the form (2) under Hörmander's condition has been obtained by Cass and Friz [7] for any $\frac{1}{4}<H<\frac{1}{2}$. While finishing the current article, an important step towards the study of regular densities in the rough case $\frac{1}{4}<H<\frac{1}{2}$ has been accomplished in [9], where integrability estimates for the Jacobian of equation (2) are established. Nevertheless, as of today, besides the result of P. Driscoll [15], when $H<\frac{1}{2}$, to the best of our knowledge, Hypothesis 1.2 is a first wide class of examples where we have an affirmative answer for the smoothness of the density. Our bound on the inverse of the Malliavin matrix together with polynomial bounds on the Hölder norm of the Malliavin derivative allows then to obtain the sub-Gaussian upper bound (4).

Notations. *Throughout this paper, unless otherwise specified we use $|\cdot|$ for Euclidean norms and $\|\cdot\|_{L^p}$ for the L^p norm with respect to the underlying probability measure $\mathbb{P}$.*

Consider a finite-dimensional vector space V. The space of V-valued Hölder continuous functions defined on $[0,1]$, with Hölder continuity exponent $\gamma\in(0,1)$, will be denoted by $\mathcal{C}^\gamma(V)$, or just $\mathcal{C}^\gamma$ when this does not yield any ambiguity. For a function $g\in\mathcal{C}^\gamma(V)$ and $0\le s<t\le 1$, we shall consider the semi-norms

$$\|g\|_{s,t,\gamma}=\sup_{s\le u<v\le t}\frac{|g_v-g_u|_V}{|v-u|^\gamma}. \tag{5}$$

The semi-norm $\|g\|_{0,1,\gamma}$ will simply be denoted by $\|g\|_\gamma$.

2. Stochastic calculus for fractional Brownian motion

For some fixed $H\in(\frac{1}{3},1)$, we consider $(\Omega,\mathcal{F},\mathbb{P})$ the canonical probability space associated with the fractional Brownian motion (in short fBm) with Hurst parameter H. That is, $\Omega=\mathcal{C}_0([0,1])$ is the Banach space of continuous functions vanishing at 0 equipped with the supremum norm, $\mathcal{F}$ is the Borel sigma-algebra and $\mathbb{P}$ is the unique probability measure on Ω such that the canonical process $B=\{B_t=(B_t^1,\dots,B_t^d),t\in[0,1]\}$ is a fractional Brownian motion with Hurst parameter H. In this context, let us recall that B is a d-dimensional centered Gaussian process, whose covariance structure is induced by equation (1). This can be equivalently stated as

$$R(t,s) := \mathbb{E}\big[B_s^j B_t^j\big] = \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big), \quad \text{for } s,t \in [0,1] \text{ and } j = 1,\dots,d.$$

In particular it can be shown, by a standard application of Kolmogorov's criterion, that B admits a continuous version whose paths are γ-Hölder continuous for any $\gamma < H$.

This section is devoted to give the basic elements of stochastic calculus with respect to B which allow to understand the remainder of the paper.

2.1. *Malliavin calculus tools*

Gaussian techniques are obviously essential in the analysis of fBm, and we proceed here to introduce some of them (see [30] for further details): let $\mathcal{E}$ be the space of $\mathbb{R}^d$-valued step functions on [0, 1], and $\mathcal{H}$ the closure of $\mathcal{E}$ for the scalar product:

$$\big\langle (\mathbf{1}_{[0,t_1]},\dots,\mathbf{1}_{[0,t_d]}), (\mathbf{1}_{[0,s_1]},\dots,\mathbf{1}_{[0,s_d]})\big\rangle_{\mathcal{H}} = \sum_{i=1}^{d} R(t_i, s_i).$$

We denote by K_H^* the isometry between $\mathcal{H}$ and $L^2([0,1])$. When $H > \frac{1}{2}$ it can be shown that $\mathbf{L}^{1/H}([0,1],\mathbb{R}^d) \subset \mathcal{H}$, and when $\frac{1}{3} < H < \frac{1}{2}$ one has

$$C^\gamma \subset \mathcal{H} \subset L^2\big([0,1]\big)$$

for all $\gamma > \frac{1}{2} - H$.

Some isometry arguments allow to define the Wiener integral $B(h) = \int_0^1 \langle h_s, \mathrm{d}B_s\rangle$ for any element $h \in \mathcal{H}$, with the additional property $\mathbb{E}[B(h_1)B(h_2)] = \langle h_1, h_2\rangle_{\mathcal{H}}$ for any $h_1, h_2 \in \mathcal{H}$. A $\mathcal{F}$-measurable real valued random variable F is then said to be cylindrical if it can be written, for a given $n \ge 1$, as

$$F = f\big(B\big(h^1\big),\dots,B\big(h^n\big)\big) = f\bigg(\int_0^1 \big\langle h_s^1, \mathrm{d}B_s\big\rangle,\dots,\int_0^1 \big\langle h_s^n, \mathrm{d}B_s\big\rangle\bigg),$$

where $h^i \in \mathcal{H}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ bounded function with bounded derivatives. The set of cylindrical random variables is denoted $\mathcal{S}$.

The Malliavin derivative is defined as follows: for $F \in \mathcal{S}$, the derivative of F is the $\mathbb{R}^d$ valued stochastic process $(\mathbf{D}_t F)_{0\le t\le 1}$ given by

$$\mathbf{D}_t F = \sum_{i=1}^{n} h^i(t) \frac{\partial f}{\partial x_i}\big(B\big(h^1\big),\dots,B\big(h^n\big)\big).$$

More generally, we can introduce iterated derivatives. If $F \in \mathcal{S}$, we set

$$\mathbf{D}^k_{t_1,\dots,t_k} F = \mathbf{D}_{t_1} \cdots \mathbf{D}_{t_k} F.$$

For any $p \ge 1$, it can be checked that the operator $\mathbf{D}^k$ is closable from $\mathcal{S}$ into $\mathbf{L}^p(\Omega; \mathcal{H}^{\otimes k})$. We denote by $\mathbb{D}^{k,p}$ the closure of the class of cylindrical random variables with respect to the norm

$$\|F\|_{k,p} = \Bigg(\mathbb{E}\big(F^p\big) + \sum_{j=1}^{k} \mathbb{E}\big(\big\|\mathbf{D}^j F\big\|^p_{\mathcal{H}^{\otimes j}}\big)\Bigg)^{1/p},$$

and

$$\mathbb{D}^\infty = \bigcap_{p\geq 1}\bigcap_{k\geq 1}\mathbb{D}^{k,p}.$$

2.2. *Differential equations driven by fBm*

Recall that we consider the following kind of equation:

$$X_t^x = x + \int_0^t V_0\big(X_s^x\big)\,\mathrm{d}s + \sum_{i=1}^d \int_0^t V_i\big(X_s^x\big)\,\mathrm{d}B_s^i, \tag{6}$$

where the vector fields $V_0, \ldots, V_n$ are C^∞-bounded.

When equation (6) is driven by a fBm with Hurst parameter $H > \frac{1}{2}$ it can be solved, thanks to a fixed point argument, with the stochastic integral interpreted in the (pathwise) Young sense (see e.g. [17]). Let us recall that Young's integral can be defined in the following way:

Proposition 2.1. *Let $f \in \mathcal{C}^\gamma$, $g \in \mathcal{C}^\kappa$ with $\gamma + \kappa > 1$, and $0 \leq s \leq t \leq 1$. Then the integral $\int_s^t g_\xi\,\mathrm{d}f_\xi$ is well-defined as limit of Riemann sums along partitions of $[s,t]$. Moreover, the following estimation is fulfilled*:

$$\left|\int_s^t g_\xi\,\mathrm{d}f_\xi\right| \leq C\|f\|_\gamma\|g\|_\kappa|t-s|^\gamma, \tag{7}$$

where the constant C only depends on γ and κ. A sharper estimate is also available:

$$\left|\int_s^t g_\xi\,\mathrm{d}f_\xi\right| \leq |g_s|\|f\|_\gamma|t-s|^\gamma + c_{\gamma,\kappa}\|f\|_\gamma\|g\|_\kappa|t-s|^{\gamma+\kappa}. \tag{8}$$

With this definition in mind, we can solve our differential system of interest, and the following moments bounds are proven in [23]:

Proposition 2.2. (*Hu-Nualart*) *Consider equation* (6) *driven by a fBm B with Hurst parameter $H > \frac{1}{2}$. Let us call X^x its unique β-Hölder continuous solution, for any $\beta < H$. Then*

(1) *When vector fields V are C^∞-bounded, we have*

$$\sup_{t\in[0,T]}\big|X_t^x\big| \leq |x| + c_{V,T}\|B\|_{0,T,\beta}^{1/\beta}.$$

(2) *If we only assume that vector fields V have linear growth, with ∇V, $\nabla^2 V$, bounded, the following estimate holds true*:

$$\sup_{t\in[0,T]}\big|X_t^x\big| \leq \big(1+|x|\big)\exp\big(c_{V,T}\|B\|_{0,T,\beta}^{1/\beta}\big). \tag{9}$$

Remark 2.3. *The framework of fractional integrals is used in* [23] *in order to define integrals with respect to B. It is however easily seen to be equivalent to the Young setting we have chosen to work with.*

When the Hurst parameter $\frac{1}{3} < H < \frac{1}{2}$, equation (6) can be solved, again by fixed point argument, with the stochastic integral interpreted in the (pathwise) rough path theory (see e.g. [17] and [26]). In this case, we obtain

Proposition 2.4. (*Besalú-Nualart* [5]) *Consider equation* (6) *driven by a fBm B with Hurst parameter $\frac{1}{3} < H < \frac{1}{2}$. Denote by X^x its unique β-Hölder continuous solution, for any $\beta < H$. If the vector fields V are C^∞-bounded, then for any $\lambda > 0$ and $\delta < H$*

$$\mathbb{E}\Big(\exp\lambda\Big(\sup_{0\leq t\leq T}|X_t - x|^\delta\Big)\Big) < \infty.$$

Once equation (6) is solved, the vector X_t^x is a typical example of random variable which can be differentiated in the Malliavin sense. In fact, fix $H \in (\frac{1}{3}, 1)$, one gets the following results (see [7] and [32] for further details):

Proposition 2.5. *Let X^x be the solution to equation (6) and suppose V_i's are C^∞-bounded vector fields on $\mathbb{R}^d$. Then for every $i = 1, \dots, d$, $t > 0$, and $x \in \mathbb{R}^d$, we have $X_t^{x,i} \in \mathbb{D}^\infty$ and*

$$\mathbf{D}_s^j X_t^x = \mathbf{J}_{0\to t}\mathbf{J}_{0\to s}^{-1} V_j(X_s), \quad j = 1, \dots, d, 0 \le s \le t,$$

where $\mathbf{D}_s^j X_t^{x,i}$ is the jth component of $\mathbf{D}_s X_t^{x,i}$, and $\mathbf{J}_{0\to t} = \frac{\partial X_t^x}{\partial x}$.

Finally the following approximation result, which can be found for instance in [16], will also be used in the sequel:

Proposition 2.6. *For $m \ge 1$ and $T > 0$, let $B^m = \{B_t^m; t \in [0, T]\}$ be the sequence of linear interpolations of B along the dyadic subdivision of $[0, T]$ of mesh m; that is if $t_i^m = i2^{-m}T$ for $i = 0, \dots, 2^m$; then for $t \in (t_i^m, t_{i+1}^m]$,*

$$B_t^m = B_{t_i^m} + \frac{t - t_i^m}{t_{i+1}^m - t_i^m}(B_{t_{i+1}^m} - B_{t_i^m}).$$

Consider X^m the solution to equation (6) restricted to $[0, T]$, where B has been replaced by B^m. Set also $\mathbf{D}_s^j X_t^m = \mathbf{J}_{0\to t}\mathbf{J}_{0\to s}^{-1} V_j(X_s^m)$, for $j = 1, \dots, d$ and $0 \le s \le t$. Then almost surely, for any $\gamma < H$ the following holds true:

$$\lim_{m\to\infty}\left(\|X^x - X^m\|_{0,T,\gamma} + \|\mathbf{D}^j X_t^x - \mathbf{D}^j X_t^m\|_{0,T,\gamma}\right) = 0. \tag{10}$$

3. Estimates for solutions of SDEs driven by fBm: The smooth case

Throughout this section, we fix $H \in (\frac{1}{2}, 1)$. Recall that X^x designates the solution to (6). This section is devoted to getting some further bounds for X_t^x and its Malliavin derivatives, under Hypothesis 1.1.

