Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2014, Vol. 50, No. 2, 327–351
DOI: 10.1214/12-AIHP529

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

The spread of a catalytic branching random walk

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Received 31 January 2012; revised 27 September 2012; accepted 6 October 2012

Abstract. We consider a catalytic branching random walk on $\mathbb{Z}$ that branches at the origin only. In the supercritical regime we establish a law of large number for the maximal position M_n: For some constant α, $\frac{M_n}{n} \to \alpha$ almost surely on the set of infinite number of visits of the origin. Then we determine all possible limiting laws for $M_n - \alpha n$ as n goes to infinity.

Résumé. Nous considérons une marche aléatoire branchant catalytique sur $\mathbb{Z}$ qui ne branche qu'à l'origine. Dans le cas surcritique, nous établissons une loi des grands nombres pour la position maximale M_n : Il existe une constante α explicite telle que $\frac{M_n}{n} \to \alpha$ presque sûrement sur l'ensemble des trajectoires pour lesquelles l'origine est visitée une infinité de fois.

Ensuite, nous déterminons toutes les lois limites possibles, lorsque $n \to +\infty$, pour la suite $M_n - \alpha n$.

MSC: 60K37

Keywords: Branching processes; Catalytic branching random walk

1. Introduction

A catalytic branching random walk (CBRW) on $\mathbb{Z}$ branching at the origin only is the following particle system:

When a particle location x is not the origin, the particle evolves as an irreducible random walk $(S_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}$ starting from x.

When a particle reaches the origin, say at time t, then a time $t+1$ it dies and gives birth to new particles positioned according to a point process $\mathcal{D}_0$. Each particle (at the origin at time t) produces new particles independently of every particle living in the system up to time t. These new particles evolve as independent copies of $(S_n)_{n\in\mathbb{N}}$ starting from their birth positions.

The system starts with an initial ancestor particle located at the origin. Denote by $\mathbb{P}$ the law of the whole system ($\mathbb{P}$ also governs the law of the underlying random walk S), and by $\mathbb{P}_x$ if the initial particle is located at x (then $\mathbb{P} = \mathbb{P}_0$).

Let $\{X_u, |u| = n\}$ denote the positions of the particles alive at time n (here $|u| = n$ means that the generation of the particle u in the *Ulam–Harris* tree is n). We assume that

$$\mathcal{D}_0 = \left\{X_u, |u| = 1\right\} \stackrel{d}{=} \left\{S_1^{(i)}, 1 \le i \le N\right\}$$

where N is an integer random variable describing the offspring of a branching particle, with finite mean $m = \mathbb{E}[N]$, and $(S_n^{(i)}, n \ge 0)_{i\ge 1}$ are independent copies of $(S_n, n \ge 0)$, and independent of N.

[1]Supported by Grant ANR-2010-BLAN-0108.
[2]Supported by ANR 2010 BLAN 0125.

Let τ be the first return time to the origin

$$\tau := \inf\{n \geq 1\colon\ S_n = 0\} \quad \text{with } \inf \varnothing = +\infty.$$

The escape probability is $q_{\text{esc}} := \mathbb{P}(\tau = +\infty) \in [0, 1)$ ($q_{\text{esc}} < 1$ because S is irreducible). Assume that we are in the *supercritical regime*, that is

$$m(1 - q_{\text{esc}}) > 1. \tag{1.1}$$

An explanation of assumption (1.1) is given in Section 7, Lemma 7.3.

Since the function defined on $(0, \infty)$ by $r \to \rho^{(r)} = m\mathbb{E}[\mathrm{e}^{-r\tau}]$ is of class C^∞, strictly decreasing, $\lim_{r\to 0} \rho^{(r)} = m\mathbb{P}(\tau < +\infty) = m(1 - q_{\text{esc}}) > 1$ and $\lim_{r\to+\infty} \rho^{(r)} = 0$, there exists a unique $r > 0$, a *Malthusian parameter* such that

$$m\mathbb{E}\big[\mathrm{e}^{-r\tau}\big] = 1. \tag{1.2}$$

Let ψ be the logarithmic moment generating function of S_1:

$$\psi(t) := \log \mathbb{E}\big[\mathrm{e}^{tS_1}\big] \in (-\infty, +\infty], \quad t \in \mathbb{R}.$$

Let $\zeta := \sup\{t > 0\colon\ \psi(t) < \infty\}$. We assume furthermore that $\zeta > 0$ and there exists some $t_0 \in (0, \zeta)$ such that

$$\psi(t_0) = r. \tag{1.3}$$

Observe that by convexity $\psi'(t_0) > 0$.

Let $M_n := \sup_{|u|=n} X_u$ be the maximal position at time n of all living particles (with convention $\sup \varnothing := -\infty$). Since the system only branches at the origin 0, we define the set of infinite number of visits of the catalyst by

$$\mathcal{S} := \Big\{\omega\colon\ \limsup_{n\to\infty} \big\{u\colon\ |u| = n,\ X_u = 0\big\} \neq \varnothing\Big\}.$$

Remark that $\mathbb{P}(\mathrm{d}\omega)$-almost surely on $\mathcal{S}^c$, for all large $n \geq n_0(\omega)$, either the system dies out or the system behaves as a finite union of some random walks on $\mathbb{Z}$, starting respectively from $X_u(\omega)$ with $|u| = n_0$. In particular, the almost sure behavior of M_n is trivial on $\mathcal{S}^c$. It is then natural to consider M_n on the set $\mathcal{S}$. Our first result on M_n is

Theorem 1.1 (Law of large numbers). *Assume* (1.1) *and* (1.3). *On the set* $\mathcal{S}$, *we have the convergence*

$$\lim_{n\to+\infty} \frac{M_n}{n} = \alpha := \frac{\psi(t_0)}{t_0} \quad \textit{a.s.}$$

In Theorem 1.1, the underlying random walk S can be periodic. In order to refine this convergence to a fluctuation result by centering M_n, we shall need to assume the aperiodicity of S. However, we cannot expect a convergence in distribution for $M_n - \alpha n$ since M_n is integer-valued whereas αn in general is not.

For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the integer part of x and $\{x\} := x - \lfloor x \rfloor \in [0, 1)$ be the fractional part of x.

Theorem 1.2. *Assume* (1.1) *and* (1.3). *Assume furthermore that* $\mathbb{E}(N^2) < \infty$ *and that* S *is aperiodic. Then there exists a constant* $c_* > 0$ *and a random variable* Λ_∞ *such that for any fixed* $y \in \mathbb{R}$,

$$\mathbb{P}(M_n - \alpha n > y) = \mathbb{E}\big[1 - \mathrm{e}^{-c_* \mathrm{e}^{-t_0 y}(\mathrm{e}^{t_0\{\alpha n + y\}} + \mathrm{o}(1))\Lambda_\infty}\big], \tag{1.4}$$

where $\mathrm{o}(1)$ *denotes some deterministic term which goes to* 0 *as* $n \to \infty$. *The random variable* Λ_∞ *is nonnegative and satisfies that*

$$\{\Lambda_\infty > 0\} = \mathcal{S} \quad \textit{a.s.} \tag{1.5}$$

Consequently for any subsequence $n_j \to \infty$ such that $\{\alpha n_j\} \to s \in [0,1)$ for some $s \in [0,1)$, we have that

$$\lim_{j\to\infty} \mathbb{P}\big(M_{n_j} - \lfloor \alpha n_j \rfloor = y\big) = \mathbb{E}\big(\mathrm{e}^{-c_* \mathrm{e}^{-t_0(y-s)} \Lambda_\infty} - \mathrm{e}^{-c_* \mathrm{e}^{-t_0(y-1-s)} \Lambda_\infty}\big) \quad (\forall y \in \mathbb{Z}). \tag{1.6}$$

Let us make some remarks on Theorem 1.2:

Remark 1.

1. *The random variable Λ_∞ is the limit of the positive fundamental martingale of Section 4. The value of constant c_* is given in (6.14) at the beginning of Section 6.*
2. *The hypothesis $\mathbb{E}(N^2) < \infty$ might be weakened to $\mathbb{E}(N \log(N+1)) < \infty$, just as the classical $L \log L$-condition (see e.g. Biggins [8]) in the branching random walk.*
3. *We do need the aperiodicity of the underlying random walk S in the proof of Theorem 1.2. However, for the particular case of the nearest neighborhood random walk (the period equals 2), we can still get a modified version of Theorem 1.2, see Remark 5 of Section 6.1.*

Theorems 1.1 and 1.2 are new, even though a lot of attention has been given to CBRW in continuous time. In papers [3–5,10,27–30] very precise asymptotics are established for the moments of $\eta_t(x)$ the number of particles located at x at time t, in every regime (sub/super/critical). Elaborate limit theorems were obtained for the critical case by Vatutin, Topchii and Yarovaya in [27–30].

Concerning on the maximal/minimal position of a branching random walk (BRW) on $\mathbb{R}$, some important progress were made in recent years, in particular a convergence in law result was proved in Aïdékon [1] when the BRW is not lattice-valued. It is expected that such convergence dos not hold in general for lattice-valued BRW, for instance see Bramson [11] where he used a centering with the integer part of some (random) sequence. In the recent studies of BRW, the spine decomposition technique plays a very important role. It turns out that a similar spine decomposition exists for CBRW (and more generally for branching Markov chains), and we especially acknowledge the paper [16] that introduced us the techniques of multiple spines, see Section 3.

We end this introduction by comparing our results to their analogue for (noncatalytic) branching random walks (see e.g. [1,2,23,25]). We shall restrict ourselves to simple random walk on $\mathbb{Z}$, that is $\mathbb{P}(S_1 = \pm 1) = \frac{1}{2}$.

For supercritical BRW ($m > 1$), almost surely on the set of nonextinction $\lim_{n\to+\infty} \frac{M_n^{(brw)}}{n} = b$, where b is the unique solution of $\psi^*(b) = \log m$, with $\psi^*(b) := \sup_t (bt - \psi(t))$ the rate function for large deviations of the simple random walk and $\psi(t) = \log\cosh(t)$. For CBRW, we can do explicit computations: Since for $x \neq 0$, $\mathbb{E}_x[\mathrm{e}^{-r\tau}] = \mathrm{e}^{-t_0|x|}$ the Malthusian parameter satisfies $r + t_0 = \log(m)$. Combined with $\log\cosh(t_0) = r$ this implies $\mathrm{e}^{t_0} = \sqrt{2m-1}$ and $\alpha = \frac{2\log(m)}{\log(2m-1)} - 1$. Numerically, for $m = 1.83$ we find $b = 0.9$ and $\alpha = 0.24$. The second order results emphasize the difference between BRW and CBRW: for BRW, $M_n^{(brw)} - bn$ is of order $\mathrm{O}(\log n)$, whereas for CBRW, $M_n - \alpha n$ is of order $\mathrm{O}(1)$, see Remark 5.

The organization of the rest of this paper is as follows: We first give in Section 2 the heuristics explaining the differences between CBRW and ordinary BRW (branching random walk). Then we proceed (in Section 3) to recall many to one/few lemmas, we exhibit a fundamental martingale (in Section 4) and prove Theorems 1.1 and 1.2 in Sections 5 and 6 respectively, with the help of sharp asymptotics derived from renewal theory. Finally, Section 7 is devoted to an extension to the case of multiple catalysts. There the supercritical assumption (1.1) appears in a very natural way.

Finally, let us denote by C, C' or C'' some unimportant positive constants whose values can be changed from one paragraph to another.

2. Heuristics

Assume for sake of simplicity that we have a simple random walk. The existence of the fundamental martingale $\Lambda_n = \mathrm{e}^{-rn} \sum_{|u|=n} \phi(X_u)$, see Section 4, such that $\{\Lambda_\infty > 0\} = \mathcal{S}$, shows that on the set of nonextinction $\mathcal{S}$, we have roughly e^{rn} particles at time n.

If we apply the usual heuristic for branching random walk (see e.g. [25], Section II.1), then we say that we have approximately e^{rn} independent random walks positioned at time n, and therefore the expected population above level $an > 0$ is roughly:

$$\mathbb{E}\left[\sum_{i=1}^{\lfloor e^{rn}\rfloor} \mathbf{1}_{(S_n^{(i)} \ge an)}\right] = \lfloor e\rfloor^{rn}\rfloor \mathbb{P}(S_n \ge an) = e^{-n(\psi^*(a)-r)(1+o(1))}$$

where $\psi^*(a) = \sup_{t\ge 0}(ta - \psi(t))$ is the large deviation rate function (for simple random walk, $e^{\psi(t)} = \mathbb{E}[e^{tS_1}] = \mathrm{ch}(t)$).