We are now ready to prove the main result of this section, which is an almost sure deterministic bound for the Malliavin derivative of X_t^x:

Theorem 3.1. *Under Hypothesis 1.1, the Malliavin derivative of the solution X_T^x to equation (6) can be bounded as follows for any $T \in [0, 1]$ (in the almost sure sense):*

$$\|\mathbf{D}X_T^x\|_\infty \le M\exp(CT), \quad \textit{with } M = \sup_{x\in\mathbb{R}^n}\sup_{\|\lambda\|\le 1}\left|\sum_{i=1}^d \lambda_i V_i(x)\right|^2, \tag{11}$$

and where the constant C linearly depends on V_0. In particular, one also has $\|\mathbf{D}X_T^x\|_{\mathcal{H}} \le M\exp(CT)$.

Proof. Let us focus on the proof of (11). Indeed, since $\|f\|_{\mathcal{H}}$ is dominated by the supremum norm when $H > \frac{1}{2}$, this will be sufficient in order to prove the second claim of our theorem. We now split our proof in two steps.

Step 1: Matricial expression for the derivative. Let us first restate Proposition 2.5 in the following form: $\mathbf{D}X_T^x$ is solution to

$$\mathbf{D}_s^j X_T^x = \mathbf{J}_{0\to T}\left(\Phi_s^* V_j\right)(x), \quad 0 \le s \le T,$$

where $\Phi_s^* V_j$ denotes the pullback action of the diffeomorphism $\Phi_s = X_s^{\cdot} : \mathbb{R}^d \to \mathbb{R}^d$ on the vector field V_j. Now, a simple application of the change of variable formula for Young type integrals yields

$$d\left(\Phi_s^* V_j\right)(x) = \left(\Phi_s^*[V_0, V_j]\right)(x)\,\mathrm{d}s + \sum_{i=1}^d \left(\Phi_s^*[V_i, V_j]\right)(x)\,\mathrm{d}B_s^i.$$

Moreover, recall that the Lie brackets $[V_i, V_j]$ can be decomposed, according to Assumption 1.1, into

$$[V_0, V_j] = \sum_{k=1}^{d} \omega_{0j}^k V_k, \quad \text{and} \quad [V_i, V_j] = \sum_{k=1}^{d} \omega_{ij}^k V_k,$$

with $\omega_{ij}^k = -\omega_{ik}^j$ for $i, j, k \geq 1$. Hence,

$$d\big(\Phi_s^* V_j\big)(x) = \sum_{k=1}^{d} \omega_{0j}^k\big(X_s^x\big)\big(\Phi_s^* V_k\big)(x)\,\mathrm{d}s + \sum_{k=1}^{d}\sum_{i=1}^{d} \omega_{ij}^k\big(X_s^x\big)\big(\Phi_s^* V_k\big)(x)\,\mathrm{d}B_s^i.$$

By denoting $\mathcal{M}_s$ the $d \times d$ matrix with columns

$$\mathcal{M}_s^j = \mathbf{D}_s^j X_T^x = \mathbf{J}_{0\to T}\big(\Phi_s^* V_j\big)(x),$$

we therefore obtain the equation

$$\mathrm{d}\mathcal{M}_s = \mathcal{M}_s\left(\omega_0\big(X_s^x\big)\,\mathrm{d}s + \sum_{i=1}^{d} \omega_i\big(X_s^x\big)\,\mathrm{d}B_s^i\right), \quad \mathcal{M}_T = V\big(X_T^x\big), \tag{12}$$

where $V(X_T^x)$ is the matrix with columns $V_j(X_T^x)$, $1 \leq j \leq d$ and where $\omega_i(X_s^x)$ is the skew symmetric matrix with entries $\omega_{ij}^k(X_s^x)$.

Step 2: Bound on $\mathcal{M}$. We now prove that the process $\mathcal{M}$ is uniformly bounded. We have

$$\begin{aligned}
&\mathrm{d}\big(\mathcal{M}_s\mathcal{M}_s^T\big)\\
&\quad= (\mathrm{d}\mathcal{M}_s)\mathcal{M}_s^T + \mathcal{M}_s\big(\mathrm{d}\mathcal{M}_s^T\big)\\
&\quad= \mathcal{M}_s\left(\omega_0\big(X_s^x\big)\,\mathrm{d}s + \sum_{i=1}^{d}\omega_i\big(X_s^x\big)\,\mathrm{d}B_s^i\right)\mathcal{M}_s^T + \mathcal{M}_s\left(\omega_0^T\big(X_s^x\big)\,\mathrm{d}s + \sum_{i=1}^{d}\omega_i^T\big(X_s^x\big)\,\mathrm{d}B_s^i\right)\mathcal{M}_s^T\\
&\quad= \mathcal{M}_s\big(\omega_0\big(X_s^x\big) + \omega_0^T\big(X_s^x\big)\big)\mathcal{M}_s^T\,\mathrm{d}s.
\end{aligned}$$

Taking into account the terminal condition $\mathcal{M}_T = V(X_T^x)$ gives then the expected result as a straightforward consequence of Gronwall's inequality. □

Once the bound (11) on $\|\mathbf{D}X_T^x\|_\infty$ is obtained, one can also retrieve some information on the Hölder norms of $\mathbf{D}X_T^x$ improving the general estimate (9). This is the content of the following proposition:

Proposition 3.2. *Consider* $\frac{1}{2} < \gamma < H$ *and set* $c_0^T = M\exp(CT)$, *which is the constant appearing in relation* (11). *Under Hypothesis* 1.1, *the Malliavin derivative of the solution* X_T^x *to equation* (6) *can be bounded as follows for any* $T \in [0, 1]$:

$$\big\|\mathbf{D}X_T^x\big\|_\gamma \leq c_{T,V,d}\big(1 + |x| + \|B\|_\gamma^{1/\gamma}\big)\|B\|_\gamma^{(1-\gamma)/\gamma}, \tag{13}$$

for a strictly positive constant $c_{T,V,d}$.

Proof. We have shown at Theorem 3.1 that $\mathbf{D}X_T^x$ is governed by equation (12), and that $\|\mathcal{M}\|_\infty \leq c_0^T$. We will now separate our proof into a local and a global estimate, and notice that the constants appearing in the computations below might change from line to line.

Step 1: Local estimate. Consider $0 \le s < t \le T$ and set $\varepsilon = t - s$. Let $u < v$ be two generic elements of $[s,t]$. Applying relation (8) to the expression of $\mathcal{M}_v - \mathcal{M}_u$ given by equation (12), we obtain

$$|\mathcal{M}_v - \mathcal{M}_u| \le c_0^T c_V |v-u| + \sum_{i=1}^d \bigl(|\mathcal{M}_u| \bigl|\omega_i\bigl(X_u^x\bigr)\bigr| \|B\|_\gamma |v-u|^\gamma + c_\gamma \bigl\|\mathcal{M}\omega_i\bigl(X^x\bigr)\bigr\|_{s,t,\gamma} \|B\|_\gamma |v-u|^{2\gamma}\bigr).$$

We thus obtain, for a constant c_V depending on the vector fields V,

$$\begin{aligned}\|\mathcal{M}\|_{s,t,\gamma} &\le c_0^T c_V |t-s|^{1-\gamma} + d c_0^T c_V \|B\|_\gamma + c_\gamma \bigl[c_0^T c_V \bigl\|X^x\bigr\|_\gamma + c_V \|\mathcal{M}\|_{s,t,\gamma}\bigr] \|B\|_\gamma |t-s|^\gamma \\ &\le c_0^T c_V \bigl(T^{1-\gamma} + d\|B\|_\gamma\bigr) + d c_0^T c_V \bigl\|X^x\bigr\|_\gamma \|B\|_\gamma |t-s|^\gamma + d c_V \|B\|_\gamma |t-s|^\gamma \|\mathcal{M}\|_{s,t,\gamma}.\end{aligned} \tag{14}$$

Take now $t - s = \varepsilon$ such that $d c_V \|B\|_\gamma \varepsilon^\gamma = 1/2$, namely $\varepsilon = [2 d c_V \|B\|_\gamma]^{-1/\gamma}$. Recall also that $\|X^x\|_\gamma \le c(1 + \|B\|_\gamma^{1/\gamma})$ according to [16]. It is then easily seen that relation (14) yields

$$\|\mathcal{M}\|_{s,t,\gamma} \le c_0^T c_{T,V,d} \bigl(1 + |x| + \|B\|_\gamma^{1/\gamma}\bigr) := a_{T,V,d,B},$$

for a strictly positive constant $c_{T,V,d}$.

Step 2: Global estimate. We consider now $s,t \in [0,T]$ such that $i\varepsilon \le s < (i+1)\varepsilon \le j\varepsilon \le t < (j+1)\varepsilon$, where ε has been defined at Step 1. Set also $t_i = s$, $t_k = k\varepsilon$ for $i+1 \le k \le j$, and $t_{j+1} = t$. Then

$$\begin{aligned}|\mathcal{M}_t - \mathcal{M}_s| &= \left|\sum_{k=i}^j \mathcal{M}_{t_{k+1}} - \mathcal{M}_{t_k}\right| \le a_{T,V,d,B} \sum_{k=i}^j (t_{k+1} - t_k)^\gamma \\ &\le a_{T,V,d,B} (j-i+1)^{1-\gamma} (t-s)^\gamma,\end{aligned} \tag{15}$$

where we have used the fact that $r \mapsto r^\lambda$ is a concave function. Note that the indices i, j above satisfy $(j-i+1) \le 2T/\varepsilon$. Plugging this into the last series of inequalities, we end up with our claim (13). □

3.1. *Log-Sobolev inequality*

In this section, we present some interesting functional inequalities which are usually studied in a Markov setting; namely, the logarithmic Sobolev inequality and Poincaré inequality. As we will see, these inequalities become available in our non-Markov case when we have uniform boundedness for the Malliavin derivative of X_T (see Theorem 3.1).

We start with the following version of logarithmic Sobolev inequality for the law of X_T.

Theorem 3.3. *Let C and M be as in Theorem 3.1. We have for all $f \in \mathrm{C}^1$ and $T \in [0,1]$,*

$$\mathbb{E} f(X_T)^2 \ln f(X_T)^2 - \bigl(\mathbb{E} f(X_T)^2\bigr) \ln\bigl(\mathbb{E} f(X_T)^2\bigr) \le 2M^2 \mathrm{e}^{2CT} T^{2H} \mathbb{E}\bigl|\nabla f(X_T)\bigr|^2$$

provided the right hand side in the above is finite.

Proof. The proof is standard by applying Clark–Ocone formula (see e.g. [6]). First recall the representation of fractional Brownian motion

$$B_t = \int_0^t K_H(t,s)\,\mathrm{d}W_s.$$

Here W is a d-dimensional Wiener process. Denote by $\mathbf{D}^W$ the Malliavin derivative with respect to the Wiener process W. We have

$$K_H^* \mathbf{D} = \mathbf{D}^W, \tag{16}$$

where K_H^* is the isometry from $\mathcal{H}$, the reproducing kernel space of B, to L^2. By Clark–Ocone formula we have

$$g(X_T)-\mathbb{E}g(X_T)=\int_0^T\mathbb{E}\big[\mathbf{D}_s^W g(X_T)|\mathcal{F}_s\big]\,\mathrm{d}W_s=\int_0^T\mathbb{E}\big[K_H^*\big(\mathbf{D}g(X_T)\big)_s|\mathcal{F}_s\big]\,\mathrm{d}W_s.$$

Hence, if we denote $M_s=\mathbb{E}(g(X_T)|\mathcal{F}_s)$, $0\le s\le T$, we have

$$\mathrm{d}M_s=\mathbb{E}\big[K_H^*\big(\mathbf{D}g(X_T)\big)_s|\mathcal{F}_s\big]\,\mathrm{d}W_s.$$

For simplicity we may assume that $g\ge\varepsilon$ for some $\varepsilon>0$, which can be removed afterwards by letting ε tend to 0. Applying Itô's formula to $M_s\ln M_s$, we get

$$\begin{aligned}\mathbb{E}g(X_T)\ln\big(g(X_T)\big)-\mathbb{E}g(X_T)\ln\big(\mathbb{E}g(X_T)\big)&=\mathbb{E}(M_T\ln M_T)-\mathbb{E}(M_0\ln M_0)\\&=\frac{1}{2}\mathbb{E}\int_0^T\frac{1}{M_s}\big|\mathbb{E}\big[K_H^*\big(\mathbf{D}g(X_T)\big)_s|\mathcal{F}_s\big]\big|^2\,\mathrm{d}s.\end{aligned}\tag{17}$$

Replace now g by f^2 in the above. By the Cauchy–Schwarz inequality,

$$\begin{aligned}\big|\mathbb{E}\big[K_H^*\big(\mathbf{D}f(X_T)^2\big)_s|\mathcal{F}_s\big]\big|^2&=4\big|\mathbb{E}\big[f(X_T)\big\langle\nabla f(X_T),K_H^*(\mathbf{D}X_T)_s\big\rangle|\mathcal{F}_s\big]\big|^2\\&\le4\mathbb{E}\big[f(X_T)^2|\mathcal{F}_s\big]\mathbb{E}\big[\big\langle\nabla f(X_T),K_H^*(DX_T)_s\big\rangle^2|\mathcal{F}_s\big].\end{aligned}$$

Substituting the above to (17), together with Theorem 3.1, we obtain the desired result. □

As a corollary of the logarithmic Sobolev inequality obtained above, we have the following Poincaré inequality (see e.g. [22, Theorem 8.6.8]).

Theorem 3.4. *Let C and M be in Theorem 3.1. We have for all $f\in\mathrm{C}^1$,*

$$\mathbb{E}f(X_T)^2-\big(\mathbb{E}f(X_T)\big)^2\le M^2\mathrm{e}^{2CT}T^{2H}\mathbb{E}\big|\nabla f(X_T)\big|^2,$$

provided the right hand side in the above is finite.

Remark 3.5. *In the above, assume further that the vector fields $V_1,\dots,V_d$ form an uniform elliptic system, we obtain the following natural expression of logarithmic Sobolev inequality and Poincaré inequality when working on a Riemannian manifold*

$$\mathbb{E}f(X_T)^2\ln f(X_T)^2-\big(\mathbb{E}f(X_T)^2\big)\ln\big(\mathbb{E}f(X_T)^2\big)\le2aM^2\mathrm{e}^{2CT}T^{2H}\sum_{i=1}^d\mathbb{E}|V_if|^2,$$

and

$$\mathbb{E}f(X_T)^2-\big(\mathbb{E}f(X_T)\big)^2\le aM^2\mathrm{e}^{2CT}T^{2H}\sum_{i=1}^d\mathbb{E}|V_if|^2.$$

3.2. *Concentration inequality*

It is classical that the boundedness of the Malliavin derivative in $\mathcal{H}$ implies the Gaussian concentration inequality, more precisely (see [35, Theorem 9.1.1] or [25]):

Lemma 3.6. *Let $F\in\mathbb{D}^{1,2}$ such that almost surely $\|\mathbf{D}F\|_{\mathcal{H}}\le C$, where C is a non random constant. Then, for every $\theta\ge0$,*

$$\mathbb{E}\big(\mathrm{e}^{\theta F}\big)\le\mathrm{e}^{\mathbb{E}(F)\theta+(1/2)C^2\theta^2}$$

and therefore for every $x \geq 0$,

$$\mathbb{P}\big(F - \mathbb{E}(F) \geq x\big) \leq \mathrm{e}^{-x^2/(2C^2)}. \tag{18}$$

As a corollary of this lemma, we deduce

Proposition 3.7. *Assume that the Hypothesis 1.1 is satisfied, then there exist C and M such that for every $T \geq 0$ and $\lambda \geq 0$,*

$$\mathbb{P}\Big(\sup_{0\leq t\leq T}\big|X_t^x\big| - \mathbb{E}\Big(\sup_{0\leq t\leq T}\big|X_t^x\big|\Big) \geq \lambda\Big) \leq \exp\Big(-\frac{\lambda^2}{2M^2\mathrm{e}^{2CT}T^{2H}}\Big).$$

Proof. Let

$$F = \sup_{0\leq t\leq T}\big|X_t^x\big|.$$

By Theorem 3.1, it is not hard to see that (see e.g. [30])

$$\|\mathbf{D}F\|_{\mathcal{H}} \leq M\mathrm{e}^{CT}T^H,$$

where M and C are the same as in Theorem 3.1. Now an easy application of Lemma 3.6 completes our proof. □

Remark 3.8. *Notice that this relation can only be obtained for $H > \frac{1}{2}$. Indeed, denote by $\mathcal{C}^0$ the space of continuous functions endowed with the supremum norm. Then the norm involved in [35, Theorem 9.1.1] is $\|\mathbf{D}X_t\|_{L^\infty(\mathcal{H})}$. This norm is dominated by $\|\mathbf{D}X_t\|_{L^\infty(\mathcal{C}^0)}$ only if $H > \frac{1}{2}$.*

3.3. *Gaussian upper bound*

One natural way to estimate the density of X_t is to apply the results in [30, Chapter 2]. More precisely, we first have the following integration by parts formula for non-degenerate random vectors.

Proposition 3.9. *Let $F = (F^1, \ldots, F^d)$ be a non-degenerate random vector whose components are in $\mathbb{D}^\infty$, and γ_F the Malliavin matrix of F. Let $G \in \mathbb{D}^\infty$ and φ be a function in the space $C_p^\infty(\mathbb{R}^d)$. Then for any multi-index $\alpha \in \{1, 2, \ldots, d\}^k$, $k \geq 1$, there exists an element $H_\alpha \in \mathbb{D}^\infty$ such that*

$$\mathbb{E}\big[\partial_\alpha\varphi(F)G\big] = \mathbb{E}\big[\varphi(F)H_\alpha\big].$$

Moreover, the elements H_α are recursively given by

$$H_{(i)} = \sum_{j=1}^d \delta\big(G\big(\gamma_F^{-1}\big)^{ij}\mathbf{D}F^j\big),$$

$$H_\alpha = H_{(\alpha_k)}(H_{(\alpha_1,\ldots,\alpha_{k-1})}),$$

and for $1 \leq p < q < \infty$ we have

$$\|H_\alpha\|_{L^p} \leq C_{p,q}\big\|\gamma_F^{-1}\mathbf{D}F\big\|_{k,2^{k-1}r}^k\|G\|_{k,q},$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

Remark 3.10. *By the estimates for H_α above, one can conclude that there exist constants $\beta, \gamma > 1$ and integers m, n such that*

$$\|H_\alpha\|_{L^p} \leq C_{p,q}\big\|\det\gamma_F^{-1}\big\|_{L^\beta}^m\|\mathbf{D}F\|_{k,\gamma}^n\|G\|_{k,q}. \tag{19}$$

Remark 3.11. *In what follows, we use $H_\alpha(F,G)$ to emphasize its dependence on F and G.*

As a consequence of the above proposition, one has

Proposition 3.12. *Let $F=(F^1,\dots,F^d)$ be a non-degenerate random vector whose components are in $\mathbb{D}^\infty$. Then the density $p_F(x)$ of F belongs to the Schwartz space, and*

$$p_F(x)=\mathbb{E}\big[\mathrm{I}_{\{F>x\}}H_{(1,2,\dots,d)}(F,1)\big].$$

In the above, $F>x$ is used to denote the event $F^i>x^i$, $i=1,2,\dots,d$.

Now we state and prove a global Gaussian upper bound for the density function of X_t^x.

Theorem 3.13. *Denote by $p_X(t,y)$ the density function of X_t^x. There exist positive constants $c_t^{(1)}, c_{t,x}^{(2)}, c_3$, and c_4 such that for all $t\in[0,\infty)$,*

$$p_X(t,y)\le c_t^{(1)}\exp\biggl(-\frac{(|y|-c_{t,x}^{(2)})^2}{c_3\mathrm{e}^{c_4t}t^{2H}}\biggr). \tag{20}$$

Here $c_t^{(1)}$ is of the form

$$c_t^{(1)}=\mathrm{O}\biggl(\frac{1}{t^\alpha}\biggr)\quad \text{as } t\downarrow 0,$$

for some positive number α; and $c_{t,x}^{(2)}$ converges to a constant as $t\downarrow 0$.

Remark 3.14. *In the above theorem, we do not prove that the value of α is indeed dH, as one should expect. This will be studied in a subsequent paper* [4] *that discusses the strict positivity and Gaussian bounds of the density, which are important ingredients toward the analysis of some hitting probabilities of X_t (cf.* [12]*). Since obtaining $\alpha=dH$ requires assumptions and involves detailed computation for orders of $\mathbf{D}^kX_t$ in t for each k as well as a more careful application of Proposition* 3.9*, for sake of conciseness we omit the details here and refer the reader to the forthcoming paper* [4].

Proof. Fix $\beta\in(\frac{1}{2},1)$. By (19) and Proposition 3.12, we have

$$p_X(t,y)\le C\big(\mathbb{P}\{X_t>y\}\big)^{1/q}\big\|\det\gamma_{X_t}^{-1}\big\|_{L^\alpha}^m\|\mathbf{D}X_t\|_{k,\gamma}^n, \tag{21}$$

for some constants $\alpha,\gamma>1$ and integers m,n. Without loss of generality, we may assume $y_i\ge 0$ for $1\le i\le d$. Since otherwise, for example $y_j<0$, we can consider the alternative expression for the density

$$p_X(t,y)=\mathbb{E}\biggl[\prod_{i\ne j}I_{\{y^i<X_t^i\}}I_{\{y^j>X_t^j\}}H_{(1,2,\dots,d)}(X_t,1)\biggr],$$

and deduce similar estimate. In the following, we provide estimate for $\|\mathbf{D}X_t\|_{k,\gamma}^n$, $\|\det\gamma_{X_t}^{-1}\|_{L^\alpha}^m$ and $\mathbb{P}\{X_t>y\}$ respectively.

To this end, first note that we have the linear stochastic differential equation satisfied by the matrix valued process $\eta_t=\gamma_{X_t}^{-1}$ (see e.g. [10, Proposition 3.8] or [23]), that is,

$$\eta_t=\alpha_0^{-1}-\sum_{\ell=1}^d\int_0^t\big[\eta_u\alpha_\ell(X_u)+\alpha_\ell^*(X_u)\eta_u\big]\,\mathrm{d}B_u^\ell-\int_0^t\big[\eta_u\beta(X_u)+\beta^*(X_u)\eta_u\big]\,\mathrm{d}u,$$

where

$$\alpha_0 = \sum_{j=1}^{d} \int_0^t \int_0^t V_j(X_r) V_j^*(X_{r'}) \left| r - r' \right|^{2H-2} \mathrm{d}r \, \mathrm{d}r',$$

and

$$\alpha_\ell(X_u) = \left(\partial_k V_\ell^i(X_u)\right)_{1 \le i,k \le d}, \qquad \beta(X_u) = \left(\partial_k V_0^i(X_u)\right)_{1 \le i,k \le d}.$$

Now by Proposition 2.2 there exist constants $C > 0$, depending on V, x and k, such that

$$\sup_{0 \le t \le T} \left| X_t^i \right| \le \left| x^i \right| + CT \| B \|_{0,T,\beta}^{1/\beta}, \tag{22}$$

$$\sup_{0 \le t \le T} \left\| \gamma_{X_t}^{-1} \right\| \le \frac{C}{T^{2Hd}} \left[1 + \mathrm{e}^{CT \| B \|_{0,T,\beta}^{1/\beta}} \right], \tag{23}$$

$$\sup_{0 \le t, r_i \le T} \left| \mathbf{D}_{r_k}^{j_k} \cdots \mathbf{D}_{r_1}^{j_1} X_t^i \right| \le C \mathrm{e}^{CT \| B \|_{0,T,\beta}^{1/\beta}}. \tag{24}$$