This expected population is of order 1 when $\psi^*(a) = r$ and therefore we would expect to have $\frac{M_n}{n} \to \gamma$ on $\mathcal{S}$, where $\psi^*(\gamma) = r$.

However, for CBRW, this is not the right speed, since the positions of the independent particles cannot be assumed to be distributed as random walks. Instead, the $\lfloor e^{rn}\rfloor$ independent particles may be assumed to be distributed as a fixed probability distribution, say ν. If $\eta_n(x) = \sum_{|u|=n} \mathbf{1}_{(X_u = x)}$ is the number of particles at location x at time n, we may assume that for a constant $C > 0$, $e^{-rn}\mathbb{E}[\eta_n(x)] \to C\nu(x)$ and thus, ν inherits from η_n the relation:

$$\nu(x) = e^{-r}\sum_y c(y)p(y,x)\big(m\mathbf{1}_{(y=0)} + \mathbf{1}_{(y\ne 0)}\big)$$

with $p(x,y)$ the random walk kernel. For simple random walk, this implies that for $|x| \ge 2$ we have $\frac{1}{2}(\nu(x+1) + \nu(x-1)) = e^r\nu(x)$ and thus $\nu(x) = Ce^{-t_0|x|}$ for $|x| \ge 2$, with $\psi(t_0) = \log\cosh(t_0) = r$.

Therefore the expected population with distance to the origin at least an is roughly

$$\begin{aligned}\mathbb{E}\Big[\sum_{|x|\ge an} \eta_n(x)\Big] &= e^{rn}\sum_{|x|\ge an} e^{-rn}\mathbb{E}\big[\eta_n(x)\big] \\ &\sim e^{rn}C\sum_{|x|\ge an} e^{-t_0|x|} \sim C' e^{rn}e^{-t_0 an}.\end{aligned}$$

This expectation is of order 1 when $a = \frac{r}{t_0} = \frac{\psi(t_0)}{t_0} = \alpha$, and this yields the right asymptotics

$$\frac{M_n}{n} \to \alpha \quad \text{a.s. on } \mathcal{S}.$$

This heuristically gives the law of large numbers in Theorem 1.1.

3. Many to one/few formulas for multiple catalysts branching random walks (MCBRW)

For a detailed exposition of many to one/few formulas and the spine construction we suggest the papers of Biggins and Kyprianou [9], Hardy and Harris [20], Harris and Roberts [22] and the references therein. For an application to the computations of moments asymptotics in the continuous setting, we refer to Döring and Roberts [16]. We state the many to one/two formulas for a CBRW with multiple catalysts and will specify the formulas in the case with a single catalyst.

3.1. *Multiple catalysts branching random walks* (*MCBRW*)

The set of catalysts is a some subset $\mathcal{C}$ of $\mathbb{Z}$. When a particle reaches a catalyst $x \in \mathcal{C}$ it dies and gives birth to new particles according to the point process

$$\mathcal{D}_x \stackrel{d}{=} \big(S_1^{(i)}, 1 \le i \le N_x\big)$$

where $(S_n^{(i)}, n \in \mathbb{N})_{i\geq 1}$ are independent copies of an irreducible random walk $(S_n, n \in \mathbb{N})$ starting form x, independent of the random variable N_x which is assumed to be integrable. Each particle in $\mathcal{C}$ produces new particles independently from the other particles living in the system. Outside of $\mathcal{C}$ a particle performs a random walk distributed as S. The CBRW (branching only at 0) corresponds to $\mathcal{C} = \{0\}$.

3.2. *The many to one formula for MCBRW*

Some of the most interesting results about first and second moments of particle occupation numbers that we obtained come from the existence of a "natural" martingale. An easy way to transfer martingales from the random walk to the branching processes is to use a slightly extended many to one formula that enables conditioning. Let

$$m_1(x) := \mathbb{E}[N_x] < \infty, \quad x \in \mathbb{Z}. \tag{3.1}$$

On the space of trees with a spine (a distinguished line of descent) one can define a probability $\mathbb{Q}$ via martingale change of probability, that satisfies

$$\mathbb{E}\Big[Z \sum_{|u|=n} f(X_u)\Big] = \mathbb{Q}\Big[Z f(X_{\xi_n}) \prod_{0\leq k\leq n-1} m_1(X_{\xi_k})\Big], \tag{3.2}$$

for all $n \geq 1$, $f : \mathbb{Z} \to \mathbb{R}_+$ a nonnegative function and Z a positive $\mathcal{F}_n$ measurable random variable, and where $(\mathcal{F}_n, n \geq 0)$ denotes the natural filtration generated by the MCBRW (it does not contain information about the spine). On the right-hand-side of (3.2) (ξ_k) is the spine, and it happens that the distribution of $(X_{\xi_n})_{n\in\mathbb{N}}$ under $\mathbb{Q}$ is the distribution of the random walk $(S_n)_{n\in\mathbb{N}}$.

Specializing this formula to CBRW for which $m_1(x) = m\mathbf{1}_{(x=0)} + \mathbf{1}_{(x\neq 0)}$ yields

$$\mathbb{E}\Big[\sum_{|u|=n} f(X_u)\Big] = \mathbb{E}\big[f(S_n) m^{L_{n-1}}\big], \tag{3.3}$$

where $L_{n-1} = \sum_{k=0}^{n-1} \mathbf{1}_{(S_k=0)}$ is the local time at level 0.

3.3. *The many to two formula for MCBRW*

Recall (3.1). Let us assume that

$$m_2(x) := \mathbb{E}\big[N_x^2\big] < \infty, \quad x \in \mathbb{Z}. \tag{3.4}$$

Then for any $n \geq 1$ and $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}_+$, we have

$$\mathbb{E}\Big[\sum_{|u|=|v|=n} f(X_u, X_v)\Big] = \mathbb{Q}\Big[f\big(S_n^1, S_n^2\big) \prod_{0\leq k<T^{\mathrm{de}}\wedge n} m_2\big(S_k^1\big) \prod_{T^{\mathrm{de}}\wedge n\leq k<n} m_1\big(S_k^1\big) m_1\big(S_k^2\big)\Big], \tag{3.5}$$

where under $\mathbb{Q}$, S^1 and S^2 are coupled random walks that start from 0 and stay coupled (in particular at the same location) until the decoupling time T^{de} and after T^{de}, they behave as independent random walks.

More precisely, we have a three component Markov process $(S_n^1, S_n^2, I_n, n \geq 0)$ where $I_n \in \{0, 1\}$ is the indicator that is one iff the random walks are decoupled: when the two random walks are coupled at time n, and at site x, the they stay coupled at time $n + 1$ with probability $\frac{m_1(x)}{m_2(x)}$. That means that the transition probability are the following:

- $\mathbb{P}(S_{n+1}^1 = y, S_{n+1}^2 = y, I_{n+1} = 0 | S_n^1 = S_n^2 = x, I_n = 0) = \frac{m_1(x)}{m_2(x)} p(x, y)$,
- $\mathbb{P}(S_{n+1}^1 = y, S_{n+1}^2 = z, I_{n+1} = 1 | S_n^1 = S_n^2 = x, I_n = 0) = (1 - \frac{m_1(x)}{m_2(x)}) p(x, y) p(x, z)$,
- $\mathbb{P}(S_{n+1}^1 = y, S_{n+1}^2 = z, I_{n+1} = 1 | S_n^1 = x_1, S_n^2 = x_2, I_n = 1) = p(x_1, y) p(x_2, z)$.

The random walks are initially coupled and at the origin. The decoupling time $T^{\mathrm{de}} = \inf\{n \geq 1:\ I_n = 1\}$ satisfies for any $k \geq 0$,

$$\mathbb{Q}\left[T^{\mathrm{de}} \geq k+1 | \sigma\left\{S_j^1, S_j^2, I_j, j \leq k\right\}\right] = \prod_{0 \leq l \leq k-1} \frac{m_1(S_l^1)}{m_2(S_l^2)} \mathbf{1}_{(I_k=0)}, \tag{3.6}$$

where we keep the usual convention $\prod_{\varnothing} \equiv 1$.

This formula is proved in [20,22] by defining a new probability $\mathbb{Q}$ on the space of trees with two spines.

An alternative proof, that makes more natural the coupling of (S^1, S^2) is to condition on the generation of the common ancestor $w = u \wedge v$ of the two nodes, then use the branching to get independence, and plug in the many to one formula in each factor. We omit the details.

4. A fundamental martingale

Martingale arguments have been used for a long time in the study of branching processes. For example, for the Galton Watson process with mean progeny m and population Z_n at time n, the sequence $W_n = \frac{Z_n}{m^n}$ is a positive martingale converging to positive finite random variable W. The Kesten–Stigum theorem implies that if $\mathbb{E}[N \log(N+1)] < +\infty$, we have the identity a.s., $\{W > 0\}$ equals the survival set. A classical proof can be found in the reference book of Athreya and Ney [6], Section I.10. A more elaborate proof, involving size-biased branching processes, may be found in Lyons–Pemantle–Peres [24].

Similarly, the law of large numbers for the maximal position M_n of branching random walks system may be proved by analyzing a whole one parameter family of martingales (see Shi [26] for a detailed exposition on the equivalent form of Kesten–Stigum's theorem for BRW). Recently, the maximal position of a branching Brownian motion with inhomogeneous spatial branching has also been studied with the help a family of martingale indexed this time by a function space (see Berestycki, Brunet, Harris and Harris [7] or Harris and Harris [21]).

We want to stress out the fact that for catalytic branching random walk, since we branch at the origin only, we only have one natural martingale, which we call the fundamental martingale.

Let $T = \inf\{n \geq 0:\ S_n = 0\}$ be the first hitting time of 0, recall that $\tau = \inf\{n \geq 1:\ S_n = 0\}$ and let

$$\phi(x) := \mathbb{E}_x\left[\mathrm{e}^{-rT}\right] \quad (x \in \mathbb{Z}), \tag{4.1}$$

where r is given in (1.2). Finally let $p(x, y) = \mathbb{P}_x(S_1 = y)$ and $Pf(x) = \sum_y p(x, y) f(y)$ be the kernel and semigroup of the random walk S.

Proposition 4.1. *Under* (1.1) *and* (1.3).

(1) *The function ϕ satisfies*

$$P\phi(x) = \mathrm{e}^r \phi(x)\left(\frac{1}{m}\mathbf{1}_{(x=0)} + \mathbf{1}_{(x \neq 0)}\right).$$

(2) *The process*

$$\Delta_n := \mathrm{e}^{-rn}\phi(S_n) m^{L_{n-1}}$$

is a martingale, where $L_{n-1} = \sum_{0 \leq k \leq n-1} \mathbf{1}_{(S_k=0)}$ is the local time at level 0.

(3) *The process*

$$\Lambda_n := \mathrm{e}^{-rn} \sum_{|u|=n} \phi(X_u)$$

is a martingale called the fundamental martingale.

(4) *If $\mathbb{E}[N^2] < +\infty$, then the process Λ_n is bounded in L^2, and therefore is a uniformly integrable martingale.*

Proof. (1) If $x \neq 0$, then $T \geq 1$, therefore, by conditioning on the first step:

$$\phi(x) = \sum_y p(x, y)\mathrm{e}^{-r}\mathbb{E}_y\big[\mathrm{e}^{-rT}\big] = \mathrm{e}^{-r} P\phi(x).$$

On the other hand, $\tau \geq 1$ so conditioning by the first step again,

$$\phi(0) = 1 = m\mathbb{E}\big[\mathrm{e}^{-r\tau}\big] = m\sum_y p(0, y)\mathrm{e}^{-r}\mathbb{E}_y\big[\mathrm{e}^{-rT}\big] = m\mathrm{e}^{-r} P\phi(0).$$

(2) Denote by $\mathcal{F}_n^S := \sigma\{S_1, \dots, S_n\}$ for $n \geq 1$. We have,

$$\begin{aligned}\mathbb{E}\big[\Delta_{n+1}|\mathcal{F}_n^S\big] &= \mathrm{e}^{-r(n+1)}m^{L_n}\mathbb{E}\big[\phi(S_{n+1})|\mathcal{F}_n^S\big] = \mathrm{e}^{-r(n+1)}m^{L_n}P\phi(S_n)\\ &= \mathrm{e}^{-r(n+1)}m^{L_n}\mathrm{e}^r\phi(S_n)\left(\frac{1}{m}\mathbf{1}_{(S_n=0)} + \mathbf{1}_{(S_n\neq 0)}\right) = \Delta_n.\end{aligned}$$

(3) Recall that $(\mathcal{F}_n)_{n\geq 0}$ denotes the natural filtration of the CBRW. By the many to one formula, if Z is $\mathcal{F}_{n-1}$ measurable positive, then

$$\begin{aligned}\mathbb{E}[\Lambda_n Z] &= \mathrm{e}^{-rn}\mathbb{E}\Big[\sum_{|u|=n}\phi(X_u)Z\Big]\\ &= \mathrm{e}^{-rn}\mathbb{E}\big[Z\phi(S_n)m^{L_{n-1}}\big] = \mathbb{E}[Z\Delta_n]\\ &= \mathbb{E}[Z\Delta_{n-1}] \quad \text{(the martingale property of } \Delta_n)\\ &= \mathbb{E}[\Lambda_{n-1}Z].\end{aligned}$$

(4) The proof is given in Section 7 in the case of multiple catalysts and uses heavily the many to two formula. □

Let us introduce $\eta_n(x)$ the number of particles located at x at time n:

$$\eta_n(x) := \sum_{|u|=n}\mathbf{1}_{(X_u=x)}.$$

Corollary 4.2. *Under* (1.1) *and* (1.3).