On the other hand we have, for some constant M_β (cf. [23]):

$$\mathbb{P}\{ \| B \|_{0,T,\beta} > r \} \le M_\beta \mathrm{e}^{-r^2/(2T^{2(H-\beta)})}. \tag{25}$$

Hence

$$\left\| \det \gamma_{X_t}^{-1} \right\|_{L^\alpha}^m < \infty \quad \text{and} \quad \| \mathbf{D} X_t \|_{k,\gamma}^n < \infty.$$

Moreover, by (23), (24) and the tail estimate (25), there exists constant $C > 0$ independent of t such that

$$\left\| \det \gamma_{X_t}^{-1} \right\|_{L^\alpha}^m \le \frac{C}{t^{2Hmd}} \left(1 + t^{m/\alpha} F(t, 1/\beta - 1, 1/\beta, C)^{m/\alpha} \right),$$

and

$$\| \mathbf{D} X_t \|_{k,\gamma}^n \le C \left(1 + t^{nH(k+1)+n/\gamma} F(t, 1/\beta - 1, 1/\beta, C)^{n/\gamma} \right).$$

In the above

$$F(t, a, b, M) = \int_0^\infty r^a \mathrm{e}^{-r^2/(2t^{2(H-\beta)})} \mathrm{e}^{Mtr^b} \, \mathrm{d}r,$$

for any $a, M > 0$ and $2 > b > 0$. Now set

$$c_t^{(1)} = \left\| \det \gamma_{X_t}^{-1} \right\|_{L^\alpha}^m \| \mathbf{D} X_t \|_{k,\gamma}^n.$$

Elementary computation shows, for some constant $\alpha > 0$

$$c_t^{(1)} \sim \mathrm{O}\left(\frac{1}{t^\alpha} \right) \quad \text{as } t \downarrow 0. \tag{26}$$

Finally we estimate $\mathbb{P}\{X_t > y\}$. By (22) and the concentration property for X_t, we conclude

$$\mathbb{P}\{X_t > y\} \le \mathbb{P}\left\{ |X_t| > |y| \right\} \le \exp\left(-\frac{(|y| - \mathbb{E}|X_t|)^2}{2M\mathrm{e}^{Ct} t^{2H}} \right) \le \exp\left(-\frac{(|y| - c_{t,x}^{(2)})^2}{2M\mathrm{e}^{Ct} t^{2H}} \right). \tag{27}$$

Now (21), (26) and (27) give us the desired upper bound for the density $p_X(t, y)$. $\square$

Remark 3.15. *There is also an upper bound for the constant* $c_t^{(1)}$ *in the above theorem as* $t \uparrow \infty$. *Indeed, by some elementary computation one can show that*

$$c_t^{(1)} \le t^{\alpha} \mathrm{e}^{t^{\beta}} \quad \text{as } t \uparrow \infty,$$

for some $\alpha, \beta > 0$.

4. Extension to the irregular case

From now on, our purpose is to extend the previous to the case of a fBm with Hurst index $\frac{1}{3} < H < \frac{1}{2}$. This requires the introduction of some rough paths tools, which is the aim of the current section. We shall use in fact the language of algebraic integration theory, which is a variant of the rough paths theory introduced in [17] (we also refer to [18] for a detailed introduction of the topic).

4.1. *Increments*

The extended pathwise integration we will deal with is based on the notion of *increments*, together with an elementary operator δ acting on them. The algebraic structure they generate is described in [17,18], but here we present directly the definitions of interest for us, for sake of conciseness. First of all, for an arbitrary real number $T > 0$, a vector space V and an integer $k \ge 1$ we denote by $\mathcal{C}_k(V)$ the set of functions $g : [0, T]^k \to V$ such that $g_{t_1 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \le k - 1$. Such a function will be called a $(k-1)$-increment, and we set $\mathcal{C}_*(V) = \bigcup_{k \ge 1} \mathcal{C}_k(V)$. We can now define the announced elementary operator δ on $\mathcal{C}_k(V)$:

$$\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V), \qquad (\delta g)_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^{k-i} g_{t_1 \cdots \hat{t}_i \cdots t_{k+1}}, \tag{28}$$

where $\hat{t}_i$ means that this particular argument is omitted. A fundamental property of δ, which is easily verified, is that $\delta\delta = 0$, where $\delta\delta$ is considered as an operator from $\mathcal{C}_k(V)$ to $\mathcal{C}_{k+2}(V)$. We denote $\mathcal{ZC}_k(V) = \mathcal{C}_k(V) \cap \operatorname{Ker} \delta$ and $\mathcal{BC}_k(V) = \mathcal{C}_k(V) \cap \operatorname{Im} \delta$.

Some simple examples of actions of δ, which will be the ones we will really use throughout the paper, are obtained by letting $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then, for any $s, u, t \in [0, T]$, we have

$$(\delta g)_{st} = g_t - g_s, \quad \text{and} \quad (\delta h)_{sut} = h_{st} - h_{su} - h_{ut}. \tag{29}$$

Furthermore, it is easily checked that $\mathcal{ZC}_k(V) = \mathcal{BC}_k(V)$ for any $k \ge 1$. In particular, the following basic property holds:

Lemma 4.1. *Let* $k \ge 1$ *and* $h \in \mathcal{ZC}_{k+1}(V)$. *Then there exists a* (*non unique*) $f \in \mathcal{C}_k(V)$ *such that* $h = \delta f$.

Proof. This elementary proof is included in [17], and will be omitted here. However, let us mention that $f_{t_1 \cdots t_k} = (-1)^{k+1} h_{0 t_1 \cdots t_k}$ is a possible choice. □

Observe that Lemma 4.1 implies that all the elements $h \in \mathcal{C}_2(V)$ such that $\delta h = 0$ can be written as $h = \delta f$ for some (non unique) $f \in \mathcal{C}_1(V)$. Thus we get a heuristic interpretation of $\delta|_{\mathcal{C}_2(V)}$: it measures how much a given 1-increment is far from being an exact increment of a function, i.e., a finite difference.

Notice that our future discussions will mainly rely on k-increments with $k \le 2$, for which we will make some analytical assumptions. Namely, we measure the size of these increments by Hölder norms defined in the following way: for $f \in \mathcal{C}_2(V)$ let

$$\|f\|_{\mu} = \sup_{s,t \in [0,T]} \frac{|f_{st}|}{|t-s|^{\mu}}, \quad \text{and} \quad \mathcal{C}_2^{\mu}(V) = \left\{ f \in \mathcal{C}_2(V); \|f\|_{\mu} < \infty \right\}. \tag{30}$$

Obviously, the usual Hölder spaces $\mathcal{C}_1^\mu(V)$ will be determined in the following way: for a continuous function $g \in \mathcal{C}_1(V)$, we simply set

$$\|g\|_\mu = \|\delta g\|_\mu, \tag{31}$$

and we will say that $g \in \mathcal{C}_1^\mu(V)$ iff $\|g\|_\mu$ is finite. Notice that $\|\cdot\|_\mu$ is only a semi-norm on $\mathcal{C}_1(V)$, but we will generally work on spaces of the type

$$\mathcal{C}_{1,a}^\mu(V) = \{g : [0,T] \to V; g_0 = a, \|g\|_\mu < \infty\}, \tag{32}$$

for a given $a \in V$, on which $\|g\|_\mu$ defines a distance in the usual way. For $h \in \mathcal{C}_3(V)$ set in the same way

$$\|h\|_{\gamma,\rho} = \sup_{s,u,t\in[0,T]} \frac{|h_{sut}|}{|u-s|^\gamma |t-u|^\rho}, \tag{33}$$

$$\|h\|_\mu = \inf\left\{\sum_i \|h_i\|_{\rho_i,\mu-\rho_i}; h = \sum_i h_i, 0 < \rho_i < \mu\right\},$$

where the last infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0,\mu)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3(V)$, and we set

$$\mathcal{C}_3^\mu(V) := \{h \in \mathcal{C}_3(V); \|h\|_\mu < \infty\}.$$

Eventually, let $\mathcal{C}_3^{1+}(V) = \bigcup_{\mu>1} \mathcal{C}_3^\mu(V)$, and notice that the same kind of norms can be considered on the spaces $\mathcal{Z}\mathcal{C}_3(V)$, leading to the definition of some spaces $\mathcal{Z}\mathcal{C}_3^\mu(V)$ and $\mathcal{Z}\mathcal{C}_3^{1+}(V)$.

With these notations in mind the following proposition is a basic result, which belongs to the core of our approach to pathwise integration. Its proof may be found in a simple form in [18].

Proposition 4.2 (The Λ-map). *There exists a unique linear map $\Lambda : \mathcal{Z}\mathcal{C}_3^{1+}(V) \to \mathcal{C}_2^{1+}(V)$ such that*

$$\delta\Lambda = \mathrm{Id}_{\mathcal{Z}\mathcal{C}_3^{1+}(V)} \quad \text{and} \quad \Lambda\delta = \mathrm{Id}_{\mathcal{C}_2^{1+}(V)}.$$

In other words, for any $h \in \mathcal{C}_3^{1+}(V)$ such that $\delta h = 0$ there exists a unique $g = \Lambda(h) \in \mathcal{C}_2^{1+}(V)$ such that $\delta g = h$. Furthermore, for any $\mu > 1$, the map Λ is continuous from $\mathcal{Z}\mathcal{C}_3^\mu(V)$ to $\mathcal{C}_2^\mu(V)$ and we have

$$\|\Lambda h\|_\mu \le \frac{1}{2^\mu - 2}\|h\|_\mu, \quad h \in \mathcal{Z}\mathcal{C}_3^\mu(V). \tag{34}$$

Let us mention at this point a first link between the structures we have introduced so far and the problem of integration of irregular functions.

Corollary 4.3. *For any 1-increment $g \in \mathcal{C}_2(V)$ such that $\delta g \in \mathcal{C}_3^{1+}$, set $\delta f = (\mathrm{Id} - \Lambda\delta)g$. Then*

$$(\delta f)_{st} = \lim_{|\Pi_{st}|\to 0} \sum_{i=0}^{n-1} g_{t_i t_{i+1}},$$

where the limit is over any partition $\Pi_{st} = \{t_0 = s, \dots, t_n = t\}$ of $[s,t]$, whose mesh tends to zero. Thus, the 1-increment δf is the indefinite integral of the 1-increment g.

4.2. *Computations in $\mathcal{C}_*$*

Let us specialize now to the case $V = \mathbb{R}$, and just write $\mathcal{C}_k^\gamma$ for $\mathcal{C}_k^\gamma(\mathbb{R})$. Then $(\mathcal{C}_*, \delta)$ can be endowed with the following product: for $g \in \mathcal{C}_n$ and $h \in \mathcal{C}_m$ let gh be the element of $\mathcal{C}_{n+m-1}$ defined by

$$(gh)_{t_1,\dots,t_{m+n+1}} = g_{t_1,\dots,t_n} h_{t_n,\dots,t_{m+n-1}}, \quad t_1,\dots,t_{m+n-1} \in [0,T]. \tag{35}$$

In this context, we have the following useful properties.