(1) *We have* $\sup_{x,n}\mathrm{e}^{-rn}\phi(x)\eta_n(x) < +\infty$ *a.s.*
(2) *If N has finite variance then there exists a constant* $0 < C < \infty$ *such that*

$$\mathbb{E}\big[\eta_n(x)\eta_m(y)\big] \leq \frac{C}{\phi(x)\phi(y)}\mathrm{e}^{r(n+m)} \quad (n, m \in \mathbb{N}, x, y \in \mathbb{Z}^d).$$

Proof. (1) Let us write $\Lambda_n = \mathrm{e}^{-rn}\sum_x \phi(x)\eta_n(x)$. Since it is a positive martingale it converges almost surely to a finite integrable positive random variable Λ_∞. Therefore $\Lambda_\infty^* := \sup \Lambda_n < +\infty$ a.s. and

$$\sup_{x,n}\mathrm{e}^{-rn}\phi(x)\eta_n(x) \leq \Lambda_\infty^*.$$

(2) Assume for example that $n \leq m$ and let $C = \sup_n \mathbb{E}[\Lambda_n^2] < +\infty$. We have, since Λ_n is a martingale,

$$\begin{aligned}\mathrm{e}^{-r(n+m)}\phi(x)\phi(y)\mathbb{E}\big[\eta_n(x)\eta_m(y)\big] &\leq \mathbb{E}[\Lambda_n\Lambda_m]\\ &= \mathbb{E}\big[\Lambda_n\mathbb{E}[\Lambda_m|\mathcal{F}_n]\big] = \mathbb{E}\big[\Lambda_n^2\big] \leq C.\end{aligned}$$

□

For the proof of the following result instead of using large deviations for L_n, we use renewal theory, in the spirit of [12,17]. Let d be the period of the return times to 0:

$$d := \gcd\{n \geq 1:\ \mathbb{P}(\tau = n) > 0\}. \tag{4.2}$$

Proposition 4.3. *Assume* (1.1) *and* (1.3). *For every* $x \in \mathbb{Z}$ *there exists a constant* $c_x \in (0,\infty)$ *and a unique* $l_x \in \{0,1,\dots,d-1\}$ *such that*

$$\lim_{n\to+\infty} \mathrm{e}^{-r(dn+l_x)}\mathbb{E}\big[\eta_{nd+l_x}(x)\big] = c_x.$$

Moreover, for any $l \not\equiv l_x (\operatorname{mod} d)$, $\eta_{nd+l}(x) = 0$ *for all* $n \geq 0$. *In particular, for* $x = 0$, $l_x = 0$ *and* $c_0 = \frac{d}{m}$.

Proof. By the many to one formula (3.2),

$$\begin{aligned} v_n(x) := \mathbb{E}\big[\eta_n(x)\big] &= \mathbb{E}\Bigg[\sum_{|u|=n} \mathbf{1}_{(X_u=x)}\Bigg] \\ &= \mathbb{Q}\big(\mathbf{1}_{(S_n=x)}\mathrm{e}^{A_0(\xi_n)}\big) \\ &= \mathbb{E}\big[\mathbf{1}_{(S_n=x)} m^{L_{n-1}}\big]. \end{aligned}$$

We decompose this expectation with respect to the value of $\tau = \inf\{n \geq 1:\ S_n = 0\}$:

$$v_n(x) = m\mathbb{E}[\mathbf{1}_{(S_n=x)}\mathbf{1}_{(\tau\geq n)}] + \sum_{1\leq k\leq n-1} \mathbb{E}\big[\mathbf{1}_{(S_n=x)} m^{L_{n-1}} \mathbf{1}_{(\tau=k)}\big].$$

By the Markov property, if $u_k := \mathbb{P}(\tau = k)$, then

$$v_n(x) = m\mathbb{P}(\tau \geq n, S_n = x) + \sum_{1\leq k\leq n-1} m u_k v_{n-k}(x) = m\mathbb{P}(\tau \geq n, S_n = x) + m v.(x) * u(n),$$

Recall that the Malthusian parameter r is defined by

$$1 = m\mathbb{E}\big[\mathrm{e}^{-r\tau}\big] = m\sum_{k\geq 1} \mathrm{e}^{-rk} u_k.$$

Hence if we let $\tilde{v}_n(x) = \mathrm{e}^{-rn} v_n(x)$ and $\tilde{u}_k = m\mathrm{e}^{-rk} u_k$ then,

$$\tilde{v}_n(x) = m\mathrm{e}^{-rn}\mathbb{P}(\tau \geq n, S_n = x) + \tilde{v}.(x) * \tilde{u}(n).$$

By the periodicity, we have $u_n = 0$ if n is not a multiple of d and for $x \in \mathbb{Z}^d$ there is a unique $l_x \in \{0,1,\dots,d-1\}$ such that $v_n(x) = 0$ if $n \not\equiv l_x (\operatorname{mod} d)$. Therefore the sequence $t_n = \tilde{v}_{nd+l}(x)$ satisfies the following renewal equation

$$t_n = y_n + t * s_n$$

with $s_n = \tilde{u}_{nd}$ and $y_n = \mathrm{e}^{-r(nd+l_x)}\mathbb{P}(\tau \geq dn + l_x, S_{dn+l_x} = x)$. Since the sequence s is aperiodic, the discrete renewal theorem (see Feller [18], Section XIII.10, Theorem 1) implies that

$$t_n \to \frac{\sum_{n=1}^{\infty} y_n}{\sum_{n=1}^{\infty} n s_n} =: c_x.$$

Remark that $\sum_{n=1}^{\infty} n s_n = \sum_{n=1}^{\infty} n\mathrm{e}^{-rnd} m u_{nd} = \frac{1}{d}$. We have

$$c_x = d\sum_{n=1}^{\infty} \mathrm{e}^{-r(nd+l_x)}\mathbb{P}(\tau \geq dn + l_x, S_{dn+l_x} = x) > 0.$$

This is exactly the desired result.

Finally for $x = 0$, $\ell_x = 0$ and $c_0 = d\sum_{n=1}^{\infty} \mathrm{e}^{-rnd}\mathbb{P}(\tau \ge dn, S_{dn} = 0) = d\sum_{n=1}^{\infty} \mathrm{e}^{-rnd}\mathbb{P}(\tau = dn) = d\mathbb{E}(\mathrm{e}^{-r\tau}) = \frac{d}{m}$ by the choice of r. This completes the proof of Proposition 4.3. □

Remark 2. *The family $(c_x)_{x\in\mathbb{Z}}$ satisfies a system of linear equations, dual to the one (see Proposition 4.1) satisfied by the function ϕ: Recalling that $p(x, y) = \mathbb{P}_x(S_1 = y)$ is the kernel of the random walk, we have the recurrence relation*

$$\mathbb{E}\big[\eta_{n+1}(x)\big] = \sum_y \mathbb{E}\big[\eta_n(y)\big]p(y, x)(m\mathbf{1}_{(y=0)} + \mathbf{1}_{(y\neq 0)}).$$

Assuming for simplicity $d = 1$ and multiplying by $\mathrm{e}^{-r(n+1)}$ and letting $n \to +\infty$, we obtain the following functional equation for the function $x \to c_x$:

$$c_x = \mathrm{e}^{-r}\sum_y c_y p(y, x)(m\mathbf{1}_{(y=0)} + \mathbf{1}_{(y\neq 0)}), \quad x \in \mathbb{Z}.$$

We end this section by the following lemma which yields the part (1.5) in Theorem 1.2.

Lemma 4.4. *Assume (1.1) and (1.3). Assume furthermore that N has finite variance. Then we have*

$$\{\varLambda_\infty > 0\} = \mathcal{S} \quad a.s.$$

Remark that in this Lemma we do not need the aperiodicity of the underlying random walk S.

Proof of Lemma 4.4. We first prove that $\mathcal{S}^c \subset \{\varLambda_\infty = 0\}$ a.s. In fact, on $\mathcal{S}^c$, either the system dies out then $\varLambda_n = 0$ for all large n, or for all large $n \ge n_0(\omega)$: $\eta_n(0) = 0$. Then, if $\eta_n = \sum_x \eta_n(x)$ is the total population, $\eta_n = \eta_{n_0}$ for all $n \ge n_0$ since the system only branches at 0. Since $\varLambda_n = \mathrm{e}^{-rn}\sum \phi(X_u) \le \mathrm{e}^{-rn}\eta_n = \mathrm{e}^{-rn}\eta_{n_0}$, we still get $\varLambda_\infty = 0$.

Let $s = \mathbb{P}(\varLambda_\infty = 0)$ and $\hat{s} := \mathbb{P}(\mathcal{S}^c)$. If we can prove $s = \hat{s}$, then the lemma follows. We shall condition on the number of children of the initial ancestor N. For $k \ge j \ge 0$, let $\varUpsilon_{k,j}$ be the event that amongst k particles of the first generation there are exactly j particles which will return to 0. Then

$$\begin{aligned} s = \mathbb{P}(\varLambda_\infty = 0) &= \sum_{k=0}^{\infty} \mathbb{P}(N = k)\sum_{j=0}^{k} \mathbb{P}\big(\varUpsilon_{k,j} \cap \{\varLambda_\infty = 0\} | N = k\big) \\ &= \sum_{k=0}^{\infty} \mathbb{P}(N = k)\sum_{j=0}^{k} \binom{k}{j} q_{\mathrm{esc}}^{k-j}(1 - q_{\mathrm{esc}})^j s^j \\ &= \sum_{k=0}^{\infty} \mathbb{P}(N = k)\big(q_{\mathrm{esc}} + s(1 - q_{\mathrm{esc}})\big)^k = f\big(q_{\mathrm{esc}} + s(1 - q_{\mathrm{esc}})\big), \end{aligned}$$

with $f(x) = \mathbb{E}[x^N]$ the generating function of the reproduction law. Exactly in the same way, we show that $\hat{s}$ satisfies the same equation as s.

It remains to check the equation $x = f(q_{\mathrm{esc}} + x(1 - q_{\mathrm{esc}}))$ has a unique solution in $[0, 1)$ ($\hat{s} \le s$ and $s < 1$ thanks to Proposition 4.1). To this end, we consider the function $g(x) := f(q_{\mathrm{esc}} + x(1 - q_{\mathrm{esc}})) - x$. The function g is strictly convex on $[0, 1]$, $g(0) = f(q_{\mathrm{esc}}) > 0$, $g(1) = 0$ and $g'(1) = m(1 - q_{\mathrm{esc}}) - 1 > 0$. Thus g has a unique zero on $[0, 1)$, proving the lemma. □

5. The law of large numbers: Proof of Theorem 1.1

5.1. *Proof of the upper bound*

Let $\theta > 0$, $x > 0$. By the many to one formula,

$$\begin{aligned}\mathbb{P}(M_n > xn) &= \mathbb{P}\biggl(\sum_{|u|=n} \mathbf{1}_{(X_u > xn)} \neq 0\biggr)\\ &\leq \mathbb{E}\biggl[\sum_{|u|=n} \mathbf{1}_{(X_u > xn)}\biggr]\\ &= \mathbb{E}\bigl[\mathbf{1}_{(S_n > nx)} m^{L_{n-1}}\bigr]\\ &\leq \mathbb{E}\bigl[\mathrm{e}^{\theta(S_n - xn)} m^{L_{n-1}}\bigr] = \mathrm{e}^{-\theta n x} h_n, \quad \text{with } h_n = \mathbb{E}\bigl[\mathrm{e}^{\theta S_n} m^{L_{n-1}}\bigr].\end{aligned}$$

As in Proposition 4.3, we are going to use the renewal theory to study the asymptotics of v_n. Let us condition on $\tau = \inf\{n \geq 1\colon\ S_n = 0\}$:

$$\begin{aligned}h_n &= \mathbb{E}\bigl[\mathrm{e}^{\theta S_n} m^{L_{n-1}} \mathbf{1}_{(\tau \geq n)}\bigr] + \sum_{1 \leq k \leq n-1} \mathbb{E}\bigl[\mathrm{e}^{\theta S_n} m^{L_{n-1}} \mathbf{1}_{(\tau = k)}\bigr]\\ &= \mathbb{E}\bigl[\mathrm{e}^{\theta S_n} \mathbf{1}_{(\tau \geq n)}\bigr] + \sum_{1 \leq k \leq n-1} m\mathbb{P}(\tau = k) h_{n-k}\\ &= z_n + mh * u(n),\end{aligned}$$

with $z_n := \mathbb{E}[\mathrm{e}^{\theta S_n} \mathbf{1}_{(\tau \geq n)}]$ and $u_n := \mathbb{P}(\tau = n)$.