Proposition 4.4. *The following differentiation rules hold true*:

(1) *Let $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_1$. Then $gh \in \mathcal{C}_1$ and $\delta(gh) = \delta g h + g \delta h$.*
(2) *Let $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then $gh \in \mathcal{C}_2$ and $\delta(gh) = -\delta g h + g \delta h$.*
(3) *Let $g \in \mathcal{C}_2$ and $h \in \mathcal{C}_1$. Then $gh \in \mathcal{C}_2$ and $\delta(gh) = \delta g h + g \delta h$.*

The iterated integrals of smooth functions on $[0, T]$ are obviously particular cases of elements of $\mathcal{C}$, which will be of interest for us. Let us recall some basic rules for these objects: consider $f \in \mathcal{C}_1^\infty$ and $g \in \mathcal{C}_1^\infty$, where $\mathcal{C}_1^\infty$ denotes the set of smooth functions on $[0, T]$. Then the integral $\int f \, dg$, which will be denoted indistinctly by $\int f \, dg$ or $\mathcal{J}(f \, dg)$, can be considered as an element of $\mathcal{C}_2^\infty$. Namely, let $\mathcal{S}_{2,T}$ denote the simplex $\{(s,t) \in [0,T]^2 : 0 \le s < t \le T\}$, for $(s,t) \in \mathcal{S}_{2,T}$ we set

$$\mathcal{J}_{st}(f \, dg) = \left(\int f \, dg \right)_{st} = \int_s^t f_u \, dg_u.$$

The multiple integrals can also be defined in the following way: given a smooth element $h \in \mathcal{C}_2^\infty$ and $(s,t) \in \mathcal{S}_{2,T}$, we set

$$\mathcal{J}_{st}(h \, dg) \equiv \left(\int h \, dg \right)_{st} = \int_s^t h_{su} \, dg_u.$$

In particular, for $f^1 \in \mathcal{C}_1^\infty$, $f^2 \in \mathcal{C}_1^\infty$ and $f^3 \in \mathcal{C}_1^\infty$ the double integral $\mathcal{J}_{st}(f^3 \, df^2 \, df^1)$ is defined as

$$\mathcal{J}_{st}\big(f^3 \, df^2 \, df^1\big) = \left(\int f^3 \, df^2 \, df^1 \right)_{st} = \int_s^t \mathcal{J}_{su}\big(f^3 \, df^2\big) \, df_u^1.$$

Now suppose that the nth order iterated integral of $f^{n+1} \, df^n \cdots df^2$, which is denoted by $\mathcal{J}(f^{n+1} \, df^n \cdots df^2)$, has been defined for $f^j \in \mathcal{C}_1^\infty$. Then, if $f^1 \in \mathcal{C}_1^\infty$, we set

$$\mathcal{J}_{st}\big(f^{n+1} \, df^n \cdots df^2 \, df^1\big) = \int_s^t \mathcal{J}_{su}\big(f^{n+1} \, df^n \cdots df^2\big) \, df_u^1, \tag{36}$$

which recursively defines the iterated integrals of smooth functions. Observe that an nth order integral $\mathcal{J}(df^n \cdots df^2 \, df^1)$ can be defined along the same lines, starting with

$$\mathcal{J}(df) = \delta f, \qquad \mathcal{J}_{st}\big(df^2 \, df^1\big) = \int_s^t \mathcal{J}_{su}\big(df^2\big) \, df_u^1 = \int_s^t \big(\delta f^2\big)_{su} \, df_u^1,$$

and so on.

The following relations between multiple integrals and the operator δ will also be useful. The reader is sent to [18] for its elementary proof.

Proposition 4.5. *Let $f \in \mathcal{C}_1^\infty$ and $g \in \mathcal{C}_1^\infty$. Then it holds that*

$$\delta g = \mathcal{J}(dg), \qquad \delta\big(\mathcal{J}(f \, dg)\big) = 0, \qquad \delta\big(\mathcal{J}(df \, dg)\big) = (\delta f)(\delta g) = \mathcal{J}(df)\mathcal{J}(dg),$$

and

$$\delta\big(\mathcal{J}\big(df^n \cdots df^1\big)\big) = \sum_{i=1}^{n-1} \mathcal{J}\big(df^n \cdots df^{i+1}\big)\mathcal{J}\big(df^i \cdots df^1\big).$$

4.3. *Weakly controlled processes*

The rough path theory allows to define and solve differential equations driven by a generic Hölder continuous path B provided enough iterated integrals of this function can be defined. We shall briefly recall how this is done, in the simplest nontrivial case of a Hölder continuity exponent $\frac{1}{3} < \gamma < \frac{1}{2}$. Observe that we keep here the notation B for the underlying path as in the fBm case for notational sake, while the theory can be applied to much more general situations.

The basic assumption one has to add in order to define our objects when $\gamma > \frac{1}{3}$ is the existence of an (abstract) double iterated integral of B with respect to itself, which can be defined as follows:

Hypothesis 4.6. *The path B is $\mathbb{R}^d$-valued γ-Hölder with $\gamma > \frac{1}{3}$ and admits a Lévy area, that is a process $\mathbf{B^2} \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d,d})$ satisfying*

$$\delta\mathbf{B^2} = \mathbf{B^1} \otimes \mathbf{B^1}, \quad \textit{i.e.}, \quad \mathbf{B}^{\mathbf{2},ij}_{sut} = \big[\mathbf{B}^{\mathbf{1},i}\big]_{su}\big[\mathbf{B}^{\mathbf{1},j}\big]_{ut},$$

for $s,u,t \in \mathcal{S}_{3,T}$ and $i,j \in \{1,\dots,d\}$. We also assume that $\mathbf{B^2}$ can be obtained in the following way: consider the sequence of linear dyadic approximation B^m of B defined like in Proposition 2.6. For $0 \le s < t \le 1$ and $i_1, i_2 \in \{1,\dots,d\}$, set $\mathbf{B}^{\mathbf{2},m,i_1i_2}_{st} = \int_{s<u_1<u_2<t} \mathrm{d}B^{m,i_1}_{u_1}\,\mathrm{d}B^{m,i_2}_{u_2}$, which is defined as a Riemann–Stieljes integral. Then we suppose that $\mathbf{B}^{\mathbf{2},m}$ converges to $\mathbf{B^2}$ in the norm of $\mathcal{C}_2^{2\gamma}$.

It should be noticed at this point that fBm satisfies the above assumption, as shown in [11,16,29]:

Proposition 4.7. *Let B be a d-dimensional fBm with $H > \frac{1}{3}$ as defined in Section 2. For the linear dyadic approximation B^m of B defined at Proposition 2.6, the increment $\mathbf{B}^{\mathbf{2},m}$ almost surely converges to an element $\mathbf{B^2}$ satisfying Hypothesis 4.6. The convergence holds in any $\mathcal{C}_2^{2\gamma}$ norm for $\gamma < H$.*

The first difference between the Young case and the situation of a Hölder continuity exponent $\frac{1}{3} < \gamma \le \frac{1}{2}$ is that a restriction has to be imposed on the class of allowed integrands with respect to B. This class is called the class of *weakly controlled processes*, and is defined as follows:

Definition 4.8. *Let z be a process in $\mathcal{C}_1^{\gamma}(\mathbb{R}^n)$ with $\frac{1}{3} < \gamma \le \frac{1}{2}$ (that is, $N := \lfloor 1/\gamma \rfloor = 2$). We say that z is a weakly controlled path based on B and starting from a if $z_0 = a$, which is a given initial condition in $\mathbb{R}^n$, and $\delta z \in \mathcal{C}_2^{\gamma}(\mathbb{R}^n)$ can be decomposed into*

$$\delta z^i = \zeta^{ii_1}\mathbf{B}^{\mathbf{1},i_1} + r^i, \quad \textit{i.e.}, \quad \big(\delta z^i\big)_{st} = \zeta^{ii_1}_s \mathbf{B}^{\mathbf{1},i_1}_{st} + r^i_{st}, \tag{37}$$

for all $(s,t) \in \mathcal{S}_{2,T}$. In the previous formula, we assume $\zeta \in \mathcal{C}_1^{\gamma}(\mathbb{R}^{n,d})$, and r is a regular part such that $r \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^n)$. The space of weakly controlled paths starting from a will be denoted by $\mathcal{Q}_{\gamma,a}(\mathbb{R}^n)$, and a process $z \in \mathcal{Q}_{\gamma,a}(\mathbb{R}^n)$ can be considered in fact as a couple (z,ζ). The natural semi-norm on $\mathcal{Q}_{\gamma,a}(\mathbb{R}^n)$ is given by

$$\mathcal{N}\big[z;\mathcal{Q}_{\gamma,a}\big(\mathbb{R}^n\big)\big] = \mathcal{N}\big[z;\mathcal{C}_1^{\gamma}\big(\mathbb{R}^n\big)\big] + \mathcal{N}\big[\zeta;\mathcal{C}_1^{\infty}\big(\mathbb{R}^{n,d}\big)\big] + \mathcal{N}\big[\zeta;\mathcal{C}_1^{\gamma}\big(\mathbb{R}^{n,d}\big)\big] + \mathcal{N}\big[r;\mathcal{C}_2^{2\gamma}\big(\mathbb{R}^n\big)\big],$$

with $\mathcal{N}[g;\mathcal{C}_1^{\kappa}]$ defined by (31) and $\mathcal{N}[\zeta;\mathcal{C}_1^{\infty}(V)] = \sup_{0\le s\le T}|\zeta_s|_V$.

Two basic steps in order to define and solve differential equations with respect to B are then:

(1) Study the decomposition of $f(z)$ as weakly controlled process, when f is a smooth function and z a weakly controlled process.
(2) Define rigorously the integral $\int z_u\,\mathrm{d}B_u = \mathcal{J}(z\,\mathrm{d}B)$ for a weakly controlled path z and compute its decomposition (37).

We shall now detail a little this program.

Let us see then how to decompose $f(z)$ as a controlled process when f is a smooth enough function, a step for which we first introduce a convention which will hold true until the end of the paper: for any smooth function $f:\mathbb{R}^n\to\mathbb{R}$, $k\ge 1$, $(i_1,\dots,i_k)\in\{1,\dots,d\}^k$ and $\xi\in\mathbb{R}^n$, we set

$$\partial^k_{i_1\dots i_k} f(\xi)=\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}(\xi). \tag{38}$$

With this notation in hand, our decomposition result is the following:

Proposition 4.9. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a C_b^2 function such that $f(a)=\hat{a}$, z a controlled process as in Definition 4.8 and set $\hat{z}=f(z)$. Then $\hat{z}\in\mathcal{Q}_{\gamma,\hat{a}}(\mathbb{R})$, and it can be decomposed into $\delta\hat{z}=\hat{\zeta}^{i_1}\mathbf{B}^{\mathbf{1},i_1}+\hat{r}$, with*

$$\hat{\zeta}^{i_1}=\partial_i f(z)\zeta^{ii_1}\quad\text{and}\quad \hat{r}=\big[\delta f(z)-\partial_i f(z)\delta z^i\big]+\partial_i f(z) r^i.$$

Furthermore,

$$\mathcal{N}\big[\hat{z};\mathcal{Q}_{\gamma,\hat{a}}(\mathbb{R})\big]\le c_{f,T}\big(1+\mathcal{N}^2\big[z;\mathcal{Q}_{\gamma,a}\big(\mathbb{R}^n\big)\big]\big). \tag{39}$$

Let us now turn to the integration of weakly controlled paths, which is summarized in the following theorem.

Theorem 4.10. *For a given $\frac{1}{3}<\gamma\le\frac{1}{2}$, let B be a process satisfying Hypothesis 4.6. Furthermore, let $m\in\mathcal{Q}_{\gamma,b}(\mathbb{R}^d)$ with $m_0=b\in\mathbb{R}^d$ and decomposition*

$$\delta m^i=\mu^{ii_1}\mathbf{B}^{\mathbf{1},i_1}+r^i,\quad \text{where } \mu\in\mathcal{C}_1^{\gamma}\big(\mathbb{R}^{d,d}\big), r\in\mathcal{C}_2^{2\gamma}\big(\mathbb{R}^d\big). \tag{40}$$

Define z by $z_0=a\in\mathbb{R}$ and

$$\delta z=m^i\mathbf{B}^{\mathbf{1},i}+\mu^{ii_1}\mathbf{B}^{\mathbf{2},i_1 i}-\Lambda\big(r^i\mathbf{B}^{\mathbf{1},i}+\delta\mu^{ii_1}\mathbf{B}^{\mathbf{2},i_1 i}\big). \tag{41}$$

Finally, set

$$\mathcal{J}_{st}(m\,\mathrm{d}B)=\int_s^t\langle m_u,\,\mathrm{d}B_u\rangle_{\mathbb{R}^d}\triangleq\delta z_{st}.$$

Then:

(1) *z is well-defined as an element of $\mathcal{Q}_{\gamma,a}(\mathbb{R})$, and coincides with the Riemann–Stieltjes integral of m with respect to B whenever these two functions are smooth.*
(2) *The semi-norm of z in $\mathcal{Q}_{\gamma,a}(\mathbb{R})$ can be estimated as*

$$\mathcal{N}\big[z;\mathcal{Q}_{\gamma,a}(\mathbb{R})\big]\le c_B\big(1+\mathcal{N}\big[m;\mathcal{Q}_{\gamma,b}\big(\mathbb{R}^d\big)\big]\big), \tag{42}$$

for a positive constant c_B which can be bounded as follows:

$$c_B\le c\big(\mathcal{N}\big[\mathbf{B}^{\mathbf{1}};\mathcal{C}_2^{\gamma}\big(\mathbb{R}^d\big)\big]+\mathcal{N}\big[\mathbf{B}^{\mathbf{2}};\mathcal{C}_2^{2\gamma}\big(\mathbb{R}^{d^2}\big)\big]\big)\quad\text{for a universal constant } c.$$

(3) *It holds*

$$\mathcal{J}_{st}(m\,\mathrm{d}x)=\lim_{|\Pi_{st}|\to 0}\sum_{q=0}^{n-1}\big[m^i_{t_q}\mathbf{B}^{\mathbf{1}}_{t_q,t_{q+1}}(i)+\mu^{ii_1}_{t_q}\mathbf{B}^{\mathbf{2},i_1 i}_{t_q,t_{q+1}}\big], \tag{43}$$

for any $0\le s<t\le T$, where the limit is taken over all partitions $\Pi_{st}=\{s=t_0,\dots,t_n=t\}$ of $[s,t]$, as the mesh of the partition goes to zero.