Assume now that $\theta > t_0$ so that $\psi(\theta) > \psi(t_0) = r$. We let

$$\tilde{h}_n := \mathrm{e}^{-n\psi(\theta)} h_n, \qquad \tilde{z}_n := \mathrm{e}^{-n\psi(\theta)} z_n, \qquad \tilde{u}_n := m\mathrm{e}^{-n\psi(\theta)} u_n.$$

On the one hand, by definition of the Malthusian parameter we have $1 = m\mathbb{E}[\mathrm{e}^{-r\tau}] = \sum m_n \mathrm{e}^{-rn} u_n$ so that $\sum_k \tilde{u}_k < 1$.

On the other hand,

$$\tilde{z}_n = \mathbb{E}\bigl[\mathrm{e}^{\theta S_n - n\psi(\theta)} \mathbf{1}_{(\tau \geq n)}\bigr] = \mathbb{P}_\theta(\tau \geq n)$$

with $\mathbb{P}_\theta$ defined by the martingale change of probability

$$\frac{\mathrm{d}\mathbb{P}_\theta}{\mathrm{d}\mathbb{P}} = \mathrm{e}^{\theta S_n - n\psi(\theta)} \quad (\text{on } \mathcal{F}_n).$$

Since under $\mathbb{P}_\theta$, $(S_n)_{n\geq 0}$ is a random walk with mean $\mathbb{E}_\theta[S_1] = \psi'(\theta) \geq \psi'(t_0) > 0$, we have

$$\tilde{z}_n \to \tilde{z}_\infty := \mathbb{P}_\theta(\tau = +\infty).$$

If we make the aperiodicity assumption $d = 1$, then by the discrete renewal theorem, we have

$$\tilde{h}_n \to \frac{\tilde{y}_\infty}{1 - \sum_k \tilde{u}_k}.$$

In the general case, we can prove exactly as in the proof of Proposition 4.3 that for every $l \in \{0, \ldots, d-1\}$ there exists a finite constant K_l such that

$$\lim_{n \to +\infty} \tilde{h}_{nd+l} \to K_l.$$

Therefore in any case, the sequence $\tilde{h}_n$ is bounded, and if $x > \frac{\psi(\theta)}{\theta}$

$$\mathbb{P}(M_n > xn) \le \mathrm{e}^{-n(\theta x - \psi(\theta))}\tilde{h}_n$$

satisfies $\sum_n \mathbb{P}(M_n > xn) < +\infty$. Hence, by Borel Cantelli's lemma

$$\limsup_{n\to+\infty} \frac{M_n}{n} \le x \quad \text{a.s.}$$

Hence, letting first $x \downarrow \frac{\psi(\theta)}{\theta}$ and then $\theta \downarrow t_0$ we obtain that

$$\limsup_{n\to+\infty} \frac{M_n}{n} \le \frac{\psi(t_0)}{t_0} = \alpha \quad \text{a.s.}$$

5.2. *Proof of the lower bound*, *under the hypothesis* $\mathbb{E}(N^2) < \infty$

The strategy of proof is as follows: Let $0 < s < 1$, $a > 0$ and consider the event $\mathcal{A}_{n,a,s}$ (with c' a positive constant): "the particles survive forever, there are at least $\frac{1}{2}c'\mathrm{e}^{rsn}$ particle alive at time sn, and one of these particle stays strictly positive until time n and reaches a position larger that $(1-s)an$ at time n."

We shall prove that for a suitable constant c', we can choose a, s such that on the set $\mathcal{S}$ of infinite number of visits to 0, for large n we are in $\mathcal{A}_{n,a,s}$. This implies that almost surely on $\mathcal{S}$, $\liminf \frac{M_n}{n} \ge a(1-s)$. Optimizing over the set of admissible couples (a, s) will yield the desired lower bound: $\liminf \frac{M_n}{n} \ge \alpha$ a.s. on $\mathcal{S}$.

Recall from Proposition 4.3 and Corollary 4.2 that

$$\lim_{n\to+\infty} \mathrm{e}^{-rdn}\mathbb{E}\big[\eta_{dn}(0)\big] = c_0, \qquad \sup_n \mathrm{e}^{-2rdn}\mathbb{E}\big[\eta_{dn}(0)^2\big] < +\infty.$$

Therefore Paley–Zygmund's inequality entails that

$$\mathbb{P}\big(\eta_{dn}(0) \ge c'\mathrm{e}^{rdn}\big) \ge c', \tag{5.1}$$

for some constant $c' > 0$. The following lemma aims at describing the a.s. behavior of $\eta_n(0)$:

Lemma 5.1. *Under* (1.1) *and* (1.3). *Almost surely on* $\mathcal{S}$,

$$\eta_{dn}(0) \ge \frac{c'}{2}\mathrm{e}^{rdn},$$

for all large n.

Proof. We shall write the proof for the aperiodic case $d = 1$. The generalization to a period $d \ge 2$ is straightforward by considering dn instead of n throughout the proof of this Lemma.

Let $\eta_n = \sum_x \eta_n(x)$ be the total population at time n. Since $0 \le \phi(x) \le 1$ we have $\Lambda_n = \mathrm{e}^{-rn}\sum_x \phi(x)\eta_n(x) \le \mathrm{e}^{-rn}\eta_n$. Furthermore, a particle living at time n has to have an ancestor at location 0 at some time $k \le n$, and if N_i is the number of children of this ancestor, then

$$\eta_n \le \sum_{1\le i\le \Gamma_n} N_i \quad \text{with } \Gamma_n = \sum_{0\le k\le n} \eta_k(0)$$

where the $(N_i)_{i\ge1}$ are independent random variables distributed as N and independent of Γ_n. Since $\mathbb{E}[N] < +\infty$, by Borel Cantelli's Lemma, there exists $i_0 = i_0(\omega)$ such that

$$N_i \le i^2 \quad \text{for } i \ge i_0.$$

Hence, almost surely for n large enough,

$$\eta_n \le \sum_{1\le i\le i_0} N_i + \Gamma_n^2 \le \sum_{1\le i\le i_0} N_i + n^2\Big(\sup_{0\le k\le n} \eta_k(0)\Big)^2.$$

By Lemma 4.4, almost surely on the survival set $\mathcal{S}$, we have $\Lambda_\infty > 0$ and thus, for n large enough $\eta_n \ge \frac{1}{2}\Lambda_\infty \mathrm{e}^{rn}$ and therefore for n large enough, on $\mathcal{S}$,

$$\sup_{0\le k\le n} \eta_k(0) > \mathrm{e}^{rn/4}.$$

Considering the stopping time (for the branching system endowed with the natural filtration)

$$T := \inf\{n:\ \eta_n(0) > \mathrm{e}^{rn/4}\}.$$

We have established that on $\mathcal{S}$, $T < \infty$ a.s. It follows from the branching property and (5.1) that

$$\mathbb{P}\big(\eta_{n+T}(0) \le c'\mathrm{e}^{rn}, \mathcal{S}\big) \le \mathbb{P}\big(\eta_n(0) \le c'\mathrm{e}^{rn}\big)^{\mathrm{e}^{rn/4}} \le \big(1-c'\big)^{\mathrm{e}^{rn/4}},$$

whose sum on n converges. By Borel–Cantelli's lemma, on $\mathcal{S}$, a.s. for all large n,

$$\eta_n(0) \ge c'\mathrm{e}^{r(n-T)} \ge \frac{c'}{2}\mathrm{e}^{rn}.$$

This proves the lemma. □

Proof of the lower bound of M_n. Let $0 < s < 1$. Define $k = k(n) := d\lfloor \frac{sn}{d} \rfloor$. By the preceding lemma, on the survival set $\mathcal{S}$, at time k, there are at least $\lfloor \frac{c'}{2}\mathrm{e}^{rk}\rfloor$ particles at 0, which move independently. Letting these particles move as the random walk S staying positive up to time $n-k$, then M_n is bigger than $\lfloor \frac{c'}{2}\mathrm{e}^{rk}\rfloor$ i.i.d. copies of S_{n-k} with $S_1 > 0, \dots, S_{n-k} > 0$. By a large deviations estimate (Theorem 5.2.1 of Dembo and Zeitouni [15], see the forthcoming Remark 3), for any fixed $a \in (0,\infty)$,

$$\mathbb{P}\big(S_{n-k} > a(1-s)n, S_1 > 0, \dots, S_{n-k} > 0\big) = \mathrm{e}^{-(1-s)n\psi^*(a)+\mathrm{o}(n)},$$

where we denote as before,

$$\psi^*(a) = \sup_{\theta>0}\big(a\theta - \psi(\theta)\big).$$

It follows that

$$\begin{aligned}
&\mathbb{P}\bigg(M_n \le (1-s)an, \eta_k(0) \ge \frac{c'}{2}\mathrm{e}^{rk}\bigg)\\
&\quad\le \big(1-\mathbb{P}\big(S_{n-k} > a(1-s)n, S_1 > 0, \dots, S_{n-k} > 0\big)\big)^{\lfloor c'/2\mathrm{e}^{rk}\rfloor}\\
&\quad= \exp\big(-\mathrm{e}^{rsn-\psi^*(a)(1-s)n+\mathrm{o}(n)}\big).
\end{aligned}$$

Choose $(a,s) \in (0,+\infty)\times(0,1)$ such that

$$rs > \psi^*(a)(1-s),$$

we apply Borel–Cantelli's lemma and get that a.s. for all large n, either $M_n > (1-s)an$ or $\eta_k(0) < \frac{c'}{2}\mathrm{e}^{rk}$. Hence on the set $\mathcal{S}$, by Lemma 5.1, a.s.,

$$\liminf_{n\to\infty} \frac{M_n}{n} \geq \gamma := \sup\bigl\{(1-s)a\colon\ (a,s) \in (0,\infty)\times(0,1), rs > \psi^*(a)(1-s)\bigr\}. \tag{5.2}$$

Recalling $r = \psi(t_0)$, then

$$\gamma = \sup_{a>0} \frac{a\psi(t_0)}{\psi^*(a)+\psi(t_0)}.$$

Let us study the derivative of $a \to \frac{a\psi(t_0)}{\psi^*(a)+\psi(t_0)}$. Recall that $\psi^*(a) = a\theta(a) - \psi(\theta(a))$ with $a = \psi'(\theta(a))$, and $(\psi^*)'(a) = \theta(a)$. Since the derivative of $a \to \frac{a\psi(t_0)}{I(a)+\psi(t_0)}$ has the same sign as $\psi^*(a) + \psi(t_0) - a(\psi^*)'(a) = \psi(t_0) - \psi(\theta(a))$, it is negative if $a > \psi'(t_0)$ (i.e. $\theta(a) > t_0$), positive if $a < \psi'(t_0)$ and vanishes at $\psi'(t_0)$. Therefore

$$\gamma = \frac{\psi(t_0)}{t_0} = \alpha,$$

which in view of (5.2) yields the lower bound of Theorem 1.1 under the hypothesis that $\mathbb{E}(N^2) < \infty$. □

Remark 3. *Mogulskii's theorem* (*Theorem* 5.2.1 *of Dembo and Zeitouni* [15]) *implies that*

$$\mathbb{P}(S_j > aj, S_1 > 0, \ldots, S_j > 0) = \mathrm{e}^{-jK(a)+\mathrm{o}(j)},$$

with

$$K(a) = \inf\left\{\int_0^1 \psi^*\bigl(\dot f(t)\bigr)\,\mathrm{d}t,\ f \in \mathcal{A}\right\},$$
$$\mathcal{A} = \bigl\{\phi \textit{ absolutely continuous},\ f(0) = 0,\ f(1) = a,\ f(s) > 0\ \forall s \in (0,1)\bigr\}.$$

Let us check that $K(a) = \psi^*(a)$. *In fact, since the function* $f(t) = at$ *is in* $\mathcal{A}$, *we have* $K(a) \leq \int_0^1 \psi^*(a)\,\mathrm{d}t = \psi^*(a)$. *On the other hand, the function* ψ^* *is convex, therefore, by Jensen's inequality, if* $\phi \in \mathcal{A}$,

$$\int_0^1 \psi^*\bigl(\dot f(t)\bigr)\,\mathrm{d}t \geq \psi^*\left(\int_0^1 \dot f(t)\,\mathrm{d}t\right) = \psi^*\bigl(f(1)-f(0)\bigr) = \psi^*(a).$$

We can thus conclude that $K(a) = \psi^*(a)$.

5.3. *Proof of the lower bound, without the hypothesis* $\mathbb{E}(N^2) < \infty$

The proof relies on a coupling for the general N with mean m: Let $N^{(L)} := \min(N, L)$ with a sufficiently large integer L such that $m_L := \mathbb{E}(N^{(L)})$ satisfies $m_L(1-q_{\mathrm{esc}}) > 1$ (this is possible since $m_L \to m$). Consider a new CBRW $(X_u^{(L)}, |u| \geq 0)$ with $N^{(L)}$ as the number of offsprings and the same random walk (S_n) as the displacements, i.e. on each step of branching at 0 we keep at most L-children and their displacements in the original CBRW. The associated maximum at generation n is denoted by $M_n^{(L)}$. Then by construction

$$M_n \geq M_n^{(L)}, \quad \text{a.s.}$$

By the lower bound for $M_n^{(L)}$ established in Section 5.2, if we denote by

$$\mathcal{S}_L := \left\{\omega\colon \limsup_{n\to\infty}\bigl\{u\colon\ |u| = n, X_u^{(L)} = 0\bigr\} \neq \varnothing\right\},$$

then a.s. on $\mathcal{S}_L$,

$$\liminf_{n\to\infty} \frac{M_n^{(L)}}{n} \ge \alpha_L,$$

with $\alpha_L = \frac{\psi(t_0(L))}{t_0(L)}$, and where $t_0(L)$ is defined in the same way as t_0 in (1.3) and (1.2) by replacing m by m_L. We remark that by continuity such solution $t_0(L)$ exists for all sufficiently large L, say $L \ge L_0$. Moreover $\alpha_L \to \alpha$ as $L \to \infty$, and $\mathcal{S}_L \subset \mathcal{S}_{L+1}$ for any $L \ge 1$. Then on the set $\widetilde{\mathcal{S}} := \bigcup_{L\ge 1} \mathcal{S}_L$, a.s. $\liminf_{n\to\infty} \frac{M_n^{(L)}}{n} \ge \alpha$. This will yield the lower bound in Theorem 1.1 once we have checked the equality:

$$\mathcal{S} = \widetilde{\mathcal{S}}, \quad \text{a.s.} \tag{5.3}$$

Let us check (5.3) in the same way as in the proof of Lemma 4.4. Plainly $\widetilde{\mathcal{S}} \subset \mathcal{S}$. To prove the reverse inclusion, we remark at first that by Lemma 4.4, $\mathcal{S}_L$ equals a.s. the nonzero set of the corresponding limit of the fundamental martingale (which is bounded in L^2), hence $\mathcal{S}_L \ne \varnothing$ for all large L. Consequently $\widetilde{\mathcal{S}} \ne \varnothing$.

Let $t := \mathbb{P}(\mathcal{S}^c)$ and $\tilde{t} := \mathbb{P}(\widetilde{\mathcal{S}}^c)$. Then $t \le \tilde{t} < 1$. As in the proof of Lemma 4.4, by conditioning on the number of offsprings N, we obtain that

$$\tilde{t} = \sum_{k=0}^{\infty} \mathbb{P}(N=k) \sum_{j=0}^{k} C_k^j q_{\mathrm{esc}}^{k-j} (1-q_{\mathrm{esc}})^j (\tilde{t})^j = f\big(q_{\mathrm{esc}} + \tilde{t}(1-q_{\mathrm{esc}})\big),$$

with $f(x) = \mathbb{E}(x^N)$. The constant t satisfies the same equation as $\tilde{t}$ and we have already proved in the proof of Lemma 4.4 the uniqueness of solutions in $[0, 1)$. Hence $t = \tilde{t}$ and (5.3) follows. This completes the proof of the lower bound in Theorem 1.1. $\square$

6. Refining the convergence: Proof of Theorem 1.2

The key of the proof of Theorem 1.2 is the following double limit of Proposition 6.1. Then we shall prove its uniform version (uniformly on the starting point of the system) in Proposition 6.2, from which Theorem 1.2 follows easily (see Section 6.3).

Proposition 6.1. *Under the assumptions in Theorem 1.2, there exists a positive constant $c_* > 0$ such that*

$$\limsup_{z\to\infty} \limsup_{n\to\infty} \left| e^{t_0 z} e^{-t_0\{\alpha n + z\}} \mathbb{P}(M_n > \alpha n + z) - c_* \right| = 0,$$

where as before $\alpha := \frac{\psi(t_0)}{t_0}$ and $\{\alpha n + z\} \in [0, 1)$ denotes the fractional part of $\alpha n + z$.

The value of c_* is given in (6.14) by $c_* = \frac{e^{-t_0}}{(1-e^{-t_0})\widetilde{\mathbb{E}}(H_1)}$ and $\widetilde{\mathbb{E}}(H_1)$ is given in equation (6.9). We also mention that we can not replace $M_n > \alpha n + z$ by $M_n \ge \alpha n + z$ in the above Proposition, since M_n is integer-valued.

The proof of Proposition 6.1 is divided into the upper and lower bounds, proved respectively in Sections 6.1 and 6.2.

6.1. *Upper bound in Proposition 6.1*

Recall that $\alpha := \frac{\psi(t_0)}{t_0}$ is the velocity of M_n. We prove the following upper bound: for all $z \in \mathbb{R}$,

$$\limsup_{n\to\infty} e^{-t_0\{\alpha n + z\}} \mathbb{P}(M_n > \alpha n + z) \le c_* e^{-t_0 z}.$$

Let us start from $\mathbb{P}(M_n > \alpha n + z) = \mathbb{P}(\exists |u| = n\colon\ X_u > \alpha n + z)$. For any $n \geq 1$ and any $|u| = n$, denote by $u_0 = \varnothing < u_1 < \cdots < u_n = u$ the shortest path relating $\varnothing$ to u such that $|u_k| = k$ for any $k \leq n$. For $|u| = n$ with $X_u > \alpha n + z > 0$ (as n is large), there exists some $k < n$ such that $X_{u_k} = 0$ and $X_{u_j} > 0$ for all $k < j \leq n$. Therefore

$$\{M_n > \alpha n + z\} = \bigcup_{0 \leq k \leq n-1} B_k \tag{6.1}$$

with

$$B_k := \bigcup_{|v|=k} A_v(k,n),$$

and

$$A_v(k,n) := \bigl\{\exists |u| = n\colon\ v = u_k, X_v = 0, X_{u_j} > 0, \forall k < j \leq n, X_{u_{n-k}} > \alpha n + z\bigr\}.$$

Denote as before by $\eta_n(x)$ the number of particles at x at time n. Then, conditioning on $\mathcal{F}_k$, B_k is an union of $\eta_k(0)$ i.i.d. events, and each event holds with probability

$$p(k,n) := \mathbb{P}\bigl(\exists |u| = n-k, X_{u_1} > 0, \ldots, X_{u_{n-k}} > 0, X_u > \alpha n + z\bigr).$$

It is easy to compute $p(k,n)$: by conditioning on the number of offspring $N = l$, $p(k,n)$ is the probability that among these l particles in the first generation there exists at least one particle which remains positive up to generation $n-k$ and lives in $(\alpha n + z, \infty)$ at $(n-k)$th generation. It follows that

$$p(k,n) = \sum_{l=0}^{\infty} \mathbb{P}(N = l)\bigl(1 - \bigl(1 - q(k,n)\bigr)^l\bigr) = 1 - f\bigl(1 - q(k,n)\bigr), \tag{6.2}$$

where $f(x) := \mathbb{E}(x^N)$ is the generating function of N and $q(k,n)$ is defined as follows:

$$q(k,n) := \mathbb{P}(S_1 > 0, \ldots, S_{n-k} > 0, S_{n-k} > \alpha n + z).$$

Let $\varepsilon > 0$ be small. By Proposition 4.3 (with $d = 1$), $\lim_{n\to\infty} \mathrm{e}^{-rn}\mathbb{E}[\eta_n(0)] = c_0 = \frac{1}{m}$. It follows that for any $n > k \geq k_0 \equiv k_0(\varepsilon)$,

$$\mathbb{P}(B_k) \leq \mathbb{E}\bigl(\eta_k(0)p(k,n)\bigr) \leq (c_0 + \varepsilon)\mathrm{e}^{rk}p(k,n).$$

Hence for any $n > k_0$,

$$\mathbb{P}(M_n > \alpha n + z) \leq \sum_{k=0}^{n-1}\mathbb{P}(B_k) \leq (c_0 + \varepsilon)\sum_{k=k_0}^{n-1}\mathrm{e}^{rk}p(k,n) + C_{k_0}\sum_{k=1}^{k_0-1}p(k,n), \tag{6.3}$$

where $C_{k_0} := \max_{1\leq k\leq k_0}\mathbb{E}(\eta_k(0))$. Recalling $f'(1) = m$ and (6.2), we deduce from the convexity of f that for all $k < n$,

$$f'\bigl(1 - q(k,n)\bigr)q(k,n) \leq p(k,n) \leq mq(k,n). \tag{6.4}$$

It is easy to see that the sum $\sum_{k=1}^{k_0-1}$ in (6.3) is negligible as $n \to \infty$. In fact, for any $1 \leq k \leq k_0$, $q(k,n) \leq \mathbb{P}(S_{n-k-1} > \alpha n + z)$. But $\mathbb{E}(S_1) = \psi'(0) < \alpha = \frac{\psi(t_0)}{t_0}$ by the (strict) convexity of ψ. Then $p(k,n) \leq mq(k,n) \to 0$ as $n \to \infty$ (exponentially fast by the large deviation principle).

To estimate the probability $q(k,n)$ for $k_0 \leq k < n$, we introduce a new probability

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\sigma\{S_0,\ldots,S_n\}} = \mathrm{e}^{t_0 S_n - n\psi(t_0)}.$$

Under $\widetilde{\mathbb{P}}$, S_1 has the mean $\psi'(t_0) > 0$. Therefore for $1 \le k \le n$ and for all $z \ge 0$,

$$\begin{aligned} q(k,n) &= \mathbb{P}(S_1 > 0, \dots, S_{n-k} > 0, S_{n-k} > \alpha n + z) \\ &= \widetilde{\mathbb{E}}\big(\mathrm{e}^{-t_0 S_{n-k} + (n-k)\psi(t_0)} 1_{(S_j > 0, \forall j \le n-k, S_{n-k} > \alpha n + z)}\big) \\ &= \mathrm{e}^{-rk} \widetilde{\mathbb{P}}\big(\mathbf{e}(\mathbf{t_0}) \ge S_{n-k} - \alpha n, S_j > 0, \forall j \le n-k, S_{n-k} > \alpha n + z\big), \end{aligned}$$

where $\mathbf{e}(\mathbf{t_0})$ denotes an independent exponential random variable with parameter t_0 and we also used the fact that $\alpha = \frac{\psi(t_0)}{t_0}$ and $r = \psi(t_0)$. Plainly in the event of the above probability term, $\mathbf{e}(\mathbf{t_0})$ must be bigger than z. Thanks to the loss of memory property of $\mathbf{e}(\mathbf{t_0})$, we get that for $1 \le k \le n$ and for all $z \ge 0$,

$$\mathrm{e}^{rk} q(k,n) = \mathrm{e}^{-t_0 z} \widetilde{\mathbb{P}}\big(S_j > 0, \forall j \le n-k, \alpha n + z < S_{n-k} \le \alpha n + z + \mathbf{e}(\mathbf{t_0})\big). \tag{6.5}$$