4.4. *Rough differential equations*

Recall that we are concerned with equations of the form (2). In our algebraic setting, we will rephrase this as follows: we will say that X^x is a solution to (2) if $X_0^x = x$, $X^x \in \mathcal{Q}_{\gamma,x}(\mathbb{R}^d)$ and for any $0 \le s \le t \le 1$ we have

$$\delta X_{st}^x = \int_s^t V_0(X_u^x)\,\mathrm{d}u + \mathcal{J}_{st}\big(V_i(X^x)\,\mathrm{d}B^i\big), \tag{44}$$

where the integral $\mathcal{J}(V_i(X^x)\,\mathrm{d}B^i)$ has to be understood in the sense of Theorem 4.10. The following existence and uniqueness result is then classical in rough paths theory.

Theorem 4.11. *Let B be a process satisfying Hypothesis 4.6 and $V_0, \dots, V_n$ a collection of vector fields which fulfill Hypothesis 1.1. Then*

(i) *Equation (44) admits a unique solution X^x in $\mathcal{Q}_{\gamma,x}(\mathbb{R}^d)$.*
(ii) *Consider the linear approximation B^m of B introduced in Hypothesis 4.6, and set $\tilde{X}^m$ for the solution of the (ordinary) differential equation (44) driven by the piecewise smooth function B^m. Then $\tilde{X}^m$ converges to X^x in $\mathcal{C}_1^\gamma$ norm.*
(iii) *Consider the sequence of dyadic partitions of Proposition 2.6, and for notational sake set $t_k = t_k^n$. Define a process X^n on the points t_k^n by $X_0^n = x$ and*

$$\delta X_{t_k t_{k+1}}^n = \frac{V_0(X_{t_k}^n)}{2^n} + V_i\big(X_{t_k}^n\big)\mathbf{B}_{t_k t_{k+1}}^{\mathbf{1},i} + V_{i_1}V_{i_2}\big(X_{t_k}^n\big)\mathbf{B}_{t_k t_{k+1}}^{\mathbf{2},i_2 i_1}. \tag{45}$$

Complete the definition of X^n on $[0,1]$ by linear interpolation. Then as $n \to \infty$, the process X^n converges to X^x in $\mathcal{C}_1^\gamma$ norm.

Proof. We refer to [26] for the proof of the existence and uniqueness part, as well as to [17] for the same result in the algebraic integration setting. Part (ii) of our proposition stems from the continuity of the Ito map, which is also stated and proved in both [17,26]. The approximation statement (iii) has first been stated by Davie [13] and then been generalized in [16].

In the sequel we will simply try to relate the decomposition of the solution to equation (44) as a controlled process and the numerical scheme given by (45), a relation which turns out to be useful in the sequel. For this, we shall denote by r any increment in $\mathcal{C}_2^{2\gamma}$ and by $r^\sharp$ any increment in $\mathcal{C}_2^{1^+}$ in the computations below, independently of their values. Observe then that, according to the right hand side of (44), the decomposition of X^x as a controlled process is given by

$$\delta X^{x,j} = \zeta^{jj_1}\mathbf{B}^{\mathbf{1},j_1} + r, \quad \text{with } \zeta^{jj_1} = V_{j_1}^j\big(X^x\big).$$

Hence, owing to Proposition 4.9, one has

$$\delta V_i\big(X^x\big) = \hat{\zeta}^{ij_1}\mathbf{B}^{\mathbf{1},j_1} + r^i, \quad \text{with } \hat{\zeta}^{ij_1} = \partial_j V_i\big(X^x\big)\zeta^{jj_1} = V_{j_1}V_i\big(X^x\big). \tag{46}$$

Now, if one desires an expansion of δX^x up to increments of regularity 1^+, consider again the right hand side of (44), and compute it by a direct application of Theorem 4.10. This yields

$$\begin{aligned}\delta X_{st}^x &= V_0\big(X_s^x\big)(t-s) + V_i\big(X_s^x\big)\mathbf{B}_{st}^{\mathbf{1},i} + \hat{\zeta}^{ij_1}\mathbf{B}_{st}^{\mathbf{2},j_1 i} + r^\sharp \\ &= V_0\big(X_s^x\big)(t-s) + V_i\big(X_s^x\big)\mathbf{B}_{st}^{\mathbf{1},i} + V_{j_1}V_i\big(X_s^x\big)\mathbf{B}_{st}^{\mathbf{2},j_1 i} + r^\sharp.\end{aligned}$$

Thanks to identity (43), it is now easily seen that (45) is a natural candidate for our numerical scheme. □

We show now how to get efficient bounds on the solution to equation (44) out of its numerical scheme. This step is understood as a warmup for the same kind of estimates concerning the Malliavin derivative of the solution.

Proposition 4.12. *Under the assumptions of Theorem 4.11, the solution to equation (44) satisfies*:

$$\|X^x\|_\gamma \le c_V\big(1+\|\mathbf{B^1}\|_\gamma^{1/\gamma}+\|\mathbf{B^2}\|_{2\gamma}^{1/2\gamma}\big), \tag{47}$$

where c_V is a constant which only depends on the vector fields $V_0,\dots,V_n$.

Remark 4.13. *This proposition is shown in [16] by identifying the signature of B with the signature of a certain finite variation process, plus some easy estimates for ordinary differential equations. We have included here a direct elementary proof of (47) for 3 main reasons*: (i) *We have not been able to find them in the literature under this form.* (ii) *It can be seen as a warmup for the estimate of our Malliavin derivative $\mathcal{M}$ included in the next section.* (iii) *Some of the estimates used throughout the proof of Proposition 4.12 will also be used in the proof of relation (59).*

Proof of Proposition 4.12. Theorem 4.11 part (iii) asserts the convergence of the approximation X^n towards X^x as $n\to\infty$. It is thus sufficient to prove relation (47) for X^n, uniformly in n. One can also be easily reduced to prove

$$\sup_{0\le i<j\le 2^n}\frac{|\delta X^n_{t_i^n t_j^n}|}{|t_j^n-t_i^n|^\gamma}\le c_V\big(1+\|\mathbf{B^1}\|_\gamma^{1/\gamma}+\|\mathbf{B^2}\|_{2\gamma}^{1/2\gamma}\big),$$

which is what we shall proceed to do. We now divide our proof in several steps.

Step 1: Expression for δX^n. Set

$$q^n_{st}=V_0\big(X^n_s\big)(t-s)+V_i\big(X^n_s\big)\mathbf{B}^{\mathbf{1},i}_{st}+V_{i_1}V_{i_2}\big(X^n_s\big)\mathbf{B}^{\mathbf{2},i_2i_1}_{st}, \tag{48}$$

so that $\delta X^n_{t_i^n t_j^n}=\sum_{l=1}^{j-1}q^n_{t_l^n t_{l+1}^n}$. For $i<j$, we also construct a dyadic partition $\{\tau_l^k;0\le k\le K,0\le l\le 2^k\}$ of the set $\{t_i^n,\dots,t_j^n\}$ inductively in the following way: set $\tau_0^0=t_i$ and $\tau_1^0=t_j$. Now, if the τ_l^k's are known, we set $\tau_{2l}^{k+1}=\tau_l^k$. Furthermore, if $\tau_{2l}^{k+1}=t_m$ and $\tau_{2l+2}^{k+1}=t_{m'}$, then take $\tau_{2l+1}^{k+1}=t_{m^*}$, with $m^*=\lfloor(m+m')/2\rfloor$. This procedure is then non trivial as long as $j-i\ge 2^k$, which means that we stop at $K=\lceil\log_2(j-i)\rceil$. Here is then a simple example of construction: consider $i=1$, $j=4$. Then we have:

$$\tau_0^0=1,\tau_1^0=4;\qquad \tau_0^1=1,\tau_1^1=2,\tau_2^1=4;\qquad \tau_0^2=1,\tau_1^2=1,\tau_2^2=2,\tau_3^2=3,\tau_4^2=4.$$

With these notations in hand, it is easily checked that the relation $\delta X^n_{t_i^n t_j^n}=\sum_{l=1}^{j-1}q^n_{t_l^n t_{l+1}^n}$ can also be written as $\delta X^n_{t_i^n t_j^n}=\sum_{k=0}^{2^K-1}q^n_{\tau_k^K\tau_{k+1}^K}$. Furthermore,

$$q^n_{\tau_{2l}^k\tau_{2l+1}^k}+q^n_{\tau_{2l+1}^k\tau_{2l+2}^k}=q^n_{\tau_l^{k-1}\tau_{l+1}^{k-1}}-\big(\delta q^n\big)_{\tau_{2l}^k\tau_{2l+1}^k\tau_{2l+2}^k},$$

and summing this equality for $k=K$ and $l=0,\dots,2^K-1$ we get

$$\sum_{l=0}^{2^K-1}q^n_{\tau_l^K\tau_{l+1}^K}=\sum_{l=0}^{2^K-1}q^n_{\tau_l^{K-1}\tau_{l+1}^{K-1}}-\sum_{l=0}^{2^K-1}\big(\delta q^n\big)_{\tau_{2l}^{K-1}\tau_{2l+1}^{K-1}\tau_{2l+2}^{K-1}}.$$

Iterating, we obtain

$$\delta X^n_{t_i^n t_j^n}=q^n_{t_i^n t_j^n}-\sum_{k=1}^{K}\sum_{l=0}^{2^k-1}\delta q^n_{\tau_{2l}^k\tau_{2l+1}^k\tau_{2l+2}^k}. \tag{49}$$

Step 2: Expression for δq^n. Denote by I the identity function on $\mathbb{R}$, so that $\delta I_{st}=t-s$. Start then from expression (49) and use Proposition 4.4 in order to get, for any s,u,t in the dyadic partition,

$$\delta q^n_{sut}=\delta V_0\big(X^n\big)_{su}(t-u)+\delta V_i\big(X^n\big)_{su}\mathbf{B}^{1,i}_{ut}+\delta\big[V_{i_1}V_{i_2}\big(X^n\big)\big]_{su}\mathbf{B}^{\mathbf{2},i_2i_1}_{ut}-V_{i_1}V_{i_2}\big(X^n_s\big)\delta\mathbf{B}^{\mathbf{2},i_2i_1}_{sut},$$

or otherwise stated thanks to convention (35),

$$\delta q^n = \delta V_0\big(X^n\big)\delta I + \delta V_i\big(X^n\big)\mathbf{B}^{1,i} + \delta\big[V_{i_1}V_{i_2}\big(X^n\big)\big]\mathbf{B}^{\mathbf{2},i_2 i_1} - V_{i_1}V_{i_2}\big(X^n\big)\delta\mathbf{B}^{\mathbf{2},i_2 i_1}. \tag{50}$$

Admit for the moment the following claim, which will be part of our induction below:

$$\delta V_i\big(X^n\big) = V_{i_1}V_i\big(X^n\big)\mathbf{B}^{\mathbf{1},i_1} + r^{i,n}, \quad \text{with } |r^{i,n}_{st}| \le c_V\big(\big|\delta X^n_{st}\big|^2 + \big\|\mathbf{B}^{\mathbf{2}}\big\|_{2\gamma}|t-s|^{2\gamma}\big). \tag{51}$$

Thus, according to the fact that $\delta\mathbf{B}^{\mathbf{2},i_2 i_1} = \mathbf{B}^{\mathbf{1},i_2}\mathbf{B}^{\mathbf{1},i_1}$, one can recast (50) into

$$\delta q^n = \delta V_0\big(X^n\big)\delta I + r^{i,n}\mathbf{B}^{1,i} + \delta\big[V_{i_1}V_{i_2}\big(X^n\big)\big]\mathbf{B}^{\mathbf{2},i_2 i_1} := \rho^{1,n} + \rho^{2,n} + \rho^{3,n}. \tag{52}$$

Notice that for $j = 1, 2, 3$, the increment $\rho^{j,n}$ lies into $\mathcal{C}_3^{3\gamma}$. Furthermore, it is readily checked that

$$\big|\rho^{1,n}_{sut}\big| \le c_V\big|\delta X^n_{su}\big||t-u|^{1+\gamma}, \qquad \big|\rho^{2,n}_{sut}\big| \le c_V\big|\delta X^n_{su}\big|^2\big\|\mathbf{B}^{\mathbf{1}}\big\|_{\gamma}|t-u|^{\gamma}, \tag{53}$$

and $|\rho^{3,n}_{sut}| \le c_V|\delta X^n_{su}|\|\mathbf{B}^{\mathbf{2}}\|_{2\gamma}|t-u|^{2\gamma}$.