Summing (6.5) over $0 \le k \le n-1$ and letting $i = n-k$, we obtain that

$$\begin{aligned} &\sum_{k=0}^{n-1} \mathrm{e}^{rk} q(k,n) \\ &\quad = \mathrm{e}^{-t_0 z} \sum_{i=1}^{n} \widetilde{\mathbb{P}}\big(S_j > 0, \forall j \le i, \alpha n + z < S_i \le \alpha n + z + \mathbf{e}(\mathbf{t_0})\big) \\ &\quad = \mathrm{e}^{-t_0 z} \big(\widetilde{\mathbb{E}}\big(U\big(\alpha n + z, \alpha n + z + \mathbf{e}(\mathbf{t_0})\big]\big) - s_n\big), \end{aligned} \tag{6.6}$$

where for any $x < y$,

$$U(y) := \sum_{k=1}^{\infty} \widetilde{\mathbb{P}}(S_j > 0, \forall 1 \le j \le k, S_k \le y), \qquad U(x,y] := U(y) - U(x), \tag{6.7}$$

and

$$s_n := \sum_{k=n}^{\infty} \widetilde{\mathbb{P}}\big(S_j > 0, \forall j \le k, \alpha n + z < S_k \le \alpha n + z + \mathbf{e}(\mathbf{t_0})\big). \tag{6.8}$$

Under $\widetilde{\mathbb{P}}$, S_j is a random walk with positive mean. Define by $T_0 := 0$, $T_j := \inf\{i > T_{j-1}\colon S_i > S_{T_{j-1}}\}$ and $H_j := S_{T_j}$ for $j \ge 1$. Then $0 < T_1 < \cdots < T_j < \cdots$ and $0 < H_1 < \cdots < H_j < \cdots$ are the strict ladder epochs and ladder heights of the random walk S (under $\widetilde{\mathbb{P}}$). The duality lemma says that for any $y > 0$,

$$U(y) = \sum_{l=1}^{\infty} \widetilde{\mathbb{P}}(H_l \le y).$$

Since $\widetilde{\mathbb{E}}[S_1^2] < +\infty$, $\widetilde{\mathbb{E}}(H_1) < \infty$ and we have the Wald identity (see [19] Feller Volume II, Chapter XVIII, Theorem 1)

$$\widetilde{\mathbb{E}}(H_1) = \widetilde{\mathbb{E}}(S_1)\widetilde{\mathbb{E}}(T_1). \tag{6.9}$$

We are going to apply the renewal theorem (see [19] Feller, p. 360) to U and prove that there exists some constant $c_H > 0$ such that

$$\lim_{x \to \infty} \mathrm{e}^{-t_0\{x\}} \widetilde{\mathbb{E}}\big(U\big(x, x + \mathbf{e}(\mathbf{t_0})\big]\big) = c_H. \tag{6.10}$$

To check (6.10), we remark that the span of H_1 equals 1 (because S is aperiodic). By the renewal theorem, for any $j \geq 1$, $U(x, x+j] \to \frac{j}{\widetilde{\mathbb{E}}(H_1)}$ as $x \to \infty$. Moreover there exists some constant $C > 0$ such that for all $y > x \geq 0$, $U(x, y] \leq C(1 + y - x)$. Let $x > 0$. Observe that almost surely,

$$U\big(x, x + \mathbf{e}(\mathbf{t_0})\big] = U\big(\lfloor x \rfloor, x + \mathbf{e}(\mathbf{t_0})\big] = \sum_{j=1}^{\infty} 1_{(j < \{x\} + \mathbf{e}(\mathbf{t_0}) < j+1)} U\big(\lfloor x \rfloor, \lfloor x \rfloor + j\big].$$

Taking expectation gives that

$$\widetilde{\mathbb{E}}\big(U\big(x, x + \mathbf{e}(\mathbf{t_0})\big]\big) = \sum_{j=1}^{\infty} \mathrm{e}^{-t_0(j - \{x\})}\big(1 - \mathrm{e}^{-t_0}\big) U\big(\lfloor x \rfloor, \lfloor x \rfloor + j\big],$$

which proves (6.10) after an application of the dominated convergence theorem, with

$$c_H := \sum_{j=1}^{\infty} \mathrm{e}^{-t_0 j}\big(1 - \mathrm{e}^{-t_0}\big) \frac{j}{\widetilde{\mathbb{E}}(H_1)} = \frac{\mathrm{e}^{-2t_0}}{(1 - \mathrm{e}^{-t_0})\widetilde{\mathbb{E}}(H_1)}. \tag{6.11}$$

Now we prove that $s_n \to 0$, where s_n is defined in (6.8). Remark that $\widetilde{E}(S_1) = \psi'(t_0) > \alpha := \frac{\psi(t_0)}{t_0}$ by convexity. Pick up some small positive constant $\delta < (\psi'(t_0) - \alpha)/2$. There exists some sufficiently small constant $b \in (0, t_0)$ such that $\widetilde{\mathbb{E}}\mathrm{e}^{-bS_1} \leq \mathrm{e}^{-b(\psi'(t_0) - \delta)}$. Then by Chebychev's inequality, for any $t > 0$ and $k \geq n$, $\widetilde{\mathbb{P}}(S_k \leq z + \alpha n + t) \leq \mathrm{e}^{bz + bt}\mathrm{e}^{b\alpha n}\widetilde{\mathbb{E}}\mathrm{e}^{-bS_k} \leq \mathrm{e}^{bz+bt}\mathrm{e}^{-\delta bk}$. It follows that

$$s_n \leq \sum_{k=n}^{\infty} \mathrm{e}^{bz}\mathrm{e}^{-\delta bk}\widetilde{\mathbb{E}}\big(\mathrm{e}^{b\mathbf{e}(\mathbf{t_0})}\big) = \frac{t_0}{(1 - \mathrm{e}^{-\delta b})(t_0 - b)}\mathrm{e}^{bz}\mathrm{e}^{-\delta bn}.$$

In particular $s_n \to 0$. This together with (6.10), (6.6) yields that for any $z \geq 0$,

$$\lim_{n\to\infty} \mathrm{e}^{-t_0\{\alpha n + z\}} \sum_{k=0}^{n-1} \mathrm{e}^{rk} q(k, n) = \mathrm{e}^{-t_0 z} c_H. \tag{6.12}$$

Now by using the lower bound of (6.4) and (6.5), for any $k < n$ and $z \geq 0$, $p(k, n) \geq f'(1 - \mathrm{e}^{-rk})q(k, n)$ because $q(k, n) \leq \mathrm{e}^{-r(k+1)}\mathrm{e}^{-t_0 z} \leq \mathrm{e}^{-rk}$. Then for any small $\delta > 0$, there exists some $k_0(\delta)$ such that $f'(1 - \mathrm{e}^{-rk}) \geq m(1 - \delta)$ for all $k \geq k_0$ (recalling $f'(1) = m$). It follows that for any $k_0 \leq k < n$ and $z \geq 0$, $p(k, n) \geq (1 - \delta)mq(k, n)$. On the other hand, $p(k, n) \leq mq(k, n)$ for any $k < n$, and $\lim_{n\to\infty} \sum_{k=0}^{k_0} q(k, n) = 0$. This in view of (6.12) implies that for any $z \geq 0$,

$$\lim_{n\to\infty} \mathrm{e}^{-t_0\{\alpha n + z\}} \sum_{k=0}^{n-1} \mathrm{e}^{rk} p(k, n) = m\mathrm{e}^{-t_0 z} c_H. \tag{6.13}$$

Applying the above limit to (6.3) gives that for any $z \geq 0$,

$$\limsup_{n\to\infty} \mathrm{e}^{-t_0\{\alpha n + z\}} \mathbb{P}(M_n > \alpha n + z) \leq (c_0 + \varepsilon) m c_H \mathrm{e}^{-t_0 z},$$

which implies the upper bound in Proposition 6.1 by letting $\varepsilon \to 0$ and

$$c_* := c_0 m c_H = \frac{c_0 m \mathrm{e}^{-2t_0}}{(1 - \mathrm{e}^{-t_0})\widetilde{\mathbb{E}}(H_1)} = \frac{\mathrm{e}^{-2t_0}}{(1 - \mathrm{e}^{-t_0})\widetilde{\mathbb{E}}(H_1)}, \tag{6.14}$$

since $c_0 = 1/m$ (the period $d = 1$) as stated in Proposition 4.1. □

Remark 4. *Let us mention an uniform estimate*: *for some constant* $C > 0$,

$$\mathbb{P}(M_n > \alpha n + z) \le C\mathrm{e}^{-t_0 z}, \quad \forall z \in \mathbb{R}, n \ge 1. \tag{6.15}$$

In fact, *there exists some constant* $C' > 0$ *such that* $\mathbb{E}(\eta_k(0)) \le C'\mathrm{e}^{rk}$ *for any* $k \ge 1$, *hence by the first inequality in* (6.3), $\mathbb{P}(M_n > \alpha n + z) \le C' \sum_{k=0}^{n-1} \mathrm{e}^{rk} p(k,n) \le C'm \sum_{k=0}^{n-1} \mathrm{e}^{rk} q(k,n)$. *Using* (6.6) *and the fact that* $\exists C'' > 0$: $U(x,y] \le C''(1+y-x)$ *for all* $x < y$, *we immediately get* (6.15).

Remark 5. *If the underlying random walk* S *is of period* $d \ge 2$, *then in* (6.1), $B_k = \varnothing$ *if* k *is not multiple of* d (*namely if* $d \not| k$). *Then instead of* $\sum_{k=0}^{n-1} \mathrm{e}^{rk} p(k,n)$, *we have to deal with* $\sum_{k=0,d|k}^{n-1} \mathrm{e}^{rk} p(k,n)$, *which in turn leads to the study of* $\sum_{k=0,d|k}^{n} \mathrm{e}^{rk} q(k,n)$. *An equality similar to* (6.6) *holds with* U *replaced by*

$$U^{(d,\ell)}(y) := \sum_{k=0}^{\infty} \widetilde{\mathbb{P}}(S_j > 0, \forall 1 \le j \le kd + \ell, S_{kd+\ell} \le y),$$

where $\ell \in \{0, \dots, d-1\}$ *comes from the rest of division of* n *by* d [ℓ *being fixed and we let* $n \to \infty$ *with* $n - 1 \equiv \ell (\mathrm{mod}\, d)$]. *Technically we are not able to prove any renewal theorem for* $U^{(d,\ell)}(y)$ *for a general random walk* S.

In the particular case when S *is a nearest neighbor random walk on* $\mathbb{Z}$, *we can use parity to handle* $U^{(d,\ell)}(y)$. *Considering for instance* $\ell = 0$ ($d = 2$). *Thanks to parity, we have that for any* $k \ge 1$ *and* $y > 0$,

$$\widetilde{\mathbb{P}}(S_j > 0, \forall 1 \le j \le 2k, S_{2k} \le y) = \widetilde{\mathbb{P}}(S_{2j} > 0, \forall 1 \le j \le k, S_{2k} \le y),$$

which implies that $U^{(2,0)}(y)$ *is the renewal function for the random walk* $(S_{2n})_{n\ge 0}$ (*under* $\widetilde{\mathbb{P}}$). *Then we can apply the standard renewal theorem to* $U^{(2,0)}(y)$. *The term* $U^{(2,1)}(y)$ *can be dealt with in the same way. Then we get a result similar to Proposition* 6.1 *and the forthcoming Proposition* 6.2, *and finally a modified version of Theorem* 1.2 *for the nearest neighbor random walk. The details are omitted.*

6.2. *Lower bound in Proposition* 6.1

Let $\varepsilon > 0$ be small. Let $\lambda \equiv \lambda(\varepsilon)$ be a large constant whose value will be determined later on. Recall (6.1). Consider

$$E_n := \bigcup_{k=0}^{n-1} B'_k,$$

with $B'_k := B_k \cap \{\eta_k(0) \le \lambda \mathrm{e}^{rk}\} := B_k \cap F_k$. Then by Cauchy–Schwarz' inequality,

$$\mathbb{P}(M_n > \alpha n + z) \ge \mathbb{P}(E_n) \ge \frac{(\sum_{0\le k<n} \mathbb{P}(B'_k))^2}{\sum_{0\le k_1,k_2<n} \mathbb{P}(B'_{k_1} \cap B'_{k_2})}. \tag{6.16}$$

Conditioning on $\mathcal{F}_k$, B_k is an union of $\eta_k(0)$ i.i.d. events,

$$\mathbb{P}(B_k|\mathcal{F}_k) = 1 - \big(1 - p(k,n)\big)^{\eta_k(0)}.$$

Let $0 \le k < n$. By (6.4) and (6.5), $p(k,n) \le m\mathrm{e}^{-r(k+1)}\mathrm{e}^{-t_0 z}$. On F_k, $\eta_k(0) \le \lambda \mathrm{e}^{rk}$ hence $p(k,n)\eta_k(0) \le \mathrm{e}^{-r-t_0 z} m\lambda$. Therefore for all $z \ge z_0(\lambda, \varepsilon)$ and for all $k < n$,

$$1 - \big(1 - p(k,n)\big)^{\eta_k(0)} \ge (1-\varepsilon) p(k,n)\eta_k(0)$$

hence

$$\mathbb{P}\big(B'_k|\mathcal{F}_k\big) \ge (1-\varepsilon) p(k,n)\eta_k(0) 1_{F_k}.$$

In particular,

$$\sum_{k=0}^{n-1}\mathbb{P}\big(B_k'\big)\ge(1-\varepsilon)\sum_{k=0}^{n-1}p(k,n)\mathbb{E}\big(\eta_k(0)1_{F_k}\big).$$

Since $\eta_k(0)\mathrm{e}^{-rk}$ is bounded in L^2, we deduce from Proposition 4.3 that we can choose (and then fix) some large λ and some $k_0\equiv k_0(\varepsilon)$ such that $\mathbb{E}(\eta_k(0)1_{F_k})\ge c_0(1-\varepsilon)\mathrm{e}^{rk}$ for all $k\ge k_0$. It follows that for all $n>k_0$,

$$\sum_{k=0}^{n-1}\mathbb{P}\big(B_k'\big)\ge c_0(1-\varepsilon)^2\sum_{k=k_0}^{n-1}\mathrm{e}^{rk}p(k,n).$$

Consequently, for all $z\ge z_0$ there exists some $n_0(z,\varepsilon)$ such that for all $n\ge n_0$,

$$\sum_{k=0}^{n-1}\mathbb{P}\big(B_k'\big)\ge c_0m(1-\varepsilon)^3\sum_{k=k_0}^{n-1}\mathrm{e}^{rk}q(k,n)\ge c_*(1-\varepsilon)^4\mathrm{e}^{-t_0z}\mathrm{e}^{t_0\{\alpha n+z\}},\tag{6.17}$$

by applying (6.12) [recalling $c_*=c_0mc_H$, $c_0=1/m$ and that for any fixed k $q(k,n)\to 0$ as $n\to\infty$]. The probability $\mathbb{P}(B_k)$ has already been estimated in the proof of upper bound of Proposition 6.1, see (6.3) and (6.13): for all $z\ge 0$ and $n\ge n_0(z,\varepsilon)$,

$$\sum_{k=1}^{n-1}\mathbb{P}\big(B_k'\big)\le\sum_{k=1}^{n-1}\mathbb{P}(B_k)\le c_*(1+\varepsilon)\mathrm{e}^{-t_0z}\mathrm{e}^{t_0\{\alpha n+z\}}.\tag{6.18}$$

Now we estimate the denominator in (6.16). Let $k_1<k_2$. On $B_{k_1}\cap B_{k_2}$, there are at least two different $v\ne v'$ at generation k_1 such that $A_v(k_1,n)$ holds and for v', there exists some descendant u (denoted by $u>v'$) at generation k_2 such that $A_u(k_2,n)$ holds. Then,

$$B_{k_1}\cap B_{k_2}\subset\bigcup_{v\ne v',|v|=|v'|=k_1}\big\{A_v(k_1,n)\cap\big\{\exists|u|=k_2,u>v'\colon\ A_u(k_2,n)\text{ holds}\big\}\big\}.$$

Since different particles branch independently, we get that

$$\mathbb{P}(B_{k_1}\cap B_{k_2}|\mathcal{F}_{k_1})\le\sum_{v\ne v',|v|=|v'|=k_1}p(k_1,n)\mathbb{E}\bigg(\sum_{|u|=k_2,u>v'}p(k_2,n)|\mathcal{F}_{k_1}\bigg).$$

Taking the expectations, we obtain that for $k_1<k_2$,

$$\mathbb{P}(B_{k_1}\cap B_{k_2})\le p(k_1,n)p(k_2,n)\mathbb{E}\big(\eta_{k_1}(0)\eta_{k_2}(0)\big)\le Cp(k_1,n)p(k_2,n)\mathrm{e}^{r(k_1+k_2)},$$

by Corollary 4.2. Therefore for all $z\ge z_0$ and $n>n_0(z,\varepsilon)$,

$$\begin{aligned}\sum_{0\le k_1,k_2<n}\mathbb{P}\big(B_{k_1}'\cap B_{k_2}'\big)&\le\sum_{k=0}^{n-1}\mathbb{P}\big(B_k'\big)+C\Bigg(\sum_{k=0}^{n-1}\mathrm{e}^{rk}p(k,n)\Bigg)^2\\&\le c_*(1+\varepsilon)\mathrm{e}^{-t_0z}\mathrm{e}^{t_0\{\alpha n+z\}}+C'\mathrm{e}^{-2t_0z},\end{aligned}$$

for some numerical constant $C'>0$. In view of (6.16), we have that for all $z\ge z_0$ and $n>n_0(z,\varepsilon)$,

$$\mathbb{P}(M_n>\alpha n+z)\ge\frac{c_*^2(1-\varepsilon)^8\mathrm{e}^{-t_0z}\mathrm{e}^{t_0\{\alpha n+z\}}}{c_*(1+\varepsilon)+C'\mathrm{e}^{-t_0z}}.$$

It follows that

$$\liminf_{z\to\infty}\liminf_{n\to\infty} \mathrm{e}^{t_0 z - t_0\{\alpha n+z\}}\mathbb{P}(M_n > \alpha n + z) \geq c_* \frac{(1-\varepsilon)^8}{1+\varepsilon}.$$

Letting $\varepsilon \to 0$, we obtain the lower bound in Proposition 6.1. The proof of Proposition 6.1 is complete. □

Recall that $\phi(x)$ is defined in (4.1) and $\phi(x) > 0$ thanks to the aperiodicity. Let us establish an uniform version of Proposition 6.1:

Proposition 6.2. *Under the assumptions in Theorem 1.2. Uniformly on $x \in \mathbb{Z}$,*

$$\limsup_{n\to\infty}\left|\frac{\mathrm{e}^{t_0 z}\mathrm{e}^{-t_0\{\alpha n+z\}}}{\phi(x)}\mathbb{P}_x(M_n > \alpha n + z) - c_*\right| \to 0,$$

as $z \to \infty$.

Proof. Assume $x \neq 0$ and let $S^* = \max_{0\leq i\leq \tau} S_i$, where τ is the first return time to 0. Then

$$\mathbb{P}_x(M_n > \alpha n + z) \leq \mathbb{P}_x\big(S^* > \alpha n + z\big) + \sum_{k=1}^{n} \mathbb{P}_x(\tau = k)\mathbb{P}(M_{n-k} > \alpha n + z).$$

Let $\varepsilon > 0$ be small (in particular $\varepsilon < c_*$). Let ℓ be some integer whose value will be fixed later on. By Proposition 6.1, there exists some $y_0(\varepsilon) > 0$ such that for all $y \geq y_0(\varepsilon)$, there exists some $j_0(y, \varepsilon)$ such that for all $j \geq j_0(y, \varepsilon)$,

$$\left|\mathrm{e}^{t_0 y}\mathrm{e}^{-t_0\{\alpha j+y\}}\mathbb{P}(M_j > \alpha j + y) - c_*\right| < \varepsilon. \tag{6.19}$$

Observe that for any $k < n$, $\mathbb{P}(M_{n-k} > \alpha n+z) = \mathbb{P}(M_{n-k} > \alpha(n-k)+z+\alpha k)$. We shall apply (6.19) to $y = \alpha k+z$ and $j = n-k$. Then for all $z \geq y_0(\varepsilon)$, there exists some $j_1(z, \ell)$ such that for all $1 \leq k \leq \ell$ and $n \geq j_1(z, \ell)$,

$$\left|\mathrm{e}^{t_0(z+\alpha k)}\mathrm{e}^{-t_0\{\alpha n+z\}}\mathbb{P}(M_{n-k} > \alpha n + z) - c_*\right| < \varepsilon. \tag{6.20}$$

We stress that $y_0(\varepsilon)$ does not depend on ℓ. Then for all $n > j_1(z, \ell)$,

$$\begin{aligned}\sum_{k=1}^{\ell} \mathbb{P}_x(\tau = k)\mathbb{P}(M_{n-k} > \alpha n + z) &\leq (c_* + \varepsilon)\mathrm{e}^{-t_0 z}\mathrm{e}^{t_0\{\alpha n+z\}}\sum_{k=1}^{\ell}\mathbb{P}_x(\tau = k)\mathrm{e}^{-\alpha t_0 k}\\ &\leq (c_* + \varepsilon)\mathrm{e}^{-t_0 z}\mathrm{e}^{t_0\{\alpha n+z\}}\phi(x),\end{aligned}$$

since $\alpha t_0 = \psi(t_0) = r$ and $\phi(x) = \mathbb{E}_x[\mathrm{e}^{-r\tau}]$. For $k > \ell$, we apply (6.15) and get that

$$\sum_{k=\ell}^{n} \mathbb{P}_x(\tau = k)\mathbb{P}(M_{n-k} > \alpha n + z) \leq \sum_{k=\ell}^{n} C\mathrm{e}^{-t_0(\alpha k+z)} = C\mathrm{e}^{-t_0 z}\frac{\mathrm{e}^{-r\ell}}{r}.$$

It follows that for any $z \geq y_0(\varepsilon)$ and any $x \in \mathbb{Z}$,

$$\limsup_{n\to\infty} \mathrm{e}^{t_0 z}\mathrm{e}^{-t_0\{\alpha n+z\}}\mathbb{P}_x(M_n > \alpha n + z) \leq (c_* + \varepsilon)\phi(x) + C\frac{\mathrm{e}^{1-r\ell}}{r}. \tag{6.21}$$

For the lower bound, we have from (6.20) that for any $z \geq y_0(\varepsilon)$ and all $n > j_1(z, \ell)$,

$$\begin{aligned}\sum_{k=1}^{\ell} \mathbb{P}_x(\tau = k)\mathbb{P}(M_{n-k} > \alpha n + z) &\geq (c_* - \varepsilon)\mathrm{e}^{-t_0 z}\mathrm{e}^{t_0\{\alpha n+z\}}\sum_{k=1}^{\ell}\mathbb{P}_x(\tau = k)\mathrm{e}^{-\alpha t_0 k}\\ &= (c_* - \varepsilon)\mathrm{e}^{-t_0 z}\mathrm{e}^{t_0\{\alpha n+z\}}\mathbb{E}_x\big[\mathrm{e}^{-r\tau}1_{(\tau\leq\ell)}\big].\end{aligned}$$

Hence for any $z \ge y_0(\varepsilon)$ and any $x \in \mathbb{Z}$,

$$\liminf_{n\to\infty} \mathrm{e}^{t_0 z}\mathrm{e}^{-t_0\{\alpha n+z\}}\mathbb{P}_x(M_n > \alpha n + z) \ge (c_* - \varepsilon)\mathbb{E}_x\big[\mathrm{e}^{-r\tau}1_{(\tau\le\ell)}\big].$$

Letting $\ell \to \infty$ in the above lim inf inequality and in (6.21) gives that for any $z \ge y_0(\varepsilon)$ and uniformly for all $x \in \mathbb{Z}$,

$$\limsup_{n\to\infty} \big|\mathrm{e}^{t_0 z}\mathrm{e}^{-t_0\{\alpha n+z\}}\mathbb{P}_x(M_n > \alpha n + z) - c_*\phi(x)\big| \le \varepsilon\phi(x), \tag{6.22}$$

proving Proposition 6.2 since ε can be arbitrarily small. □

6.3. *Proof of Theorem* 1.2

The part (1.5) of Theorem 1.2 was already proved in Lemma 4.4. We now prove (1.4).