Step 3: An induction procedure. Let us consider an integer $\ell \ge 1$ and the quantity:

$$N^n_\ell = \sup_{0\le i<j\le \ell} \frac{|\delta X^n_{t^n_i t^n_j}|}{|t^n_j - t^n_i|^{\gamma}}.$$

We localize now our study to an interval of the form $[a, a+\eta]$ with an arbitrary positive number a, and η small enough. We will prove that if η is of order $(1 + \|\mathbf{B}^{\mathbf{1}}\|_{\gamma}^{1/2\gamma} + \|\mathbf{B}^{\mathbf{2}}\|_{2\gamma}^{1/2\gamma})^{-1}$, then $N_\ell \le c_V(1 + \|\mathbf{B}^{\mathbf{1}}\|_{\gamma} + \|\mathbf{B}^{\mathbf{2}}\|_{2\gamma}^{1/2})$ by induction.

The case $\ell = 1$ being trivial, let us assume that the hypothesis is true up to a given $\ell \ge 1$. Take now $1 \le i \le \ell$ and $j = \ell + 1$. According to (49), write

$$\delta X^n_{t^n_i t^n_j} - q^n_{t^n_i t^n_j} = -\sum_{k=1}^{K}\sum_{l=0}^{2^k-1} \delta q^n_{\tau^k_{2l}\tau^k_{2l+1}\tau^k_{2l+2}}.$$

In the right-hand side of this decomposition, all the points $\tau^k_{2l}, \tau^k_{2l+1}$ are of the form t^n_p with $p \le \ell$. Thus (52), our bounds (53) on $\rho^{j,n}$ and the induction hypothesis entail

$$\big|\delta X^n_{t^n_i t^n_j} - q^n_{t^n_i t^n_j}\big| \le c_V\big(N_\ell + N_\ell^2\big\|\mathbf{B}^{\mathbf{1}}\big\|_{\gamma} + N_\ell\big\|\mathbf{B}^{\mathbf{2}}\big\|_{2\gamma}\big)\big|t^n_j - t^n_i\big|^{3\gamma}. \tag{54}$$

Furthermore, it is obvious from (48) that

$$\frac{|q^n_{t^n_i t^n_j}|}{|t^n_j - t^n_i|^{\gamma}} \le c_V\big(1 + \big\|\mathbf{B}^{\mathbf{1}}\big\|_{\gamma} + \big\|\mathbf{B}^{\mathbf{2}}\big\|_{2\gamma}\big|t^n_j - t^n_i\big|^{\gamma}\big). \tag{55}$$

Hence, putting together the last two inequalities, taking into account that we work on an interval of size η and that we have chosen $j = \ell + 1$, we end up with the following induction relation: $N_{\ell+1} \le F_\eta(N_\ell)$, where the function $F_\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$F_\eta(\xi) = \xi \vee c_V\big[\big(1 + \big\|\mathbf{B}^{\mathbf{1}}\big\|_{\gamma} + \big\|\mathbf{B}^{\mathbf{2}}\big\|_{2\gamma}\eta^{\gamma}\big) + \big(1 + \big\|\mathbf{B}^{\mathbf{2}}\big\|_{2\gamma}\xi + \big\|\mathbf{B}^{\mathbf{1}}\big\|_{\gamma}\xi^2\big)\eta^{2\gamma}\big].$$

By separating the cases $\eta^\gamma\xi \le 1$ and $\eta^\gamma\xi > 1$, one can also prove that $F_\eta(\xi) \le \xi \vee \varphi_\eta(\xi)$, with

$$\varphi_\eta(\xi) = c_V\big(\big\|\mathbf{B}^{\mathbf{2}}\big\|_{2\gamma}\eta^{3\gamma} + \big\|\mathbf{B}^{\mathbf{1}}\big\|_{\gamma}\eta^{2\gamma}\big)\xi^2 + \big(1 + \big\|\mathbf{B}^{\mathbf{1}}\big\|_{\gamma} + \big\|\mathbf{B}^{\mathbf{2}}\big\|_{2\gamma}\eta^{\gamma}\big) := a\xi^2 + c.$$

In order to obtain a bound of the form $N_\ell \le M$ which remains valid for all ℓ, it is now sufficient to have the interval $[0, M]$ left invariant by φ_η.

We let the reader check the following elementary fact: whenever $4ac$ is of order 1, the interval $[0,M]$ is left invariant by the application $\xi \mapsto a\xi^2 + c$, with M of order c. Applying this to φ_η, we obtain that

$$\eta \asymp \left[1 + \|\mathbf{B^1}\|_\gamma + \|\mathbf{B^2}\|_{2\gamma}^{1/2}\right]^{-1/\gamma} \quad \Longrightarrow \quad N_\ell \lesssim \|\mathbf{B^1}\|_\gamma + \|\mathbf{B^2}\|_{2\gamma}^{1/2}. \tag{56}$$

With this result in hand, one can then go back to expression (49) and deduce relation (51) by induction.

Step 4: Conclusion. We have thus obtained that on any interval of length η given by (56), we have $\|X^n\|_\gamma \lesssim \|\mathbf{B^1}\|_\gamma + \|\mathbf{B^2}\|_{2\gamma}^{1/2}$. Our claim (47) is now easily deduced by dividing an arbitrary interval $[s,t]$ into subintervals of length η as in (15). □

4.5. *Estimates for the Malliavin derivative*

We are now interested in extending Proposition 3.2 beyond the Young setting. Recall thus that we are concerned with equation (12), which can be written in our algebraic integration setting as

$$\delta\mathcal{M}_{st} = \int_s^t \mathcal{M}_u \omega_0\left(X_u^x\right) \mathrm{d}u + \mathcal{J}_{st}\left(\mathcal{M}\omega_i\left(X^x\right) \mathrm{d}B^i\right), \quad 0 \le s < t \le T, \tag{57}$$

with final condition $\mathcal{M}_T = V(X_T^x)$. Then we have the following equivalent of Theorem 4.11:

Proposition 4.14. *Theorem 4.11 holds true for equation (57) under Hypotheses 1.1 and 4.6. The discretization scheme for $\mathcal{M}$ can be written as:*

$$\begin{aligned} &\delta\mathcal{M}^n_{t_k^n t_{k+1}^n} \\ &\quad = \mathcal{M}^n_{t_k^n}\left[\frac{\omega_0(X^n_{t_k^n})}{2^n}\delta I_{t_k^n t_{k+1}^n} + \omega_i\left(X^n_{t_k^n}\right)\mathbf{B}^{\mathbf{1},i}_{t_k^n t_{k+1}^n} + \left(\omega_{i_1}\omega_{i_2}\left(X^n_{t_k^n}\right) + V_{i_1}\omega_{i_2}\left(X^n_{t_k^n}\right)\right)\mathbf{B}^{\mathbf{2},i_2 i_1}_{t_k^n t_{k+1}^n}\right]. \end{aligned} \tag{58}$$

Proof. As in the case of X^x, we only justify expression (58). Note that, according to equation (57), we have $\delta\mathcal{M}_{st} = \zeta_s^l \mathbf{B}^{\mathbf{1},l}_{st} + r$, with $\zeta_s^l = \mathcal{M}_s \omega_l(X_s^x)$. Hence, invoking Proposition 4.4 and relation (46), we get

$$\delta[\mathcal{M}\omega_{i_2}] = \mathcal{M}\left[\omega_{i_1}\omega_{i_2}\left(X^x\right) + V_{i_1}\omega_{i_2}\left(X^x\right)\right]\mathbf{B}^{\mathbf{1},i_1} + r.$$

Expanding now the right hand side of equation (57) with the help of Theorem 4.10, one easily gets

$$\delta\mathcal{M} = \mathcal{M}\left[\omega_0\left(X^x\right)\delta I + \omega_i\left(X^x\right)\mathbf{B}^{\mathbf{1},i} + \left(\omega_{i_1}\omega_{i_2}\left(X^x\right) + V_{i_1}\omega_{i_2}\left(X^x\right)\right)\mathbf{B}^{\mathbf{2},i_2 i_1}\right] + r^\sharp,$$

which gives the desired justification of our scheme (58). □

We are now ready to state and prove our bounds for the process $\mathcal{M}$:

Proposition 4.15. *Let $\mathcal{M}$ be the unique solution to (57) under Hypotheses 1.1 and 4.6. Then*

(i) *The bound (11) on $\|\mathcal{M}\|_\infty$ still holds true in our irregular context.*
(ii) *$\|\mathcal{M}\|_\gamma$ satisfies the relation:*

$$\|\mathcal{M}\|_\gamma \le c_V\left(1 + \|\mathbf{B^1}\|_\gamma^{1/\gamma} + \|\mathbf{B^2}\|_{2\gamma}^{1/2\gamma}\right). \tag{59}$$

Proof. The proof of Theorem 3.1 is still valid in our rough context, which entails our first claim. Once a bound on $\|\mathcal{M}\|_\infty$ is available, $\|\mathcal{M}\|_\gamma$ can be bounded by considering the Davie type scheme (58), along the same lines as for X^x. In order to give some details about this step in a reasonably concise manner, we shall assume from now that $V_0 \equiv 0$, so that the numerical scheme (58) becomes $\delta\mathcal{M}^n_{t_k t_{k+1}} = q^{\mathcal{M},n}_{t_k t_{k+1}}$, with:

$$q^{\mathcal{M},n}_{t_k^n t_{k+1}^n} = \mathcal{M}^n_{t_k^n}\left[\omega_i\left(X^n_{t_k^n}\right)\mathbf{B}^{\mathbf{1},i}_{t_k^n t_{k+1}^n} + \left(\omega_{i_1}\omega_{i_2}\left(X^n_{t_k^n}\right) + V_{i_1}\omega_{i_2}\left(X^n_{t_k^n}\right)\right)\mathbf{B}^{\mathbf{2},i_2 i_1}_{t_k^n t_{k+1}^n}\right]. \tag{60}$$

Let us also mention that, owing to the fact that $\|\mathcal{M}\|_\infty \le c_T$ with $c_T = M\exp(CT)$, the process $\hat{\mathcal{M}}^n$ defined by $\hat{\mathcal{M}}^n_s \equiv \mathcal{M}^n_s \mathbf{1}_{(\|\mathcal{M}^n_s\| \le 2c_T)}$ also converges to $\mathcal{M}$ as $n \to \infty$. For notational sake we will write $\mathcal{M}^n$ instead of $\hat{\mathcal{M}}^n$ in the sequel, but we can assume without loss of generality that:

$$\|\mathcal{M}^n\|_\infty \le 2c_T, \quad \text{uniformly in } n \ge 1. \tag{61}$$