Let $\varepsilon, \delta > 0$ be small. For any $k \ge 1$, there exists some integer $\ell_k = \ell_k(\varepsilon)$ such that

$$\mathbb{P}\Big(\max_{|u|=k} |X_u| \le \ell_k\Big) \ge 1 - \varepsilon.$$

Recalling the martingale Λ_n defined in Proposition 4.1. Since a.s. $\Lambda_n \to \Lambda_\infty$, there exists some $k_1 = k_1(\varepsilon, \delta)$ such that for any $k \ge k_1$,

$$\mathbb{P}\big((1-\delta)\Lambda_\infty \le \Lambda_k \le (1+\delta)\Lambda_\infty\big) \ge 1 - \varepsilon.$$

By (6.22), there exists some $z_0(\delta)$ such that for all $z \ge z_0(\delta)$ and for all $x \in \mathbb{Z}$, there exists some $n_0(z, x, \delta)$ such that for all $j \ge n_0(z, x, \delta)$,

$$\big|\mathrm{e}^{t_0 z}\mathrm{e}^{-t_0\{\alpha j+z\}}\mathbb{P}_x(M_j > \alpha j + z) - c_*\phi(x)\big| \le \delta\phi(x). \tag{6.23}$$

Elementarily there exists some $s_0(\delta) > 0$ such that $1 - s \ge \mathrm{e}^{-(1+\delta)s}$ for all $0 \le s < s_0(\delta)$. Let $k_2 = k_2(\delta, y)$ be some integer satisfying $(c_* + \delta)\mathrm{e}^{-t_0(\alpha k_2+y-1)} < s_0(\delta)$. Define $k := k_1 + k_2 + \lfloor \frac{z_0(\delta)}{\alpha} \rfloor + 1$. Let $n_1 := \max_{x\in\mathbb{Z},|x|\le\ell_k} n_0(z, x, \delta) + k$. Considering $n \ge n_1$. Conditioning on $\mathcal{F}_k$ and on the set $\{\max_{|u|=k} |X_u| \le \ell_k\}$, the particles in the kth generation move independently, hence for any $n > n_1$,

$$\mathbb{P}(M_n > \alpha n + y|\mathcal{F}_k) = 1 - \prod_{x\in\mathbb{Z},|x|\le Lk} \mathbb{P}_x(M_{n-k} \le \alpha n + y)^{\eta_k(x)}. \tag{6.24}$$

Applying (6.23) to $j = n - k$, $z = \alpha k + y$ yields that for any $|x| \le \ell_k$ (and $x \in \mathbb{Z}$),

$$(c_* - \delta)\phi(x)\mathrm{e}^{-t_0(\alpha k+y)+t_0\{\alpha n+y\}} \le \mathbb{P}_x(M_{n-k} > \alpha n + y) \le (c_* + \delta)\phi(x)\mathrm{e}^{-t_0(\alpha k+y)+t_0\{\alpha n+y\}}.$$

Since $1 - s \ge \mathrm{e}^{-(1+\delta)s}$ for all $0 \le s < s_0(\delta)$, we deduce from (6.24) that on the set $\{\max_{|u|=k} |X_u| \le \ell_k\}$,

$$\begin{aligned}
&\mathbb{P}(M_n > \alpha n + y|\mathcal{F}_k)\\
&\quad\le 1 - \exp\Big(-\sum_{x\in\mathbb{Z},|x|\le\ell_k} (c_*+\delta)(1+\delta)\phi(x)\eta_k(x)\mathrm{e}^{-t_0(\alpha k+y)}\mathrm{e}^{t_0\{\alpha n+y\}}\Big)\\
&\quad= 1 - \exp\big(-(c_*+\delta)(1+\delta)\Lambda_k\mathrm{e}^{-t_0 y}\mathrm{e}^{t_0\{\alpha n+y\}}\big).
\end{aligned} \tag{6.25}$$

Then by taking the expectation, we get

$$\begin{aligned}
&\mathbb{P}(M_n > \alpha n + y)\\
&\quad\le \mathbb{E}\big(1 - \exp\big(-(c_*+\delta)(1+\delta)\Lambda_k\mathrm{e}^{-t_0 y}\mathrm{e}^{t_0\{\alpha n+y\}}\big)\big) + \mathbb{P}\Big(\max_{|u|=k} |X_u| > \ell_k\Big)\\
&\quad\le \mathbb{E}\big(1 - \exp\big(-(c_*+\delta)(1+\delta)^2\Lambda_\infty\mathrm{e}^{-t_0 y}\mathrm{e}^{t_0\{\alpha n+y\}}\big)\big) + 2\varepsilon,
\end{aligned}$$

where the factor 2 in 2ε comes from Λ_k which is replaced by $(1+\delta)\Lambda_\infty$. Since ε and δ can be arbitrarily small, we get the upper bound in (1.4). The lower bound in (1.4) can be proved in the same way.

Finally, let $y \in \mathbb{Z}$. Observe that for any $n_j \ge 1$, $\mathbb{P}(M_{n_j} - \lfloor \alpha n_j \rfloor \ge y+1) = \mathbb{P}(M_{n_j} - \lfloor \alpha n_j \rfloor > y + \{\alpha n_j\}) = \mathbb{P}(M_{n_j} - \alpha n_j > y)$. We apply (1.4) to y and $y-1$, (1.6) follows immediately. This completes the proof of Theorem 1.2. □

7. Extension to multiple catalysts branching random walk (MCBRW)

Recall Section 3.1 for the definition of MCBRW. Let us assume that the set of catalysts $\mathcal{C}$ is a finite subset of $\mathbb{Z}$. By forgetting/erasing the time spent between the catalysts, we obtain an underlying Galton–Watson process which is multitype with the moment matrix

$$\begin{aligned} M_{xy} &:= \text{mean number of particles born at } x \text{ that reach site } y \\ &= m_1(x)\mathbb{P}_x(\tau = \tau_y, \tau < \infty) \quad (x, y \in \mathcal{C}), \end{aligned}$$

where $m_1(x) = \mathbb{E}[N_x]$ is the mean offspring at site x, $\tau_y := \inf\{n \ge 1\colon S_n = y\}$ is the first return time at y, and $\tau = \tau_{\mathcal{C}} = \inf_{y\in\mathcal{C}} \tau_y$ is the first return time to $\mathcal{C}$.

We assume to be in the supercritical regime, that is $\rho > 1$, where ρ is the maximal eigenvalue of matrix M, which by assumption is irreducible. We let $\rho^{(r)}$ be the maximum eigenvalue of the matrix

$$M_{xy}^{(r)} := m_1(x)\mathbb{E}_x\big[\mathrm{e}^{-r\tau}\mathbf{1}_{(\tau=\tau_y,\tau<\infty)}\big] \quad (x, y \in \mathcal{C}).$$

The function $r \to \rho^{(r)}$ is continuous, strictly decreasing, C^∞ on $(0, +\infty)$, $\rho^{(0)} = \rho > 1$ and $\lim_{r\to+\infty} \rho^{(r)} = 0$ since $M_{xy}^{(r)} \le m_1(x)\mathrm{e}^{-r}$. Therefore there exists a unique $r > 0$, a *Malthusian parameter*, such that $\rho^{(r)} = 1$. We shall fix this value of r in the sequel.

Let $v = v^{(r)}$ be a right eigenvector of $M^{(r)}$ associated to $\rho^{(r)} = 1$: For any $x \in \mathcal{C}$, $v(x) > 0$ and

$$v(x) = \sum_{a\in\mathcal{C}} m_1(x)\mathbb{E}_x\big[\mathrm{e}^{-r\tau}\mathbf{1}_{(\tau=\tau_a,\tau<\infty)}\big]v(a) \quad (x \in \mathcal{C}).$$

Let us denote by $p(x, y) = \mathbb{E}_x[S_1 = y]$ and $Pf(x) = \sum_y p(x, y)f(y)$ the random walk kernel and semigroup. Let us consider the hitting times

$$T_x := \inf\{n \ge 0\colon S_n = x\}, \qquad T_{\mathcal{C}} = \inf_{x\in\mathcal{C}} T_x = \inf\{n \ge 0\colon S_n \in \mathcal{C}\}.$$

Lemma 7.1. *The function*

$$\phi(x) := \sum_{a\in\mathcal{C}} v(a)\mathbb{E}_x\big[\mathrm{e}^{-rT_{\mathcal{C}}}\mathbf{1}_{(T_{\mathcal{C}}=T_a,T_{\mathcal{C}}<\infty)}\big]$$

is a solution of

$$P\phi(x) = \mathrm{e}^r\phi(x)\left(\frac{1}{m_1(x)}\mathbf{1}_{(x\in\mathcal{C})} + \mathbf{1}_{(x\notin\mathcal{C})}\right).$$

Proof. The proof is similar to that of Proposition 4.1 by using the Markov property of the random walk. The details are omitted. □

We are now ready to introduce the *fundamental martingale*.

Lemma 7.2.

(1) *For the CBRW process with multiple catalysts, the process*

$$\Lambda_n := \mathrm{e}^{-rn} \sum_{|u|=n} \phi(X_u)$$

is a martingale.

(2) *For the random walk, the process*

$$\Delta_n := \mathrm{e}^{-rn} \phi(S_n) \prod_{x \in \mathcal{C}} m_1(x)^{L_{n-1}^x}$$

is a martingale where $L_{n-1}^x = \sum_{0 \le k \le n-1} \mathbf{1}_{(S_k = x)}$ *is the local time at level* x *at time* $n-1$.

(3) *If* N *has finite variance, then the process* Λ_n *is bounded in* L^2 *and therefore a uniformly integrable martingale.*

Proof. Based on the many-to-one formula, the parts (1) and (2) can be proved in the same way as in Proposition 4.1. Let us only give the details of the proof of (3). To compute the second moment, we use the many to two formula (3.5) of Section 3

$$\begin{aligned} \mathbb{E}\big[\Lambda_n^2\big] &= \mathrm{e}^{-2rn} \mathbb{E}\Bigg[\sum_{|u|=|v|=n} \phi(X_u)\phi(X_v)\Bigg] \\ &= \mathrm{e}^{-2rn} \mathbb{Q}\Bigg[\phi\big(S_n^1\big)\phi\big(S_n^2\big) \prod_{0 \le k < T \wedge n} m_2\big(S_k^1\big) \prod_{T \wedge n \le k < n} m_1\big(S_k^1\big) m_1\big(S_k^2\big)\Bigg]. \end{aligned}$$

Recall (3.6). We have that

$$\begin{aligned} \mathbb{E}\big[\Lambda_n^2\big] &= \mathrm{e}^{-2rn} \mathbb{Q}\Bigg[\phi(S_n)^2 \prod_{x \in \mathcal{C}} m_1(x)^{L_{n-1}^x}\Bigg] \\ &\quad + \mathrm{e}^{-2rn} \sum_{1 \le k \le n-1} \mathbb{Q}\Bigg[\prod_{0 \le l \le k-2} \frac{m_1(S_l)}{m_2(S_l)} \bigg(1 - \frac{m_1(S_{k-1})}{m_2(S_{k-1})}\bigg) \mathbb{E}_{S_{k-1}}[\Delta_{n-(k-1)}]^2 \mathrm{e}^{2r(n-(k-1))}\Bigg]. \end{aligned}$$

Observe that since $0 \le \phi \le 1$ we have $0 \le \phi(x)^2 \le \phi(x)$, and combine it with $\mathbb{E}_x[\Delta_p] = \phi(x) m_1(x) \le C$ and $\frac{m_1(x)}{m_2(x)} \le 1$ to obtain the upper bound

$$\mathbb{E}\big[\Lambda_n^2\big] \le 1 + C^2 \sum_{1 \le k \le n-1} \mathrm{e}^{-2r(k-1)} \le C' < \infty,$$

which completes the proof of this Lemma. □

We are now able to give an explanation of the supercritical regime assumption of the introduction.

Lemma 7.3. *When there is only one catalyst at the origin, the supercritical regime is* $m(1 - q_{\mathrm{esc}}) > 1$.

Proof. Indeed, M is then a one dimensional matrix and $\rho = M_{00} = m\mathbb{P}(\tau < +\infty) = m(1 - q_{\mathrm{esc}})$. □

We end this section by stating the law of large numbers. Intuitively, if $\overline{c}$ is the rightmost catalyst, the maximal position at time n comes from particles born at location $\overline{c}$.

Proposition 7.4 (Law of large numbers). *Assume the supercritical regime and* (1.3). *Then, on the set of nonextinction* $\mathcal{S}$ *we have*

$$\lim_{n \to +\infty} \frac{M_n}{n} = \frac{r}{t_0}, \quad a.s.,$$

with r the Malthusian parameter defined by $\rho^{(r)} = 1$ and $t_0 > 0$ such that $\psi(t_0) = r$.

Proof. First observe that the heuristics do not change at all since by applying the optional stopping theorem to the martingale $e^{t_0 S_n - nr}$ to the time T, we obtain that for $x > \overline{c}$

$$e^{t_0 x} = e^{t_0 \overline{c}} \mathbb{E}_x\left[e^{-rT}\right],$$

and thus

$$\phi(x) = v(\overline{c}) \mathbb{E}_x\left[e^{-rT_c}\right] = v(\overline{c}) e^{t_0(x - \overline{c})},$$

and we approximate the expected number of particles above level an in the same way, and hence obtain the same guess for the asymptotics.

Furthermore, the proofs are *mutatis mutandis* the same as the one given in Section 5. The only difference would come from the use of renewal theorems: we get a system of renewal equations, e.g., for

$$\left(\mathbb{E}_a\left(e^{\theta S_n} \prod_{b \in \mathcal{C}} m_1(b)^{L_{n-1}^b}\right)\right)_{a,b \in \mathcal{C}}$$

as $n \to \infty$, which can be dealt with an application of a matrix version of renewal theorems (see [13,14]). We feel free to omit the details. □

Acknowledgments

We are very grateful to two anonymous referees for their careful readings and helpful comments on the first version of this paper.

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