We now divide the remainder of the proof in two steps:

Step 1: Algebraic properties of $q^{\mathcal{M},n}$. We follow the computations of Proposition 4.12, adapted to the increment $q^{\mathcal{M},n}$ defined by (60). This yields an equivalent of relation (50) of the form $\delta q^{\mathcal{M}^n} = \sum_{j=1}^4 \rho^{\mathcal{M},j,n}$, with

$$\rho^{\mathcal{M},1,n} = r^{\mathcal{M}^n}\omega_{i_1}(X^n)\mathbf{B}^{\mathbf{1},i_1}, \qquad \rho^{\mathcal{M},2,n} = \mathcal{M}^n\omega_{i_2}(X^n)\big[\mathbf{B}^{\mathbf{1},i_2}\cdot\delta\omega_{i_1}(X^n)\big]\mathbf{B}^{\mathbf{1},i_1},$$

and

$$\rho^{\mathcal{M},3,n} = \mathcal{M}^n r^{\omega_{i_1},n}\mathbf{B}^{\mathbf{1},i_1}, \qquad \rho^{\mathcal{M},4,n} = \delta\big(\mathcal{M}^n\omega_{i_1}\omega_{i_2}(X^n) + \mathcal{M}^n V_{i_1}\omega_{i_2}(X^n)\big)\mathbf{B}^{\mathbf{2},i_2 i_1},$$

with the following conventions in mind:

(i) We assume (this will be part of our induction procedure, and this step is left to the patient reader) that $r^{\mathcal{M}^n}$ verifies the relation $|r^{\mathcal{M}^n}_{su}| \le c_V(|\delta\mathcal{M}^n_{su}|^2 + \|\mathbf{B}^\mathbf{2}\|_{2\gamma}|u-s|^{2\gamma})$.

(ii) The increment $\mathbf{B}^{\mathbf{1},i_2}\cdot\delta\omega_{i_1}(X^n)$ is defined by $[\mathbf{B}^{\mathbf{1},i_2}\cdot\delta\omega_{i_1}(X^n)]_{su} = \mathbf{B}^{\mathbf{1},i_2}_{su}[\delta\omega_{i_1}(X^n)]_{su}$.

(iii) According to Proposition 4.12, we have $|r^{\omega_{i_1},n}_{su}| \le c_V(|\delta X^n_{su}|^2 + \|\mathbf{B}^\mathbf{2}\|_{2\gamma}|u-s|^{2\gamma})$.

Step 2: Induction procedure. Like for the estimation of X^n, let us consider an integer $\ell \ge 1$ and the quantity:

$$N^n_\ell = \sup_{0\le i<j\le \ell}\frac{|\delta\mathcal{M}^n_{t^n_i t^n_j}|}{|t^n_j - t^n_i|^\gamma}.$$

We localize our study to an interval of the form $[a, a+\eta]$ with an arbitrary positive number a, and η small enough. We shall consider the same η as for X^n, namely η of order $(1 + \|\mathbf{B}^\mathbf{1}\|_\gamma^{1/2\gamma} + \|\mathbf{B}^\mathbf{2}\|_{2\gamma}^{1/2\gamma})^{-1}$, and prove that $N_\ell \le c_V(1 + \|\mathbf{B}^\mathbf{1}\|_\gamma + \|\mathbf{B}^\mathbf{2}\|_{2\gamma}^{1/2})$ by induction.

A first step in this direction is to obtain an equivalent of relation (54). To this aim, we gather our information on the terms $\rho^{\mathcal{M},j,n}$, invoke the fact that we already know that $\|\delta X^n\|_\gamma \le c_V(1 + \|\mathbf{B}^\mathbf{1}\|_\gamma + \|\mathbf{B}^\mathbf{2}\|_{2\gamma}^{1/2})$ on the small interval $[a, a+\eta]$ and recall inequality (61). After some elementary and tedious computations, we end up with:

$$\begin{aligned}\big|\delta\mathcal{M}^n_{t^n_i t^n_j} - q^{\mathcal{M},n}_{t^n_i t^n_j}\big| \le c_V\big(&N_\ell + N_\ell^2\|\mathbf{B}^\mathbf{1}\|_\gamma + N_\ell\|\mathbf{B}^\mathbf{2}\|_{2\gamma} \\ &+ \|\mathbf{B}^\mathbf{1}\|_\gamma^3 + \|\mathbf{B}^\mathbf{1}\|_\gamma^2\|\mathbf{B}^\mathbf{2}\|_{2\gamma}^{1/2} + \|\mathbf{B}^\mathbf{1}\|_\gamma\|\mathbf{B}^\mathbf{2}\|_{2\gamma} + \|\mathbf{B}^\mathbf{2}\|_{2\gamma}^{3/2}\big)\big|t^n_j - t^n_i\big|^{3\gamma}.\end{aligned}$$

Furthermore, one can get rid of the products $\|\mathbf{B}^\mathbf{1}\|_\gamma^2\|\mathbf{B}^\mathbf{2}\|_{2\gamma}^{1/2}$ and $\|\mathbf{B}^\mathbf{1}\|_\gamma\|\mathbf{B}^\mathbf{2}\|_{2\gamma}$ by means of Young's inequality with appropriate exponents, which yields:

$$\big|\delta\mathcal{M}^n_{t^n_i t^n_j} - q^{\mathcal{M},n}_{t^n_i t^n_j}\big| \le c_V\big(N_\ell + N_\ell^2\|\mathbf{B}^\mathbf{1}\|_\gamma + N_\ell\|\mathbf{B}^\mathbf{2}\|_{2\gamma} + \|\mathbf{B}^\mathbf{1}\|_\gamma^3 + \|\mathbf{B}^\mathbf{2}\|_{2\gamma}^{3/2}\big)\big|t^n_j - t^n_i\big|^{3\gamma}. \tag{62}$$

Taking into account the fact that $\mathcal{M}^n$ is bounded, the following equivalent of relation (55) also holds true:

$$\frac{|q^n_{t^n_i t^n_j}|}{|t^n_j - t^n_i|^\gamma} \le c_V\big(1 + \|\mathbf{B}^\mathbf{1}\|_\gamma + \|\mathbf{B}^\mathbf{2}\|_{2\gamma}\big|t^n_j - t^n_i\big|^\gamma\big). \tag{63}$$

We can now gather our bounds (62) and (63) and perform the same kind of manipulations as in the proof of Proposition 4.12 (Step 3) in order to get the relation $N_{\ell+1} \le \xi \vee \varphi_\eta(N_\ell)$, with $\varphi_\eta(\xi) = a\xi^2 + c$ and

$$a = \|\mathbf{B}^\mathbf{2}\|_{2\gamma}\eta^{3\gamma} + \|\mathbf{B}^\mathbf{1}\|_\gamma\eta^{2\gamma}, \qquad c = 1 + \|\mathbf{B}^\mathbf{1}\|_\gamma + \|\mathbf{B}^\mathbf{2}\|_{2\gamma}\eta^\gamma + \|\mathbf{B}^\mathbf{1}\|_\gamma^3\eta^{2\gamma} + \|\mathbf{B}^\mathbf{2}\|_{2\gamma}^{3/2}\eta^{2\gamma}.$$

The following elementary facts can now be checked easily: whenever η is of order $(1 + \|\mathbf{B^1}\|_\gamma^{1/2\gamma} + \|\mathbf{B^2}\|_{2\gamma}^{1/2\gamma})^{-1}$, we have ac of order 1 and $[0, M]$ is left invariant by φ_η, with $M = c_V(1 + \|\mathbf{B^1}\|_\gamma + \|\mathbf{B^2}\|_{2\gamma}^{1/2})$. This enables the proof of relation (59) exactly in the same way as in the proof of Proposition 4.12. □

4.6. *Density upper bound*

We finish this section by extending the density estimate and functional inequalities obtained in the smooth case (when $H > \frac{1}{2}$) to the irregular case (when $\frac{1}{3} < H < \frac{1}{2}$). We first show that in the rough case, we can obtain the smoothness of density of X_t under Hypothesis 1.2. Indeed, for this purpose, we only need to show the following integrability of Malliavin matrix.

Theorem 4.16. *Fix $H \in (\frac{1}{3}, \frac{1}{2})$. Assume Hypothesis 1.2. Let γ_X be the Malliavin matrix of X_1^x, we have $|\det \gamma_X|^{-1} \in L^\infty(\mathbb{P})$. Therefore X_1^x admits a smooth density.*

Proof. It suffices to show that there exists $C > 0$ such that for all $v \in \mathbb{R}^d$,

$$v^T \gamma_X v \geq C|v|^2.$$

Recall that

$$\gamma_X^{ij} = \langle \mathbf{D}_\cdot X^i, \mathbf{D}_\cdot X^j \rangle_{\mathcal{H}}$$

and that the $d \times d$ matrix $\mathcal{M}$ is such that its jth column $\mathcal{M}^j$ is given by

$$\mathcal{M}_t^j = \mathrm{D}_t^j X_1.$$

We have

$$\begin{aligned} v^T \gamma_X v &= \sum_{j=1}^d \| v^T \mathcal{M}_\cdot^j \|_{\mathcal{H}}^2 \\ &\geq C \sum_{j=1}^d \int_0^1 \| v^T \mathcal{M}_s^j \|^2 \, \mathrm{d}s = C \int_0^1 v^T \mathcal{M}_s \mathcal{M}_s^T v \, \mathrm{d}s. \end{aligned}$$

In the above, we used that $\mathcal{H} \subset L^2([0, 1])$.

On the other hand one can show, arguing as in the proof of Theorem 3.1 that

$$v^T \mathcal{M}_s (\mathcal{M}_s)^T v \geq C|v|^2$$

uniformly for some constant $C > 0$, if we assume Hypothesis 1.2. The proof is therefore completed. □

Another consequence of the above boundedness of the Malliavin Matrix γ_X is the following.

Theorem 4.17. *Fix $H \in (\frac{1}{3}, \frac{1}{2})$. Assume Hypothesis 1.2. Let $p_X(t, y)$ denote the density function of X_t^x. There exist constants $c_t^{(1)}$ and $c_t^{(2)}$ such that*

$$p_X(t, y) \leq c_t^{(1)} \exp\bigl(-|y - x|^\delta\bigr), \quad y \in \mathbb{R}^d,$$

for any $\delta < H$.

Proof. The proof is similar to that of Theorem 3.13. The tail estimate for $\mathbb{P}\{X_t > y\} = \mathbb{P}\{X_t - x > y - x\}$ is derived by Proposition 2.4 and Txebychev inequality. Thanks for Theorem 4.16, $\det \gamma_X^{-1}$ is uniformly bounded almost surely by some constant, and hence its L^p norm is bounded by some constant. Finally, we apply Proposition 4.15 to obtain L^p integrability of $\mathbf{J}_{0\to t}$. Then $\|\mathbf{D}X_t\|_{k,\gamma}$ is controlled by the L^p norm of $\mathbf{J}_{0\to t}$ by [21]. The proof is thus completed. □

Remark 4.18. *We are not able to obtain log-Sobolev inequality as in Theorem 3.3 in the rough case, since when $H < \frac{1}{2}$ the Hilbert norm $\mathcal{H}$ is not controlled by $L^\infty([0,1])$. On the other hand, by reproducing the proof in Theorem 3.3 together with the following interpolation inequality*

$$\|\cdot\|_{\mathcal{H}} \le C\big(\|\cdot\|_\infty + \|\cdot\|_\gamma\big),$$

and estimates in Proposition 4.15, we are able to prove the following version of a log-Sobolev *type inequality: for $T \in [0,1]$ we have*

$$\mathbb{E}f(X_T)^2 \ln f(X_T)^2 - \big(\mathbb{E}f(X_T)^2\big)\ln\big(\mathbb{E}f(X_T)^2\big) \le C_{p,T}\big(\mathbb{E}\big|\nabla f(X_T)\big|^{2p}\big)^{1/p}.$$

For all $p > 1$. Here $C_{p,T}$ is a universal constant independent of f.

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